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Research Article

The Morbidity of Multivariable Grey Model MGM$(1, m)$

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This paper proposes the morbidity of the multivariable grey prediction MGM$(1, m)$ model. Based on the morbidity of the differential equations, properties of matrix, and Gerschgorin Panel Theorem, we analyze the factors that affect the morbidity of the multivariable grey model and give a criterion to justify the morbidity of MGM$(1, m)$. Finally, an example is presented to illustrate the practicality of our results.

1. Introduction

In recent decades, grey system theory, as well as fuzzy set theory [1] and rough set theory [2], is one of the most widely used theories to study uncertain problems. The grey system theory which was introduced by Deng [3], characterized by few data and poor information, has been successfully utilized in uncertain problems. On account of their enormous applications in agriculture, economics, management, and engineering, the grey system attracts many scientific research workers and scholars devoted to various aspect of those fields.

Grey forecasting models, an important part of grey systems, have been widely adopted to predict practical problems due to their simple calculating process and higher forecasting accuracy [4, 5]. However, some researchers put forward that the tiny changes of the initial data can result in the estimation errors, which is called the morbidity of the grey models. The research on morbidity and stability problems occupies an important part in grey forecasting system. Zheng et al. [6] pointed out that there existed morbidity in grey prediction models and analyzed the reasons in earlier times. Dang et al. [7] showed the possibility of the morbidity problem could only exist in GM$(1, 1)$ when the first item of original sequence was unequal to zero while other items were equal to zero approximatively. Wei [8] resolved the morbidity problem for the grey model with the accumulating method based on the condition number theory. Xiao and Li [9] studied the effects of the multiple transformation to the condition number of the non-equigap GM$(1, 1)$ model.

Except for the research on the morbidity of GM$(1, 1)$ model, there are also some studies concentrating on the morbidity of other grey models. Xiao and Guo [10] and Zeng and Xiao [11] researched on the morbidity problem of GM$(2, 1)$ which had two characteristic values. Wang et al. [12] summarized the main factor that affected the morbidity of GM$(1, 1, t^{\alpha})$ and suggested that there existed morbidity in some cases. Cui et al. [13, 14] found that there was no morbidity in NGM$(1, 1, k)$ and grey Verhulst model; the solution of those models will not make significant drift for the original data series of systems if there exist minor errors in collecting process.

Compared to the morbidity of grey models group, there is a little attention on the morbidity of multivariate grey prediction model MGM$(1, m)$. The MGM$(1, m)$ model was proposed by Zhai et al. [15] and has been developed rapidly and caught the attention of many researchers. Zou [16] applied a step by step optimum new information modeling method to build multivariable nonequidistance information grey model. Xiong et al. [17] optimized the background value and set the multiple linear regression model based on MGM in order to eliminate the fluctuations or random errors of the original data. Guo et al. [18] constructed SMGM$(1, m)$ through coupling self-memory principle of dynamic system to MGM; examples showed that it had superior predictive performance over other traditional grey prediction models.

Does the possibility of the morbidity in MGM exist? How to identify the morbidity of the multivariable grey model has become an important aspect in the process of constructing the MGM$(1, m)$ model. This paper discusses the possibility of the MGM$(1, m)$ model and the remainder of the paper is organized as follows: Section 2 introduces the morbidity of matrix equations and analyzes the factors that affect the condition number of special matrix. Section 3 provides a criterion to justify the morbidity of MGM$(1, m)$. Section 4 gives an example to illustrate the practicality of our results. Some conclusions are presented in Section 5.

2. The Morbidity of Equations

Considering the differential equation $Ax = b$, A is nonsingular matrix, b is the constant variable, and x is the solution of the equation.

Definition 1 (see [19]). If A or b has a small change and causes a larger change in the solution of the equation $Ax = b$, the equation is said to be morbidity equation.

Definition 2 (see [19]). Suppose that A is a square matrix with full rank. The condition number of A is

$$\operatorname{cond}(A)_v = \left\|A^{-1}\right\|_v \cdot \|A\|_v, \quad v = 1, 2, \ldots, \infty. \tag{1}$$

If A is a real symmetric matrix, then the condition number of A is

$$\operatorname{cond}(A) = \frac{\left|\lambda_{\max}(A)\right|}{\left|\lambda_{\min}(A)\right|}, \tag{2}$$

where $\lambda_{\max}$ is the maximal eigenvalue of the matrix and $\lambda_{\min}$ is the minimal eigenvalue of matrix. If $\operatorname{cond}(A) \in (1, 10)$, A is well conditioned. If $\operatorname{cond}(A) \in [10, 100)$, A is slightly ill-conditioned. If $\operatorname{cond}(A) \in [100, 1000)$, A is moderately ill-conditioned. If $\operatorname{cond}(A) \in [1000, \infty)$, A is strongly ill-conditioned.

In the process of parameters identification of multivariable grey model, we usually use the least square method to estimate the parameters, so there exist least square problems in the parameters.

Assuming that $C \in R^{m\times n}$, $y \in R^m$, and C is the parameters matrix of the grey model. If there exists a vector $x_0 \in R^n$, making $\|Cx - y\|_2$ achieve the minimum of the function, which is

$$\left\|Cx_0 - y\right\|_2 = \min_{x\in R^n} \left\|Cx - y\right\|_2, \tag{3}$$

then x_0 is the solution of the linear equation $Cx = y$, which is the estimated parameter of the grey model.

Suppose that $f(x) = \|Cx - y\|^2 = (Cx - y)^T(Cx - y) = x^T C^T Cx - x^T D^T y - y^T Dx + y^T y$. By the extremum condition of the equation, we have

$$\frac{df(x)}{dx} = 2C^T Cx - 2C^T y = 0. \tag{4}$$

Then we obtain the solution $C^T Cx = C^T y$, which is also the least square solution of the equation $Cx = y$.

In the multivariable grey prediction models, the data matrix C is usually the long matrix; it is not easy to solve its condition number. It should be noted that $C^T C$ is a real symmetric matrix, the condition number is easy to obtain. Therefore, we often justify the morbidity of the multivariable grey model by the condition number of $C^T C$.

3. The Morbidity of MGM$(1, m)$

3.1. Grey MGM$(1, m)$ Model. The multiple variable grey prediction model abbreviated as MGM$(1, m)$ is one of the frequently used grey forecasting models. The MGM$(1, m)$ model constructing process is presented below.

Definition 3. Assume that the data sequence

$$X_j^{(0)} = \left(x_j^{(0)}(1), x_j^{(0)}(2), \ldots, x_j^{(0)}(m)\right)^T, \quad j = 1, 2, \ldots, m \tag{5}$$

is the original nonnegative data matrix. The data matrix

$$X_j^{(1)} = \left(x_j^{(1)}(1), x_j^{(1)}(2), \ldots, x_j^{(1)}(n)\right)^T, \quad j = 1, 2, \ldots, m \tag{6}$$

is the first-order accumulated generating matrix of $X^{(0)}$, where

$$x_j^{(1)}(k) = \sum_{i=1}^{k} x_j^{(0)}(i). \tag{7}$$

The adjacent neighbour average sequence of $X^{(1)}$ is

$$Z_j^{(1)} = \left(z_j^{(1)}(1), z_j^{(1)}(2), \ldots, z_j^{(1)}(n)\right), \tag{8}$$

where $z_j^{(1)}(k) = 0.5(x_j^{(1)}(k) + x_j^{(1)}(k-1))$, $k = 2, 3, \ldots, n$.

The first-order differential equations of the multivariable grey model MGM$(1, m)$ are as follows:

$$\frac{dx_1^{(1)}}{dt} = \alpha_{11}x_1^{(1)} + \alpha_{12}x_2^{(1)} + \cdots + \alpha_{1m}x_m^{(1)} + \beta_1$$

$$\frac{dx_2^{(1)}}{dt} = \alpha_{21}x_1^{(1)} + \alpha_{22}x_2^{(2)} + \cdots + \alpha_{2m}x_m^{(1)} + \beta_2$$

$$\vdots$$

$$\frac{dx_m^{(1)}}{dt} = \alpha_{m1}x_1^{(1)} + \alpha_{m2}x_2^{(1)} + \cdots + \alpha_{mm}x_m^{(m)} + \beta_m. \tag{9}$$

Note that

$$A = (\alpha_{ij})_{m\times m}, \quad \beta = (\beta_1, \beta_2, \ldots, \beta_m)^T, \tag{10}$$

and (9) can be noted as

$$\frac{dX^{(1)}(t)}{dt} = AX^{(1)}(t) + \beta. \tag{11}$$

Applying the least square method to the first-order differential equation

$$\frac{dX^{(1)}(t)}{dt} = AZ^{(1)}(t) + \beta, \tag{12}$$

we obtain the estimated parameters

$$\widehat{Q} = \begin{pmatrix} \widehat{A}' \\ \widehat{\beta}' \end{pmatrix} = (P^TP)^{-1} P^T (Y_1, Y_2, \ldots, Y_m), \tag{13}$$

where

$$\widehat{Q} = \begin{pmatrix} \widehat{\alpha}_{11} & \widehat{\alpha}_{21} & \cdots & \widehat{\alpha}_{m1} \\ \widehat{\alpha}_{12} & \widehat{\alpha}_{22} & \cdots & \widehat{\alpha}_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{\alpha}_{1m} & \widehat{\alpha}_{2m} & \cdots & \widehat{\alpha}_{mm} \\ \widehat{\beta}_1 & \widehat{\beta}_2 & \cdots & \widehat{\beta}_m \end{pmatrix},$$

$$P = \begin{pmatrix} z_1^{(1)}(2) & z_2^{(1)}(2) & \cdots & z_m^{(1)}(2) & 1 \\ z_1^{(1)}(3) & z_2^{(1)}(3) & \cdots & z_m^{(1)}(3) & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ z_1^{(1)}(n) & z_2^{(1)}(n) & \cdots & z_m^{(1)}(n) & 1 \end{pmatrix}, \tag{14}$$

and $Y_j = (x_j^{(0)}(2), x_j^{(0)}(3), \ldots, x_j^{(0)}(n))^T$, $j = 1, 2, \ldots, m$.

3.2. The Morbidity of MGM. In this part, we give a criterion to justify the morbidity of MGM$(1, m)$.

Lemma 4 (Gerschgorin Panel Theorem). *If $A \in C^{n\times n}$ and $A = (a_{ij})$, then every eigenvalue of A is contained in the plane, which is*

$$\lambda \in \bigcup_{i=1}^{n} D_i, \tag{15}$$

where D_i is the panel centred by a_{ii} in the complex plane and

$$D_i = \left\{ z \in C \mid |z - a_{ii}| \le \sum_{j=1, j\ne i}^{n} |a_{ij}| \right\}, \quad i = 1, 2, \ldots, n. \tag{16}$$

Theorem 5. *Suppose that $X_j^{(0)}(1), X_j^{(0)}(2), \ldots, X_j^{(0)}(n)$ are data vectors, and $X_j^{(1)}(n)$ is the first-order accumulated generating vector. If every consecutive neighbour $z_j^{(1)}(k) \ge 1$ ($j = 1, 2, \ldots, m$), then the multivariable grey model MGM$(1, m)$ is morbidity.*

Proof. In the process of estimating the parameters of A, β, by least square method, we calculate the matrix of P^TP, which is

$$P^TP = \begin{pmatrix} \sum_{k=2}^{n}(z_1^{(1)}(k))^2 & \cdots & \sum_{k=2}^{n} z_1^{(1)}(k) z_m^{(1)}(k) & \sum_{k=2}^{n} z_1^{(1)}(k) \\ \sum_{k=2}^{n} z_1^{(1)}(k) z_2^{(1)}(k) & \cdots & \sum_{k=2}^{n} z_2^{(1)}(k) z_m^{(1)}(k) & \sum_{k=2}^{n} z_2^{(1)}(k) \\ \vdots & \ddots & \vdots & \vdots \\ \sum_{k=2}^{n} z_m^{(1)}(k) z_1^{(1)}(k) & \cdots & \sum_{k=2}^{n}(z_m^{(1)}(k))^2 & \sum_{k=2}^{n} z_m^{(1)}(k) \\ \sum_{k=2}^{n} z_1^{(1)}(k) & \cdots & \sum_{k=2}^{n} z_m^{(1)}(k) & n-1 \end{pmatrix}. \tag{17}$$

From

$$(P^TP)^T = P^TP, \tag{18}$$

we know P^TP is a symmetric matrix; since P is invertible, we deduce that all the eigenvalues of the matrix P^TP are positive real numbers and P^TP is positive definite matrix. Therefore, the condition number of matrix P^TP can be represented by the maximal eigenvalue and minimal eigenvalue of the matrix.

Set $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ as the eigenvalues of P^TP. By Gerschgorin Panel Theorem, we have

$$D_1 = \left\{ \lambda_1 \in R^+ \mid \left| \lambda_1 - \sum_{k=2}^{n}(z_1^{(1)}(k))^2 \right| \le \sum_{j=2}^{m}\left(\sum_{k=2}^{n} z_1^{(1)}(k) z_j^{(1)}(k)\right) + \sum_{k=2}^{n} z_1^{(1)}(k) \right\},$$

$$D_2 = \left\{ \lambda_2 \in R^+ \mid \left| \lambda_2 - \sum_{k=2}^{n}(z_2^{(1)}(k))^2 \right| \le \sum_{j=1, j\ne 2}^{m}\left(\sum_{k=2}^{n} z_2^{(1)}(k) z_j^{(1)}(k)\right) + \sum_{k=2}^{n} z_2^{(1)}(k) \right\},$$

$$\vdots$$

$$D_{n-1} = \left\{ \lambda_{n-1} \in R^{+} \mid \left| \lambda_{n-1} - (n-1) \right| \le \sum_{j=1}^{m} \sum_{k=2}^{n} z_j^{(1)}(k) \right\}. \quad (19)$$

It is easy to see that all the eigenvalues of $P^T P$ are contained in the $D_1 \cup D_2 \cup \cdots \cup D_{n-1}$; that is to say, every eigenvalue of $P^T P$ is contained in the panel.

If all the adjacent neighbour average sequences $z_i^{(1)}(k) \ge 1$ and the chosen sample is the minimal permitted data in grey system, then we conclude that $\sum_{k=2}^{n}(z_i^{(1)}(k))^2$ is larger than $n-1$, and the maximal eigenvalue and minimal eigenvalue are contained in different circles, and the centres of circles are far from each other. Therefore, the maximal eigenvalue and minimal eigenvalue are far away from each other on the number line. From the definition of the ill-conditioned matrix, we deduce that the multivariable grey model MGM$(1, m)$ is morbidity. This completes the proof. □

4. Example

In what follows, we give an example to illustrate the practicality of our results. The data are the price indexes of financial intermediation and real estate in 1981–1984, and data resource is the China statistical yearbook. Set $X_1^{(0)}$ and $X_2^{(0)}$ as the price index of financial intermediation and price index of the real estate, respectively; the data are shown in Table 1. As usual, we chose 4 group samples which are the minimum permitted data in grey models.

We construct MGM$(1, 2)$ model to simulate and predict the data vectors. By the definition of P, we obtain

$$P = \begin{pmatrix} 1.339 & 1.133 & 1 \\ 1.7855 & 1.2125 & 1 \\ 2.3015 & 1.415 & 1 \end{pmatrix},$$
$$P^T P = \begin{pmatrix} 10.2778 & 6.9386 & 5.4260 \\ 6.9386 & 4.7561 & 3.7605 \\ 5.426 & 3.7605 & 3 \end{pmatrix}. \quad (20)$$

By Theorem 5, there exists morbidity in MGM$(1, 2)$ model. In fact, all the eigenvalues of $P^T P$ are

$$\lambda_1 = 0.0011,$$
$$\lambda_2 = 0.1249, \quad (21)$$
$$\lambda_3 = 17.9079.$$

Table 1: The data vectors.

	$k=1$	$k=2$	$k=3$	$k=4$
$X_1^{(0)}$	1.102	1.576	1.995	2.608
$X_2^{(0)}$	1.084	1.182	1.243	1.587

It is clear that cond$(A) \ge 1000$, and there exists morbidity in the model. It proves that our criterion is a useful way to justify the morbidity of MGM$(1, m)$ model.

5. Conclusions

This paper discusses the morbidity of the multivariable grey model. From the morbidity of the differential equations, we analyze the factors that affect the morbidity of MGM$(1, m)$ model. By Gerschgorin Panel Theorem and the knowledge of matrix, we give a criterion to justify the morbidity of MGM$(1, m)$. An example is given to illustrate the maneuverability of our results.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] L. A. Zadeh, "Fuzzy algorithms," *Information and Control*, vol. 12, no. 2, pp. 94–102, 1968.

[2] Z. Pawlak, "Rough sets," *International Journal of Computer & Information Science*, vol. 11, no. 5, pp. 341–356, 1982.

[3] J. L. Deng, "Control problems of grey systems," *Systems Control Letters*, vol. 1, no. 5, pp. 288–294, 1982.

[4] Y. H. Wang, K. Qu, and Z. H. Wang, "A kind of nonlinear strengthening operators for predicting the output value of China's marine electric power industry," *The Journal of Grey System*, vol. 28, no. 2, pp. 35–52, 2016.

[5] S. F. Liu, J. Forrest, and Y. J. Yang, "Advances in grey systems research," *The Journal of Grey System*, vol. 25, no. 2, pp. 1–18, 2013.

[6] Z. N. Zheng, Y. Y. Wu, and H. L. Bao, "Morbidity problem in grey model," *Chinese Journal of Management Science*, vol. 9, no. 5, pp. 38–44, 2001.

[7] Y. G. Dang, Z. X. Wang, and S. F. Liu, "Study on morbidity problem in grey model," *Systems Engineering Theory Practice*, vol. 28, no. 1, pp. 156–160, 2008.

[8] Y. Wei, "Morbidity research on grey forecast model," *Communications in Computer and Information Science*, vol. 224, no. 1, pp. 294–298, 2011.

[9] X. P. Xiao and F. Q. Li, "Research on the stability of non-equigap grey control model under multiple transformations," *Kybernetes*, vol. 38, no. 10, pp. 1701–1708, 2009.

[10] X. P. Xiao and J. H. Guo, "The morbidity peoblem of GM(2, 1) model based on vector transformation," *The Journal of Grey System*, vol. 26, no. 3, pp. 1–11, 2014.

[11] X. Y. Zeng and X. P. Xiao, "Research on morbidity problem of accumulating method," *Journal of Systems Engineering and Electronics*, vol. 28, no. 4, pp. 542–572, 2006.

[12] Z. X. Wang, Y. G. Dang, and S. F. Liu, "The morbidity of GM(1, 1) power model," *System Engineering Theory Practice*, vol. 33, no. 7, 1859.

[13] J. Cui, S. F. Liu, N. M. Xie, and B. Zeng, "Study on morbidity of grey Verhulst forecasting model," *Systems Engineering—Theory Practice*, vol. 34, no. 2, pp. 416–420, 2014.

[14] J. Cui, Y. G. Dang, and S. F. Liu, "Study on morbidity of NGM(1, 1, k) model based on conditions of matrix," *Control and Decision*, vol. 25, no. 7, pp. 1050–1054, 2010.

[15] J. Zhai, J. M. Sheng, and Y. J. Feng, "The grey model MGM(1, n) and its application," *Systems Engineering—Theory Practice*, vol. 17, no. 5, pp. 109–113, 1997.

[16] R. B. Zou, "The Non-equidistant new information optimizing MGM(1, n) based on a step by step optimum constructing background value," *Applied Mathematics & Information Sciences*, vol. 6, no. 3, pp. 745–750, 2012.

[17] P. P. Xiong, Y. G. Dang, X. H. Wu, and X. M. Li, "Combined model based on optimized multi-variable grey model and multiple linear regression," *Journal of Systems Engineering and Electronics*, vol. 22, no. 4, pp. 615–620, 2011.

[18] X. J. Guo, S. F. Liu, L. F. Wu, Y. B. Gao, and Y. J. Yang, "A multi-variable grey model with a self-memory component and its application on engineering prediction," *Engineering Applications of Artificial Intelligence*, vol. 42, pp. 82–93, 2015.

[19] J. L. Chen and X. H. Chen, *Special Matrices*, Tsinghua university press, Beijing, China, 2001.