The Momentum Kernel of Gauge and Gravity Theories

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based on the works

[0908.1923], [1004.1392], [0907.1425] and [1003.2403]

done in collaboration with

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I just put 1.795372 and 2.204628 together.
And what does that mean?
Four!
(Doctor Who)

It is crucial for experimental and theoretical reasons to have efficient methods for evaluating amplitudes of physical processes in quantum field theory

- multilegs and multiloop amplitudes for LHC physics
- Quantum gravity: perturbative ultraviolet nature of $\mathcal{N} = 8$ supergravity

Unfortunately the number of individual Feynman graphs rises dramatically with the number of external legs or loop order, and tensor reduction methods increase the number of terms even more.

A huge number of cancellation are needed to get the result leading to

- instabilities due to large numerical cancellations in matrix elements
- obfuscation of the fundamental structure of the interactions: gauge invariance, ultraviolet divergences, infrared singularities, hidden symmetries

Explicit amplitude computations display rather unexpectedly simple structures

- One-loop multi-photon in QED and multi-graviton amplitudes in $\mathcal{N} = 8$ supergravity amplitudes share the *same* no-triangle property [Bjerrum-Bohr, Vanhove],[Badger,Bjerrum-Bohr,Vanhove]
- Simpler than expected subleading color contribution at one-loop for Higgs + 2 jets process [Badger, Campbell, Ellis, Williams]
- Better UV behaviour of subleading color factor amplitudes at multi-loop order in $\mathcal{N} = 4$ SYM [Berkovits, Green, Russo, Vanhove]
- Need amplitudes to get UV behaviour of $\mathcal{N} = 8$ supergravity in $D = 4$

All these simplifications hints on simple structures than the diagrammatics from Feynman rules suggest

Part I

Tree-level amplitudes

Tree-level amplitudes in gauge theory

Tree-level gauge theory amplitudes are decomposed as in color ordered factors

$$\mathcal{A}_n^{\text{tree}} = g_{\text{YM}}^{n-2} \sum_{\sigma \in \mathfrak{S}_n/\mathbb{Z}_n} \text{Tr}(\lambda^{\sigma(1)} \cdots \lambda^{\sigma(n)})\, A_n(\sigma(1), \cdots, \sigma(n))$$

All the information are in the $n!$ color ordered partial amplitude

$$A_n^\sigma := A_n(\sigma(1), \cdots, \sigma(n)); \qquad \sigma \in \mathfrak{S}_n$$

The color ordered amplitudes are not independent and satisfy relations kinematic relations

- Cyclicity property

$$A_n(1,\ldots,n) = A_n(2,n,\ldots,1)$$

Reduces the number of independent amplitudes to $(n-1)!$

- Reflection property

$$A_n(1,\ldots,n) = (-1)^n A_n(n,\ldots,1)$$

Reduces further to $\frac{1}{2}(n-1)!$ independent amplitudes

- Photon decoupling identity

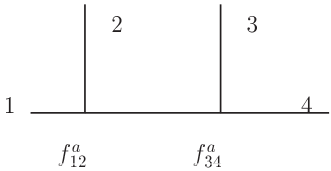
$$\sum_{\sigma\in\mathfrak{S}_{n-1}} A_n(1,\sigma(2),\ldots,\sigma(n)) = 0$$

- Changing basis from the fundamental to the adjoint we get the alternative expansion [Del duca, Frizzo, Maltoni]

$$\mathcal{A}_n^{\text{tree}} = g_{\text{YM}}^{n-2} \sum_{\sigma \in \mathfrak{S}_{n-2}} c_{1|\sigma(2\cdots n-1)|n} A_n(1, \sigma(2), \cdots, \sigma(n-1), n)$$

- The sum is over the $(n-2)!$ permutations and the coefficients are given by the product of the structure constant

$$c_{1|23|4} = f_{12}^{a} f_{34}^{a}$$



One can get a further reduction using kinematic constraints

Tree-level amplitudes in gauge theory

Instead of considering the sum of the mutilple field theory graph individualy we treat the field theory amplitudes and the infinite tension limit $\alpha' \to 0$ of the tree-level string amplitudes

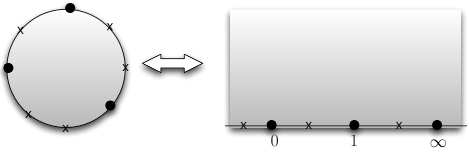
$$A_{\text{SYM}}(\sigma(1),\ldots,\sigma(n)) = \lim_{\alpha'\to 0} \mathfrak{A}(\sigma(1),\ldots,\sigma(n))$$

$$\mathfrak{A}(\sigma(1),\ldots,\sigma(n)) = \left\langle U^{(1)}(z_1)U^{(n-1)}(z_{n-1})U^{(n)}(z_n) \prod_{i=2}^{n-2} \int_{\text{ordered}} d^2z_i V^{(i)} \right\rangle$$

where $U(z)$ and $V(z)$ are vertex operators and $\langle\cdots\rangle$ is the path integral over the world-sheet fields.

This can applied to any string theory formalism (Bosonic, RNS, Green-Schwarz, pure spinor, ...) in any spacetime dimensions

Open string tree-level amplitude on the disc



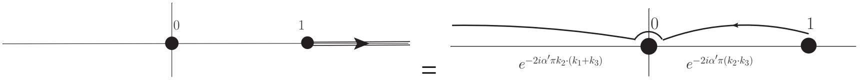
$PSL(2,\mathbb{R})$ invariance $z_1 = 0$, $z_{n-1} = 1$ and $z_n = +\infty$. (3 marked points)

$$\mathfrak{A}(1,\dots,n) = \int_{x_1<\cdots<x_n} \prod_{i=2}^{n-2} dx_i \prod_{1\leqslant i<j\leqslant n} (x_i - x_j)^{2\alpha' k_i\cdot k_j} \sum_{(\zeta_j)\in\{0,1,x_i\}} L_k \prod_{i=2}^{n-2} \frac{1}{x_j - \zeta_j}$$

- The L_k factor encodes the information on the theory we consider: scalar or vector in the adjoint, etc
- The integral has branch cuts when one exchanges to order of the external states

Monodromies from contour deformation

Contour deformation [Bjerrum-bohr, Damgaard, Vanhove]



- The real and imaginary part of the monodromy relations lead to a set of linear system of equations

$$\mathfrak{A}_n(\beta_1,\ldots,\beta_r,1,\alpha_1,\ldots,\alpha_s,n)=(-1)^r\times$$
$$\times\Re\mathrm{e}\Big[\prod_{1\leqslant i<j\leqslant r}e^{(\beta_i\cdot\beta_j)}\sum_{\sigma\subset OP\{\alpha\}\cup\{\beta^T\}}\prod_{i=1}^{r}\prod_{j=1}^{s}e^{(\alpha_i,\beta_j)}\mathfrak{A}_n(1,\{\sigma\},n)\Big]$$
$$0=\Im\mathrm{m}\Big[\prod_{1\leqslant i<j\leqslant r}e^{(\beta_i\cdot\beta_j)}\sum_{\sigma\subset OP\{\alpha\}\cup\{\beta^T\}}\prod_{i=1}^{r}\prod_{j=1}^{s}e^{(\alpha_i,\beta_j)}\mathfrak{A}_n(1,\{\sigma\},n)\Big]$$

$\exp(\alpha,\beta)=\exp(2i\pi\alpha' k_\alpha\cdot k_\beta)$ if $\Re\mathrm{e}(z_\beta-z_\alpha)>0$ or 1 otherwise

Momentum kernel

- This leads to the following set of constraints on the field theory amplitudes for all $\beta \in \mathfrak{S}_{n-2}$

$$\sum_{\sigma\in\mathfrak{S}_{n-2}} \mathcal{S}[\sigma(2,\dots,n-1)|\beta(2,\dots,n-1)]_{k_1}\mathcal{A}_n(n,\sigma(2,\dots,n-1),1) = 0\,,$$

- $\mathcal{S}$ is a momentum kernel

$$\mathcal{S}[i_1,\dots,i_k|j_1,\dots,j_k]_p \equiv \prod_{t=1}^{k}(p\cdot k_{i_t} + \sum_{q>t}^{k}\theta(i_t,i_q)\,k_{i_t}\cdot k_{i_q})\,,$$

- The momentum kernel matrix has rank $(n-3)!$
- Basis of color ordered amplitudes of dimension $(n-3)!$

Momentum kernel

- With the momentum kernel the amplitudes in Yang-Mills and Gravity take the symmetric form

$$
\begin{aligned}
\mathcal{A}_n^{\text{YM}} &= A^{\text{vector}} \otimes \mathcal{S} \otimes A^{\text{scalar}} \\
\mathcal{M}_n^{\text{Grav}} &= A^{\text{vector}} \otimes \mathcal{S} \otimes A^{\text{vector}}
\end{aligned}
$$

- The form with the minimal number of terms is

$$
\mathcal{A}_n^{\text{YM/Grav}} = (-1)^{n-3} \sum_{\sigma,\gamma \in \mathfrak{S}_{n-3}} \mathcal{S}[\gamma(2,\ldots,n-2)|\sigma(2,\ldots,n-2)]_{k_1}
$$

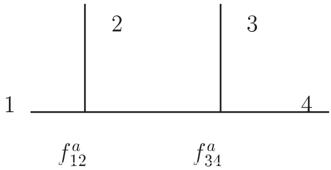
$$
\times \mathcal{A}_n^{\text{vector}}(1,\sigma(2,\ldots,n-2),n-1,n)\widetilde{\mathcal{A}}_n^{\text{scalar/vector}}(n-1,n,\gamma(2,\ldots,n-2),1)
$$

Scalar amplitudes

- The Yang–Mills amplitudes are expressed as

$$\mathcal{A}_n^{\mathrm{YM}} = A^{\mathrm{vector}} \otimes \mathcal{S} \otimes A^{\mathrm{scalar}}$$

- The scalar amplitudes are φ^3 graphs for cubic interaction f_{ab}^c



$$A^{\mathrm{scalar}}(1234) = \frac{f_{12}^a f_{34}^a}{(k_1+k_2)^2} + \frac{f_{13}^a f_{24}^a}{(k_1+k_3)^2}$$

Lorentz numerators

- The amplitude displays a striking symmetry between the kinematic (vector) part of the interaction and the color factor

$$\mathcal{A}_n^{YM} = A^{\text{vector}} \otimes \mathcal{S} \otimes A^{\text{scalar}}$$

- Expanding the scalar amplitude we get the [Del duca et al.] form

$$\mathcal{A}_n^{\text{tree}} = g_{\text{YM}}^{n-2} \sum_{\sigma \in \mathfrak{S}_{n-2}} c_{1|\sigma(2\cdots n-1)|n} A_n^{\text{vector}}(1, \sigma(2), \cdots, \sigma(n-1), n)$$

- expanding the vector part we get the dual form

$$\mathcal{A}_n^{\text{tree}} = g_{\text{YM}}^{n-2} \sum_{\sigma \in \mathfrak{S}_{n-2}} n_{1|\sigma(2\cdots n-1)|n} A_n^{\text{scalar}}(1, \sigma(2), \cdots, \sigma(n-1), n)$$

- $n_{1|\ldots|n}$ are combination of the [Bern, Carrasco, Johansson] numerator factors

Lorentz numerators

- Using the BCJ numerators one can rewrite the tree amplitudes as [Bern, Carrasco, Johansson]
- The tree level amplitude takes the form

$$\mathcal{A}_n^{\text{tree}} = \sum_i \frac{n_i c^i}{\prod_{r=1}^{n-3} p_r^2}$$

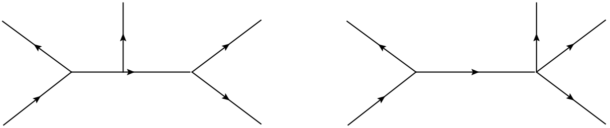
- Gravity amplitude are constructed as

$$\mathcal{M}_n^{\text{tree}} = \sum_i \frac{n_i n^i}{\prod_{r=1}^{n-2} p_r^2}$$

Lorentz numerators: the five-point case

- We consider the for color ordered gauge amplitudes

$$A_5^{\text{vector}}(\sigma(1),\dots,\sigma(5)) = \sum_{i=1}^{5} \frac{n_{r_i}}{p_{1,i}^2 p_{2,i}^2}$$



Lorentz numerators: the five-point case

- We consider the for color ordered gauge amplitudes

$$A_5^{\text{vector}}(\sigma(1),\dots,\sigma(5)) = \sum_{i=1}^{5} \frac{n_{r_i}}{p_{1,i}^2 p_{2,i}^2}$$

- The numerator factors are *not* gauge invariant

- The monodromy S-kernel relations off the color ordered amplitudes

$$0 = (s_{13}+s_{23})A_5^{\text{vector}}(1,2,3,4,5) - s_{35}A_5^{\text{vector}}(1,2,4,3,5) + s_{13}A_5^{\text{vector}}(1,3,2,4,5)$$

Lorentz numerators: the five-point case

- We consider the for color ordered gauge amplitudes

$$A_5^{\text{vector}}(\sigma(1),\dots,\sigma(5)) = \sum_{i=1}^{5} \frac{n_{r_i}}{p_{1,i}^2 p_{2,i}^2}$$

- The system is solved by the generalized dual Jacobi relations

$$X_{ijk} = n_i - n_j + n_k = P_n(s_{ij}); \qquad c_i - c_j + c_k = 0$$

- The n_i are not uniquely defined by the pairing $n_i c_i$ summed over the graph gives the gauge invariant amplitudes
- $P_n = 0$ is the [Bern, Carrasco, Johansson] solution
- $P_n \neq 0$ is from the freedom in resolving the higher point vertices [Tye, Zhang], [Bjerrum-Bohr, Damgaard, Søndergaard, Vanhove]

Part II

Loop amplitudes

Amplitudes from dressed φ^3 cubic vertices

[Bern, Carrasco, Johansson] have proposed the generalize to loop order the numerator parametrisation of gauge and gravity tree amplitudes amplitude

On n-points, L-loop skeleton graphs, $\gamma \in \Gamma_n^L$, with only cubic φ^3 vertices dressed by

- n_γ: lorentz factor build from momenta and polarisations
- c_γ: a color factor

$$\mathcal{A}_n^L = g_{\text{YM}}^{2(L-1)+n} \sum_{\gamma \in \Gamma_n^L} \int d^{3L-3}\ell_\alpha \frac{n_\gamma c^\gamma}{\prod_{r=1}^{3(L-1)+n} p_r^2};$$

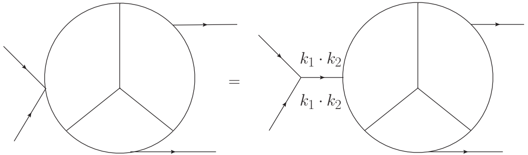
$$\mathcal{M}_n^L = \kappa_{(D)}^{2(L-1)+n} \sum_{\gamma \in \Gamma_n^L} \int d^{3L-3}\ell_\alpha \frac{n_\gamma n^\gamma}{\prod_{r=1}^{3(L-1)+n} p_r^2}$$

Amplitudes from dressed φ^3 cubic vertices

Gauge invariance, supersymmetry, crossing symmetry of the amplitudes requires *contact terms* given by higher point vertices

Such contact terms are resolved by inserting appropriate factors

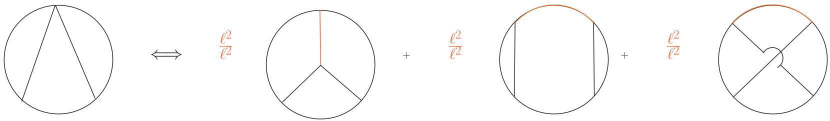
$$1 = \frac{k_1 \cdot k_2}{k_1 \cdot k_2} :$$



the same manipulations for internal vertices as well lead to graphs build on φ^3 skeletons with dressing numerator factors characteristic of the theory one considers.

Parametrisation of the $\mathcal{N} = 4$ SYM amplitudes

The contact terms are resolved into cubic vertices as follows



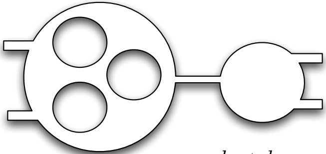
- The contact terms are needs for the total amplitude to be gauge invariant
- Important for the counting of the fermionic zero modes and the ultraviolet behaviour of the amplitude in $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ SUGRA

  [Berkovits, Green, Russo,Vanhove], [Green, Bjornsson], [Vanhove]

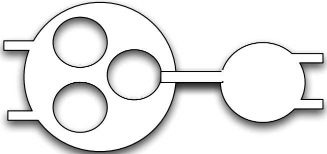
Parametrisation of the $\mathcal{N} = 4$ SYM amplitudes

$$\mathcal{A}^{(D)}_{4,L} = g_{\rm YM}^{2L+2} \sum_{\gamma \in \Gamma_4^L} \int \prod_{i=1}^{L} \frac{d^D \ell_i}{(2\pi)^D} \frac{1}{S_j} \frac{n_\gamma c^\gamma}{\prod_{r=1}^{3L+1} p_r^2},$$

It was shown in [Berkovits, Green, Russo, Vanhove] that the *sub-leading* color contribution of the four-point L-loop amplitude in $\mathcal{N} = 4$ SYM has a *better* ultraviolet behaviour than the leading (planar) color factor contribution



$$\mathrm{Tr}(T_1T_2T_3T_4)\,\frac{k_1+k_2}{k_1\cdot k_2}$$



$$\mathrm{Tr}([T_1,T_2])\mathrm{Tr}([T_3,T_4])\frac{k_1+k_2}{k_1\cdot k_2} = 0$$

$$\lim_{\ell\sim\Lambda\gg 1} \mathcal{A}^{(D)}_{4,L} \sim \partial^2\, \mathrm{tr}F^4 \times \Lambda^{(D-4)L-4} + \partial^4\,(\mathrm{tr}F^2)^2 \times \Lambda^{(D-4)L-6} \qquad L \geqslant 4$$

Parametrisation of the $\mathcal{N}=8$ SUGRA amplitudes

- We recycle the previous parametrisation in the $\mathcal{N}=8$ amplitudes

$$\mathcal{M}_{4,L}^{(D)} = \kappa_{(D)}^{2L+2} \sum_{\gamma\in\Gamma_4^L} \int \prod_{i=1}^{L} \frac{d^D\ell_i}{(2\pi)^D} \frac{1}{S_j} \frac{n_\gamma \widetilde{n}_\gamma}{\prod_{r=1}^{3L+1} p_r^2}\,.$$

- We use left/right symmetric expression

$$n_i\widetilde{n}_i \sim |\partial^4 F^4\, t(\ell,k)|^2 \sim \partial^8 R^4\, t(\ell,k) t(\ell,k)$$

$$\lim_{\ell\sim\Lambda\gg1} \mathcal{M}_{4,L}^{(D)} \sim \Lambda^{(D-2)L-14}\, \partial^8 R^4$$

The ultraviolet behaviour of $\mathcal{N}=8$ supergravity

- For $L \leqslant 4$ the UV behaviour is

$$[\mathcal{M}^{(D)}_{4;L}] \sim \Lambda^{(D-4)L-6}\, \partial^{2L} R^4 \qquad 2 \leqslant L \leqslant 4$$

- identical to the one of $\mathcal{N}=4$ SYM
- After 4-loop only $\partial^8 R^4$ is factorized $\beta_L = 4$ for $L \geqslant 4$

$$[\mathcal{M}^{(D)}_{4;L}] \sim \Lambda^{(D-2)L-14}\, \partial^{8} R^4 \qquad L \geqslant 4$$

- worse than the one of $\mathcal{N}=4$ SYM

- At five-loop order the 4-point amplitude in [vanhove, 2010]
  - $\mathcal{N}=4$ SYM divergences for $5 < 26/5 \leqslant D$
  - $\mathcal{N}=8$ SUGRA divergences for $24/5 \leqslant D$
- Would imply a *seven-loop* divergence in $D=4$ with counter-term $\partial^8 R^4$

- Same conclusion by [Green, Bjornsson, 2010] using pure spinor field theory arguments

- Green: explicitly checked by field theory or string theory computation
- Black 'allowed': Allowed by not explicitly checked
- Red: First possible ultraviolet divergence. Coefficient has not been evaluated.

	R^4	$\partial^4 R^4$	$\partial^6 R^4$	$\partial^8 R^4$	$\partial^{10} R^4$	$\partial^{12} R^4$
D=11	– –	– –	– –	– –	– –	L=2 yes
D=10	– –	– –	– –	– –	L=2 yes	– –
D=9	– –	– –	– –	L=2 yes	– –	– –
D=8	L=1 yes	– –	L=2 yes	– –	– –	L=3 yes
D=7	– –	L=2 yes	– –	– –	– –	– –
D=6	– –	– –	L=3 yes	– –	L=4 yes	– –
D=5	L=2 no	– –	L=4 no	– –	– –	L=6
D=4	L=3 no	L=5 no	L=6 no	L=7 !	L=8	L=9

An allowed E_7 invariant 7-loop counterterm in $D = 4$ constructed in [Bossard, Howe, Stelle, Vanhove]

Outlook

We have analyzed the parametrisation of amplitudes in gauge and gravity

- We have showed that the color ordered tree level amplitude satisfy kinematic constraints that imply a minimal basis: generalization at loop orders?
  - The *S* kernel provides that best possible reorganization of the tree-level amplitude in gravity and gauge theory [Bjerrum-Bohr, Damgaard, Vanhove]
  - Higher-loop Generalization ?
- BCJ numerators parametrisation and φ^3 skeleton graphs: systematic for low supersymmetry theories?