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THE MOORE-PENROSE INVERSE OF A PARTITIONED MATRIX $M = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$

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I. INTRODUCTION

If A is an $m \times n$ matrix over the complex field, then the Moore-Penrose inverse of A, denoted A^+, is an $n \times m$ matrix such that

(1.1) $AA^+A = A$

(1.2) $A^+AA^+ = A^+$

(1.3) $(AA^+)^* = AA^+$

(1.4) $(A^+A)^* = A^+A$.

Any matrix which satisfies equation (1.i) is called an (i)-inverse of A. A generalized inverse of A will indicate a matrix X satisfying some of the conditions (1.1) through (1.4).

If

$$M = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$$

and M is invertible, then M^{-1} is lower block triangular. It is natural then to ask the following question: For an $m \times n$ partitioned matrix

$$M = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}, \tag{1.5}$$

when is the Moore-Penrose inverse also lower block triangular? C. MEYER has given necessary and sufficient conditions for this question in [2].

We first give a formula for computing M^+, and then we obtain Meyer's result as a corollary to this general expansion. We also examine some other cases which occur rather naturally.

In [3], Meyer considers square matrices which are upper triangular and he determines conditions for a generalized inverse to be upper triangular. Moreover, he gives explicit formulas for determining these inverses in some special cases.

Throughout our paper, we shall restrict our attention (except for a fleeting reference to i-inverses) to the Moore-Penrose inverse. We shall use the following well-known facts in our work [e.g., see 4].

(1.6) $A^+ = A^*(AA^*)^+ = (A^*A)^+ A^*$

(1.7) $(AA^*)^+ = (A^+)^* A^+$

(1.8) If $N(A)$ denotes the null column space of A, then $N(A) \subset N(B)$ if and only if $B = BA^+A$.

II. TWO LEMMAS

In order to prove our theorem, we shall need the following lemmas.

Lemma 1. *For M partitioned as in (1.5), we have*

$$M^+ = \begin{pmatrix} A^+ & B^*L^+ \\ 0 & C^*L^+ \end{pmatrix},$$

$L = BB^* + CC^*$, *if and only if* $AB^* = 0$.

Proof. Assume

$$M^+ = \begin{pmatrix} A^+ & B^*L^+ \\ 0 & C^*L^+ \end{pmatrix}.$$

Then by (1.1), we obtain

(2.1) $BA^+A + LL^+B = B$.

By the definition of L, we have $N(L) \subseteq N(B^*)$, so $LL^+B = B$ by (1.8). Then (2.1) implies $BA^+A = 0$, and hence $BA^+ = 0$. But $BA^+ = 0$ is equivalent to $AB^* = 0$, so the necessity is complete.

For the sufficiency, we will use relation (1.6). We have $M^+ = M^*(MM^*)^+$, so

$$\begin{pmatrix} A & 0 \\ B & C \end{pmatrix}^+ = \begin{pmatrix} A^* & B^* \\ 0 & C^* \end{pmatrix} \begin{pmatrix} AA^* & 0 \\ 0 & BB^* + CC^* \end{pmatrix}^+ \quad \text{since} \quad AB^* = 0\,.$$

Thus

$$M^+ = \begin{pmatrix} A^*(AA^*)^+ & B^*L^+ \\ 0 & C^*L^+ \end{pmatrix},$$

which gives the desired result.

Lemma 2. *If M is partitioned as in (1.5), then*

$$M^{+} = \begin{pmatrix} K^{+}A^{*} & K^{+}B^{*} \\ 0 & C^{+} \end{pmatrix}$$

where $K = A^{*}A + B^{*}B$ if and only if $B^{*}C = 0$.

Proof. For the necessity, use (1.1) to obtain $BK^{+}K + CC^{+}B = B$, which implies $B^{*}C = 0$. For the sufficiency, again we employ (1.6).

III. THE MOORE-PENROSE INVERSE OF $M = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$

We first determine the Moore-Penrose inverse of M given in (1.5).

Theorem. *Let*

$$M = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}.$$

Then

$$M^{+} = \begin{pmatrix} K^{+}A^{*} - K^{+}B^{*}CF & K^{+}B^{*} - K^{+}B^{*}CH \\ F & H \end{pmatrix},$$

where

$$\begin{aligned}
K &= A^{*}A + B^{*}B\,, \\
D &= -AK^{+}B^{*}C\,, \\
E &= C - BK^{+}B^{*}C\,, \\
T &= D^{*}D + E^{*}E\,, \\
S &= K^{+}B^{*}C(I - T^{+}T)\,, \\
F &= T^{+}D^{*} + (I - T^{+}T)(I + S^{*}S)^{-1}\, C^{*}BK^{+}(K^{+}A^{*} - K^{+}B^{*}CT^{+}D^{*})\,,
\end{aligned}$$

and

$$H = T^{+}E^{*} + (I - T^{+}T)(I + S^{*}S)^{-1}\, C^{*}BK^{+}(K^{+}B^{*} - K^{+}B^{*}CT^{+}E^{*})\,.$$

Proof. Cline [1] has shown that if $UV^{*} = 0$, then

$$(U + V)^{+} = U^{+} + (I - U^{+}V)\left[G^{+} + (I - G^{+}G)\, QV^{*}(U^{+})^{*}\, U^{+}(I - VG^{+})\right],$$

where $G = V - UU^{+}V$, $Q = [I + (I - G^{+}G)\, V^{*}(U^{+})^{*}\, U^{+}V(I - G^{+}G)]^{-1}$. Now, let

$$U = \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix}.$$

Then

$$\begin{pmatrix} A & 0 \\ B & C \end{pmatrix} = U + V$$

and $UV^* = 0$. Hence, Cline's theorem is applicable.

By Lemma 2,

$$U^+ = \begin{pmatrix} K^+A^* & K^+B^* \\ 0 & 0 \end{pmatrix},$$

where $K = A^*A + B^*B$. Thus,

$$G = \begin{pmatrix} 0 & -AK^+B^*C \\ 0 & C - BK^+B^*C \end{pmatrix}.$$

Let $D = -AK^+B^*C$ and $E = C - BK^+B^*C$; then we have

$$G = \begin{pmatrix} 0 & D \\ 0 & E \end{pmatrix}.$$

Therefore, by Lemma 1 and the fact $G^+ = (G^{*+})^*$, we get

$$G^+ = \begin{pmatrix} 0 & 0 \\ T^+D^* & T^+E^* \end{pmatrix},$$

where $T = D^*D + E^*E$. Hence,

$$I - G^+G = \begin{pmatrix} I & 0 \\ 0 & I - T^+T \end{pmatrix}.$$

Thus,

$$U^+V(I - G^+G) = \begin{pmatrix} 0 & K^+B^*C(I - T^+T) \\ 0 & 0 \end{pmatrix}$$

and

$$Q = \begin{pmatrix} I & 0 \\ 0 & (I + S^*S)^{-1} \end{pmatrix},$$

where $S = K^+B^*C(I - T^+T)$, and

$$I - VG^+ = \begin{pmatrix} I & 0 \\ -CT^+D^* & I - CT^+E^* \end{pmatrix}.$$

Now

$$U^+(I - VG^+) = \begin{pmatrix} K^+A^* - K^+B^*CT^+D^* & K^+B^* - K^+B^*CT^+E^* \\ 0 & 0 \end{pmatrix},$$

so

$$G^+ + (I - G^+G)\, QV^*(U^+)^*\, U^+(I - VG^+) = \begin{pmatrix} 0 & 0 \\ F & H \end{pmatrix},$$

where

$$F = T^+D^* + (I - T^+T)(I + S^*S)^{-1}\, C^*BK^+(K^+A^* - K^+B^*CT^+D^*)$$

$$H = T^+E^* + (I - T^+T)(I + S^*S)^{-1}\, C^*BK^+(K^+B^* - K^+B^*CT^+E^*),$$

and

$$I - U^+V = \begin{pmatrix} I & -K^+B^*C \\ 0 & I \end{pmatrix}.$$

Therefore

$$(I - U^+V)[G^+ + (I - G^+G)\, QV^*U^{+*}U^+(I - VG^+)] =$$

$$= \begin{pmatrix} -K^+B^*CF & -K^+B^*CH \\ F & H \end{pmatrix},$$

and finally we get

$$\begin{pmatrix} A & 0 \\ B & C \end{pmatrix}^+ = (U + V)^+ = \begin{pmatrix} K^+A^* - K^+B^*CF & K^+B^* - K^+B^*CH \\ F & H \end{pmatrix}.$$

In [2, p. 748, Theorem 6], C. Meyer has given a formula for (1)-inverses of partitioned upper block triangular matrices. Our theorem also accomplishes this task, since the Moore-Penrose inverse is clearly a (1)-inverse. However, since (1)-inverses are not unique, our results are, in general, different from those of Meyer. For example, if

$$M = \begin{pmatrix} \frac{1}{2} & 1 \\ 0 & 0 \end{pmatrix},$$

then Meyer's theorem yields a (1)-inverse,

$$M^- = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix},$$

while our theorem yields

$$M^+ = \begin{pmatrix} \frac{2}{5} & 0 \\ \frac{4}{5} & 0 \end{pmatrix}.$$

At this point, we note the following identities, whose proofs are straightforward.

(3.1) $T = C^*E$

(3.2) If $R = I + S^*S$, then $T^+TR^{-1} = R^{-1}T^+T$

(3.3) $D^*A + E^*B = 0$

(3.4) $F = T^+D^* + R^{-1}S^*(K^+A^* - K^+B^*CT^+D^*)$

(3.5) $H = T^+E^* + R^{-1}S^*(K^+B^* - K^+B^*CT^+E^*)$.

We shall assume throughout the remainder of the paper that M is partitioned as in (1.5). Moreover, we now consider necessary and sufficient conditions for M^+ to be upper block triangular, lower block triangular, and list at the end of the paper some special forms.

Corollary 1. $M^+ = \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix}$ *if and only if* $S^*K^+A^* = 0$ *and* $D^* = 0$, *where* S, K, *and* D *are as defined in the theorem.*

Proof. From the Theorem, we can see that

$$M^+ = \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix} \Leftrightarrow F = 0\,.$$

But from (3.4), we have

$$(3.6)\quad RF = RT^+D^* + S^*K^+A^* - S^*K^+B^*CT^+D^*.$$

By the definition of F, we have

$$TF = TT^+D^* = D^* \quad \text{since} \quad N(T) \subset N(D)\,.$$

Thus $F = 0$ implies $D^* = 0$.
From (3.6), we get

$$S^*K^+A^* = 0 \quad \text{and} \quad D^* = 0 \Leftrightarrow F = 0\,.$$

This completes the proof.

Note.

$$F = 0 \Rightarrow T = E^*E \Rightarrow T^+E^* = E^+ \Rightarrow H = E^+ + R^{-1}S^*(K^+B^* - K^+B^*CE^+)\,.$$

Corollary 2 [2, p. 746, Theorem 4].

$$M^+ = \begin{pmatrix} X & 0 \\ Y & Z \end{pmatrix}$$

if and only if $N(A) \subset N(B)$ and $N(C^*) \subset N(B^*)$. *In this case, we have*

$$M^+ = \begin{pmatrix} A^+ & 0 \\ -C^+BA^+ & C^+ \end{pmatrix}.$$

Proof. From the theorem, we see that

$$M^+ = \begin{pmatrix} X & 0 \\ Y & Z \end{pmatrix}$$

if and only if $K^+B^* = K^+B^*CH$. If $K^+B^* = K^+B^*CH$, then $KK^+B^* = KK^+B^*CH$, and we have $B^* = B^*CH$. Now $TH = TT^+E^* = E^*$ implies $C^*EH = E^*$. Thus $C^*CH = C^*$, and using (1.6) we get $C^+CH = C^+$, and $CH = CC^+$. Hence $B^* = B^*CH = B^*CC^+$, and $N(C^*) \subseteq N(B^*)$.

It can be shown that $H = T^+E^*$ and $F = T^+D^*$ if $K^+B^* = K^+B^*CH$. Thus

$$FA + HB = T^+D^*A + T^+E^*B = T^+(D^*A + E^*B) = 0$$

by (3.3), and we get

(3.7) $FA = -HB$.

Note next that $BK^+A^*A - BK^+B^*CFA = BK^+A^*A - BK^+B^*C(-HB)$ by (3.7). This last term is the same as $BK^+K = B$. Finally, $(BK^+A^*A - BK^+B^*CFA)\,A^+A = BA^+A$ yields $B = BA^+A$, which is equivalent to $N(A) \subseteq N(B)$.

On the other hand, it is straightforward to verify that when $N(A) \subseteq N(B)$ and $N(C^*) \subseteq N(B^*)$, then

$$M^+ = \begin{pmatrix} A^+ & 0 \\ -C^+BA^+ & C^+ \end{pmatrix}.$$

We note that if M is invertible (i.e. A and C are invertible), then

$$M^{-1} = \begin{pmatrix} A^{-1} & 0 \\ -C^{-1}BA^{-1} & C^{-1} \end{pmatrix}.$$

Suppose $A = 0$. Then M^+ is lowerblock triangular if and only if $B = 0$. There are many special cases which can be derived from corollary 2.

In conclusion, the following special results can be obtained.

(3.8) If $K^+B^* = K^+B^*CH$, then $K^+A^* - K^+B^*CF = A^+$,

$F = -C^+BA^+$, $HC = T^+T$, $H = C^+$, and $S = 0$.

(3.9) $M^+ = \begin{pmatrix} A^+ & B^+ \\ 0 & C^+ \end{pmatrix}$ if and only if $B^*C = 0$ and $AB^* = 0$.

(3.10) $M^+ = \begin{pmatrix} A^+ & D^+ \\ 0 & C^+ - C^+BD^+ \end{pmatrix}$, where $D = B - CC^+B$ if and only if

$AB^* = 0$ and $C^+B = C^+BD^+B$.

(3.11) $M^+ = \begin{pmatrix} A^+ & PC^+ \\ 0 & C^+ - C^+BPC^+ \end{pmatrix}$ where $P = Q^{-1}(C^+B)^*$ and

$Q = I + (C^+B)^*\,(C^+B)$ if and only if $AB^* = 0$ and $N(C^*) \subset N(B^*)$.

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