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The Monge–Ampère equation on almost complex manifolds

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Abstract We study the Dirichlet problem for the Monge–Ampère equation on almost complex manifolds. We obtain the existence of the unique smooth solution in strictly pseudoconvex domains.

Keywords Monge–Ampère equation · Almost complex manifold · J-plurisubharmonic function · Maximal function

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Let (M, J) be an almost complex manifold of a real dimension $2n$ (the definitions are given in Sect. 1). Pali proved (in [7]) that, as it is in the case of complex geometry, for plurisubharmonic functions the $(1, 1)$ current $i\partial\bar{\partial}u$ is positive.[1] So for a smooth plurisubharmonic function u we have well defined Monge–Ampère operator $(i\partial\bar{\partial}u)^n \geq 0$ and we can study the complex Monge–Ampère equation

$$(i\partial\bar{\partial}u)^n = f dV, \tag{1}$$

where $f \geq 0$ and dV is a (smooth) volume form.

Let $\Omega \Subset M$ be a strictly pseudoconvex domain of class $\mathcal{C}^\infty$. In this article we study the following Dirichlet problem for the Monge–Ampère equation:

$$\begin{cases} u \in \mathcal{PSH}(\Omega) \cap \mathcal{C}^\infty(\bar{\Omega}) \\ (i\partial\bar{\partial}u)^n = dV \;\text{ in }\; \Omega \\ u = \varphi \;\text{ on }\; \partial\Omega, \end{cases} \tag{2}$$

where $\varphi \in \mathcal{C}^\infty(\bar{\Omega})$. The main theorem is the following:

[1] But it can be not closed!.

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Theorem 1 *There is a unique smooth plurisubharmonic solution u of the problem* (2).

In [3] the theorem above was proved for $\Omega \subset \mathbb{C}^n$ with J_{st}. Note that even in the integrable case it is not enough to assume that $\partial\Omega$ is strictly pseudoconvex.[2] Indeed, if Ω is the blow-up of a strictly pseudoconvex domain in $\mathbb{C}^n$ in one point, then $\partial\Omega$ is strictly pseudoconvex. But if $u \in \mathcal{PSH}(\Omega) \cap \mathcal{C}^\infty(\bar{\Omega})$, then the form $(i\partial\bar{\partial}u)^n$ is not a volume form.

In case of J not integrable McDuff constructed a domain Ω with a non connected strictly pseudoconvex boundary (see [6]).[3] One can prove the theorem above (in almost the same way) for Ω not necessary strictly pseudoconvex but $\partial\Omega$ strictly pseudoconvex and $dV \le (i\partial\bar{\partial}\varphi)^n$. It is however not clear for the author, whether there is an example of such φ in McDuff's example (or in any other not strictly pseudoconvex domain with a strictly pseudoconvex boundary).

In the last section we explain how Theorem 1 gives the theorem of Harvey and Lawson about existing a continuous solution of the Dirichlet Problem for maximal functions. We even improve their result by proving that the solution is Lipschitz (if the boundary condition is regular enough).

1 Notion

We say that (M, J) is an almost complex manifold if M is a manifold and J is an ($\mathcal{C}^\infty$ smooth) endomorphism of the tangent bundle TM, such that $J^2 = -\mathrm{id}$. The real dimension of M is even in that case.

We have then a direct sum decomposition $T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$, where $T_{\mathbb{C}}M$ is a complexification of TM,

$$T^{1,0}M = \{X - iJX : X \in TM\}$$

and

$$T^{0,1}M = \{X + iJX : X \in TM\} \left(= \left\{\zeta \in T_{\mathbb{C}}M : \bar{\zeta} \in T^{1,0}M\right\}\right).$$

Let $\mathcal{A}^k$ be the set of k-forms, i.e. the set of sections of $\bigwedge^k (T_{\mathbb{C}}M)^\star$ and let $\mathcal{A}^{p,q}$ be the set of (p,q)-forms, i.e. the set of sections of $\bigwedge^p (T^{1,0}M)^\star \otimes_{(\mathbb{C})} \bigwedge^q (T^{0,1}M)^\star$. Then we have a direct sum decomposition $\mathcal{A}^k = \bigoplus_{p+q=k} \mathcal{A}^{p,q}$. We denote the projections $\mathcal{A}^k \to \mathcal{A}^{p,q}$ by $\Pi^{p,q}$.

If $d : \mathcal{A}^k \to \mathcal{A}^{k+1}$ is (the $\mathbb{C}$-linear extension of) the exterior differential, then we define $\partial : \mathcal{A}^{p,q} \to \mathcal{A}^{p+1,q}$ as $\Pi^{p+1,q} \circ d$ and $\bar{\partial} : \mathcal{A}^{p,q} \to \mathcal{A}^{p,q+1}$ as $\Pi^{p,q+1} \circ d$.

We say that an almost complex structure J is integrable, if any of the following (equivalent) conditions is satisfied:

(i) $d = \partial + \bar{\partial}$;
(ii) $\bar{\partial}^2 = 0$;
(iii) $[\zeta, \xi] \in T^{0,1}M$ for vector fields $\zeta, \xi \in T^{0,1}M$.

By the Newlander–Nirenberg Theorem J is integrable if and only if it is induced by a complex structure.

[2] We say that $\partial\Omega$ is strictly pseudoconvex if it is locally equal to $\{\rho = 0\}$, for some smooth function ρ such that $i\partial\bar{\partial}\rho > 0$, $\{\rho < 0\} \subset \Omega$ and $\nabla\rho \neq 0$.

[3] If Ω is strictly pseudoconvex, then $\partial\Omega$ is connected (see [2]).

In the paper $\zeta_1, \dots, \zeta_n$ is always a (local) frame of $T^{1,0}$. Let us put for a smooth function u

$$u_p = \zeta_p u, \quad u_{p\bar q} = \zeta_p \bar\zeta_q u = u_{\bar q p} + \left[\zeta_p, \bar\zeta_q\right] u, \quad \text{etc.}$$

and

$$A_{p\bar q} = A_{p\bar q}(u) = u_{p\bar q} - \left[\zeta_p, \bar\zeta_q\right]^{0,1} u,$$

where for any $X \in T_{\mathbb{C}}M$ a vector $X^{0,1} \in T^{0,1}M$ is such that $X^{1,0} := X - X^{0,1} \in T^{1,0}M$. Then for a smooth function u we have (see [7]):

$$i\partial\bar\partial u = i \sum A_{p\bar q} \zeta_p^\star \wedge \bar\zeta_q^\star,$$

where $\zeta_1^\star, \dots, \zeta_n^\star, \bar\zeta_1^\star, \dots, \bar\zeta_n^\star$ is a base of $(T_{\mathbb{C}}M)^\star$ dual to the base $\zeta_1, \dots, \zeta_n, \bar\zeta_1, \dots, \bar\zeta_n$ of $T_{\mathbb{C}}M$.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. We say that a (smooth) function $\lambda : \mathbb{D} \to M$ is J-holomorphic or simpler holomorphic if $\lambda'(\frac{\partial}{\partial \bar z}) \in T^{0,1}M$. The following proposition from [5], where it is stated for $C^{k',\alpha}$ class of J, shows that there exists plenty of such disks:

Proposition 1.1 *Let $0 \in M \subset \mathbb{R}^{2n}$, $k, k' \geq 1$. For $v_0, v_1, \dots, v_k \in \mathbb{R}^{2n}$ close enough to 0, there is a holomorphic function $\lambda : \mathbb{D} \to M$, such that $\lambda(0) = v_0$ and $\frac{\partial^l \lambda}{\partial x^l} = v_l$, for $l = 1, \dots, k$. Moreover, we can choose λ with $\mathcal{C}^1$ dependence on parameters $(v_0, \dots, v_k) \in (\mathbb{R}^{2n})^{k+1}$, where for holomorphic functions we consider $\mathcal{C}^{k'}$ norm.*

We can locally normalize coordinates with respect to a given holomorphic disc λ, that is we can assume that $\lambda(z) = (z, 0) \in \mathbb{C}^n$ and $J = J_{\mathrm{st}}$ on $\mathbb{C} \times \{0\} \subset \mathbb{C}^n$, where J_{st} is the standard almost complex structure in $\mathbb{C}^n$ (see section 1.2 in [1]) and moreover we can assume that for every J-holomorphic μ such that $\mu(0) = 0$ we have $\triangle\mu(0) = 0$ (see [8]).

An upper semi-continuous function u on an open subset of M is said to be J-plurisubharmonic or simpler plurisubharmonic, if a function $u \circ \lambda$ is subharmonic for every holomorphic function λ. We denote the set of plurisubharmonic functions on $\Omega \subset M$ by $\mathcal{PSH}(\Omega)$. For a smooth function u it means that a matrix $(A_{p\bar q})$ is nonnegative. Recently Harvey and Lawson proved that an upper semicontinuous locally integrable function u is plurisubharmonic iff a current $i\partial\bar\partial u$ is positive. We say that a function $u \in \mathcal{C}^{1,1}(\Omega)$ is strictly plurisubharmonic if for every $K \Subset \Omega$ there is $m > 0$ such that $\omega \leq im\partial\bar\partial u$ a.e. in K, where ω is any hermitian metric[4] on Ω. If $u \in \mathcal{C}^2(\Omega)$ then the following conditions are equivalent:

(i) u is strictly plurisubharmonic;
(ii) $i\partial\bar\partial u > 0$;
(iii) u is plurisubharmonic and $(i\partial\bar\partial u)^n > 0$.

We say that a domain $\Omega \Subset M$ is strictly pseudoconvex of class $\mathcal{C}^\infty$ (respectively of class $\mathcal{C}^{1,1}$) if there is a strictly plurisubharmonic function ρ of class $\mathcal{C}^\infty$ (respectively of class $\mathcal{C}^{1,1}$) in a neighbourhood of $\bar\Omega$, such that $\Omega = \{\rho < 0\}$ and $\nabla\rho \neq 0$ on $\partial\Omega$. In that case we say that ρ is a defining function for Ω.

Let $z_0 \in M$. The basic example of a (strictly) plurisubharmonic function in a neighbourhood of z_0 is $u(z) = (\mathrm{dist}(z, z_0))^2$ (where dist is a distance in some Rimannian metric). Domains $\Omega_\varepsilon = \{u < \varepsilon\}$ are strictly pseudoconvex of class $\mathcal{C}^\infty$ for $\varepsilon > 0$ small enough and they make a fundamental neighbourhood system for z_0.

[4] Hermitian metric is a smooth positive (1, 1) form.

2 Comparison principle

In this section $\Omega \Subset M$ is a domain not necessary strictly pseudoconvex but such that there is a bounded function $\rho \in \mathcal{C}^2 \cap \mathcal{PSH}(\Omega)$.

In the pluripotential theory in $\mathbb{C}^n$, the comparison principle is a very effective tool. We give here the basic version for J-plurisubharmonic functions.

Proposition 2.1 (comparison principle)*If $u, v \in \mathcal{C}^2(\bar{\Omega})$ are such that u is a plurisubharmonic function, $(i\partial\bar{\partial}u)^n \geq (i\partial\bar{\partial}v)^n$ on the set $\{i\partial\bar{\partial}v > 0\}$ and $u \leq v$ on $\partial\Omega$, then $u \leq v$ in $\bar{\Omega}$.*

Proof First, let us assume that $(i\partial\bar{\partial}u)^n > (i\partial\bar{\partial}v)^n$ on the set $\{i\partial\bar{\partial}v \geq 0\}$ and a function $u - v$ takes its maximum in $z_0 \in \Omega$. By Proposition 1.1 for small $\zeta \in T^{1,0}_{z_0}M$ there is a holomorphic disk λ such that $\lambda(0) = z_0$ and $\frac{\partial\lambda}{\partial x}(0) - iJ\frac{\partial\lambda}{\partial x}(0) = \zeta$. Hence at z_0

$$\partial\bar{\partial}(v-u)(\zeta,\bar{\zeta}) = \triangle\left((v-u)\circ\lambda\right)(0) \geq 0$$

so we have $i\partial\bar{\partial}u \leq i\partial\bar{\partial}v$ and then we obtain $(i\partial\bar{\partial}u)^n \leq (i\partial\bar{\partial}v)^n$ which is the contradiction with our first assumption.

In the general case we put $u' = u + \varepsilon(\rho - \sup_{\bar{\Omega}}\rho)$ and the lemma follows from the above case (with u' instead of u). □

In Sect. 4 we use a slight stronger version of the proposition above.

Proposition 2.2 *Suppose that $u, v \in \mathcal{C}^2(\bar{\Omega})$ are such that u is a plurisubharmonic function and $(i\partial\bar{\partial}u)^n \geq (i\partial\bar{\partial}v)^n$ on the set*

$$\left\{i\partial\bar{\partial}v > 0\right\}.$$

Then for any $H \in \mathcal{PSH}$, an inequality

$$\overline{\lim_{z\to z_0}}\,(u + H - v) \leq 0$$

for any $z_0 \in \partial\Omega$ implies $u + H \leq v$ on Ω.

Proof Let $z_0 \in \Omega$ be a point where a function $f = u + H - v$ attains a maximum and λ is a holomorphic disk such that $\lambda(0) = z_0$. Because $H \circ \lambda$ is a subharmonic function one can find a sequence t_k of nonzero complex numbers such that

$$\lim_{k\to\infty} t_k = 0$$

and

$$4H\circ\lambda(0) \leq H\circ\lambda(t_k) + H\circ\lambda(it_k) + H\circ\lambda(-t_k) + H\circ\lambda(-it_k).$$

Hence

$$\begin{aligned}&\triangle\left((v-u)\circ\lambda\right)(0)\\&\quad\geq \overline{\lim_{k\to\infty}}\frac{4f\circ\lambda(0) - f\circ\lambda(t_k) - f\circ\lambda(it_k) - f\circ\lambda(-t_k) - f\circ\lambda(-it_k)}{|t_k|^2} \geq 0.\end{aligned}$$

Therefore we can obtain our result exactly as in the proof of the previous proposition. □

3 A priori estimate

In this section we will prove a $\mathcal{C}^{1,1}$ estimate for the smooth solution u of the problem (2). By the general theory of elliptic equations (see for example [3]) we obtain from this the $\mathcal{C}^{k,\alpha}$ estimate and then the existence of a smooth solution. The uniqueness follows from the comparison principle.

Our proofs are close to these in [3] but more complicated because of the noncommutativity of some vector fields.

3.1 Some technical preparation

In this section we assume that $\Omega \Subset M$ is strictly pseudoconvex of class $\mathcal{C}^\infty$ with the defining function ρ. Let us fix a hermitian metric ω on M. From now all norms, gradient and hessian are taken with respect to this metric or more precisely with respect to a Rimannian metric which is given by $g(X, Y) = \omega(X, JY)$ for vector fields X, Y.

Let $f \in \mathcal{C}^\infty(\bar{\Omega})$ be such that $dV = f\omega^n$. Then locally our Monge–Ampère equation $(i\partial\bar{\partial}u)^n = dV$ has a form:

$$\det(A_{p\bar{q}}) = \tilde{f} = gf,$$

where $g = \det(-i\omega(\zeta_p, \bar{\zeta}_q))$. So if vectors $\zeta_1, \ldots, \zeta_n$ are orthonormal (i.e. $\omega(\zeta_p, \bar{\zeta}_q) = i\delta_{pq}$), then $g = 1$.

The following elliptic operator is very useful

$$L = L_\zeta = A^{p\bar{q}} \left(\zeta_p \bar{\zeta}_q - [\zeta_p, \bar{\zeta}_q]^{0,1} \right).$$

Note that for X, Y vector fields we have

$$X(\log \tilde{f}) = A^{p\bar{q}} X A_{p\bar{q}},$$
$$XY(\log \tilde{f}) = A^{p\bar{q}} XY A_{p\bar{q}} - A^{p\bar{j}} A^{i\bar{q}} (Y A_{i\bar{j}})(X A_{p\bar{q}}),$$

where $(A^{p\bar{q}})$ is the inverse of the matrix $(\overline{A_{p\bar{q}}})$.

In the lemmas we specify exactly how a priori *estimates* depend on ρ, f and φ. We should emphasize that they also depend strongly on M, J, ω, M' and $m(\rho)$, where M' is some fixed domain such that $\Omega \Subset M' \Subset M$ and $m(\rho)$ is defined as the smallest constant $m > 0$ such that $\omega \le mi\partial\bar{\partial}\rho$ on Ω. The notion $C = C(A)$ really means that C depends on an upper bound for A.

In the proofs below C is a constant under control, but it can change from a line to a next line.

3.2 Uniform estimate

Lemma 3.1 *We have* $\|u\|_{L^\infty(\Omega)} \le C$, *where* $C = C(\|\rho\|_{L^\infty(\Omega)}, \|f\|_{L^\infty(\Omega)}, \|\varphi\|_{L^\infty(\Omega)})$.

Proof From the comparison principle and the maximum principle we have

$$\|f^{1/n}\|_{L^\infty(\Omega)} m(\rho)\rho + \inf_{\partial\Omega} \varphi \le u \le \sup_{\partial\Omega} \varphi.$$

□

3.3 Gradient estimate

In the next two lemmas we shall prove a priori estimate for the first derivative.

Lemma 3.2 *We have*

$$\|u\|_{\mathcal{C}^{0,1}(\partial\Omega)} \leq C,$$

where $C = C(\|\rho\|_{\mathcal{C}^{0,1}(\Omega)}, \|f\|_{L^\infty(\Omega)}, \|\varphi\|_{\mathcal{C}^{1,1}(\Omega)})$.

Proof We can choose $A > 0$ such that $Ai\partial\bar{\partial}\rho + i\partial\bar{\partial}\varphi \geq f^{1/n}\omega$ and $Ai\partial\bar{\partial}\rho \geq i\partial\bar{\partial}\varphi$. Thus by the comparison principle and the maximum principle we have

$$\varphi + A\rho \leq u \leq \varphi - A\rho$$

for A large enough. So on the boundary we have

$$|\nabla u| \leq |\nabla A\rho| + |\nabla\varphi|.$$

□

Lemma 3.3 *We have*

$$\|u\|_{\mathcal{C}^{0,1}(\Omega)} \leq C,$$

where $C = C(\|\rho\|_{\mathcal{C}^{0,1}(\Omega)}, \|f^{1/n}\|_{\mathcal{C}^{0,1}}, \|u\|_{\mathcal{C}^{0,1}(\partial\Omega)})$.

Proof Let us consider the function $v = \psi|\nabla u|^2$, where a smooth plurisubharmonic function ψ will be determined later. Let us assume that v takes its maximum in $z_0 \in \Omega$. We can choose $\zeta_1, \ldots, \zeta_n$, such that they are orthonormal in a neighbourhood of z_0, and the matrix $A_{p\bar{q}}$ is diagonal at z_0. From now on all formulas are assumed to hold at z_0.

We have $Xv = 0$, hence $X(|\nabla|^2) = -|\nabla u|^2 X \log\psi$. We can calculate

$$\begin{aligned}
L(v) &= L(\psi)|\nabla u|^2 + \psi L(|\nabla u|^2) + A^{p\bar{p}}\left(\psi_p(|\nabla u|^2)_{\bar{p}} + \psi_{\bar{p}}(|\nabla u|^2)_p\right)\\
&= |\nabla u|^2 A^{p\bar{p}}\left(\psi_{p\bar{p}} - \left[\zeta_p, \bar{\zeta_p}\right]^{0,1}\psi - 2\frac{|\psi_p|^2}{\psi}\right) + \psi L(|\nabla u|^2),\\
L(|\nabla u|^2) &= A^{p\bar{p}}\left((|\nabla u|^2)_{p\bar{p}} - [\zeta_p, \bar{\zeta_p}]^{0,1}|\nabla u|^2\right)\\
&= A^{p\bar{p}}\sum_k\left(u_{p\bar{p}k}u_{\bar{k}} + u_k u_{p\bar{p}\bar{k}} + |u_{pk}|^2 + |u_{\bar{p}k}|^2\right.\\
&\quad\left. - \left[\zeta_p, \bar{\zeta_p}\right]^{0,1} u_k u_{\bar{k}} - u_k\left[\zeta_p, \bar{\zeta_p}\right]^{0,1} u_{\bar{k}}\right),\\
&\quad A^{p\bar{p}}(u_{p\bar{p}k} - \left[\zeta_p, \bar{\zeta_p}\right]^{0,1} u_k)\\
&= A^{p\bar{p}}(u_{kp\bar{p}} - \zeta_k\left[\zeta_p, \bar{\zeta_p}\right]^{0,1} u + \zeta_p\left[\bar{\zeta_p}, \zeta_k\right] u + \left[\zeta_p, \zeta_k\right]\bar{\zeta_p} u)\\
&= (\log f)_k + A^{p\bar{p}}\left(\zeta_p\left[\bar{\zeta_p}, \zeta_k\right] u + \bar{\zeta_p}\left[\zeta_p, \zeta_k\right] u + \left[\left[\zeta_p, \zeta_k\right], \bar{\zeta_p}\right] u\right.\\
&\quad\left. - \left[\left[\zeta_p, \bar{\zeta_p}\right]^{0,1}, \zeta_k\right] u\right).
\end{aligned}$$

Then we have

$$\begin{aligned}
&|A^{p\bar{p}}(u_{p\bar{p}k} - [\zeta_p, \bar{\zeta_p}]^{0,1} u_k)|\\
&\quad\leq C\left(\frac{\|f^{1/n}\|_{\mathcal{C}^{0,1}}}{f^{1/n}} + A^{p\bar{p}}\left(\sum_s(|u_{ps}| + |u_{p\bar{s}}|) + |\nabla u|\right)\right)
\end{aligned}$$

and similarly

$$|A^{p\bar{p}}(u_{p\bar{p}\bar{k}} - [\zeta_p, \bar{\zeta_p}]^{0,1}u_{\bar{k}})| \leq C\left(\frac{\|f^{1/n}\|_{\mathcal{C}^{0,1}}}{f^{1/n}} + A^{p\bar{p}}\left(\sum_s(|u_{ps}| + |u_{p\bar{s}}|) + |\nabla u|\right)\right)$$

so for the proper choice of ψ (we can get $\psi = e^{A\rho} + B$ for A, B large enough) we have $L(v)(0) > 0$ and this is a contradiction with the maximality of v. □

3.4 $\mathcal{C}^{1,1}$ estimate

Let us fix a point $P \in \partial\Omega$. Now we give the $\mathcal{C}^{1,1}$ estimate in a point P (which does not depend on P). The estimate of $XYu(P)$, where X, Y are tangent to $\partial\Omega$, follows from the gradient estimate.

Lemma 3.4 *Let $N \in T_PM$ be orthogonal to $\partial\Omega$ such that $N\rho = -1$ and let X be a vector field on a neighbourhood of P tangent to $\partial\Omega$ on $\partial\Omega$. We have*

$$|NXu(P)| \leq C,$$

where $C = C(\|\rho\|_{\mathcal{C}^{0,1}(\Omega)}, \|f^{1/n}\|_{\mathcal{C}^{0,1}}, \|\varphi\|_{\mathcal{C}^{2,1}(\Omega)}, \|X\|_{\mathcal{C}^{0,1}}, \|u\|_{\mathcal{C}^{0,1}(\Omega)})$.

Proof Let $X_1, X_2, \ldots, X_n$ be (real) vector fields on U a neighbourhood of P, tangent at P to $\partial\Omega$, such that $X_1, JX_1, \ldots, X_n, JX_n$ is a frame. Consider the function

$$v = X(u-\varphi) + B\rho + \sum_{k=1}^{n} |X_k(u-\varphi)|^2 - A(\mathrm{dist}(P,\cdot))^2.$$

Let $V \Subset U$ be a neighbourhood of P and $S = V \cap \Omega$. For A large enough $v \leq 0$ on ∂S.

Our goal is to show that for B large enough we have $v \leq 0$ on $\bar{S}$. Let $z_0 \in S$ be a point where v attains a maximum and let $\zeta_1, \ldots, \zeta_n$ be orthonormal and such that $(A_{p\bar{q}})$ is diagonal. From now on all formulas are assumed to hold at z_0. Let us calculate:

$$m(\rho)L(\rho) \geq \sum A^{p\bar{p}}$$

and

$$L(-X\varphi - A(\mathrm{dist}(P,\cdot))^2) \geq -C\sum A^{p\bar{p}},$$

hence for B large enough

$$L(B\rho - X\varphi - A(\mathrm{dist}(P,\cdot))^2) \geq \frac{B}{2m(\rho)}\sum A^{p\bar{p}}.$$

To estimate $L(Xu + \sum_{k=1}^{n}|X_k(u-\varphi)|^2)$ let us first consider $Y \in \{X, X_1, \ldots X_n\}$ and calculate

$$\begin{aligned} L(Yu) &= A^{p\bar{q}}\left(\zeta_p\bar{\zeta}_q Yu - \left[\zeta_p, \bar{\zeta}_q\right]^{0,1} Yu\right) \\ &= Y\log f + A^{p\bar{q}}\left(\zeta_p\left[\bar{\zeta}_q, Y\right]u + [\zeta_p, Y]\bar{\zeta}_q u - \left[\left[\zeta_p, \bar{\zeta}_q\right]^{0,1}, Y\right]u\right). \end{aligned}$$

There are $\alpha_{q,k}, \beta_{q,k} \in \mathbb{C}$ such that

$$\left[\bar{\zeta}_q, Y\right] = \sum_{k=1}^{n} \alpha_{q,k}\bar{\zeta}_k + \beta_{q,k}X_k$$

and so

$$A^{p\bar{q}}\zeta_p\left[\bar{\zeta}_q, Y\right]u = \sum_q \alpha_{q,q} + \sum_{k=1}^{n} A^{p\bar{p}}\beta_{p,k}\zeta_p X_k u + A^{p\bar{p}}Z_p u,$$

where Z_p are vector fields under control. This gives us

$$|A^{p\bar{q}}\zeta_p\left[\bar{\zeta}_q, Y\right]u| \le CA^{p\bar{p}}\left(1 + \sum_k |\beta_{p,k}\zeta_p X_k u|\right).$$

In a similar way we can estimate $A^{p\bar{q}}[\zeta_p, Y]\bar{\zeta}_q u$ and we obtain

$$|L(Yu)| \le CA^{p\bar{p}}\left(1 + \sum_k |\zeta_p X_k u|\right).$$

Therefore

$$\begin{aligned} &L(Xu + \sum_k |X_k(u-\varphi)|^2) \\ &\ge A^{p\bar{p}}\sum_{k=1}^{n}\left(\zeta_p X_k(u-\varphi)\right)\left(\bar{\zeta}_p X_k(u-\varphi)\right) - CA^{p\bar{p}}\left(1 + \sum_k |\zeta_p X_k u|\right) \\ &\ge A^{p\bar{p}}\sum_k |\zeta_p X_k u|^2 - CA^{p\bar{p}}(1 + \sum_k |\zeta_p X_k u|). \end{aligned}$$

Now for B large enough, since $L(v)(z_0) > 0$, we have contradiction with maximality of v. Hence $v \le 0$ on S and so $NXu(P) \le C$. □

Lemma 3.5 *Let X be a vector field orthogonal to $\partial\Omega$ at P. We have*

$$\|XXu(P)\| \le C,$$

where

$$C = C(\|\rho\|_{\mathcal{C}^{2,1}(\Omega)}, \|f^{1/n}\|_{\mathcal{C}^{0,1}}, \|f^{-1}\|_{L^\infty(\Omega)}, \|\varphi\|_{\mathcal{C}^{3,1}(\Omega)}, \|X\|_{\mathcal{C}^{0,1}}, \|u\|_{\mathcal{C}^{0,1}(\Omega)}).$$

Proof By the previous Lemma it is enough to prove that

$$|\zeta|^2 \le C\left(\zeta\bar{\zeta} - \left[\zeta, \bar{\zeta}\right]^{0,1}\right)u(P)$$

for every vector field $\zeta \in T^{1,0}M$ tangent (at P) to $\partial\Omega$.

Because our argue are local we can assume that $P = 0 \in \mathbb{C}^n$. Let $\zeta_1, \zeta_2, \ldots \zeta_n \in T^{1,0}$ be an orthonormal frame in a neighbourhood of 0 such that $\zeta_k\rho = -\delta_{kn}$. We can assume that $\zeta_1 = \zeta$. By the strictly pseudoconvexity we have $(\zeta\bar{\zeta} - [\zeta, \bar{\zeta}]^{0,1})\rho(P) \neq 0$, so we can also assume that $(\zeta\bar{\zeta} - [\zeta, \bar{\zeta}]^{0,1})\varphi(P) = 0$.

From the strictly pseudoconvexity and using the Proposition 1.1 (for $k = 2$) we can choose J-holomorphic disk λ such that $\lambda(0) = 0$, $\frac{\partial\lambda}{\partial z}(0) = a\zeta$ and

$$\rho \circ \lambda(z) = b|z|^2 + O(|z|^3) \tag{3.1}$$

for some $a, b > 0$. In particular we have

$$|z|^2 \leq C\mathrm{dist}(\lambda(z), \bar{\Omega}). \tag{3.2}$$

Indeed, for $a > 0$ small enough, by the proposition 1.1 (for $k = 1$) there is a J-holomorphic disk $\tilde{\lambda}$ such that $\tilde{\lambda}(0) = 0$ and $\frac{\partial \tilde{\lambda}}{\partial z}(0) = a\zeta$. Then (by changing coordinates) we can assume $(\zeta J)(0) = 0$, so for any J-holomorphic disk λ such that $\lambda(0) = 0$ and $\frac{\partial \lambda}{\partial z}(0) = a\zeta$ we have $\frac{\partial^2 \lambda}{\partial x^2}(0) = -J\frac{\partial^2 \lambda}{\partial x \partial y}(0) = -\frac{\partial^2 \lambda}{\partial y^2}(0)$. Now if we put

$$\frac{\partial^2 \lambda}{\partial x^2}(0) = a^2\left(0, 4\frac{\partial^2 \rho}{\partial x_1^2}(0) - 2\left(\zeta\bar{\zeta} - \left[\zeta, \bar{\zeta}\right]^{0,1}\right)\rho(0), -4\frac{\partial^2 \rho}{\partial x_1 \partial x_2}(0)\right)$$

we obtain (3.1) with $b = a^2(\zeta\bar{\zeta} - [\zeta, \bar{\zeta}]^{0,1})\rho(0)$.

Once again changing coordinates we may assume $\lambda(z_1) = (z_1, 0), \zeta_k(0) = \frac{\partial}{\partial z_k}$ for $k = 1, \ldots, n$ and for every J-holomorphic disk μ such that $\mu(0) = 0$ we have

$$\frac{\partial^2 \mu}{\partial z \partial \bar{z}}(0) = 0. \tag{3.3}$$

We can find a holomorphic cubic polynomial p_1 and a complex number α such that

$$\begin{aligned}\varphi(z) &= \varphi(0) + \varphi'(0)(z) \\ &\quad + \mathrm{Re}\left(\sum_{p=1}^{n} \frac{\partial^2 \varphi}{\partial z_1 \partial \bar{z}_p} z_1 \bar{z}_p + p_1(z) + \alpha z_1 |z_1|^2\right) + O(|z_1|^4 + |z_2|^2 + \ldots + |z_n|^2).\end{aligned}$$

By (3.2) we can choose another cubic polynomial p_2 and numbers $\beta_1, \beta_2, \ldots, \beta_n \in \mathbb{C}$, $\beta_1 > 0$ such that

$$\mathrm{Re} z_n = \mathrm{Re}\left(\sum_{p=1}^{n} \beta_p z_1 \bar{z}_p + p_2(z)\right) + O(|z_1|^3 + |z_2|^2 + \ldots + |z_n|^2) \text{ on } \partial\Omega.$$

Then we obtain

$$\varphi(z) + \varphi(0) = \varphi'(0)(z) + \mathrm{Re}\left(\sum_{p=2}^{n} a_p z_1 \bar{z}_p + p_3(z)\right) + O(|z_2|^2 + \ldots + |z_n|^2)$$

for some numbers $a_2, \ldots, a_n \in \mathbb{C}$ and a new cubic polynomial p_3. Hence

$$u(z) - u(0) = \mathrm{Re}\left(\sum_{p=2}^{n} a_p z_1 \bar{z}_p + p_4(z)\right) + O(|z_2|^2 + \ldots + |z_n|^2) \tag{3.4}$$

for $z \in \partial\Omega$ and same polynomial p_4.

Let $B > 1$[5] and $D = B^{-1}\max\{|a_2|, \ldots, |a_n|\}$. By the Proposition 1.1 (again for $k = 2$) there is a family of J-holomorphic disks $g_w : \mathbb{D} \to \mathbb{C}^n$, $w \in \mathbb{C}^{n-1}$, $g_w = (g_w^1, \ldots, g_w^n)$ such that

[5] Constants C below do not depend on the constant B.

$$g_w(0) = (0, w),$$
$$\frac{\partial g_w}{\partial z}(0) = (1, -\frac{a_2}{B}, \ldots, -\frac{a_n}{B}),$$
$$\|g_w - \lambda\|_{\mathcal{C}^4} \le C(|w| + D)$$

and a function $G : \mathbb{D} \times \mathbb{C}^{n-1} \to \mathbb{C}^n$ given by $G(z, w) = g_w(z)$ is of class $\mathcal{C}^4$. Then we have

$$|g_w(z) - (w_1 + z, w_2 - \frac{a_2 z}{B}, \ldots, w_n - \frac{a_n z}{B})| < C|z|^2(|w| + D) \tag{3.5}$$

for $z \in \mathbb{D}$ and by (3.2)

$$|z| < C(\sqrt{|w|} + D) \tag{3.6}$$

if $g_w(z) \in \Omega$.

We can choose domains $U \subset \mathbb{D}$, $V \subset \mathbb{C}^{n-1}$, $W \subset \mathbb{C}^n$ such that $0 \in W$, $G(\partial U \times V) \cap \bar{\Omega} = \emptyset$ and $G|_{U\times V}$ is a diffeomorphism onto W.

Let

$$h(g_w(z)) = \text{Re}\, p_w(z) + AB|w|^2 + \varepsilon\rho,$$

where $A, \varepsilon > 0$ and p_w is a holomorphic cubic polynomial in one variable such that

$$\text{Re}\, p_4(g_w(z)) = \text{Re}\, p_w(z) + a_w|z|^2 + \text{Re}\, b_w z|z|^2 + O(|z|^4)$$

for some $a_w \in \mathbb{R}$, $b_w \in \mathbb{C}$. Note that $|b_w| < C(|w| + D)$ and by (3.3) $|a_w| < C|w|$. Thus enlarging A (if necessary) and using also (3.6) we obtain

$$\text{Re}\, p_4(g_w(z)) \le h(g_w(z)) + \frac{1}{2}D^2|z|^2 \tag{3.7}$$

on $\partial\Omega$.

By inequalities (3.5) and (3.6) we have

$$\begin{aligned} &2\sum_{k=2}^{n} \text{Re}\, a_k g_w^1(z) g_w^k(z) \\ &= \sum_{k=2}^{n} B\left(|-\frac{a_k g_w^1(z)}{B} - g_w^k(z)|^2 - |\frac{a_k g_w^1(z)}{B}|^2 - |g_w^k(z)|^2\right) \\ &\le B(|w|^2 - D^2|z|^2 + C|z|^4(|w|^2 + D^2)) \le CB|w|^2 - \frac{1}{2}D^2|z|^2 \end{aligned}$$

for B large enough. By an above estimate, (3.4), and (3.7) we obtain that if A is large enough then $h \ge u - u(0)$ on $\partial\Omega \cap W$. Again enlarging A we can assume $h \ge u - u(0)$ on ∂S where $S = \Omega \cap W$. Since $i\partial\bar{\partial}h$ is under control for ε enough small we get an inequality

$$\left(i\partial\bar{\partial}h\right)^n < \left(i\partial\bar{\partial}u\right)^n$$

on the set $S \cap \{i\partial\bar{\partial}h > 0\}$. This by the comparison principle gives us $h \ge u - u(0)$ on S. Note that $h_N \ge u_N$, $\varphi_{1\bar{1}} = 0$ and $\varphi_N = h_N - \varepsilon\rho_N = h_N + \varepsilon$, so we can conclude that

$$u_{1\bar{1}} = u_{1\bar{1}} - \varphi_{1\bar{1}} = (\varphi_N - u_N)\,\rho_{1\bar{1}} \ge \varepsilon\rho_{1\bar{1}}.$$

□

Finally we will obtain the interior $\mathcal{C}^{1,1}$ estimate, which together with previous lemmas gives us a full $\mathcal{C}^{1,1}$ estimate. By a standard argumentation this ends the proof of Theorem 1.

Lemma 3.6 *We have*

$$\|Hu\|_{L^\infty(\Omega)} \le C, \tag{3.8}$$

where Hu is a Hessian of u and

$$C = C\left(\|\rho\|_{\mathcal{C}^{0,1}(\Omega)}, \|f^{\frac{1}{2n}}\|_{\mathcal{C}^{1,1}}, \|u\|_{\mathcal{C}^{0,1}(\Omega)}, \|Hu\|_{L^\infty(\partial\Omega)}\right).$$

Proof Let us define M as the biggest eigenvalue of the Hessian Hu. We will show that the function

$$\Lambda = \psi e^{K|\nabla u|^2} M,$$

where a smooth plurisubharmonic function $\psi > 1$ and a small positive number K will be determined later, does not attain maximum in Ω. Because a function u is plurisubharmonic this will give (3.8).

Assume that a maximum of the function Λ is attained at a point $z_0 \in \Omega$ (otherwise we are done). There are $\zeta_1, \dots, \zeta_n \in T^{1,0}_{z_0}M$ orthonormal at z_0 such that the matrix $(A_{p\bar{q}})$ is diagonal at z_0. Let $X \in T_{z_0}M$ be such that $\|X\| = 1$ and $M = H(X, X)$. We can normalize coordinates near z_0 such that $z_0 = 0 \in \mathbb{C}^n$, $X = \frac{\partial}{\partial x_1}(0)$ and $J(z, 0) = J_{st}$ for small $z \in \mathbb{C}$. Let us extend X as $\frac{\partial}{\partial x_1}$ and then in a natural way we can extend $\zeta_1, \dots, \zeta_n$ to some neighbourhood U of 0 such that $[\zeta_k, X](0) = 0$ and $[\zeta_k, \bar{\zeta}_k](0) = 0$ for $k = 1, \dots, n$. Indeed, on $U \cap \mathbb{C} \times \{0\}$ we can put ζ_k as the same linear combination of vectors $\frac{\partial}{\partial z_1} \dots, \frac{\partial}{\partial z_n}$ as in 0. Then for some small $a > 0$ we can take (for ζ_k not tangent to $\mathbb{C} \times \{0\}$) J-holomorphic disks $d_k : \mathbb{D} \to U$ such that $d_k(0) = 0$ and $\frac{\partial d_k}{\partial z}(0) = a\zeta_k(0)$, and on the image of d_k we can put $\zeta_k(w) = a^{-1}\frac{\partial d_k}{\partial z}(d_k^{-1}(w))$. On the end we extend the vector fields on whole U.

Let

$$v = \psi e^{K|\nabla u|^2} \frac{Hu\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1}\right)}{|\frac{\partial}{\partial x_1}|^2} = \Psi e^{K|\nabla u|^2}(u_{x_1x_1} + Tu) \text{on} U,$$

where $\Psi = \frac{\psi}{|\frac{\partial}{\partial x_1}|^2}$ and T is a vector field (which is under control), then also a function v has a maximum at 0 (in particular $L(v) \le 0$). Let us put $\mu = u_{x_1x_1} + Tu$ (then $|\frac{\partial}{\partial x_1}(0)|^2\mu(0) = M(0)$). Note that we have $XYu \le C\mu$ for vector fields X, Y (which are under control). Assume $\mu > 1$ (otherwise we have $\Lambda < C$, so we are done).

From now all formulas are assumed to hold at 0. We estimate $L(v)$ from below:

$$L(v) = L\left(\Psi e^{K|\nabla u|^2}\right)\mu + \Psi e^{K|\nabla u|^2} L(\mu) - 2A^{p\bar{p}} \frac{(\Psi e^{K|\nabla u|^2})_p (\Psi e^{K|\nabla u|^2})_{\bar{p}}\mu}{\Psi e^{K|\nabla u|^2}}$$

To estimate the first term let us calculate

$$\begin{aligned}
&L(\Psi e^{K|\nabla u|^2}) \\
&\quad = e^{K|\nabla u|^2} A^{p\bar{p}}(\Psi_{p\bar{p}} + 2K\mathrm{Re}(\Psi_p(|\nabla u|^2)_{\bar{p}}) + K\Psi(|\nabla u|^2)_{p\bar{p}} \\
&\qquad + K^2\Psi|(|\nabla u|^2)_p|^2), \; A^{p\bar{p}}(|\nabla u|^2)_{p\bar{p}} \\
&\quad = A^{p\bar{p}} \sum_k ((\zeta_p\bar{\zeta}_p\eta_k u)u_{\bar{k}} + u_k(\zeta_p\bar{\zeta}_p\bar{\eta}_k u) + (\bar{\zeta}_p\eta_k u)(\zeta_p\bar{\eta}_k u) + (\zeta_p\eta_k u)(\bar{\zeta}_p\bar{\eta}_k u))
\end{aligned}$$

$$
\begin{aligned}
&= \sum_k ((\log \tilde{f})_k u_{\bar{k}} + (\log \tilde{f})_{\bar{k}} u_k) \\
&+ A^{p\bar{p}} \sum_k ((\zeta_p[\zeta_{\bar{p}}, \eta_k]u)u_{\bar{k}} + ([\zeta_p, \eta_k]u_{\bar{p}})u_{\bar{k}} + (\zeta_p[\bar{\zeta}_p, \bar{\eta}_k]u)u_k + ([\zeta_p, \bar{\eta}_k]u_{\bar{p}})u_k) \\
&+ A^{p\bar{p}} \sum_k \left(2K\mathrm{Re}\left(\eta_k \left[\zeta_p, \bar{\zeta}_p\right]^{0,1} uu_{\bar{k}}\right) + \left(\bar{\zeta}_p \eta_k u\right)\left(\zeta_p \bar{\eta}_k u\right) + \left(\zeta_p \eta_k u\right)\left(\bar{\zeta}_p \bar{\eta}_k u\right)\right),
\end{aligned}
$$

where $\eta_1, \ldots \eta_n$ is an orthonormal frame such that $\eta_k(0) = \zeta_k(0)$. Therefore we have

$$
A^{p\bar{p}}(|\nabla u|^2)_{p\bar{p}} \geq -C + A^{p\bar{p}} \frac{1}{2} \sum_k \left(\left(\bar{\zeta}_p \zeta_k u\right)\left(\zeta_p \bar{\zeta}_k u\right) + \left(\zeta_p \zeta_k u\right)\left(\bar{\zeta}_p \bar{\zeta}_k u\right) - C\right),
$$

hence

$$
\begin{aligned}
L(\Psi e^{K|\nabla u|^2}) \geq & -Ce^{K|\nabla u|^2} \\
& + e^{K|\nabla u|^2} A^{p\bar{p}} \left(\Psi_{p\bar{p}} - CK|\Psi_p| \sum_k \left(|u_{p\bar{k}}| + |u_{pk}| + 1\right)\right) \\
& + e^{K|\nabla u|^2} A^{p\bar{p}} \Psi (\frac{1}{2}K - CK^2) \sum_k \left(|u_{p\bar{k}}|^2 + |u_{pk}|^2\right).
\end{aligned}
$$

Let us start the calculation for the second term

$$
\begin{aligned}
L(\mu) &= L\left(u_{x_1 x_1}\right) + L(Tu), \\
L(Tu) &\leq T(\log \tilde{f}) - C(\mu + 1) \sum A^{p\bar{p}}, \\
L(u_{x_1 x_1}) &= \left(\log \tilde{f}\right)_{x_1 x_1} + A^{p\bar{p}} A^{q\bar{q}} |X(\zeta_p \bar{\zeta}_q - \left[\zeta_p, \bar{\zeta}_q\right]^{0,1})u|^2 \\
&\quad + A^{p\bar{p}} \left(\zeta_p \left[\bar{\zeta}_p, X\right] Xu + \left[\zeta_p, X\right] \bar{\zeta}_p Xu + X\zeta_p \left[\bar{\zeta}_p, X\right] u + X \left[\zeta_p, X\right] \bar{\zeta}_p u\right. \\
&\quad \left. + XX \left[\zeta_p, \bar{\zeta}_p\right]^{0,1} u\right) \\
&= (\log \tilde{f})_{x_1 x_1} + A^{p\bar{p}} A^{q\bar{q}} |X\left(\zeta_p \bar{\zeta}_q - \left[\zeta_p, \bar{\zeta}_q\right]^{0,1}\right) u|^2 \\
&\quad + A^{p\bar{p}} \left(\left[\zeta_p, \left[\bar{\zeta}_p, X\right]\right] Xu + X[\zeta_p, [\bar{\zeta}_p, X]]u + \left[X, \left[\bar{\zeta}_p, X\right]\right] \zeta_p u\right. \\
&\quad \left. + \left[X, [\zeta_p, X]\right] \bar{\zeta}_p u\right) + A^{p\bar{p}} \left(X\left[X, \left[\zeta_p, \bar{\zeta}_p\right]^{0,1}\right] u + [X, [\zeta_p, \bar{\zeta}_p]^{0,1}]Xu\right) \\
&\geq -C(\mu + 1) \sum A^{p\bar{p}}
\end{aligned}
$$

and we obtain

$$
L(\mu) \geq -C\mu \sum A^{p\bar{p}}.
$$

Now we come to the last term

$$
\begin{aligned}
& -2A^{p\bar{p}} \frac{\left(\Psi e^{K|\nabla u|^2}\right)_p \left(\Psi e^{K|\nabla u|^2}\right)_{\bar{p}}}{\Psi e^{K|\nabla u|^2}} \\
& = -2A^{p\bar{p}} e^{K|\nabla u|^2} \left(\frac{|\Psi_p|^2}{\Psi} + 2K\mathrm{Re}(\Psi_p(|\nabla u|^2)_{\bar{p}}) + K^2 \Psi (|\nabla u|^2)_{\bar{p}} (|\nabla u|^2)_p\right) \\
& \geq -2e^{K|\nabla u|^2} A^{p\bar{p}} \left(\frac{|\Psi_p|^2}{\Psi} + CK|\Psi_p| \sum_k (|u_{p\bar{k}}| + |u_{pk}|) + K^2 \sum_k (|u_{p\bar{k}}|^2 + |u_{pk}|^2)\right).
\end{aligned}
$$

Therefore there is a constant $C_0 > 1$ which is under control such that

$$L(v) \geq \mu e^{K|\nabla u|^2} A^{p\bar{p}} \left(\Psi_{p\bar{p}} + \Psi \left(\frac{1}{2} K - C_0 K^2 \right) \sum_k (|u_{p\bar{k}}|^2 + |u_{pk}|^2) - 2 \frac{|\Psi_p|^2}{\Psi} \right)$$
$$- C_0 \mu e^{K|\nabla u|^2} A^{p\bar{p}} (1+K)(1+|\Psi_p|) \left(1 + \sum_k \left(|u_{p\bar{k}}| + |u_{pk}| \right) \right).$$

For a proper choice of K and ψ ($K = \frac{1}{2C_0}$, $\psi = e^{A\phi} + 4e^A$ where $\frac{1}{2} < \phi < 1$ is strictly plurisubharmonic in a neighborhood of $\bar{\Omega}$ and A is large enough) we can conclude that $L(v) > 0$ and this is a contradiction with the maximality of v. □

4 Maximal plurisubharmonic functions

We say that a function $u \in \mathcal{PSH}(\Omega)$ is maximal if for every function $v \in \mathcal{PSH}(\Omega)$ such that $v \leq u$ outside a compact subset of Ω we have $v \leq u$ in Ω.

Now we want to find the solution to the following Dirichlet problem:

$$\begin{cases} u \in \mathcal{PSH}(\Omega) \cap \mathcal{C}(\bar{\Omega}) \\ u \text{ is maximal} \\ u = \varphi \quad \text{on} \quad \partial\Omega, \end{cases} \tag{4.1}$$

where Ω is a strictly pseudoconvex domain of class $\mathcal{C}^{1,1}$ and φ is a continuous function on $\partial\Omega$.

Proposition 4.1 *If $\varphi \in \mathcal{C}^{1,1}(\bar{\Omega})$, then there is a unique solution $u \in \mathcal{C}^{0,1}(\bar{\Omega})$ of the problem* (4.1) *and*

$$\|u\|_{\mathcal{C}^{0,1}(\bar{\Omega})} \leq C = C \left(\|\rho\|_{\mathcal{C}^{0,1}(\Omega)}, m(\rho), \|\varphi\|_{\in \mathcal{C}^{1,1}} \right).$$

Proof The uniqueness is a consequence of the definition.

To prove the existence assume that ρ is smooth. There are an increasing sequence φ_k of smooth functions such that φ_k tends to φ in $\mathcal{C}^{1,1}$ norm. By Theorem 1 there is a solution u_k of the following Dirichlet Problem

$$\begin{cases} u_k \in \mathcal{PSH}(\Omega) \cap \mathcal{C}^{\infty}(\bar{\Omega}) \\ (i\partial\bar{\partial}u_k)^n = \frac{1}{k^n} (i\partial\bar{\partial}\rho)^n \quad \text{in} \quad \Omega \\ u_k = \varphi_k \quad \text{on} \quad \partial\Omega. \end{cases}$$

By Lemma 3.3 $\|u_k\|_{\mathcal{C}^{0,1}(\bar{\Omega})} \leq C(\|\rho\|_{\mathcal{C}^{0,1}(\Omega)}, m(\rho), \|\varphi\|_{\in \mathcal{C}^{1,1}})$. Now we can put

$$u := \lim_{k \to \infty} u_k.$$

It is enough to show that u is a maximal function. Let a function $v \in \mathcal{PSH}(\Omega)$ be smaller than u outside a compact subset of Ω. From the comparison principle (Proposition 2.2) we obtain

$$v + \frac{\rho}{k} - \sup_{\partial\Omega} (\varphi - \varphi_k) \leq u_p$$

for $p \geq k$. Taking the limit we conclude that $v \leq u$ in Ω.

In the general case we can assume that φ is a plurisubharmonic function on Ω (by adding $A\rho$ for A enough large). We can approximate Ω by an increasing sequence of smooth strictly pseudoconvex domains Ω_k such that $\bigcup_k \Omega_k = \Omega$ and $\|\rho\|_{\mathcal{C}^{0,1}(\Omega)}, m(\rho)$ are under control, where ρ_k are strictly plurisubharmonic smooth defining functions for Ω_k. Let u_k be a solution of the following Dirichlet Problem

$$\begin{cases} u_k \in \mathcal{PSH}(\Omega_k) \cap \mathcal{C}(\bar{\Omega}_k) \\ u_k \text{ is maximal} \\ u_k = \varphi \quad \text{on} \quad \partial\Omega_k. \end{cases}$$

Then $u_k \geq \varphi$, hence it is an increasing sequence and again we can put

$$u := \lim_{k\to\infty} u_k.$$

If v is as above, for every $\varepsilon > 0$ we have $v - \varepsilon \leq u_k$ outside a compact set for k large enough. So we obtain $v \leq u$ and conclude that u is a maximal function as in the statement. □

Note that in the above proposition it is not enough to assume that φ is $\mathcal{C}^{1,\alpha}$ regular for some $\alpha < 1$. Indeed, one can show that if Ω is strictly pseudoconvex, $P \in \partial\Omega$ and $\varphi(z) \leq \varphi(P) - (\text{dist}(z, p))^{1+\alpha}$, then a solution of (4.1) is not Hölder continuous with the exponent greater than $\frac{1+\alpha}{2}$.

Theorem 4.2 (Harvey, Lawson [4]) *There is a unique solution u of the problem* (4.1).

Proof Let φ_k be an increasing sequence of smooth functions on $\bar{\Omega}$ such that $\lim_{k\to\infty} \varphi_k = \varphi$. By Proposition 4.1 there is a sequence u_k of solutions of (4.1) with boundary conditions φ_k (instead of φ). Because

$$u_k \leq u_p \leq u_k + \sup_{\partial\Omega}(\varphi - \varphi_k)$$

for $p \geq k$, the sequence u_k is a Cauchy sequence in $\mathcal{C}(\bar{\Omega})$. Similar as in the previous proof we can conclude that its limit u is a solution of the problem (4.1). □

Note that we can also prove the above theorem directly from Theorem 1.

The following proposition shows that being a continuous maximal plurisubharmonic function is a local property.

Proposition 4.3 *Let* $\Omega \subset M$ *and* $u \in \mathcal{PSH}(\Omega)$. *Then*

(i) If u is maximal then $u|_U$ *is maximal for every* $U \subset \Omega$;
(ii) If Ω *is such that there is a bounded strictly plurisubharmonic function* $\rho \in \mathcal{C}^2(\Omega)$, *u is continuous and every point in* Ω *has a neighbourhood U such that* $u|_U$ *is maximal, then u is maximal.*

Proof (i) Suppose that $v \in \mathcal{PSH}(U)$ is such that $v \leq u$ outside a compact subset of U. Then $\max\{u, v\} \in \mathcal{PSH}(\Omega)$ and we obtain $v \leq \max\{u, v\} \leq u$ on U.

(ii) We can assume that $\rho < 0$. Let $\varepsilon > 0$, $v \in \mathcal{PSH}(\Omega)$ and let $z_0 \in \Omega$ be a point where a function $v + \varepsilon\rho - u$ attains its maximum. By (i) there is a strictly pseudoconvex domain $\tilde{\Omega} \subset \Omega$ with a smooth plurisubharmonic defining function $\tilde{\rho}$ such that $z_0 \in \tilde{\Omega}$ and $u|_{\tilde{\Omega}}$ is maximal. Note that there is $\tilde{\varepsilon} > 0$ such that a function $\rho - \tilde{\varepsilon}\tilde{\rho}$ is plurisubharmonic in some neighbourhood of $\text{cl}(\tilde{\Omega})$. Hence a function $\tilde{v} = \max\{v + \varepsilon\rho, v + \varepsilon(\rho - \tilde{\rho}\tilde{\varepsilon})\}$ is also plurisubharmonic and $\tilde{v} - u$ attains a maximum only in some compact subset of $\tilde{\Omega}$, which is impossible because $u|_{\tilde{\Omega}}$ is maximal. As ε and v were arbitrary we can conclude that the function u is maximal. □

In [4] the authors consider problem (4.1) for $\mathcal{F}(\mathcal{J})$-harmonic functions which they define in a different way than we define maximal functions but we will see that these concepts agree.

Let $\Omega \subset M$ and $u \in \mathcal{C} \cap \mathcal{PSH}(\Omega)$. We say that u is $\mathcal{F}(\mathcal{J})$-harmonic if for every $U \subset \Omega$ and for every smooth strictly plurisubharmonic function $\phi \le u$ on U we have $\phi < u$ on U. One can show (using the comparison principle) that $\mathcal{C}^2$ $\mathcal{F}(\mathcal{J})$-harmonic functions are exactly $\mathcal{C}^2$ solutions of (1) with $f = 0$.

Proposition 4.4 *Let Ω and u be as above. Then*

(i) If u is maximal then u is $\mathcal{F}(\mathcal{J})$-harmonic;
(ii) If Ω is such that there is a bounded strictly plurisubharmonic function $\rho \in \mathcal{C}^2(\Omega)$ and u is $\mathcal{F}(\mathcal{J})$-harmonic, then u is maximal.

Proof The first assertion follows from definitions. To proof (ii) we can assume, by Proposition 4.3, that Ω is a smooth strictly pseudoconvex domain with defining function ρ such that $u \in \mathcal{C}(\bar{\Omega})$. Let $\varepsilon > 0$. By Theorem 4.2 there is a continuous maximal plurisubharmonic function u_0 equal to u on $\partial\Omega$. By Theorem 1 there is a smooth strictly plurisubharmonic function u_1 such that $u - \varepsilon < u_1 < u$ on a boundary and

$$(i\partial\bar{\partial}u_1)^n = \frac{1}{2}\varepsilon^n (i\partial\bar{\partial}\rho)^n.$$

Then using the comparison principle (Proposition 2.2) we obtain

$$u_0 + \varepsilon\rho - \varepsilon \le u_1 \le u \le u_0$$

and thus we get $u = u_0$. □

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