# The momentum mapping of the affine real symplectic group 

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In this paper we explain how the cocycle of the momentum map of the action of the affine symplectic group on $\mathbb{R}^{2 n}$ gives rise to a coadjoint orbit of the odd real symplectic group with a modulus.

## 1 Basic setup

Consider the set $\operatorname{AfSp}\left(\mathbb{R}^{2 n}, \omega\right)$ of invertible affine real symplectic mappings

$$
(A, a):\left(\mathbb{R}^{2 n}, \omega\right) \rightarrow\left(\mathbb{R}^{2 n}, \omega\right): v \mapsto A v+a .
$$

Using the multiplication $(A, a) \cdot(B, b)=(A B, A b+a)$, which corresponds to composition of affine linear mappings, $\operatorname{AfSp}\left(\mathbb{R}^{2 n}, \omega\right)$ is a group. Identifying $(A, a)$ with the matrix $\left(\begin{array}{cc}A & a \\ 0 & 1\end{array}\right), \operatorname{AfSp}\left(\mathbb{R}^{2 n}, \omega\right)$ becomes a closed subgroup of $\operatorname{Sp}\left(\mathbb{R}^{2 n}, \omega\right) \times \mathbb{R}^{2 n}$. Thus $\operatorname{AfSp}\left(\mathbb{R}^{2 n}, \omega\right)$ is a Lie group. Its Lie algebra is $\operatorname{afsp}\left(\mathbb{R}^{2 n}, \omega\right)=\left\{(X, x) \in \operatorname{sp}\left(\mathbb{R}^{2 n}, \omega\right) \times \mathbb{R}^{2 n}\right\}$ with Lie bracket

$$
\begin{equation*}
[(X, x),(Y, y)]=([X, Y], X y-Y x) \tag{1}
\end{equation*}
$$

$\operatorname{AfSp}\left(\mathbb{R}^{2 n}, \omega\right)$ acts on $\left(\mathbb{R}^{2 n}, \omega\right)$ by

$$
\begin{equation*}
\Phi: \operatorname{AfSp}\left(\mathbb{R}^{2 n}, \omega\right) \times\left(\mathbb{R}^{2 n}, \omega\right) \rightarrow\left(\mathbb{R}^{2 n}, \omega\right):((A, a), v) \mapsto A v+a \tag{2}
\end{equation*}
$$

Since the symplectic form $\omega$ on $\mathbb{R}^{2 n}$ is invariant under translation, for every $(A, a) \in \operatorname{AfSp}\left(\mathbb{R}^{2 n}, \omega\right)$ the affine linear mapping $\Phi_{(A, a)}$ preserves $\omega$. The infinitesimal generator $X^{(X, x)}$ of the action $\Phi$ in the direction $(X, x) \in$ $\operatorname{afsp}\left(\mathbb{R}^{2 n}, \omega\right)$ is the vector field $X^{(X, x)}(v)=X v+x$ on $\mathbb{R}^{2 n}$. We now show
Claim 1.1. The $\operatorname{AfSp}\left(\mathbb{R}^{2 n}, \omega\right)$ action $\Phi$ (2) is Hamiltonian.
Proof. For every $(Y, y) \in \operatorname{afsp}\left(\mathbb{R}^{2 n}, \omega\right)$ let

$$
\begin{equation*}
J^{(Y, y)}: \mathbb{R}^{2 n} \rightarrow \mathbb{R}: v \mapsto J^{Y}(v)+\omega^{\sharp}(y) v=\frac{1}{2} \omega(Y v, v)+\omega(y, v) \tag{3}
\end{equation*}
$$

and set

$$
\begin{equation*}
J: \mathbb{R}^{2 n} \rightarrow \operatorname{afsp}\left(\mathbb{R}^{2 n}, \omega\right)^{*} \tag{4}
\end{equation*}
$$

[^0]where $J(v)(Y, y)=J^{(Y, y)}(v)$. Then for every $(X, x) \in \operatorname{afsp}\left(\mathbb{R}^{2 n}, \omega\right)$, every $v \in \mathbb{R}^{2 n}$, and every $w \in T_{v} \mathbb{R}^{2 n}=\mathbb{R}^{2 n}$
\[

$$
\begin{aligned}
\mathrm{d} J^{(X, x)}(v) w & =\left(T_{v} J(X, x)\right) w=\omega(X v, w)+\omega(x, w) \\
& =\omega(X v+x, w)=\omega\left(X^{(X, x)}(v), w\right),
\end{aligned}
$$
\]

that is, $X^{(X, x)}=X_{J^{(X, x)}}$. Hence the action $\Phi$ is Hamiltonian.
The above argument shows that the map $J$ (4) is the momentum map of the Hamiltonian action $\Phi$ (2). The following discussion is motivated by theorem 11.34 of Souriau [5, p.143].
Lemma 1.2. The mapping

$$
\begin{equation*}
\sigma: \operatorname{AfSp}\left(\mathbb{R}^{2 n}, \omega\right) \rightarrow \operatorname{afsp}\left(\mathbb{R}^{2 n}, \mathbb{R}\right)^{*}: g \mapsto J\left(\Phi_{g}(v)\right)-\operatorname{Ad}_{g^{-1}}^{T} J(v) \tag{5}
\end{equation*}
$$

does not depend on $v \in \mathbb{R}^{2 n}$.
Proof. For each $\eta \in \operatorname{afsp}\left(\mathbb{R}^{2 n}, \mathbb{R}\right)$ we have

$$
\begin{aligned}
\mathrm{d}\left(J \circ \Phi_{g}\right)^{\eta}(v) & =T_{v} \Phi_{g} X^{\eta}(v) \_\omega(v)=X^{\operatorname{Ad}_{g} \eta}(v) \_\omega(v) \\
& =\mathrm{d} J^{\operatorname{Ad}_{g} \eta}(v)=\mathrm{d}\left(\operatorname{Ad}_{g^{-1}}^{T} J\right)^{\eta}(v),
\end{aligned}
$$

that is, $\mathrm{d}\left(J \circ \Phi_{g}-\operatorname{Ad}_{g^{-1}}^{T} J\right)(v)=0$. Since $\mathbb{R}^{2 n}$ is connected the function $J \circ \Phi_{g}-\operatorname{Ad}_{g^{-1}}^{T} J: \mathbb{R}^{2 n} \rightarrow \mathbb{R}$ is constant.
Corollary 1.2A For every $g, g^{\prime} \in \operatorname{AfSp}\left(\mathbb{R}^{2 n}, \omega\right)$

$$
\begin{equation*}
\sigma\left(g g^{\prime}\right)=\sigma(g)+\operatorname{Ad}_{g^{-1}}^{T} \sigma\left(g^{\prime}\right) \tag{6}
\end{equation*}
$$

Proof. We compute.

$$
\begin{aligned}
\sigma\left(g g^{\prime}\right) & =J \circ \Phi_{g g^{\prime}}-\operatorname{Ad}_{\left(g g^{\prime}\right)^{-1}}^{T} J \\
& =\left(J \circ \Phi_{g}-\operatorname{Ad}_{g^{-1}}^{T} J\right) \circ \Phi_{g^{\prime}}+\operatorname{Ad}_{g^{-1}}^{T}\left(J \circ \Phi_{g^{\prime}}-\operatorname{Ad}_{\left(g^{\prime}\right)^{-1}}^{T} J\right) \\
& =\sigma(g)+\operatorname{Ad}_{g^{-1}}^{T} \sigma\left(g^{\prime}\right) .
\end{aligned}
$$

Evaluating $\sigma$ (5) at $\exp t \eta$ and then at $\zeta \in \operatorname{afsp}\left(\mathbb{R}^{2 n}, \mathbb{R}\right)$ gives

$$
\begin{equation*}
(\sigma(\exp t \eta)) \zeta=J^{\zeta}\left(\Phi_{\exp t \eta}(v)\right)-\left(\operatorname{Ad}_{\exp -t \eta}^{T} J(v)\right) \zeta \tag{7}
\end{equation*}
$$

Differentiating (7) with respect to $t$ and then setting $t$ equal to 0 gives

$$
\left(T_{e} \sigma \eta\right) \zeta=\mathrm{d} J^{\zeta}(v) X^{\eta}(v)+\left(\operatorname{ad}_{\eta}^{T} J(v)\right) \zeta
$$

$$
\begin{equation*}
=L_{X^{\eta}} J^{\zeta}(v)+J(v) \operatorname{ad}_{\eta} \zeta=\left\{J^{\zeta}, J^{\eta}\right\}(v)-J^{[\zeta, \eta]}(v) . \tag{8}
\end{equation*}
$$

Let $\Sigma^{\sharp}: \operatorname{afsp}\left(\mathbb{R}^{2 n}, \mathbb{R}\right) \rightarrow \operatorname{afsp}\left(\mathbb{R}^{2 n}, \mathbb{R}\right)^{*}$ be the linear mapping $\eta \mapsto$ $\Sigma^{\sharp}(\eta)=-\left(T_{e} \sigma\right) \eta \in \operatorname{afsp}\left(\mathbb{R}^{2 n}, \mathbb{R}\right)^{*}$. Equation (8) may be written as

$$
\begin{equation*}
\left\{J^{\eta}, J^{\zeta}\right\}(v)=J^{[\eta, \zeta]}(v)+\Sigma(\eta, \zeta), \tag{9}
\end{equation*}
$$

where $\Sigma(\eta, \zeta)=\Sigma^{\sharp}(\eta) \zeta$. From equation (19) it follows that the bilinear map $\Sigma$ is skew symmetric.

Lemma 1.3. $\Sigma$ is an $\operatorname{afsp}\left(\mathbb{R}^{2 n}, \mathbb{R}\right)$ cocycle, that is, for every $\xi$, $\eta$, and $\zeta \in \operatorname{afsp}\left(\mathbb{R}^{2 n}, \mathbb{R}\right)$

$$
\begin{equation*}
\Sigma(\zeta,[\xi, \eta])=\Sigma([\zeta, \xi], \eta)+\Sigma(\xi,[\zeta, \eta]) . \tag{10}
\end{equation*}
$$

Proof. Since $\left(C^{\infty}\left(\mathbb{R}^{2 n}\right),\{\},\right)$ is a Lie algebra

$$
\left\{J^{\zeta},\left\{J^{\xi}, J^{\eta}\right\}\right\}=\left\{\left\{J^{\zeta}, J^{\xi}\right\}, J^{\eta}\right\}+\left\{J^{\xi},\left\{J^{\zeta}, J^{\eta}\right\}\right\} .
$$

Using equation (9) the above equation reads

$$
\begin{aligned}
\left\{J^{\zeta}, J^{[\xi, \eta]}\right\} & +\left\{J^{\zeta}, \Sigma(\xi, \eta)\right\}=\left\{J^{[\zeta, \xi]}, J^{\eta}\right\}+\left\{\Sigma(\zeta, \xi), J^{\eta}\right\} \\
& +\left\{J^{\xi}, J^{[\zeta, \eta]}\right\}+\left\{J^{\zeta}, \Sigma(\zeta, \eta)\right\} .
\end{aligned}
$$

Using (9) again gives

$$
J^{[\zeta,[\xi, \eta]}+\Sigma(\zeta,[\xi, \eta])=J^{[[\zeta, \xi], \eta]}+\Sigma([\zeta, \xi], \eta)+J^{[\xi,[\zeta, \eta]}+\Sigma(\xi,[\zeta, \eta]),
$$

since $\Sigma(\xi, \eta), \Sigma(\zeta, \xi)$ and $\Sigma(\zeta, \eta)$ are constant functions on $\mathbb{R}^{2 n}$. By linearity and the Jacobi identity $J^{[\zeta,[\zeta \eta \eta]}=J^{[\zeta \zeta, \xi], \eta]}+J^{[\xi,[\zeta, \eta]]}$ equation (10) holds.
Claim 1.4. The momentum map $J$ (4) has the cocycle

$$
\begin{equation*}
\Sigma: \operatorname{afsp}\left(\mathbb{R}^{2 n}, \omega\right) \times \operatorname{afsp}\left(\mathbb{R}^{2 n}, \omega\right) \rightarrow \mathbb{R}:((Y, y),(Z, z)) \mapsto \omega(y, z) . \tag{11}
\end{equation*}
$$

Proof. We compute.

$$
\begin{aligned}
\left\{J^{(Y, y)}, J^{(Z, z)}\right\}(v) & =\left(L_{X(Z, z)} J^{(Y, y)}\right)(v)=\mathrm{d} J^{(Y, y)}(v) X^{(Z, z)}(v) \\
& =\omega(Y v, Z v+z)+\omega(y, Z v+z) \\
& =\omega(Y v, Z v)+\omega(Y v, z)+\omega(y, Z v)+\omega(y, z) .
\end{aligned}
$$

Now

$$
\frac{1}{2} \omega([Y, Z] v, v)=\frac{1}{2} \omega((Y Z-Z Y) v, v)=\frac{1}{2} \omega(Y Z v, v)-\frac{1}{2} \omega(Z Y v, v)
$$

$$
=-\frac{1}{2} \omega(Z v, Y v)+\frac{1}{2} \omega(Y v, Z v)=\omega(Y v, Z v) .
$$

Thus

$$
\begin{aligned}
\left\{J^{(Y, y)}, J^{(Z, z)}\right\}(v) & =\frac{1}{2} \omega([Y, Z] v, v)+\omega(Y z-Z y, v)+\omega(y, z) \\
& =J^{[Y, Z]}(v)+\omega(Y z-Z y, v)+\omega(y, z) \\
& =J^{[(Y, y),(Z, z)]}(v)+\omega(y, z) .
\end{aligned}
$$

Define the map

$$
\begin{gather*}
\Psi: \operatorname{AfSp}\left(\mathbb{R}^{2 n}, \omega\right) \times \operatorname{afsp}\left(\mathbb{R}^{2 n}, \omega\right)^{*} \rightarrow \operatorname{afsp}\left(\mathbb{R}^{2 n}, \omega\right)^{*} \\
(g, \alpha) \longmapsto \operatorname{Ad}_{g^{-1}}^{T} \alpha+\sigma(g), \tag{12}
\end{gather*}
$$

where $\sigma$ is given by equation (5).
Claim 1.5 The map $\Psi$ (12) is an action of $\operatorname{AfSp}\left(\mathbb{R}^{2 n}, \omega\right)$ on $\operatorname{afsp}\left(\mathbb{R}^{2 n}, \omega\right)^{*}$.
Proof. We compute. For $g, g^{\prime} \in \operatorname{AfSp}\left(\mathbb{R}^{2 n}, \omega\right)$ and $\alpha \in \operatorname{afsp}\left(\mathbb{R}^{2 n}, \omega\right)^{*}$ we have

$$
\begin{aligned}
\Psi_{g g^{\prime}} \alpha= & \operatorname{Ad}_{\left(g g^{\prime}\right)^{-1}}^{T} \alpha+\sigma\left(g g^{\prime}\right) \\
= & \operatorname{Ad}_{g^{-1}}^{T}\left(\operatorname{Ad}_{\left(g^{\prime}\right)^{-1}}^{T} \alpha\right)+\sigma(g)+\operatorname{Ad}_{g^{-1}}^{T} \sigma\left(g^{\prime}\right) \\
& \quad \text { using corollary 1.2A } \\
= & \operatorname{Ad}_{g^{-1}}^{T}\left(\operatorname{Ad}_{\left(g^{\prime}\right)^{-1}}^{T} \alpha+\sigma\left(g^{\prime}\right)\right)+\sigma(g) \\
= & \operatorname{Ad}_{g^{-1}}^{T}\left(\Psi_{g^{\prime}} \alpha\right)+\sigma(g)=\Psi_{g}\left(\Psi_{g^{\prime}} \alpha\right) .
\end{aligned}
$$

Claim 1.6 The momentum mapping $J$ (4) is coadjoint equivariant under the action $\Psi$ (12).
Proof. We compute. For every $g \in \operatorname{AfSp}\left(\mathbb{R}^{2 n}, \omega\right)$ and every $w \in \mathbb{R}^{2 n}$

$$
\begin{aligned}
\Psi_{g}(J(w)) & =\operatorname{Ad}_{g^{-1}}^{T} J(w)+\sigma(g), \text { using (12) } \\
& =\operatorname{Ad}_{g^{-1}}^{T} J(w)+J\left(\Phi_{g}(w)\right)-\operatorname{Ad}_{g^{-1}}^{T} J(w), \text { using (5) } \\
& =J\left(\Phi_{g}(w)\right) .
\end{aligned}
$$

## 2 Extension

Following Wallach [6] we find a central extension of Lie algebra $\operatorname{afsp}\left(\mathbb{R}^{2 n}, \omega\right)$ the dual of whose adjoint map is

$$
\begin{align*}
T_{e} \Psi(X, x) \alpha & =\left.\frac{\mathrm{d}}{\mathrm{~d} t}\right|_{t=0} \Psi_{\exp t(X, x)} \alpha=-\operatorname{ad}_{(X, x)}^{T} \alpha+T_{e} \sigma(X, x) \\
& =-\operatorname{ad}_{(X, x)}^{T} \alpha+\Sigma^{\sharp}(X, x), \tag{13}
\end{align*}
$$

where $(X, x) \in \operatorname{sp}\left(\mathbb{R}^{2 n}, \omega\right) \times \mathbb{R}^{2 n}=\operatorname{afsp}\left(\mathbb{R}^{2 n}, \omega\right)$.
Consider the Lie algebra $\widehat{\mathfrak{g}}=\left\{(X, v, \xi) \in \operatorname{afsp}\left(\mathbb{R}^{2 n}, \omega\right) \times \mathbb{R}\right\}$ whose Lie bracket is

$$
\begin{equation*}
[(X, v, \xi),(Y, w, \eta)]=([X, Y], X w-Y v, \omega(v, w)) \tag{14}
\end{equation*}
$$

From (14) it follows that $(0,0,1)$ lies in the center of $\widehat{\mathfrak{g}}$, that is, $[(0,0,1),(X, v, \xi)]$ $=(0,0,0)$. Also

$$
\begin{equation*}
[(X, v, 0),(Y, v, 0)]=([X, Y], X w-Y v, 0)+\omega(v, w)(0,0,1) \tag{15}
\end{equation*}
$$

Thus the Lie algebra $\widehat{\mathfrak{g}}$ is a central extension of the Lie algebra $\operatorname{afsp}\left(\mathbb{R}^{2 n}, \omega\right)$ by the cocycle $\omega$. Since we can write (14) as

$$
\begin{equation*}
\operatorname{ad}_{[X, x, \xi]}[Y, y, \eta]=\operatorname{ad}_{[X, x]}[Y, y]+\Sigma(\xi, \eta) \tag{16}
\end{equation*}
$$

$\widehat{\mathfrak{g}}$ is the sought for Lie algebra.

## 3 The odd real symplectic group

We now find a connected linear Lie group $\widehat{G}$ whose Lie algebra is $\widehat{\mathfrak{g}}$. Consider the group $\widehat{G} \subseteq \operatorname{AfSp}\left(\mathbb{R}^{2 n}, \frac{1}{2} \omega\right) \times \mathbb{R}$ with multiplication

$$
((A, v), r) \cdot((B, w), s)=\left((A B, A w+v), r+s+\frac{1}{2} \omega\left(A^{-1} v, w\right)\right)
$$

The map

$$
\widehat{\pi}: \widehat{G} \rightarrow \operatorname{AfSp}\left(\mathbb{R}^{2 n}, \omega\right):(A, v, r) \mapsto(A, v)
$$

is a surjective group homomorphism, whose kernel is the normal subgroup $\widehat{Z}=\{(I, 0, r) \in \widehat{G} \mid r \in \mathbb{R}\}$, which is the center of $\widehat{G}$. Note that $\pi_{1}(\widehat{G})=$ $\pi_{1}\left(\operatorname{Sp}\left(\mathbb{R}^{2 n}, \omega\right)\right)=\mathbb{Z} . \widehat{G}$ is a Lie group with Lie algebra $\widehat{\mathfrak{g}}$, whose Lie bracket is given by (14).
Claim 3.1. The group $\widehat{G}$ is isomorphic to the odd real symplectic group.
Proof. Consider the map

$$
\rho: \widehat{G} \rightarrow \mathrm{Gl}\left(\mathbb{R}^{2 n+2}, \mathbb{R}\right):(A, v, r) \mapsto\left(\begin{array}{ccc}
1 & 0 & 0  \tag{17}\\
v & A & 0 \\
r & \frac{1}{2} \omega^{\sharp}\left(A^{-1} v\right) & 1
\end{array}\right) .
$$

The map $\rho$ is an injective homomorphism. Here we have $\frac{1}{2} \omega^{\sharp}\left(A^{-1} v\right)=$ $-\frac{1}{2}\left(v^{T} J A\right)$, since for every $z \in \mathbb{R}^{2 n}$

$$
\begin{aligned}
\frac{1}{2} \omega^{\sharp}\left(A^{-1} v\right) z & =\frac{1}{2} \omega\left(A^{-1} v, z\right)=\frac{1}{2} \omega(v, A z), \text { because } A \in \operatorname{Sp}\left(\mathbb{R}^{2 n}, \frac{1}{2} \omega\right) \\
& =-\frac{1}{2} \omega(A z, v)=-\frac{1}{2}\left(v^{T} J A\right) z .
\end{aligned}
$$

The following calculation shows that $\rho$ is a homomorphism.

$$
\begin{array}{r}
\rho((A, v, r) \cdot(B, w, s))=\rho\left(A B, A w+v, r+s+\frac{1}{2} \omega\left(A^{-1} v, w\right)\right) \\
=\left(\begin{array}{ccc}
1 & 0 & 0 \\
v+A w & A B & 0 \\
r+s+\frac{1}{2} \omega^{\sharp}\left(A^{-1} v\right) w & \frac{1}{2} \omega^{\sharp}\left((A B)^{-1}(v+A w)\right) & 1
\end{array}\right) . \tag{18}
\end{array}
$$

Now

$$
\begin{align*}
& \frac{1}{2} \omega^{\sharp}\left((A B)^{-1}(v+A w)\right)=\frac{1}{2} \omega^{\sharp}\left(B^{-1}\left(A^{-1} v\right)\right)+\frac{1}{2} \omega^{\sharp}\left(B^{-1} w\right) \\
& \quad=B^{T} \frac{1}{2} \omega^{\sharp}\left(A^{-1} v\right)+\frac{1}{2} \omega^{\sharp}\left(B^{-1} w\right), \text { since } B \in \operatorname{Sp}\left(\mathbb{R}^{2 n}, \frac{1}{2} \omega\right) \\
& \quad=\frac{1}{2} \omega^{\sharp}\left(A^{-1} v\right) B+\frac{1}{2} \omega^{\sharp}\left(B^{-1} w\right) . \tag{19}
\end{align*}
$$

But

$$
\begin{array}{r}
\rho(A, v, r) \rho(B, w, s)=\left(\begin{array}{ccc}
1 & 0 & 0 \\
v & A & 0 \\
r & \frac{1}{2} \omega^{\sharp}\left(A^{-1} v\right) & 1
\end{array}\right)\left(\begin{array}{ccc}
1 & 0 & 0 \\
w & B & 0 \\
s & \frac{1}{2} \omega^{\sharp}\left(B^{-1} w\right) & 1
\end{array}\right) \\
=\left(\begin{array}{ccc}
1 & 0 & 0 \\
v+A w & A B & 0 \\
r+s+\frac{1}{2} \omega^{\sharp}\left(A^{-1} v\right) w & \frac{1}{2} \omega^{\sharp}\left(A^{-1} v\right) B+\frac{1}{2} \omega^{\sharp}\left(B^{-1} w\right) & 1
\end{array}\right) . \tag{20}
\end{array}
$$

Using (19) we see that the right hand sides of equations (18) and (20) are equal, that is,

$$
\rho((A, v, r) \cdot(B, w, s))=\rho(A, v, r) \rho(B, w, s)
$$

Thus the map $\rho$ (17) is a group homomorphism. The map $\rho$ is injective, for if $\rho(A, v, r)=\left(I_{2 n}, 0,0\right)$, then $v=0$ and $r=0$. So $(A, v, r)=\left(I_{2 n}, 0,0\right)$, the identity element of $\widehat{G}$.

Since $\widehat{G}$ is a subgroup of $\operatorname{AfSp}\left(\mathbb{R}^{2 n}, \frac{1}{2} \omega\right) \times \mathbb{R}$, it follows that if $(A, v, r) \in$ $\widehat{G}$, then $A \in \operatorname{Sp}\left(\mathbb{R}^{2 n}, \frac{1}{2} \omega\right)$. Thus the image of the map $\rho$ is contained in $\operatorname{Sp}\left(\mathbb{R}^{2 n+2}, \Omega\right)$. Here the matrix of the symplectic form $\Omega$ with respect to the basis $\left\{e_{0}, e_{1}, \ldots, e_{n}, f_{1}, \ldots, f_{n}, f_{n+1}\right\}$ of $\mathbb{R}^{2 n+2}$ is $\left(\begin{array}{ccc}0 & 0 & 1 \\ 0 & \frac{1}{2} J & 0 \\ -1 & 0 & 0\end{array}\right)$, and $J=$ $\left(\begin{array}{cc}0 & I_{n} \\ -I_{n} & 0\end{array}\right)$ is the matrix of the symplectic form $\omega$ with respect to the basis
$\left\{e_{1}, \ldots, e_{n}, f_{1}, \ldots, f_{n}\right\}$ of $\mathbb{R}^{2 n}$. The image of the map $\rho$ (17) is the odd real symplectic group $\operatorname{Sp}\left(\mathbb{R}^{2 n+2}, \Omega\right)_{f_{n+1}}=\left\{\mathcal{A} \in \operatorname{Sp}\left(\mathbb{R}^{2 n+2}, \Omega\right) \mid \mathcal{A} f_{n+1}=f_{n+1}\right\}$. Consequently, $\widehat{G}$ is isomorphic to $\operatorname{Sp}\left(\mathbb{R}^{2 n+2}, \Omega\right)_{f_{n+1}}$.

The Lie algebra $\operatorname{sp}\left(\mathbb{R}^{2 n+2}, \Omega\right)_{f_{n+1}}$ of the Lie group $\operatorname{Sp}\left(\mathbb{R}^{2 n+2}, \Omega\right)_{f_{n+1}}$ is

$$
\left\{\left.\widehat{X}=\left(\begin{array}{ccc}
0 & 0 & 0 \\
x & X & 0 \\
\xi & \frac{1}{2} \omega^{\sharp}(x) & 0
\end{array}\right) \in \operatorname{Sp}\left(\mathbb{R}^{2 n+2}, \Omega\right) \right\rvert\, x \in \mathbb{R}^{2 n}, X \in \operatorname{sp}\left(\mathbb{R}^{2 n}, \frac{1}{2} \omega\right), \text { and } \xi \in \mathbb{R}\right\}
$$

with Lie bracket

$$
\left.[\widehat{X}, \widehat{Y}]=\left[\begin{array}{ccc}
0 & 0 & 0  \tag{21}\\
x & X & 0 \\
\xi & \frac{1}{2} \omega^{\sharp}(x) & 0
\end{array}\right),\left(\begin{array}{ccc}
0 & 0 & 0 \\
y & Y & 0 \\
\eta & \frac{1}{2} \omega^{\sharp}(y) & 0
\end{array}\right)\right]=\left(\begin{array}{ccc}
X_{y}-Y x & 0 & 0 \\
\omega(x, y) & \frac{1}{2} \omega^{\sharp}(X, Y] & 0 \\
0
\end{array}\right) .
$$

Here $\frac{1}{2} \omega^{\sharp}(x)=-\frac{1}{2} x^{T} J$, since for each $z \in \mathbb{R}^{2 n}$

$$
\frac{1}{2} \omega^{\sharp}(x) z=\frac{1}{2} \omega(x, z)=-\frac{1}{2} \omega(z, x)=\left(-\frac{1}{2} x^{T} J\right) z .
$$

The map

$$
\mu: \widehat{\mathfrak{g}} \rightarrow \operatorname{sp}\left(\mathbb{R}^{2 n+2}, \Omega\right)_{f_{n+1}}:(X, x, \xi) \mapsto \widehat{X}=\left(\begin{array}{ccc}
0 & 0 & 0 \\
x & X & 0 \\
\xi & \frac{1}{2} \omega^{\sharp}(x) & 0
\end{array}\right)
$$

is a Lie algebra isomorphism, because it is a bijective linear map and

$$
\begin{aligned}
& \mu([(X, x, \xi),(Y, y, \eta)])=\mu([X, Y], X y-Y x, \omega(x, y)) \\
& \quad=\left(\begin{array}{ccc}
x y-Y x & 0 & 0 \\
\omega(x, y) & \left.\frac{1}{2} \omega^{\sharp}(X, Y]-Y x\right) & 0 \\
0
\end{array}\right)=[\widehat{X}, \widehat{Y}]=[\mu(X, x, \xi), \mu(Y, y, \eta)] .
\end{aligned}
$$

This verifies that the Lie algebra of the Lie group $\widehat{G}$ has Lie bracket given by (14), because the group $\widehat{G}$ is isomorphic to $\operatorname{Sp}\left(\mathbb{R}^{2 n+2}, \Omega\right)_{f_{n+1}}$.
Claim 3.2. The action

$$
\begin{gather*}
\widehat{\Phi}: \operatorname{Sp}\left(\mathbb{R}^{2 n+2}, \Omega\right)_{f_{n+1}} \times\left(\mathbb{R}^{2 n}, \omega\right) \rightarrow\left(\mathbb{R}^{2 n}, \omega\right): \\
\quad\left(\left(\begin{array}{ccc}
1 & 0 & 0 \\
v & A & 0 \\
r & \frac{1}{2} \omega^{\sharp}\left(A^{-1} v\right) & 1
\end{array}\right), w\right) \mapsto A w+v \tag{22}
\end{gather*}
$$

is Hamiltonian.
Proof. The infinitesimal generator $X^{\widehat{X}}$ of the action $\widehat{\Phi}$ in the direction $\widehat{X} \in \operatorname{sp}\left(\mathbb{R}^{2 n+2}, \Omega\right)_{f_{n+1}}$ is the vector field $X^{\widehat{X}}(w)=X(w)+x$ on $\mathbb{R}^{2 n}$. For every $\widehat{Y}=\left(\begin{array}{ccc}0 & 0 & 0 \\ y & \frac{Y}{2} & 0 \\ \eta & \frac{1}{2} \omega^{\sharp}(y) & 0\end{array}\right) \in \operatorname{sp}\left(\mathbb{R}^{2 n+2}, \Omega\right)_{f_{n+1}}$ let

$$
\begin{equation*}
J^{\widehat{Y}}: \mathbb{R}^{2 n} \rightarrow \mathbb{R}: w \mapsto \frac{1}{2} \omega(Y w, w)+\omega(y, w)+\eta . \tag{23}
\end{equation*}
$$

Then

$$
\mathrm{d} \widehat{J} \widehat{Y}(v) w=T_{v} \widehat{J}(\widehat{Y}) w=\omega(Y v, w)+\omega(y, w)=\omega\left(X^{\widehat{Y}}(v), w\right) .
$$

Thus $X^{\widehat{Y}}=X_{J_{\hat{Y}}}$. So the action $\widehat{\Phi}(22)$ is Hamiltonian. Since the mapping $\widehat{Y} \mapsto J^{\widehat{Y}}(w)$ is linear for every $w \in \mathbb{R}^{2 n}$, the action $\widehat{\Phi}(22)$ has a momentum mapping

$$
\begin{equation*}
\widehat{J}: \mathbb{R}^{2 n} \rightarrow \operatorname{sp}\left(\mathbb{R}^{2 n+2}, \Omega\right)_{f_{n+1}}^{*}, \tag{24}
\end{equation*}
$$

with $\widehat{J}(w) \widehat{Y}=\widehat{J}^{\widehat{Y}}(w)$.
Claim 3.3. The momentum mapping $\widehat{J}(24)$ of the $\operatorname{Sp}\left(\mathbb{R}^{2 n+2}, \Omega\right)_{f_{n+1}}$ action $\widehat{\Phi}$ (22) is coadjoint equivariant, that is,

$$
\begin{equation*}
\widehat{J}\left(\widehat{\Phi}_{g}(w)\right)=\operatorname{Ad}_{g^{-1}}^{T} \widehat{J}(w) \tag{25}
\end{equation*}
$$

for every $g \in \operatorname{Sp}\left(\mathbb{R}^{2 n+2}, \Omega\right)_{f_{n+1}}$ and every $w \in \mathbb{R}^{2 n}$.
Proof. It is enough to show that

$$
\begin{equation*}
\left\{\widehat{J}^{\widehat{Y}}, \widehat{J}^{\widehat{Z}}\right\}=\widehat{J}[\widehat{Y}, \widehat{Z}], \text { for every } \widehat{Y}, \widehat{Z} \in \widehat{\mathfrak{g}} \tag{26}
\end{equation*}
$$

because (26) is the infinitesimalization of the coadjoint equivariance condition (25) and $\operatorname{Sp}\left(\mathbb{R}^{2 n+2}, \Omega\right)_{f_{n+1}}$ is generated by elements which lie in the image of the exponential mapping, since it is connected. We compute

$$
\begin{aligned}
\widehat{J}^{[\widehat{Y}, \widehat{Z}]}(w)= & \frac{1}{2} \omega([Y, Z] w, w)+\omega(Y z-Z y, w)+\omega(y, z), \\
& \quad \text { using equations (21) and (23) } \\
= & \omega(Y w, Z w)+\omega(Y w, z)+\omega(y, Z w)+\omega(y, z) \\
= & \omega(Y w, Z w+z)+\omega(y, Z w+z)=\mathrm{d} \widehat{J}^{\widehat{Y}}(w) X^{\widehat{Z}}(w) \\
= & \left(L_{X_{\widehat{J}} \widehat{Z}} \widehat{J}^{\widehat{Y}}\right)(w)=\left\{\widehat{J}^{\widehat{Y}}, \widehat{J}^{\widehat{Z}}\right\}(w) .
\end{aligned}
$$

## 4 Coadjoint orbit

In this section using results of [4] we algebraically classify the coadjoint orbit $\mathcal{O}\left(\widehat{J}\left(e_{1}\right)\right)$ of the odd real symplectic group $\operatorname{Sp}\left(\mathbb{R}^{2 n+2}, \Omega\right)_{f_{n+1}}$ through $\widehat{J}\left(e_{1}\right) \in \operatorname{sp}\left(\mathbb{R}^{2 n+2}, \Omega\right)_{f_{n+1}}^{*}$. We show that this coadjoint orbit has a modulus.

First we note that the action $\widehat{\Phi}(22)$ of $\operatorname{Sp}\left(\mathbb{R}^{2 n+2}, \Omega\right)_{f_{n+1}}$ on $\mathbb{R}^{2 n}$ is transitive. Thus to determine the $\operatorname{Sp}\left(\mathbb{R}^{2 n+2}, \Omega\right)_{f_{n+1}}$ coadjoint orbit through $\widehat{J}(w)$
for a fixed $w \in \mathbb{R}^{2 n}$, it suffices to determine the coadjoint orbit $\mathcal{O}\left(\widehat{J}\left(e_{1}\right)\right)$ through $\widehat{J}\left(e_{1}\right) \in \operatorname{sp}\left(\mathbb{R}^{2 n+2}, \Omega\right)_{f_{n+1}}^{*}$. We have

$$
\widehat{Y}=\left(\begin{array}{c|c|c|c}
0 & 0 & 0 & 0 \\
\hline y^{1} & A & B & 0 \\
\hline y^{2} & C & -A^{T} & 0 \\
\hline \eta & \frac{1}{2}\left(y^{2}\right)^{T} & -\frac{1}{2}\left(y^{1}\right)^{T} & 0
\end{array}\right),
$$

where $Y=\left(Y_{i j}\right)=\left(\begin{array}{cc}A & B \\ C & -A^{T}\end{array}\right) \in \operatorname{sp}\left(\mathbb{R}^{2 n}, \frac{1}{2} \omega\right)$ with $A, B$, and $C \in \operatorname{gl}\left(\mathbb{R}^{n}, \mathbb{R}\right)$, where $B=B^{T}$ and $C=C^{T} ; y^{T}=\left(\left(y^{1}\right)^{T} \mid\left(y^{2}\right)^{T}\right)=\left(y_{1}, \ldots, y_{n} \mid y_{n+1}, \ldots, y_{2 n}\right)$ $\in \mathbb{R}^{2 n}$; and $\eta \in \mathbb{R}$. Then using (23) we get

$$
\begin{aligned}
\widehat{J}\left(e_{1}\right) \widehat{Y} & =\frac{1}{2} e_{1}^{T}\left(\begin{array}{cc}
C & -A^{T} \\
-A & -B
\end{array}\right) e_{1}+e_{1}^{T}\binom{y^{2}}{-y^{1}}+\eta \\
& =\frac{1}{2} Y_{n+1,1}+y_{n+1}+\eta .
\end{aligned}
$$

Therefore

$$
\begin{equation*}
\widehat{J}\left(e_{1}\right)=\frac{1}{2} E_{n+1,1}^{*}+\frac{1}{2} E_{n+1,0}^{*}+E_{2 n+1,1}^{*}+E_{2 n+1,0}^{*} \tag{27}
\end{equation*}
$$

With respect to the basis $\left\{E_{i j}^{*}\right\}_{i, j=0}^{2 n+1}$, which is dual to the standard basis $\left\{E_{i j}\right\}_{i, j=0}^{2 n+1}$ of $\operatorname{gl}\left(\mathbb{R}^{2 n+2}, \mathbb{R}\right)$ where $E_{i j}=\left(\delta_{i k} \delta_{j \ell}\right)$, we have

$$
\widehat{J}\left(e_{1}\right)=\left(\begin{array}{c|c|c|c}
0 & 0 & 0 & 0  \tag{28}\\
\hline 0 & 0 & 0 & 0 \\
\hline r & D & 0 & 0 \\
\hline 1 & 2 r^{T} & 0 & 0
\end{array}\right) \in \operatorname{sp}\left(\mathbb{R}^{2 n+2}, \Omega\right)_{f_{n+1}}^{*} .
$$

Here $r^{T}=(1 / 2,0, \ldots, 0) \in \mathbb{R}^{n}$ and $D=\operatorname{diag}(1 / 2,0, \ldots, 0) \in \operatorname{gl}\left(\mathbb{R}^{n}, \mathbb{R}\right)$.
We now use results of [4] to algebraically characterize the coadjoint orbit $\mathcal{O}\left(\widehat{J}\left(e_{1}\right)\right)$. The affine cotype $\nabla$ represented by the tuple $\left(\mathbb{R}^{2 n+2}, Z, f_{n+1} ; \Omega\right)$ corresponds to the coadjoint orbit $\mathcal{O}\left(\widehat{J}\left(e_{1}\right)\right)$, see proposition 4 of [4]. Here

$$
Z=\widehat{J}\left(e_{1}\right)^{T}=\left(\begin{array}{c|c|c|c}
0 & 0 & r^{T} & 1 \\
\hline 0 & 0 & D & 2 r \\
\hline 0 & 0 & 0 & 0 \\
\hline 0 & 0 & 0 & 0
\end{array}\right) \in \operatorname{sp}\left(\mathbb{R}^{2 n+2}, \Omega\right) .
$$

Since $Z^{2}=0$, the cotype $\nabla$ is nilpotent and has height 1 . The parameter of $\nabla$ is 1 . Thus by proposition 7 of [4] the affine cotype $\nabla=\nabla_{1}(0), 1+\Delta$. Here $\nabla_{1}(0), 1$ is the nilpotent indecomposable cotype of height 1 and modulus

1 , which is represented by the tuple $\left(\mathbb{R}^{2},\left(\begin{array}{cc}0 & 1 \\ 0 & 0\end{array}\right), f_{1} ;\left(\begin{array}{cc}0 & 1 \\ -1 & 0\end{array}\right)\right)$, and $\Delta$ is a semisimple type, see [2]. Since $\Delta$ is a nilpotent of height 0 , it equals 0 .

We give another argument which proves the above assertion. The tuples $\left(\mathbb{R}^{2 n+2}, Z, f_{n+1} ; \Omega\right)$ and $\left(\mathbb{R}^{2 n+2}, P Z P^{-1}, f_{n+1} ; \Omega\right)$ are equivalent, when $P \in$ $\operatorname{Sp}\left(\mathbb{R}^{2 n+2}, \Omega\right)_{f_{n+1}}$ and thus represent the same cotype $\nabla$. Let $P=\left(\begin{array}{ccc}1 & 0 & 0 \\ d & I_{2 n} & 0 \\ f & -\frac{1}{2} d^{T} J & 1\end{array}\right)$ $\in \operatorname{Sp}\left(\mathbb{R}^{2 n+2}, \Omega\right)_{f_{n+1}}$. We compute.

$$
\begin{aligned}
& P Z P^{-1}=\left(\begin{array}{ccc}
1 & 0 & 0 \\
d & I_{2 n} & 0 \\
f & -\frac{1}{2} d^{T} J & 1
\end{array}\right)\left(\begin{array}{ccc}
0 & \widetilde{r}^{T} & 1 \\
0 & \widetilde{D} & \widetilde{s} \\
0 & 0 & 0
\end{array}\right)\left(\begin{array}{ccc}
1 & 0 & 0 \\
-d & I_{2 n} & 0 \\
-f & \frac{1}{2} d^{T} J & 1
\end{array}\right) \\
& \text { where } \widetilde{r}^{T}=\left(0 \mid r^{T}\right), \widetilde{s}^{T}=(2 J \widetilde{r})^{T}=(2 r \mid 0), \text { and } \\
& \widetilde{D}=\left(\begin{array}{ll}
0 & D \\
0 & 0
\end{array}\right) \\
&=\left(\begin{array}{ccc}
1 & 0 & 0 \\
d & I_{2 n} & 0 \\
f & -\frac{1}{2} d^{T} J & 1
\end{array}\right)\left(\begin{array}{cccc}
-\widetilde{r^{T}} d-f & \widetilde{r}^{T}+\frac{1}{2} d^{T} J & 1 \\
-\widetilde{D} d-f \widetilde{s} & \widetilde{D}+\frac{1}{2} \widetilde{s} \otimes d^{T} J & \widetilde{s} \\
0 & 0 & 0
\end{array}\right) .
\end{aligned}
$$

Choose $d=-\widetilde{s}$ and set $f=0$. Then

$$
\begin{aligned}
& -r^{T} d-f=\widetilde{r}^{T} \widetilde{s}=\left(0 \mid r^{T}\right)\left(\frac{2 r}{0}\right)=0 \\
& \widetilde{r}^{T}+\frac{1}{2} d^{T} J=\left(0 \mid r^{T}\right)+\frac{1}{2}\left(-2 r^{T} \mid 0\right)\left(\begin{array}{cc}
0 & I \\
-I & 0
\end{array}\right)=\left(0 \mid r^{T}\right)+\left(0 \mid-r^{T}\right)=0 \\
& -\widetilde{D} d-f \widetilde{s}=-\widetilde{D} \widetilde{s}=\left(\begin{array}{c|c}
0 & D \\
\hline 0 & 0
\end{array}\right)\left(\frac{2 r}{0}\right)=0 \\
& \widetilde{D}+\frac{1}{2} \widetilde{s} \otimes d^{T} J=\widetilde{D}+\frac{1}{2} \widetilde{s} \otimes(0 \mid-2 r)=\widetilde{D}-\frac{1}{2} e_{1} \otimes f_{1}^{T}=0 .
\end{aligned}
$$

Therefore

$$
P Z P^{-1}=\left(\begin{array}{ccc}
1 & 0 & 0 \\
-\widetilde{s} & I_{n} & 0 \\
0 & \frac{1}{2} \widetilde{s}^{T} J & 1
\end{array}\right)\left(\begin{array}{lll}
0 & 0 & 1 \\
0 & 0 & \widetilde{s} \\
0 & 0 & 0
\end{array}\right)=\left(\begin{array}{lll}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right) .
$$

## References

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