*The Monge–Ampère Equation in Magnetohydrodynamics**

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Communicated by J. B. DIAZ

1. Introduction. With the choice of the stream function and the pressure as independent variables, Martin [1], [2] has shown that the solution of problems in two-dimensional steady flow may be reduced to finding solutions of a Monge–Ampère partial differential equation. The method is based on forming proper combinations of the continuity equation and the two momentum equations so that the resultant equations may be interpreted as integrability conditions for three functions which, together with certain flow quantities, satisfy a Pfaffian system; the solution of this latter system may be made equivalent to the problem of integrating a Monge–Ampère equation, the solution of which leads to a parametric representation of the flow.

It is the purpose of the present study to show that Martin's investigation may be extended to a particular class of two-dimensional steady magnetohydrodynamic flows of a perfectly conducting fluid; namely, flows subjected to magnetic fields for which at most one component of the magnetic induction does not vanish. For this special class of flows, the governing equations simplify to a great extent, and, in fact, the three possible cases may be studied virtually simultaneously; structurally, they lead to the same differential equation, the only difference being in the definition of one of the independent variables, P, the sum of the fluid and magnetic pressures. With the choice of the stream function and the total pressure P as independent variables, it is seen that these special magnetic flows are governed by equations structurally identical to the equations considered by Martin; consequently, his analysis may be extended directly with but minor modifications.

2. The equations of motion. The equations which govern the two dimensional steady flow of an ideal, inviscid, perfectly conducting compressible fluid,

* This research was supported by a National Aeronautics and Space Administration general research grant to the University of Wisconsin.

subjected to an applied magnetic field $\mathbf{B} = (B_1, B_2, B_3)$, may be written as [3]:

$$p = \exp[(s - s_0)/c_v]\rho^\gamma \tag{2.1}$$

$$(\rho u)_x + (\rho v)_y = 0 \tag{2.2}$$

$$\rho(uu_x + vu_y) + p_x = -[B_3B_{3x} + B_2(B_{2x} - B_{1y})]\mu^{-1} \tag{2.3}$$

$$\rho(uv_x + vv_y) + p_y = -[B_3B_{3y} - B_1(B_{2x} - B_{1y})]\mu^{-1} \tag{2.4}$$

$$B_2B_{3y} + B_1B_{3x} = 0 \tag{2.5}$$

$$(vB_1 - uB_2)_y = 0 \tag{2.6}$$

$$(vB_1 - uB_2)_x = 0 \tag{2.7}$$

$$(uB_3)_x + (vB_3)_y = 0 \tag{2.8}$$

$$B_{1x} + B_{2y} = 0 \tag{2.9}$$

$$us_x + vs_y = 0, \tag{2.10}$$

where p, ρ, s, s_0, γ, $(u, v, 0)$, c, μ, $b_i^2 = B_i^2/\mu\rho$ $(i = 1, 2, 3)$ are, respectively, the pressure, density, specific entropy, specific entropy at some reference state, ratio of specific heats at constant pressure c_p and at constant volume c_v, velocity vector, local speed of sound, permeability and square of the Alfvén speed; all dependent variables are assumed to be at most functions of (x, y), and the subscripts x and y denote partial differentiation with respect to these spatial variables.

In each of the three cases for which at most one component of the induction vector does not vanish, it is possible to find an integral of the system of Equations (2.2)–(2.10), so that only the continuity equation, two momentum equations and, if necessary, the entropy equation need be considered. Since the details of the derivation have been given previously [4], it will suffice to merely state the results in the appropriate sections of the present paper.

3. The case $\mathbf{B} = (B_1, 0, 0)$. The body of the paper will be devoted to this case with the requisite modifications for the other cases indicated in later sections. When only the first component of the induction vector does not vanish, the system of Equations (2.2)–(2.10) reduces to

$$(\rho u)_x + (\rho v)_y = 0 \tag{3.1}$$

$$\rho v v_y + \left(p + \frac{B_1^2}{2\mu}\right)_y = 0 \tag{3.2}$$

$$\rho(uu_x + vu_y) + p_x = 0 \tag{3.3}$$

and the subsidiary relations $B_1 = B_1(y)$, $v = v(y)$ only and

$$B_1 v = A, \quad \text{a constant.} \tag{3.4}$$

Equation (2.10) merely states that the entropy is constant along the streamlines of the flow.

Equations (3.1)–(3.3) may be considered to be an underdetermined system of three partial differential equations for the four unknown functions u, v, p and ρ. It is convenient to study this system as it stands rather than immediately introducing an additional relation sufficient to make the system determinate.

Adding u times Equation (3.1) to Equation (3.3) and v times Equation (3.1) to Equation (3.2) gives the following two equations

$$(p + \rho u^2)_x + (\rho uv)_y = 0$$

$$\left(p + \frac{B_1^2}{2\mu} + \rho v^2\right)_y + v(\rho u)_x = 0.$$

Since B_1 and v are independent of x, these equations may be written more conveniently as

$$\left(p + \frac{B_1^2}{2\mu} + \rho u^2\right)_x + (\rho uv)_y = 0 \tag{3.5}$$

$$\left(p + \frac{B_1^2}{2\mu} + \rho v^2\right)_y + (\rho uv)_x = 0. \tag{3.6}$$

Equations (3.1), (3.5)–(3.6) may be interpreted as integrability conditions which allow the introduction of functions ψ, $\bar{\xi}$ and $\bar{\eta}$ which satisfy

$$d\psi\,(x, y) = -\rho v\,dx + \rho u\,dy \tag{3.7}$$

$$d\bar{\xi}\,(x, y) = -\rho uv\,dx + \left(p + \frac{B_1^2}{2\mu} + \rho u^2\right) dy \tag{3.8}$$

$$d\bar{\eta}\,(x, y) = -\left(p + \frac{B_1^2}{2\mu} + \rho v^2\right) dx + \rho uv\,dy. \tag{3.9}$$

By the use of Equation (3.7), Equations (3.8)–(3.9) may also be written as

$$d\bar{\xi} = u\,d\psi + \left(p + \frac{B_1^2}{2\mu}\right) dy \tag{3.10}$$

$$d\bar{\eta} = v\,d\psi - \left(p + \frac{B_1^2}{2\mu}\right) dx. \tag{3.11}$$

$\psi(x, y)$ is the usual stream function, and, since $d\psi = 0$ for an element of streamline, $d\bar{\xi}$ and $d\bar{\eta}$ are the x- and y- components of the (total) pressure force across an element of streamline; thus, the total changes in $\bar{\xi}$ and $\bar{\eta}$ along a fixed boundary give the components of the total force exerted. The negatives of $\bar{\xi}$ and $\bar{\eta}$ are frequently referred to as drag and lift functions.

From Equation (2.10), $s(p, \rho)$ is a function of ψ alone, say $F(\psi)$, which may be assumed to be determined from the boundary conditions, so a relation between (p, ρ, ψ) throughout the flow may be assumed known. In the case of conventional gas dynamics, p and ψ are used as new independent variables instead of (x, y) with ρ considered known from the aforementioned relation.

For magnetic flows of the special type considered in this paper, this procedure will be modified in order to most simply include the magnetic effects. Letting

$$\xi = \bar{\xi} - \left(p + \frac{B_1^2}{2\mu}\right)y$$

$$\eta = \bar{\eta} + \left(p + \frac{B_1^2}{2\mu}\right)x$$

it follows that Equations (3.10)–(3.11) may be written as

$$d\xi = u\,d\psi - y\,dP \tag{3.12}$$

$$d\eta = v\,d\psi + x\,dP \tag{3.13}$$

with $P = p + (B_1^2/2\mu)$.

For perfectly conducting fluids, it is clear that (ψ, P) provide a natural choice for new independent variables to be used instead of (x, y), and it is convenient to replace $(\bar{\xi}, \bar{\eta})$ by (ξ, η), using the definition given above. This choice of independent variables leads to equations structurally identical to the equations considered by Martin, and his analysis may be extended with but minor modifications. Of course, flows for which there exists a functional relation between P and ψ, i.e., $P = P(\psi)$, must be considered as special cases and treated separately. Flows related to this special case have been discussed elsewhere [5], [6], but a further discussion will be given in section 9 of the present paper.

With $\xi = \xi(\psi, P)$, $\eta = \eta(\psi, P)$, Equations (3.12)–(3.13) yield

$$x = \eta_P \qquad y = -\xi_P \tag{3.14}$$

$$u = \xi_\psi \qquad v = \eta_\psi\,. \tag{3.15}$$

Further, Equation (3.7) gives

$$uy_\psi - vx_\psi = \frac{1}{\rho}$$

$$uy_P - vx_P = 0,$$

and replacing x, y, u and v by their equivalents from Equations (3.14)–(3.15) gives

$$\xi_\psi \xi_{\psi P} + \eta_\psi \eta_{\psi P} + \frac{1}{\rho} = 0 \tag{3.16}$$

$$\xi_\psi \xi_{PP} + \eta_\psi \eta_{PP} = 0. \tag{3.17}$$

In these simultaneous differential equations for (ξ, η), ρ is (temporarily) regarded as an arbitrary function of (ψ, P); after these equations have been solved, x, y, u and v are given by Equations (3.14)–(3.15), which give a parametric representation of the flow.

In other words, the problem of finding a solution of the underdetermined system of Equations (3.1)–(3.3) for four unknown functions has been replaced by the problem of finding a solution of the underdetermined system of Equations (3.16)–(3.17) for three unknown functions; this latter system becomes determinate once the function $\rho(\psi, P)$ is specified. Further, a solution of the original system may then be constructed from the solutions for ξ and η.

Assuming

$$J = \frac{\partial(x, y)}{\partial(\psi, P)} = \xi_{P\psi}\eta_{PP} - \eta_{P\psi}\xi_{PP} \neq 0, \tag{3.18}$$

Equation (3.14) gives $\psi = \psi(x, y)$, $P = P(x, y)$, which, on substitution into Equation (3.15) and the relation $\rho = \rho(\psi, P)$, gives u, v and ρ as functions of (x, y). It is easy to show that u, v, ρ and P so obtained satisfy Equations (3.1)–(3.3). For this purpose, the following relations are needed:

$$\begin{aligned} Ju_x &= \xi_{\psi P}^2 - \xi_{\psi\psi}\xi_{PP} \\ Ju_y &= \xi_{\psi P}\eta_{P\psi} - \xi_{\psi\psi}\eta_{PP} \\ Jv_x &= \eta_{\psi P}\xi_{P\psi} - \eta_{\psi\psi}\xi_{PP} = 0 \\ Jv_y &= \eta_{\psi P}^2 - \eta_{\psi\psi}\eta_{PP} \\ J\rho_x &= \xi_{P\psi}\rho_P - \xi_{PP}\rho_\psi \\ J\rho_y &= \eta_{P\psi}\rho_P - \eta_{PP}\rho_\psi \\ JP_x &= \xi_{P\psi} \\ JP_y &= \eta_{P\psi}\ . \end{aligned} \tag{3.19}$$

In addition, by differentiating Equation (3.16) with respect to P and Equation (3.17) with respect to ψ, the following relation obtains

$$\rho_P = \rho^2[\xi_{\psi P}^2 + \eta_{\psi P}^2 - \xi_{\psi\psi}\xi_{PP} - \eta_{\psi\psi}\eta_{PP}]. \tag{3.20}$$

Substituting the relations from Equation (3.19) into the system of Equations (3.1)–(3.3), it follows from Equations (3.15)–(3.17) and (3.20) that u, v, ρ and P satisfy Equations (3.1)–(3.3).

Finally, solving Equations (3.17)–(3.18) simultaneously for ξ_{PP} and η_{PP} gives

$$\xi_{PP} = \rho J\eta_\psi \qquad \eta_{PP} = -\rho J\xi_\psi \tag{3.21}$$

so that

$$x_P^2 + y_P^2 = \rho^2 q^2 J^2$$

with $q^2 = u^2 + v^2$. Thus, the condition $x_P^2 + y_P^2 > 0$, which says the directional derivative of P along a streamline must remain finite, and Equation (3.18) are equivalent provided $\rho \neq 0$, $q \neq 0$.

These conclusions are, of course, virtually identical to the conclusions obtained by Martin and may be summarized in the analog of his Theorem 1, *viz.*, *The solution of the underdetermined system of Equations* (3.1)–(3.3) *and the solution of the underdetermined system of Equations* (3.16)–(3.17), *containing the arbitrary function* $\rho = \rho(\psi, P)$, *are equivalent problems provided the streamlines are not curves of constant* P, *the directional derivative of the* (*total*) *pressure along the streamlines remains finite, the density does not vanish and the flow has no stagnation points.*

The specification of the density function $\rho = \rho(\psi, P)$ would be equivalent to assuming that along each streamline the density is a function of P alone, with the functional dependence being allowed to vary from one streamline to another.

Equation (3.16) may be written as

$$uu_P + vv_P + \frac{1}{\rho} = 0$$

which may be integrated to give a generalized Bernoulli theorem

$$\frac{q^2}{2} - \frac{q_0^2}{2} = -\int_{P_0}^{P} \frac{dP}{\rho} \tag{3.22}$$

$$P_0 = P_0(\psi), \qquad q_0 = q_0(\psi),$$

where P_0 is some reference (total) pressure, which may vary from streamline to streamline, and $P_0(\psi)$ and $q_0(\psi)$ are arbitrary functions.

In other words, the system of Equations (3.16)–(3.17) may be replaced by the system of Equation (3.17) and

$$\xi_\psi^2 + \eta_\psi^2 = q^2, \tag{3.23}$$

where $q = q(\psi, P)$ is given by Equation (3.22); the function q may be taken arbitrarily save that $qq_P < 0$, *i.e.*,

$$q_0 = q(\psi, P_0(\psi)), \qquad \rho = -\frac{1}{qq_P} \tag{3.24}$$

Differentiating Equation (3.22) twice with respect to P gives

$$q_P^2 + qq_{PP} = \frac{1}{\rho^2}\,\rho_P$$

In the conventional case of an ideal gas, the corresponding term of the right-hand side is $\rho_p = 1/c^2$ since the derivative is taken with ψ constant. Restricting attention to an ideal gas and noting from Equation (3.4) that

$$P = p + \frac{A^2}{2\mu v^2},$$

it follows that

$$dp = \left[1 + \frac{A^2}{\mu\eta_\psi^3}\,\eta_{\psi P}\right] dP$$

along curves of constant ψ. Consequently

$$q_P^2 + qq_{PP} = \frac{1}{\rho^2 c^2}\left[1 + \frac{A^2}{\mu\eta_\psi^3}\eta_{\psi P}\right]$$

which gives

$$M^2 = \frac{1 + \dfrac{qq_{PP}}{q_P^2}}{1 + \dfrac{A^2\eta_{\psi P}}{\mu\eta_\psi^3}}, \tag{3.25}$$

where $M = q/c$ is the usual Mach number, so that the flow will be supersonic if

$$q_{PP} > \frac{A^2 v_P q_P^2}{\mu v^3 q}$$

and subsonic if the inequality is reversed; in the nonmagnetic case, the right-hand side is zero.

For a perfect gas (Equation (2.1)), Equation (3.22) may be written as

$$\frac{q^2}{2} - \frac{q_0^2}{2} = -\frac{\exp[(s - s_0)c_v]p^{1-n}}{1-n} - \frac{B_1^2}{2\mu\rho} - \int \frac{b_1^2\, d\rho}{2\rho}$$

with $n = 1/\gamma$; for $P_0 = 0$, q_0 would be the ultimate velocity magnitude attainable along a streamline at zero pressure and magnetic induction.

Since $v = v(y)$ only, the vorticity $\omega = v_x - u_y$ may be written as

$$J\omega = \xi_{\psi\psi}\eta_{PP} - \xi_{\psi P}\eta_{P\psi}\ .$$

The condition $v_x = 0$ gives $\eta_{P\psi}\xi_{P\psi} = \eta_{\psi\psi}\xi_{PP}$ so that

$$J\omega = \xi_{\psi\psi}\eta_{PP} - \eta_{\psi\psi}\xi_{PP}\ . \tag{3.26}$$

Substituting Equation (3.21) into Equation (3.26) gives

$$\omega = -\rho q q_\psi = q_\psi / q_P\ . \tag{3.27}$$

For a perfect gas with uniform ultimate speed, Equation (3.27) gives

$$\omega = \frac{p}{(\gamma - 1)c_v}\frac{ds}{d\psi} - \left(p + \frac{B_1^2}{2\mu}\right)_\psi$$

and since P and ψ are independent variables, this reduces to

$$\omega = \frac{p}{(\gamma - 1)c_v}\frac{ds}{d\psi}. \tag{3.28}$$

Thus, if q_0 is constant, the vorticity is proportional to the pressure along a streamline; in the nonmagnetic case, this result was found by Crocco.

Finally, from the definitions of $\bar{\xi}$ and $\bar{\eta}$

$$\begin{aligned}\bar{\xi} &= \xi - P\xi_P \\ \bar{\eta} &= \eta - P\eta_P\ .\end{aligned} \tag{3.29}$$

4. The Munk–Prim substitution principle. As shown above, the integration of the original system of Equations (3.1)–(3.3) is equivalent to the integration of the system

$$\xi_\psi^2 + \eta_\psi^2 = q^2, \qquad \xi_\psi \xi_{PP} + \eta_\psi \eta_{PP} = 0, \qquad q = q(\psi, P).$$

Clearly, a change of parameter $\psi = \psi(\psi^*)$ along the curves of constant P leaves the form of this system unchanged; the new system is

$$\left(\frac{\partial \xi^*}{\partial \psi^*}\right)^2 + \left(\frac{\partial \eta^*}{\partial \psi^*}\right)^2 = (q^*)^2$$

$$\frac{\partial \xi^*}{\partial \psi^*}\frac{\partial^2 \xi^*}{\partial P^2} + \frac{\partial \eta^*}{\partial \psi^*}\frac{\partial^2 \eta^*}{\partial P^2} = 0,$$

where $\xi^* = \xi(\psi(\psi^*), P)$, $\eta^* = \eta(\psi(\psi^*), P)$

$$q^* = \left|\frac{d\psi}{d\psi^*}\right| q(\psi(\psi^*), P).$$

Further

$$x^* = \eta_P^* = \eta_P\,, \qquad y^* = -\xi_P^* = -\xi_P$$

$$u^* = \frac{\partial \xi^*}{\partial \psi^*} = \frac{d\psi}{d\psi^*}\xi_\psi = \frac{d\psi}{d\psi^*}u$$

$$v^* = \frac{\partial \eta^*}{\partial \psi^*} = \frac{d\psi}{d\psi^*}\eta_\psi = \frac{d\psi}{d\psi^*}v$$

$$\bar{\xi}^* = \xi^* - P\xi_P^* = \bar{\xi}$$

$$\bar{\eta}^* = \eta^* - P\eta_P^* = \bar{\eta}$$

$$B_1^* = \left|\frac{d\psi}{d\psi^*}\right|^{-1} B_1\,,$$

so that the new flow has the same streamlines, (total) pressure distribution, lift and drag functions as the original system; however, the velocity components u and v and the magnetic field must change, and, from Equation (3.20),

$$\rho^* = (d\psi/d\psi^*)^{-2}\rho.$$

Thus, there is the following substitution principle:

From a given flow, a new flow having the same streamlines and total pressure distribution P (but not the same fluid pressure distribution p) may be obtained by multiplying the magnitudes of the velocity vectors tangent to a given streamline by the factor λ, the density by the factor λ^{-2} and the magnetic field by the factor λ^{-1}, where the factor λ is allowed to vary from streamline to streamline.

This substitution principle provides an extension to the two-dimensional flow of a perfectly conducting fluid of a principle first formulated by Munk and Prim [7].

Smith [8] has considered three-dimensional flows of a perfectly conducting fluid for the case of parallel and transverse orientations of the magnetic field and has extended the substitution principle in certain special cases.

5. The Monge–Ampère partial differential equation. When Equations (3.17) and (3.23) are solved simultaneously for η_ψ and η_{PP}, it follows that

$$\eta_\psi = (q^2 - \xi_\psi^2)^{1/2}, \qquad \eta_{PP} = -\frac{\xi_\psi \xi_{PP}}{(q^2 - \xi_\psi^2)^{1/2}} \tag{5.1}$$

Eliminating η by partial differentiation, these two equations lead to the following Monge–Ampère equation for $\xi = \xi(\psi, P)$:

$$2qq_P\xi_\psi\xi_{\psi P} - qq_\psi\xi_\psi\xi_{PP} - (qq_{PP} + q_P^2)\xi_\psi^2 + q^3q_{PP} + q^2(\xi_{\psi\psi}\xi_{PP} - \xi_{\psi P}^2) = 0. \tag{5.2}$$

From the symmetry of Equations (3.17) and (3.23), $\eta = \eta(\psi, P)$ is also a solution of Equation (5.2).

Consequently, the problem of solving the system of Equations (3.17) and (3.23) is equivalent to finding the solution of Equation (5.2), which is identical to the equation derived by Martin [1], save for the difference in the meaning of the symbol P.

When a solution ξ of Equation (5.2) has been found, η_P may be determined up to an arbitrary additive constant from the line integral

$$\eta_P = \int [\eta_{P\psi}\, d\psi + \eta_{PP}\, dP] \tag{5.3}$$

after which η is given by the line integral

$$\eta = \int [\eta_\psi\, d\psi + \eta_P\, dP].$$

Martin has investigated the existence and uniqueness of flows when the fluid velocity q in Equation (5.2) is a prescribed function of (ψ, P), which from the previous considerations is equivalent to prescribing the associated fluid velocity $q_0 = q_0(\psi)$ and the density function $\rho = \rho(\psi, P)$. For the purposes of the present paper, it will be sufficient to state his results.

If $C : x = x(P)$, $y = y(P)$ is an analytic arc with $x'^2 + y'^2 > 0$ holding for $P_0 \leqq P \leqq P_1$, if the fluid velocity $q(\psi, P)$ is regular analytic for $|\psi - \psi_0| < \delta$, $P_0 \leqq P \leqq P_1$ with $q(\psi_0, P) \neq 0$, the arc C is a streamline for exactly one flow (of prescribed sense on C) with (total) pressure P and fluid velocity $q(\psi_0, P)$ at the point $(x(P), y(P))$. The streamlines and the hodographs of the flow in which C is imbedded are presented parametrically by Equations (3.14)–(3.15) *in which the functions $\xi(\psi, P)$, $\eta(\psi, P)$ permit the expansions, Equation* (5.4), *valid for sufficiently small $|\psi - \psi_0|$, the coefficients h, k being regular analytic functions of P in the interval $P_0 \leqq P \leqq P_1$.*

The prime denotes differentiation with respect to P, and the mentioned expansions are

$$\begin{aligned} \xi &= h_0 + h_1(\psi - \psi_0) + h_2(\psi - \psi_0)^2 + \cdots \\ \eta &= k_0 + k_1(\psi - \psi_0) + k_2(\psi - \psi_0)^2 + \cdots \\ \hat{q}^2 &= \hat{q}_0 + \hat{q}_1(\psi - \psi_0) + \cdots, \qquad \hat{q}_0 \neq 0. \end{aligned} \tag{5.4}$$

Martin has shown how the coefficients in these expansions may be determined step by step.

In order to illustrate these conclusions, the special case of irrotational flow will be considered. Leaving aside cases where the density or velocity vanish, it follows from Equation (3.27) that a necessary and sufficient condition for irrotationality is $q_\psi = 0$. In order to have a solution linear in $\psi - \psi_0$, *viz.*,

$$\xi = h_0 + h_1(\psi - \psi_0), \qquad \eta = k_0 + k_1(\psi - \psi_0), \tag{5.5}$$

it is necessary that $q_\psi = 0$; conversely, it will be shown there exist solutions, Equation (5.5), among the irrotational flows. Again, the conclusions are virtually the same as those obtained by Martin except for the meaning of the symbol P; thus, it will be sufficient to merely state his results incorporating the requisite modifications.

With

$$q = q(P) \tag{5.6}$$

and $\hat{q}_{n-1} = h_n = k_n = 0$ for $n = 2, 3, \cdots$, in the expansions given by Equations (5.4), it is found that

$$h_1 = q \cos \theta, \qquad k_1 = q \sin \theta \tag{5.7}$$

and

$$\theta = \theta(P) = \theta_0 + \int_{P_0}^{P} \left[\frac{q''}{q}\right]^{1/2} dP, \tag{5.8}$$

where $\theta = \text{arc tan } (y'/x')$. This gives

$$\begin{aligned} x &= k_0' + \zeta'(\psi - \psi_0) \cos (\theta - \mu^*) \\ y &= -h_0' + \zeta'(\psi - \psi_0) \sin (\theta - \mu^*) \end{aligned} \tag{5.9}$$

$$\begin{aligned} u &= u(P) = q(P) \cos \theta(P) \\ v &= v(P) = q(P) \sin \theta(P), \end{aligned} \tag{5.10}$$

where $\theta(P)$ and $q(P)$ are the functions given in Equations (5.8) and (5.6) and ζ denotes the arc length of the hodograph curve

$$u = u(P), \qquad v = v(P)$$

in the hodograph plane. The angle μ^* is defined by the condition

$$\cot \mu^* = \frac{q\theta'}{q'}. \tag{5.11}$$

From Equations (5.8) and (3.25), it follows that

$$\cot^2 \mu^* = M^2 - 1 + \frac{A^2}{\mu \eta_\psi^3} \eta_{\psi P} M^2 = M^2 - 1 + \frac{A^2 k_1' M^2}{\mu k_1^3} \tag{5.12}$$

$$\zeta' = \zeta'(P) = -\frac{\csc \mu^*}{\rho q}$$

In analogy with conventional gas dynamics, the angle $\mu^* = \mu^*(P)$ may be called a generalized Mach angle.

Setting P = constant in Equations (5.9)–(5.10), it follows that the curves of constant total pressure are straight lines along which all velocity vectors have the same magnitude and make the same angle μ^* with the curves of constant total pressure. These straight lines might be called generalized Mach lines.

With ψ = constant in Equations (5.9), the streamlines are given parametrically in terms of P; all streamlines have the same hodograph, so these flows might be called generalized Prandtl–Meyer flows. The problem is completed by determining (h_0, k_0), which may be accomplished as indicated by Martin.

6. The functions $\bar{\xi}$ and $\bar{n}$. With (ψ, y) as independent variables instead of (ψ, P) and $\bar{\xi} = \bar{\xi}(\psi, y)$ as unknown function, Equations (3.7) and (3.10) give

$$d\bar{\xi} = u\, d\psi + P\, dy$$

$$d\psi = -\rho v x_\psi\, d\psi + \rho(u - v x_y)\, dy$$

and, therefore,

$$u = \bar{\xi}_\psi, \qquad P = \bar{\xi}_y, \qquad x_\psi = -\frac{1}{\rho v}, \qquad x_y = \frac{u}{v}.$$

Since $u^2 + v^2 = q^2$, the last two equations may be written as

$$x_\psi = \frac{q q_P}{(q^2 - \bar{\xi}_\psi^2)^{1/2}}, \qquad x_y = \frac{\bar{\xi}_\psi}{(q^2 - \bar{\xi}_\psi^2)^{1/2}}. \tag{6.1}$$

Eliminating x in Equations (6.1) gives the following quasi-linear equation

$$q^2 \bar{\xi}_{\psi\psi} - [q^3 q_{PP} - (q q_{PP} + q_P^2)\bar{\xi}_\psi^2]\bar{\xi}_{yy} - q q_\psi \bar{\xi}_\psi + q q_P \bar{\xi}_y \bar{\xi}_{\psi y} = 0, \tag{6.2}$$

where $q = q(\psi, \bar{\xi}_y)$. In Equation (6.2), which could have been obtained directly from Equation (5.2) by applying an Ampère contact transformation, it is assumed that P is to be replaced by $\bar{\xi}_y$. Given a solution $\bar{\xi} = \bar{\xi}(\psi, y)$ of Equation (6.2), the Cartesian equation of the streamlines is given by

$$x = x(\psi, y) = \int [x_\psi\, d\psi + x_y\, dy],$$

where (x_ψ, x_v) are given by Equation (6.1).

Finally,

$$P = \bar{\xi}_v, \qquad u = \bar{\xi}_\psi$$
$$v = (q^2 - \bar{\xi}_\psi^2)^{1/2}.$$

7. The special case B = (0, B₂, 0). When only the second component of the induction vector does not vanish, the system of Equations (2.2)–(2.10) reduces to

(7.1) $$(\rho u)_x + (\rho v)_y = 0$$

(7.2) $$\rho u u_x + \left(p + \frac{B_2^2}{2\mu}\right)_x = 0$$

(7.3) $$\rho(u v_x + v v_y) + p_y = 0$$

and the subsidiary conditions $B_2 = B_2(x)$, $u = u(x)$ and

(7.4) $$B_2 u = A_1, \quad \text{a constant.}$$

Adding u times Equation (7.1) to Equation (7.2) and v times Equation (7.1) to Equation (7.3) gives

$$\left(p + \frac{B_2^2}{2\mu} + \rho u^2\right)_x + u(\rho v)_y = 0$$
$$(p + \rho v^2)_y + (\rho u v)_x = 0.$$

Since B_2 and u are independent of y, these equations may be written more conveniently as

(7.5) $$\left(p + \frac{B_2^2}{2\mu} + \rho u^2\right)_x + (\rho u v)_y = 0$$

(7.6) $$\left(p + \frac{B_2^2}{2\mu} + \rho v^2\right)_y + (\rho u v)_x = 0.$$

These equations are identical in structure to Equations (3.5)–(3.6); consequently, the majority of the analysis of the preceding sections also applies to this case if P is redefined as $P = p + (B_2^2/2\mu)$.

The discussions used to derive Equation (3.25) for the Mach number must be modified. Along curves of constant ψ,

$$dp = \left[1 + \frac{A_1^2}{\mu \xi_\psi^3} \xi_{\psi P}\right] dP$$

so that the analog of Equation (3.25) becomes

$$M^2 = \frac{1 + \dfrac{q q_{PP}}{q_P^2}}{1 + \dfrac{A_1^2 \xi_{\psi P}}{\mu \xi_\psi^3}}.$$

Consequently, Equation (5.12), defining the generalized Mach angle, would be changed to

$$\cot^2 \mu^* = M^2 - 1 + \frac{A_1^2 \xi_{\psi P}}{\mu \xi_\psi^3} M^2 = M^2 - 1 + \frac{A_1^2 h_1' M^2}{\mu h_1^3}.$$

8. The special case B = (0, 0, B₃). When only the third component of the induction vector does not vanish, the system of Equations (2.2)–(2.10) reduces to

$$(\rho u)_x + (\rho v)_y = 0 \tag{8.1}$$

$$\rho(uu_x + vu_y) + \left(p + \frac{B_3^2}{2\mu}\right)_x = 0 \tag{8.2}$$

$$\rho(uv_x + vv_y) + \left(p + \frac{B_3^2}{2\mu}\right)_y = 0 \tag{8.3}$$

$$(uB_3)_x + (vB_3)_y = 0 \tag{8.4}$$

$$us_x + vs_y = 0 \tag{8.5}$$

From Equations (8.1) and (8.4), it is clear that B_3/ρ is constant along a streamline, which is the usual frozen flux relation, so that it is possible to write

$$B_3 = \rho S(s)$$

where S is an arbitrary function of the entropy; thus, Equation (8.4) may be suppressed, and the usual combination of Equations (8.1)–(8.3) lead to the system:

$$\left(p + \frac{B_3^2}{2\mu} + \rho u^2\right)_x + (\rho uv)_y = 0$$

$$\left(p + \frac{B_3^2}{2\mu} + \rho v^2\right)_y + (\rho uv)_x = 0$$

which is identical in structure to the system of Equations (3.5)–(3.6). Again, most of the analysis of the preceding sections applies if P is redefined as

$$P = p + \frac{B_3^2}{2\mu}.$$

Along curves of constant ψ, it is easily seen that

$$dP = \frac{a^2}{c^2}\, dp$$

where $a = (b_3^2 + c^2)^{1/2}$, the limiting case of the fast wave speed; consequently, the analog of Equation (3.25) is

$$N^2 - 1 = \frac{qq_{PP}}{q_P^2},$$

where $N = q/a$, a magnetic Mach number. Further, the magnetic Mach angle would be defined by

$$\cot^2 \mu^* = N^2 - 1$$

or

$$\sin \mu^* = \frac{1}{N}$$

This result agrees with conclusions found earlier [4].

Finally, the generalized Bernoulli relation may be integrated explicitly to give [4]:

$$\frac{q^2}{2} - \frac{q_0^2}{2} = b_3^2 - \frac{c^2}{\gamma - 1} \equiv \frac{(2 - \gamma)b_3^2}{\gamma - 1} - \frac{a^2}{\gamma - 1}.$$

9. The special case P = P(ψ). The preceding analysis must be modified if there is a functional relation between P and ψ. Equations (3.7), (3.12)–(3.13) must be replaced by

$$d\xi = (u - yP')\, d\psi \tag{9.1}$$

$$d\eta = (v + xP')\, d\psi \tag{9.2}$$

$$d\psi = -\rho v\, dx + \rho u\, dy, \tag{9.3}$$

where the prime denotes differentiation with respect to ψ. Thus, ξ and η are functions of ψ alone, and taking ψ and β, where β denotes arc length measured along a streamline from some reference point on it, as independent variables, it follows from Equation (9.1)–(9.3) that

$$\xi' = u - yP', \qquad \eta' = v + xP' \tag{9.4}$$

$$uy_\beta - vx_\beta = 0 \tag{9.5}$$

$$uy_\psi - vx_\psi = \rho^{-1}. \tag{9.6}$$

Differentiating Equation (9.4) with respect to β gives

$$u_\beta - y_\beta P' = 0, \qquad v_\beta + x_\beta P' = 0 \tag{9.7}$$

which, provided $P' \neq 0$, may be used to eliminate x_β, y_β from Equation (9.5) to give

$$uu_\beta + vv_\beta = 0; \quad \text{thus}$$

$$u^2 + v^2 = q^2, \qquad q = q(\psi). \tag{9.8}$$

Using Equations (9.4) to eliminate u, v from Equation (9.8) gives

$$\left(x - \frac{\eta'}{P'}\right)^2 + \left(y + \frac{\xi'}{P'}\right)^2 = \frac{q^2}{P'^2} \tag{9.9}$$

which shows that the streamlines are circles with center $(\eta'/P', -\xi'/P')$ and radius $r = r(\psi) = q/P'$. It is easy to show that these circles are concentric, and the conclusions of this section may be summarized by modifying Martin's Theorem 2.

If the streamlines are curves of constant total pressure and $\rho = \rho(\psi)$, *the streamlines are either concentric circles* ($P \neq$ *constant*) *or are parallel straight lines* ($P =$ *constant*). *In the first case, any two of* P, q, r *may be prescribed as functions of the radius* r *to determine the third as a function of* r. *In the second case,* ρ, q *may be taken as arbitrary functions of the distance* n *of the straight line from the origin.*

The special case for which $\mathbf{B} = (0, 0, B_3)$ has been investigated as regards the existence of solutions $p = p(\psi)$ [5]; for this case $p = p(\psi)$ implies $B_3 = B_3(\psi)$ (and, of course, $\rho = \rho(\psi)$) so that $P = P(\psi)$, and it was shown that circles and their limiting cases of straight lines were possible streamline configurations. It is now clear the streamlines must be either concentric circles or parallel straight lines.

The special cases for which only B_1 or B_2 appears have also been investigated for solutions of the type $p = p(\psi)$ [6]; however $\mathbf{B} = \mathbf{B}(\psi)$ in these cases only if $\mathbf{B}$ is constant.

10. Unsteady one-dimensional flow. The equations governing one-dimensional unsteady flow subjected to a transverse magnetic field may be written as

$$\rho_t + (\rho u)_x = 0 \tag{10.1}$$

$$\rho(u_t + uu_x) + \left(p + \frac{B_2^2}{2\mu}\right)_x = 0 \tag{10.2}$$

$$B_{2t} + (uB_2)_x = 0 \tag{10.3}$$

$$s_t + us_x = 0. \tag{10.4}$$

From Equations (10.1) and (10.3), it follows that Equation (10.3) may be suppressed and

$$B_2 = \rho S(s), \tag{10.5}$$

where S is an arbitrary function.

Adding u times Equation (10.1) to Equation (10.2) gives

$$(\rho u)_t + \left(p + \frac{B_2^2}{2\mu} + \rho u^2\right)_x = 0. \tag{10.6}$$

Equations (10.1) and (10.6) permit the introduction of functions ψ and ξ by the relations

$$d\psi = \rho(dx - u\,dt) \tag{10.7}$$

$$d\xi = \rho u\,dx - \left(p + \frac{B_2^2}{2\mu} + \rho u^2\right) dt = u\,d\psi - P\,dt, \tag{10.8}$$

where $P = (B_2^2/2\mu) + p$.

Letting $\xi = \bar{\xi} + Pt$ gives

$$d\xi = u\,d\psi + t\,dP \tag{10.9}$$

From Equation (10.7), with ψ and P as independent variables

$$\begin{aligned} x_\psi - ut_\psi &= \rho^{-1} \\ x_P - ut_P &= 0. \end{aligned} \tag{10.10}$$

Since $u = \xi_\psi$, $t = \xi_P$ from Equation (10.9), Equations (10.10) give

$$\begin{aligned} x_\psi - \xi_\psi \xi_{P\psi} &= \rho^{-1} \\ x_P - \xi_\psi \xi_{PP} &= 0, \end{aligned} \tag{10.11}$$

where ρ is an arbitrary function. Eliminating x from Equations (10.11) gives

$$\xi_{\psi\psi}\xi_{PP} - \xi_{\psi P}^2 = \left(\frac{1}{\rho}\right)_P . \tag{10.12}$$

After a solution $\xi = \xi(\psi, P)$ of Equation (10.12) has been found, the flow is represented by

$$x = x(\psi, P) = \int (x_\psi\, d\psi + x_P\, dP) = \int \{(\xi_\psi \xi_{P\psi} + \rho^{-1})\, d\psi + \xi_\psi \xi_{PP}\, dP\}$$

$$u = \xi_\psi(\psi, P), \qquad t = \xi_P(\psi, P).$$

Helliwell [9] has studied a related equation. In this connection, see also Giese [10].

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Date Communicated: April 18, 1967