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THE MOSER'S ITERATIVE METHOD FOR A CLASS OF ULTRAPARABOLIC EQUATIONS

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We adapt the iterative scheme by Moser, to prove that the weak solutions to an ultraparabolic equation, with measurable coefficients, are locally bounded functions. Due to the strong degeneracy of the equation, our method differs from the classical one in that it is based on some ad hoc Sobolev type inequalities for solutions.

Keywords: Ultraparabolic equations; measurable coefficients; Moser's iterative method.

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1. Introduction

We consider the second order partial differential equation

$$Lu(x,t) \equiv \sum_{i,j=1}^{m_0} \partial_{x_i}(a_{ij}(x,t)\partial_{x_j}u(x,t)) + \sum_{i,j=1}^{N} b_{ij}x_i\partial_{x_j}u(x,t) - \partial_t u(x,t) = 0\,, \quad (1.1)$$

where $(x,t) = (x_1,\ldots,x_N,t) = z$ denotes the point in $\mathbb{R}^{N+1}$, and $1 \le m_0 \le N$. Equation (1.1) arises in the theory of stochastic processes as well as in the kinetic theory; in particular, it contains the (spatially inhomogeneous) Fokker–Planck–Landau equation.

We aim to adapt the iterative scheme introduced by Moser in [1, 2] for the uniformly parabolic equations, to prove that the weak solutions to (1.1) are locally bounded functions. Moser's method is based on a combination of a Caccioppoli type estimate with the classical embedding Sobolev inequality. Due to the strong degeneracy of the operator L, some new difficulties arise in treating (1.1). Indeed, the natural extension of the Caccioppoli estimates gives an L^2_{loc} bound only of the first order derivatives $\partial_{x_j}u$, for $j = 1,\ldots,m_0$, but it does not give any information on the other spatial directions.

Actually, the various extensions of the Moser's iteration technique available in literature (see, for instance, [3–6]) rely on the implicit assumption that the Sobolev inequality (adapted to the suitable functional setting) is the necessary starting point of the procedure. This argument fails in our case, since the Caccioppoli estimates provide an incomplete information. In order to overcome this problem, we prove a Sobolev type inequality only for the solutions to (1.1). Our idea is to represent u in terms of a *parametrix* of L, which is the fundamental solution of the following operator

$$L_0 = \Delta_{m_0} + Y\,, \tag{1.2}$$

where Δ_{m_0} is the Laplace operator in the variables $x_1, \dots, x_{m_0}$ and Y is the first order part of L:

$$Y = \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j} - \partial_t\,. \tag{1.3}$$

With a slight abuse of notation, we shall refer to L_0 as *principal part of* L. Then, if u is a solution to (1.1), we have

$$L_0 u = (L_0 - L)u = \sum_{i=1}^{m_0} \partial_{x_i} F_i\,, \tag{1.4}$$

where

$$F_i = \sum_{j=1}^{m_0} (\delta_{ij} - a_{ij}) \partial_{x_j} u\,, \quad i = 1, \dots, m_0\,.$$

Since the F_i's depend only on the first order derivatives $\partial_{x_j} u$, $j = 1, \dots, m_0$, the Caccioppoli inequality yields an H^{-1}_{loc}-estimate of the right hand side of (1.4). Thus, by using the fundamental solution of L_0, we get the needed L^p_{loc} estimate of the solution. This argument seems quite natural, since the classical Sobolev inequality can be proved by representing any function $u \in H^1$ as a convolution with the fundamental solution of the Laplace operator.

We next state our assumptions and main results.

[H.1] *The coefficients* a_{ij}, $1 \le i, j \le m_0$, *are real valued, measurable functions of* z. *Moreover* $a_{ij} = a_{ji}$, $1 \le i, j \le m_0$, *and there exists a positive constant* μ *such that*

$$\mu^{-1} |\xi|^2 \le \sum_{i,j=1}^{m_0} a_{ij}(z) \xi_i \xi_j \le \mu |\xi|^2\,,$$

for every $z \in \mathbb{R}^{N+1}$ *and* $\xi \in \mathbb{R}^{m_0}$. *The matrix* $B = (b_{ij})_{i,j=1,\dots,N}$ *is constant.*

[H.2] L_0 *is hypoelliptic* (*i.e. every distributional solution of* $L_0 u = 0$ *is a* C^∞ *function*) *and* δ_λ*-homogeneous of degree two with respect to some dilations group* $(\delta_\lambda)_{\lambda > 0}$ *in* $\mathbb{R}^{N+1}$ (*see* (2.6) *below*).

We explicitly remark that, although [H.2] is expressed in terms of L_0, it is a requirement on the coefficients b_{ij} of the operator L. Indeed, a well-known criterion for the hypoellipticity is the Hörmander's condition [7]. In our setting, it reads:

$$\text{rank Lie}(\partial_{x_1}, \dots, \partial_{x_{m_0}}, Y)(z) = N+1\,, \quad \forall\, z \in \mathbb{R}^{N+1}\,,$$

where Lie $(\partial_{x_1}, \dots, \partial_{x_{m_0}}, Y)$ denotes the Lie algebra generated by the first order differential operators (vector fields) $\partial_{x_1}, \dots, \partial_{x_{m_0}}, Y$. Then the hypoellipticity of L_0 (as well as the dilations group $(\delta_\lambda)_{\lambda>0}$) depends *only* on m_0 and on the first order part of L. In Sec. 2, we recall a known structure condition on the matrix B equivalent to [H.2].

Let us remark that if L is an uniformly parabolic operator (i.e. $m_0 = N$ and $B \equiv 0$), then [H.2] is clearly satisfied. Indeed, the principal part of L simply is the heat operator, which is hypoelliptic and homogeneous with respect to the parabolic dilations $\delta_\lambda(x,t) = (\lambda x, \lambda^2 t)$.

We give the definition of solution to (1.1). We denote by $D = (\partial_{x_1}, \dots, \partial_{x_N})$, $\langle \cdot, \cdot \rangle$ respectively the gradient and the inner product in $\mathbb{R}^N$. Besides, D_{m_0} is the gradient with respect to the variables $x_1, \dots, x_{m_0}$.

Definition 1.1. A weak solution of (1.1) in a subset Ω of $\mathbb{R}^{N+1}$ is a function u such that u, $D_{m_0}u$, $Yu \in L^2_{\text{loc}}(\Omega)$ and

$$\int_\Omega -\langle ADu, D\phi \rangle + \phi Y u = 0\,, \quad \forall\, \phi \in C_0^\infty(\Omega)\,. \tag{1.5}$$

As we shall see in Sec. 2, the natural geometry underlying operator L is determined by a suitable homogeneous Lie group structure on $\mathbb{R}^{N+1}$. Our main results below reflect this non-Euclidean background. Let "$\circ$" denote the Lie product on $\mathbb{R}^{N+1}$ defined in (2.3), and consider the cylinder

$$R_1 = \{(x,t) \in \mathbb{R}^N \times \mathbb{R} \,|\, |x| < 1, |t| < 1\}\,.$$

For every $z_0 \in \mathbb{R}^{N+1}$ and $r > 0$, we set

$$R_r(z_0) \equiv z_0 \circ (\delta_r(R_1)) = \{z \in \mathbb{R}^{N+1} | z = z_0 \circ \delta_r(\zeta), \zeta \in R_1\}\,. \tag{1.6}$$

We have

Theorem 1.2. *Let u be a non-negative weak solution of* (1.1) *in* Ω. *Let $z_0 \in \Omega$ and r, ϱ, $0 < \frac{r}{2} \le \varrho < r$, be such that $\overline{R_r(z_0)} \subseteq \Omega$. Then there exists a positive constant c which depends on μ and on the homogeneous dimension Q (cf.* (2.7)*) such that, for every $p > 0$, it holds*

$$\sup_{R_\varrho(z_0)} u^p \le \frac{c}{(r-\varrho)^{Q+2}} \int_{R_r(z_0)} u^p\,. \tag{1.7}$$

Estimate (1.7) *also holds for every $p < 0$ such that $u^p \in L^1(R_r(z_0))$.*

Remark 1.3. Sub and super-solutions also verify estimate (1.7) for suitable values of p (see Corollary 4.3). More precisely, (1.7) holds for

(i) $p \geq 1$ or $p < 0$, if u is a non-negative weak sub-solution of (1.1) such that $u^p \in L^1(R_r(z_0))$;

(ii) $p \in]0, \frac{1}{2}[$, if u is a non-negative weak super-solution of (1.1). In this case, the constant c in (1.7) also depends on p.

A direct consequence of Theorem 1.2 is the local boundedness of weak solutions to (1.1).

Corollary 1.4. *Let u be a weak solution of* (1.1) *in Ω. Let z_0, ϱ, r as in Theorem* 1.2. *Then, we have*

$$\sup_{R_\varrho(z_0)} |u| \leq \left(\frac{c}{(r-\varrho)^{Q+2}} \int_{R_r(z_0)} |u|^p \right)^{\frac{1}{p}}, \quad \forall\, p \geq 1\,, \tag{1.8}$$

where $c = c(Q, \mu)$.

The interest in the above class of operators is motivated by the following applications.

Example 1.5. Consider the following kinetic equation

$$\partial_t f - \langle v, \nabla_x f \rangle = \mathcal{Q}(f)\,, \quad t \geq 0, x \in \mathbb{R}^n, v \in \mathbb{R}^n\,, \tag{1.9}$$

where $n \geq 1$ and $\mathcal{Q}(f)$ is the so-called "collision operator" which can take either a linear or a non linear form. The solution f corresponds at each time t to the density of particles at the point x with velocity v. If

$$\mathcal{Q}(f) = \triangle_v f\,,$$

then (1.9) becomes the prototype of the linear Fokker–Planck equation (see, for instance, [8, 9]) and it can be written in the form (1.1) by choosing $m_0 = n$, $N = 2n$ and

$$B = \begin{pmatrix} 0 & -I_n \\ 0 & 0 \end{pmatrix},$$

where I_n is the identity $n \times n$ matrix. In this case the Lie group is given by the Galilean change of coordinates $(v, x, t) \cdot (v', x', t') = (v + v', x + x' + t'v, t + t')$ and the dilations group is $\delta_\lambda(v, x, t) = (\lambda v, \lambda^3 x, \lambda^2 t)$.

In the Boltzmann–Landau equation (see [10–12])

$$\mathcal{Q}(f) = \sum_{i,j=1}^{n} \partial_{v_i}(a_{ij}(\cdot, f)\partial_{v_j} f)\,,$$

the coefficients a_{ij} actually depend on the unknown function through some integral expressions.

We also recall that equations of the form (1.9) arise in mathematical finance (see [13–15]).

Example 1.6. The equation

$$\partial_t u = \sum_{i,j=1}^{n} \partial_{x_i}(a_{ij}\partial_{x_j} u) + \sum_{i=1}^{n} x_i \partial_{y_i} u + \sum_{i=1}^{n} y_i \partial_{s_i} u\,,$$

$$(x, y, s, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}\,,$$

arises in the theory of degenerate diffusion processes (see, for instance, [16, 17]). The above operator satisfies hypotheses [H.1]–[H.2] and the corresponding Lie group has step two. The dilations are given by $\delta_\lambda(x, y, s, t) = (\lambda x, \lambda^3 y, \lambda^5 s, \lambda^2 t)$.

A further motivation comes from the theory of partial differential equations. As said above, in the case of a_{ij} constant coefficients, the smoothness of the solutions has been pointed out by Kolmogorov [18] and by Hörmander [7]. A systematic study of this class of operators has been carried out by Kupcov [19], Lanconelli and one of us [20].

The Levi parametrix method has been used by Weber [21], Il'in [22], Eidelman [16] and Polidoro [23], [24] to deal with Hölder continuous coefficients a_{ij}. In these hypotheses, Schauder type estimates have been proved by Satyro [25], Lunardi [26], Manfredini [27]. Besides, the regularity properties of the weak solutions to (1.1) have been studied by Bramanti, Cerutti and Manfredini [28], Manfredini and Polidoro [29], Polidoro and Ragusa [30], assuming a weak continuity condition on the coefficients a_{ij} (they are supposed to be in a suitable vanishing mean oscillation space).

A boundary value problem for a nonlinear equation of the form (1.1) has been considered by Lanconelli and Lascialfari in [31], by Lascialfari and Morbidelli in [32]. Their results have been proved by combining the Kakutani–Ky Fan fixed point theorem with the above interior estimates. However, the dependence of the *a priori* estimates on the regularity of the coefficients a_{ij} forces some restrictive conditions on the nonlinearity of the operator.

The Moser's method extends the techniques previously used in the elliptic case [33, 34] and which are equivalent to the ones due to De Giorgi [35]. These classical results have been generalized in many directions (see [36–40]). The first extensions of Moser's technique to a non-Euclidean framework are contained in [4, 41]. We also recall that the technique introduced by Nash [42] and developed in [43], has been used in [44], in the framework of subelliptic operators on Lie groups. The main goal in the above quoted papers is the uniform Hölder continuity of the solutions, which is a basic tool in the study of the non linear problem. In a future study we plan to complete the Moser's procedure for operator (1.1) by proving a weak Harnack inequality, which has not been established yet. We also recall that Theorem 1.2 has been used in [45] to obtain a pointwise global upper bound for the fundamental solution of (1.1).

The paper is organized as follows. In Sec. 2, we recall some known facts on the principal part L_0 and we collect some preliminaries. In Sec. 3, we prove some Caccioppoli and Sobolev type inequalities. Section 4 is devoted to the proof of

Theorem 1.2 and Corollary 1.4, by the Moser's iteration scheme. In order to restore the analogy with the classical result by Moser, in Sec. 5, we show that, for $p < 0$, (1.7) holds if we replace $R_r(z_0)$, $R_\varrho(z_0)$ by $R_r(z_0) \cap \{t < t_0\}$, $R_\varrho(z_0) \cap \{t < t_0\}$ respectively.

2. Preliminaries

In this section we recall some known facts about the principal part L_0 of L, and we show some preliminary results. We rewrite operator L in (1.1) in the compact form

$$L = \text{div}\,(AD) + Y\,, \tag{2.1}$$

where $A = (a_{ij})_{1\le i,j\le N}$, $a_{ij} \equiv 0$ if $i > m_0$ or $j > m_0$, and Y is defined in (1.3). We also set

$$A_0 = \begin{pmatrix} I_{m_0} & 0 \\ 0 & 0 \end{pmatrix},$$

where I_{m_0} is the identity matrix in $\mathbb{R}^{m_0}$. Then the principal part of L takes the form

$$L_0 = \text{div}\,(A_0 D) + Y\,.$$

Operator L_0 has the remarkable property of being invariant with respect to a Lie product in $\mathbb{R}^{N+1}$. More precisely, we let

$$E(s) = \exp(-sB^T)\,, \quad s \in \mathbb{R}\,, \tag{2.2}$$

and we denote by ℓ_ζ, $\zeta \in \mathbb{R}^{N+1}$, the left translation $\ell_\zeta(z) = \zeta \circ z$ in the group law

$$(x,t) \circ (\xi,\tau) = (\xi + E(\tau)x, t+\tau)\,, \quad (x,t),(\xi,\tau) \in \mathbb{R}^{N+1}\,, \tag{2.3}$$

then we have

$$L \circ \ell_\zeta = \ell_\zeta \circ L\,.$$

We recall that, by [20, Propositions 2.1 and 2.2], hypothesis [H.2] is equivalent to assume that for some basis on $\mathbb{R}^N$, the matrix B has the canonical form

$$\begin{pmatrix} 0 & B_1 & 0 & \cdots & 0 \\ 0 & 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & B_r \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{2.4}$$

where B_k is a $m_{k-1} \times m_k$ matrix of rank m_k, $k = 1, 2, \ldots, r$ with

$$m_0 \ge m_1 \ge \cdots m_r \ge 1\,, \quad \text{and} \quad \sum_{k=0}^{r} m_k = N\,.$$

In this case the dilations associated to L_0 are given by

$$\delta_\lambda = \mathrm{diag}(\lambda I_{m_0}, \lambda^3 I_{m_1}, \dots, \lambda^{2r+1} I_{m_r}, \lambda^2), \quad \lambda > 0, \tag{2.5}$$

where I_{m_k} denotes the $m_k \times m_k$ identity matrix. We can write explicitly the second assertion in Hypothesis [H.2] as

$$L_0 \circ \delta_\lambda = \lambda^2 (\delta_\lambda \circ L_0), \quad \forall\, \lambda > 0. \tag{2.6}$$

In the sequel we shall always assume that B has the canonical form (2.4).

We denote by $\Gamma_0(\cdot,\zeta)$ the fundamental solution of L_0 in (1.2) with pole in $\zeta \in \mathbb{R}^{N+1}$. An explicit expression of $\Gamma_0(\cdot,\zeta)$ has been constructed in [7] and [19]:

$$\Gamma_0(z,\zeta) = \Gamma_0(\zeta^{-1} \circ z, 0), \quad \forall\, z,\zeta \in \mathbb{R}^{N+1}, z \neq \zeta,$$

where

$$\Gamma_0((x,t),(0,0)) = \begin{cases} \dfrac{(4\pi)^{-\frac{N}{2}}}{\sqrt{\det C(t)}} \exp\left(-\dfrac{1}{4} \langle C^{-1}(t)x, x\rangle \right), & \text{if } t > 0, \\ 0, & \text{if } t \leq 0, \end{cases}$$

and

$$C(t) = \int_0^t E(s) A_0 E^T(s) ds,$$

($E(\cdot)$ is the matrix defined in (2.2)). Note that hypothesis [H.2] implies that $C(t)$ is strictly positive for every positive t (see [20, Proposition A.1]). In view of the invariance properties of L_0, it is not difficult to show that

$$\Gamma_0(\delta_\lambda(z), 0) = \lambda^{-Q} \Gamma_0(z, 0), \quad \forall\, z \in \mathbb{R}^{N+1} \setminus \{0\}, \lambda > 0,$$

where

$$Q = m_0 + 3m_1 + \cdots + (2r+1)m_r. \tag{2.7}$$

The natural number $Q+2$ is usually called the *homogeneous dimension of* $\mathbb{R}^{N+1}$ *with respect to* $(\delta_\lambda)_{\lambda>0}$. This denomination is proper since the Jacobian $J\delta_\lambda$ equals λ^{Q+2}. Let $\|\cdot\|$ denote a δ_λ-*homogeneous norm*[a] in $\mathbb{R}^{N+1}$. The following bound holds

$$\Gamma_0(z,\zeta) \leq c \|\zeta^{-1} \circ z\|^{-Q}, \tag{2.8}$$

for some positive constant c.

[a]For instance, a δ_λ-homogeneous norm is given by

$$\|z\| \equiv \left(\sum_{j=1}^N x_j^{\alpha_j} + |t|^{\frac{(2r+1)!}{2}} \right)^{\frac{1}{(2r+1)!}}$$

where $\alpha_j = (2r+1)!$ if $1 \leq j \leq m_0$ and

$$\alpha_j = \frac{(2r+1)!}{2k+1}, \quad \text{if } 1 + \sum_{i=0}^{k-1} m_i \leq j \leq \sum_{i=0}^{k} m_i, 1 \leq k \leq r.$$

We define the L_0-*potential* of the function $f \in L^1(\mathbb{R}^{N+1})$ as follows

$$\Gamma_0(f)(z) = \int_{\mathbb{R}^{N+1}} \Gamma_0(z,\zeta) f(\zeta) d\zeta\,, \quad z \in \mathbb{R}^{N+1}\,. \tag{2.9}$$

This definition is well posed, indeed, for every $T > 0$, we have

$$\int_{\mathbb{R}^N \times [-T,T]} |\Gamma_0(f)(z)| dz$$

(inverting the order of integration)

$$\le \int_{\mathbb{R}^{N+1}} |f(\zeta)| d\zeta \int_{\mathbb{R}^N \times [-T,T]} \Gamma_0(z,\zeta) dz \le 2T \int_{\mathbb{R}^{N+1}} |f(\zeta)| d\zeta = 2T \|f\|_{L^1(\mathbb{R}^{N1})}\,.$$

Let us also recall some classical potential estimates (cf., for instance, [46]).

Theorem 2.1. *Let $\alpha \in]0, Q+2[$ and let $G \in C(\mathbb{R}^{N+1} \setminus \{0\})$ be a δ_λ-homogeneous function of degree $\alpha - Q - 2$. If $f \in L^p(\mathbb{R}^{N+1})$ for some $p \in]1, +\infty[$, then the function*

$$G_f(z) \equiv \int_{\mathbb{R}^{N+1}} G(\zeta^{-1} \circ z) f(\zeta) d\zeta\,,$$

is defined almost everywhere and there exists a constant $c = c(Q,p)$ such that

$$\|G_f\|_{L^q(\mathbb{R}^{N+1})} \le c \max_{\|z\|=1} |G(z)|\, \|f\|_{L^p(\mathbb{R}^{N+1})}\,,$$

where q is defined by

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q+2}\,.$$

For reader's convenience, we state separately some potential estimates which will be used in the sequel. These estimates are essentially contained in the previous theorem. We also remark that, by the homogeneity properties of Γ_0, the potential $\Gamma_0(D_{m_0} f)$ is well-defined for any $f \in L^2(\mathbb{R}^{N+1})$, at least in the distributional sense, that is

$$\Gamma_0(D_{m_0} f)(z) \equiv -\int_{\mathbb{R}^{N+1}} D^{(\zeta)}_{m_0} \Gamma_0(z,\zeta) f(\zeta) d\zeta\,. \tag{2.10}$$

In (2.10), the superscript in $D^{(\zeta)}_{m_0}$ indicates that we are differentiating with respect to the variable ζ.

Corollary 2.2. *Let $f \in L^2(\mathbb{R}^{N+1})$. There exists a positive constant $c = c(Q)$ such that*

$$\|\Gamma_0(f)\|_{L^{2\tilde\kappa}(\mathbb{R}^{N+1})} \le c \|f\|_{L^2(\mathbb{R}^{N+1})}\,, \tag{2.11}$$

$$\|\Gamma_0(D_{m_0} f)\|_{L^{2\kappa}(\mathbb{R}^{N+1})} \le c \|f\|_{L^2(\mathbb{R}^{N+1})}\,, \tag{2.12}$$

where $\tilde\kappa = 1 + \frac{4}{Q-2}$ and $\kappa = 1 + \frac{2}{Q}$.

Proof. Estimate (2.11) is an immediate consequence of Theorem 2.1 and of the homogeneity of Γ_0. In order to prove (2.12), it suffices to observe that, by (2.10), we have

$$\Gamma_0(D_{m_0}f)(z) = \int_{\mathbb{R}^{N+1}} (\tilde{D}\Gamma_0)(\zeta^{-1} \circ z, 0)f(\zeta)d\zeta\,, \tag{2.13}$$

where $\tilde{D}$ is a first order differential operator δ_λ-homogeneous of degree one. Hence (2.12) follows applying Theorem 2.1 with $G = (\tilde{D}\Gamma_0)(\cdot, 0)$ and $\alpha = 1$.

In order to show (2.13), let us denote by D_{m_k}, $k = 1, \dots, r$, the gradient with respect to the variables x_j for

$$1 + \sum_{i=0}^{k-1} m_i \le j \le \sum_{i=0}^{k} m_i\,.$$

We remark that the matrix B in (2.4) is nilpotent and we have

$$E(s) = \sum_{k=0}^{r} \frac{(-s)^k}{k!}(B^T)^k\,, \quad s \in \mathbb{R}\,. \tag{2.14}$$

Thus, by (2.14) and expression (2.3) of the product law, we deduce

$$\begin{aligned} D_{m_0}^{(\zeta)}\Gamma_0(z,\zeta) &= D_{m_0}^{(\zeta)}\Gamma_0(\zeta^{-1} \circ z, 0) \\ &= \left(-D_{m_0}\Gamma_0(\cdot,0) - \sum_{k=1}^{r} \frac{(-1)^k}{k!}(t-\tau)^k D_{m_k}\Gamma_0(\cdot,0)B_k^T \cdots B_1^T\right)(\zeta^{-1} \circ z). \end{aligned}$$

□

Definition 2.3. A weak sub-solution of (1.1) in a domain Ω is a function u such that u, $D_{m_0}u$, $Yu \in L^2_{\text{loc}}(\Omega)$ and

$$\int_\Omega -\langle ADu, D\phi\rangle + \phi Yu \ge 0\,, \quad \forall\, \phi \in C_0^\infty(\Omega), \phi \ge 0\,. \tag{2.15}$$

A function u is a weak super-solution of (1.1) if $-u$ is a sub-solution.

Remark 2.4. If u is a sub and super-solution of (1.1) in Ω then it is a solution, i.e. (1.5) holds. Indeed, for every given $\phi \in C_0^\infty(\Omega)$, we may consider $\psi \in C_0^\infty(\Omega)$ such that $\psi \ge 0$ and $\phi + \psi \ge 0$ in Ω. Therefore (1.5) follows by applying (2.15) to $\pm u$.

Roughly speaking, the next lemma states that we can use the fundamental solution Γ_0 as a test function in the definition of sub and super-solution.

Lemma 2.5. *Let v be a weak sub-solution of* (1.1) *in Ω. For every $\phi \in C_0^\infty(\Omega)$, $\phi \ge 0$, and for almost every $z \in \mathbb{R}^{N+1}$, we have*

$$\int_\Omega -\langle ADv, D(\Gamma_0(z,\cdot)\phi)\rangle + \Gamma_0(z,\cdot)\phi Yv \ge 0\,.$$

An analogous result holds for weak super-solutions.

Proof. For every $\varepsilon > 0$, we set

$$\chi_\varepsilon(z,\zeta) = \chi\left(\frac{\|\zeta^{-1} \circ z\|}{\varepsilon}\right), \quad z,\zeta \in \mathbb{R}^{N+1},$$

where $\chi \in C^1([0,+\infty[,[0,1])$ is such that $\chi(s) = 0$ for $s \in [0,1]$, $\chi(s) = 1$ for $s \geq 2$ and $0 \leq \chi' \leq 2$. By (2.15), for every $\varepsilon > 0$ and $z \in \mathbb{R}^{N+1}$, we have

$$\begin{aligned} 0 &\leq \int_\Omega -\langle ADv, D(\Gamma_0(z,\cdot)\chi_\varepsilon(z,\cdot)\phi)\rangle + \Gamma_0(z,\cdot)\chi_\varepsilon(z,\cdot)\phi Yv \\ &= -I_{1,\varepsilon}(z) + I_{2,\varepsilon}(z) - I_{3,\varepsilon}(z), \end{aligned}$$

where

$$\begin{aligned} I_{1,\varepsilon}(z) &= \int_\Omega \langle ADv, D(\Gamma_0(z,\cdot))\rangle \chi_\varepsilon(z,\cdot)\phi, \\ I_{2,\varepsilon}(z) &= \int_\Omega \Gamma_0(z,\cdot)\chi_\varepsilon(z,\cdot)(-\langle ADv, D\phi\rangle + \phi Yv), \\ I_{3,\varepsilon}(z) &= \int_\Omega \langle ADv, D\chi_\varepsilon(z,\cdot)\rangle \Gamma_0(z,\cdot)\phi. \end{aligned}$$

Keeping in mind the proof of Corollary 2.2, it is clear that the integral which defines $I_{1,\varepsilon}(z)$ is a potential and it is convergent for almost every $z \in \mathbb{R}^{N+1}$. Thus, since

$$|\langle ADv, D(\Gamma_0(z,\cdot))\rangle \chi_\varepsilon(z,\cdot)\phi| \leq |\langle ADv, D(\Gamma_0(z,\cdot))\rangle \phi| \in L^1, \quad \forall\, \varepsilon > 0,$$

by the dominated convergence theorem, we get

$$\lim_{\varepsilon\to 0^+} I_{1,\varepsilon}(z) = \int_\Omega \langle ADv, D(\Gamma_0(z,\cdot))\rangle \phi, \quad \text{for a.e. } z \in \mathbb{R}^{N+1}.$$

Analogously, we have

$$\lim_{\varepsilon\to 0^+} I_{2,\varepsilon}(z) = \int_\Omega \Gamma_0(z,\cdot)(-\langle ADv, D\phi\rangle + \phi Yv), \quad \text{for a.e. } z \in \mathbb{R}^{N+1}.$$

In order to conclude, it suffices to prove that

$$\lim_{\varepsilon\to 0^+} I_{3,\varepsilon}(z) = 0, \quad \text{for a.e. } z \in \mathbb{R}^{N+1}. \tag{2.16}$$

By [H.1] and Cauchy–Schwartz inequality, we have

$$I_{3,\varepsilon}(z) \leq \frac{2\mu}{\varepsilon} \int_{\|\zeta^{-1}\circ z\| \leq 2\varepsilon} |D_{m_0} v(\zeta)| \Gamma_0(z,\zeta)\phi d\zeta$$

(by estimate (2.8) of Γ_0, for $\alpha \in]0,1[$)

$$\leq 2c\mu\varepsilon^\alpha \int_{\mathbb{R}^{N+1}} |D_{m_0} v(\zeta)| \|\zeta^{-1} \circ z\|^{-Q-1-\alpha} \phi(\zeta) d\zeta \longrightarrow 0, \quad \text{as } \varepsilon \to 0^+,$$

since the last integral is convergent for a.e. $z \in \mathbb{R}^{N+1}$. ☐

Lemma 2.6. *Let $f \in C^2 \cap \mathrm{Lip}(\mathbb{R})$ be a monotone non-decreasing function. If f is convex (respectively concave) and u is a weak sub-solution (respectively super-solution) of (1.1), then $v = f(u)$ is a weak sub-solution (respectively super-solution) of (1.1).*

Proof. The proof is standard and we only consider the case of a sub-solution. Since $f \in C^2$ and it is Lipschitz continuous, then v, $D_{m_0}v$, $Yv \in L^2_{\mathrm{loc}}$. By a standard density argument, we express (2.15) in terms of

$$\phi = f'(u)\psi\,, \quad \psi \in C_0^\infty, \psi \geq 0\,.$$

We remark that $\phi \geq 0$ because f is non-decreasing, thus we obtain

$$\begin{aligned} 0 &\leq \int -\langle ADu, D\phi\rangle + \phi Y \\ &= \int -\langle ADu, Du\rangle f''(u)\psi + f'(u)(-\langle ADu, D\psi\rangle + \psi Yu) \end{aligned}$$

(since $f'' \geq 0$)

$$\leq \int -\langle ADv, D\psi\rangle + \psi Yv\,.$$

□

3. Caccioppoli and Sobolev Type Inequalities

In this section we prove some Caccioppoli and some Sobolev type inequalities for the non-negative solutions to (1.1). We recall the notation (1.6) and, by simplicity, we shall write R_r instead of $R_r(0)$.

Theorem 3.1 (Caccioppoli type inequalities). *Let u be a non-negative weak solution of (1.1) in R_1. Let $p \in \mathbb{R}$, $p \neq 0$, $p \neq 1/2$ and let ϱ, r be such that $\frac{1}{2} \leq \varrho < r \leq 1$. If $u^p \in L^2(R_r)$ then $D_{m_0}u^p \in L^2(R_\varrho)$ and there exists a constant c, only dependent on the homogeneous dimension Q, such that*

$$\|D_{m_0}u^p\|_{L^2(R_\varrho)} \leq \frac{c\sqrt{\mu(\mu+\varepsilon)}}{\varepsilon(r-\varrho)}\|u^p\|_{L^2(R_r)}\,, \quad \text{where} \quad \varepsilon = \frac{|2p-1|}{4p}\,. \tag{3.1}$$

Proof. We consider the case $p < 1$, $p \neq 0$, $p \neq 1/2$. We first assume that u is uniformly positive, that is $u \geq u_0$ for some constant $u_0 > 0$. Let $v = u^p$. Since u is a weak solution to $Lu = 0$ and $u \geq u_0$, then v, $D_{m_0}v$, $Yv \in L^2(R_r)$. For every $\psi \in C_0^\infty(R_1)$ we consider the function $\phi = u^{2p-1}\psi^2$. Note that ϕ and $D_{m_0}\phi \in L^2(R_1)$, then we can use ϕ as a test function in (1.5). We find

$$\begin{aligned} 0 &= \frac{p}{2}\int_{R_1} \langle ADu, D\phi\rangle - \phi Yu \\ &= \frac{p}{2}\int_{R_1} (2p-1)u^{2p-2}\psi^2\langle ADu, Du\rangle + 2\psi u^{2p-1}\langle ADu, D\psi\rangle - u^{2p-1}\psi^2 Yu \\ &= \int_{R_1} \left(1 - \frac{1}{2p}\right)\psi^2\langle ADv, Dv\rangle + v\psi\langle ADv, D\psi\rangle - \frac{\psi^2}{4}Y(v^2) \end{aligned}$$

(using the identity

$$\psi^2 Y(v^2) = Y(\psi^2 v^2) - 2v^2\psi Y\psi$$

and applying the divergence theorem)

$$= \int_{R_1} \left(1 - \frac{1}{2p}\right) \psi^2 \langle ADv, Dv\rangle + v\psi\langle ADv, D\psi\rangle + \frac{v^2\psi}{2} Y\psi\,.$$

Setting $\varepsilon = \frac{|2p-1|}{4p}$ and using the estimate

$$v\psi|\langle ADv, D\psi\rangle| \le \varepsilon\psi^2\langle ADv, Dv\rangle + \frac{v^2}{4\varepsilon}\langle AD\psi, D\psi\rangle\,,$$

we finally obtain

$$\varepsilon \int_{R_1} \psi^2\langle ADv, Dv\rangle \le \frac{1}{4}\int_{R_1} v^2 \left(\frac{1}{\varepsilon}\langle AD\psi, D\psi\rangle + 2|\psi Y\psi|\right)\,. \tag{3.2}$$

The thesis follows by making a suitable choice of the function ψ in (3.2). More precisely, we set

$$\psi(x,t) = \chi(\|(x,0)\|)\chi(|t|^{\frac{1}{2}})\,, \tag{3.3}$$

where $\chi \in C^\infty(\mathbb{R}, [0,1])$ is such that

$$\chi(s) = 1 \text{ if } s \le \varrho\,, \quad \chi(s) = 0 \text{ if } s \ge r\,, \quad |\chi'| \le \frac{2}{r-\varrho}\,.$$

We observe that

$$|\partial_t\psi|, |\partial_{x_j}\psi| \le \frac{c_1}{r-\varrho}\,, \quad j = 1,\dots,N \tag{3.4}$$

where c_1 is a dimensional constant. Then, accordingly to (3.2), we obtain

$$\begin{aligned} \frac{\varepsilon}{\mu}\int_{R_\varrho} |D_{m_0}u^p|^2 &\le \varepsilon\int_{R_r} \psi^2\langle ADu^p, Du^p\rangle \\ &\le \frac{1}{4}\int_{R_r} u^{2p}\left(\frac{c_1\mu}{\varepsilon(r-\varrho)^2} + \frac{2c_1}{r-\varrho}\right) \le \frac{c_2}{(r-\varrho)^2}\left(1+\frac{\mu}{\varepsilon}\right)\int_{R_r} u^{2p}\,, \end{aligned} \tag{3.5}$$

and this proves (3.1).

The previous argument can be straightforwardly adapted to the case of a non-negative weak solution to (1.1). Indeed, we may consider estimate (3.5) for the solution $u + \frac{1}{n}$, $n \in \mathbb{N}$,

$$\frac{\varepsilon}{\mu}\int_{R_\varrho} \left|D_{m_0}\left(u+\frac{1}{n}\right)^p\right|^2 \le \frac{c_2}{(r-\varrho)^2}\left(1+\frac{\mu}{\varepsilon}\right)\int_{R_r}\left(u+\frac{1}{n}\right)^{2p}\,,$$

and we let n go to infinity. The passage to the limit in the first integral is allowed since

$$\left| D_{m_0} \left(u + \frac{1}{n} \right)^p \right| = p \left(u + \frac{1}{n} \right)^{p-1} |D_{m_0} u| \uparrow |D_{m_0} u^p| \,, \quad \forall\, p < 1, n \longrightarrow \infty \,.$$

In the second integral we rely on the assumption $u^p \in L^2(R_r)$.

We next consider the case $p \geq 1$. For any $n \in \mathbb{N}$, we define the function $g_{n,p}$ on $]0, +\infty[$ as follows

$$g_{n,p}(s) = \begin{cases} s^p \,, & \text{if } 0 < s \leq n \,, \\ n^p + p n^{p-1}(s - n) \,, & \text{if } s > n \,, \end{cases}$$

then we let

$$v_{n,p} = g_{n,p}(u) \,. \tag{3.6}$$

Note that

$$g_{n,p} \in C^1(\mathbb{R}^+) \,, \quad g'_{n,p} \in L^\infty(\mathbb{R}^+) \,,$$

thus, since u is a weak solution of (1.1), we have

$$v_{n,p} \in L^2_{\text{loc}} \,, \quad D_{m_0} v_{n,p} \in L^2_{\text{loc}} \,, \quad Y v_{n,p} \in L^2_{\text{loc}} \,.$$

We also note that the function

$$g''_{n,p}(s) = \begin{cases} p(p-1)s^{p-2} \,, & \text{if } 0 < s < n \,, \\ 0 \,, & \text{if } s \geq n \,, \end{cases}$$

is the weak derivative of $g'_{n,p}$, then $D_{m_0} g'_{n,p}(u) = g''_{n,p}(u) D_{m_0} u$ (for the detailed proof of this assertion, we refer to [47, Theorem 7.8]). Hence, by using

$$\phi = g_{n,p}(u) g'_{n,p}(u) \psi^2 \,, \quad \psi \in C_0^\infty(R_1)$$

as a test function in (1.5), we find

$$\begin{aligned} 0 &= \int_{R_1} \langle ADu, D\phi \rangle - \phi Y u \\ &= \int_{R_1} (g'_{n,p}(u)^2 + g''_{n,p}(u) g_{n,p}(u)) \psi^2 \langle ADu, Du \rangle \\ &\quad + g_{n,p}(u) g'_{n,p}(u) (2\psi \langle ADu, D\psi \rangle - \psi^2 Y u) \end{aligned}$$

(since $g''_{n,p}(u) \geq 0$)

$$\begin{aligned} &\geq \int_{R_1} \psi^2 \langle ADv_{n,p}, Dv_{n,p} \rangle + 2 v_{n,p} \psi \langle ADv_{n,p}, D\psi \rangle - \frac{\psi^2}{2} Y(v_{n,p}^2) \\ &= \int_{R_1} \psi^2 \langle ADv_{n,p}, Dv_{n,p} \rangle + 2 v_{n,p} \psi \langle ADv_{n,p}, D\psi \rangle + v_{n,p}^2 \psi Y \psi \,. \end{aligned}$$

Therefore, if $\varepsilon = \frac{|2p-1|}{4p}$, we get

$$\varepsilon \int_{R_1} \psi^2 \langle ADv_{n,p}, Dv_{n,p} \rangle \leq \frac{1}{4} \int_{R_1} v_{n,p}^2 \left(\frac{1}{\varepsilon} \langle AD\psi, D\psi \rangle + \frac{1}{2} |\psi Y \psi| \right) \,.$$

Since $0 < v_{n,p} \le u^p$ and

$$\langle ADv_{n,p}, Dv_{n,p}\rangle \uparrow \langle ADu^p, Du^p\rangle\,, \quad \text{as } n \to \infty\,,$$

we get from the above inequality

$$\varepsilon \int_{R_1} \psi^2 \langle ADu^p, Du^p\rangle \le \frac{1}{4}\int_{R_1} u^{2p}\left(\frac{1}{\varepsilon}\langle AD\psi, D\psi\rangle + \frac{1}{2}|\psi Y\psi|\right)$$

and we conclude the proof as in the previous case. □

We next state a result which extends Theorem 3.1 to super and sub-solutions. We omit the proof, since it follows the same lines of Theorem 3.1. Note that, by our method, we obtain some estimates only for $p < 1/2$ or $p \ge 1$.

Proposition 3.2. *Let u be a non-negative weak sub-solution of* (1.1) *in R_1. Let ϱ, r, $\frac{1}{2} \le \varrho < r \le 1$, and $p \ge 1$ or $p < 0$. If $u^p \in L^2(R_r)$ then $D_{m_0}u^p \in L^2(R_\varrho)$ and there exists a constant c, only dependent on the homogeneous dimension Q, such that*

$$\|D_{m_0}u^p\|_{L^2(R_\varrho)} \le \frac{c\sqrt{\mu(\mu+\varepsilon)}}{\varepsilon(r-\varrho)}\|u^p\|_{L^2(R_r)}\,, \quad \text{where} \quad \varepsilon = \frac{|2p-1|}{4p}\,.$$

The same statement holds when u is a non-negative weak super-solution of (1.1) *and $p \in]0, 1/2[$.*

Theorem 3.3 (Sobolev type inequalities for super and sub-solutions). *Let v be a non-negative weak sub-solution of L in R_1. Then $v \in L^{2\kappa}_{\text{loc}}(R_1)$, $\kappa = 1 + \frac{2}{Q}$, and there exists a constant c, only dependent on Q and μ, such that*

$$\|v\|_{L^{2\kappa}(R_\varrho)} \le \frac{c}{r-\varrho}(\|v\|_{L^2(R_r)} + \|D_{m_0}v\|_{L^2(R_r)})\,, \tag{3.7}$$

for every ϱ, r with $\frac{1}{2} \le \varrho < r \le 1$.

The same statement holds for non-negative super-solutions.

Proof. Let v be a non-negative sub-solution of L. We represent v in terms of the fundamental solution Γ_0. To this end, we consider the cut-off function ψ introduced in (3.3). For every $z \in R_\varrho$, we have

$$\begin{aligned} v(z) &= v\psi(z) \\ &= \int_{R_r} [\langle A_0 D(v\psi), D\Gamma_0(z,\cdot)\rangle - \Gamma_0(z,\cdot)Y(v\psi)](\zeta)d\zeta = I_1(z) + I_2(z) + I_3(z)\,, \end{aligned} \tag{3.8}$$

where

$$I_1(z) = \int_{R_r} [\langle A_0 D\psi, D\Gamma_0(z,\cdot)\rangle v](\zeta)d\zeta - \int_{R_r} [\Gamma_0(z,\cdot)vY\psi](\zeta)d\zeta\,,$$

$$I_2(z) = \int_{R_r} [\langle (A_0 - A)Dv, D\Gamma_0(z,\cdot)\rangle\psi](\zeta)d\zeta - \int_{R_r} [\Gamma_0(z,\cdot)\langle ADv, D\psi\rangle](\zeta)d\zeta\,,$$

$$I_3(z) = \int_{R_r} [\langle ADv, D(\Gamma_0(z,\cdot)\psi)\rangle - \Gamma_0(z,\cdot)\psi Yv](\zeta)d\zeta\,.$$

Since the function v is a weak sub-solution of L, it follows from Lemma 2.5 that $I_3 \leq 0$, then

$$0 \leq v(z) \leq I_1(z) + I_2(z) \quad \text{for a.e. } z \in R_\varrho\,.$$

To prove our claim it is sufficient to estimate v by a sum of L_0-potentials.

We start by estimating I_1. Denote by I_1' and I_1'' the first and the second integral in I_1, respectively. Then I_1' can be estimate by (2.12) of Corollary 2.2 as follows

$$\|I_1'\|_{L^{2\kappa}(R_\varrho)} \leq c\|vD_{m_0}\psi\|_{L^2(\mathbb{R}^{N+1})} \leq \frac{c}{r-\varrho}\|v\|_{L^2(R_r)}\,,$$

where the last inequality follows from (3.4). To estimate I_1'' we use (2.11):

$$\|I_1''\|_{L^{2\kappa}(R_\varrho)} \leq \text{meas}(R_\varrho)^{2/Q}\|I_1''\|_{L^{2\tilde{\kappa}}(R_\varrho)} \leq c\|vY\psi\|_{L^2(\mathbb{R}^{N+1})} \leq \frac{c}{r-\varrho}\|v\|_{L^2(R_r)}\,.$$

We can use the same technique to prove that

$$\|I_2\|_{L^{2\kappa}(R_\varrho)} \leq \frac{c}{r-\varrho}\|D_{m_0}v\|_{L^2(R_r)}\,,$$

for some constant $c = c(Q,\mu)$, thus our first claim is proved.

A similar argument proves the thesis when v is a L-super-solution. In this case, we introduce the following auxiliary operator

$$\tilde{L}_0 = \text{div}\,(A_0 D) + \tilde{Y}\,, \quad \tilde{Y} \equiv -\langle x, BD\rangle - \partial_t\,.$$

If R is a domain which is symmetric with respect to the time variable t, for any $z = (x,t) \in R$, we denote $\hat{z} = (x,-t) \in R$ and $w(z) = v(\hat{z})$. We remark that

$$Dw(z) = Dv(\hat{z})\,, \quad \tilde{Y}w(z) = -Yv(\hat{z})$$

for almost every $z \in R$. Then, since v is a L-super-solution, we have

$$\int_R [-\langle A(\hat{z})Dw, D\phi\rangle - \phi\tilde{Y}w](z)dz$$
$$= \int_R (-\langle A(\hat{z})Dv(\hat{z}), D\phi(z)\rangle + \phi(z)Yv(\hat{z}))dz \leq 0\,, \tag{3.9}$$

for every $\phi \in C_0^\infty(R)$, $\phi \geq 0$.

Next, we represent w in terms of the fundamental solution $\tilde{\Gamma}_0$ of $\tilde{L}_0$. For ψ as in (3.3) and $z \in R_\varrho$, we have

$$w(z) = w\psi(z)$$
$$= \int_{R_r} [\langle A_0 D(w\psi), D\tilde{\Gamma}_0(z,\cdot)\rangle - \tilde{\Gamma}_0(z,\cdot)\tilde{Y}(v\psi)](\zeta)d\zeta = I_1(z) + I_2(z) + I_3(z)\,,$$

where

$$I_1(z) = \int_{R_r} [(\langle A_0 D\psi, D\tilde{\Gamma}_0(z,\cdot)\rangle - \tilde{\Gamma}_0(z,\cdot)\tilde{Y}\psi)w](\zeta)d\zeta\,,$$

$$I_2(z) = \int_{R_r} \langle (A_0 + A(\hat{\zeta}))Dw(\zeta), D(\tilde{\Gamma}_0(z,\cdot)\psi)(\zeta)\rangle d\zeta$$

$$- \int_{R_r} [\tilde{\Gamma}_0(z,\cdot)\langle A_0 Dw, D\psi\rangle](\zeta)d\zeta\,,$$

$$I_3(z) = \int_{R_r} [-\langle A(\hat{z})Dw, D(\tilde{\Gamma}_0(z,\cdot)\psi)\rangle - \tilde{\Gamma}_0(z,\cdot)\psi\tilde{Y}w](\zeta)d\zeta\,.$$

Since w satisfies (3.9), by Lemma 2.5, we have $I_3(z) \le 0$, for a.e. $z \in R_\varrho$. As in the previous case, we conclude the proof of (3.7) by using the potential estimates of the Corollary 2.2, that still hold for the function $\tilde{\Gamma}_0$. Thus we have

$$\|v\|_{L^{2\kappa}(R_\varrho)} = \|w\|_{L^{2\kappa}(R_\varrho)} \le \frac{c}{r-\varrho}(\|w\|_{L^2(R_r)} + \|D_{m_0}w\|_{L^2(R_r)})$$

$$= \frac{c}{r-\varrho}(\|v\|_{L^2(R_r)} + \|D_{m_0}v\|_{L^2(R_r)})$$

for some constant $c = c(Q,\mu)$ and this completes the proof. ∎

4. Iteration

In this section we use the classical Moser's iteration scheme to prove Theorem 1.2. We begin with some preliminary remarks.

Remark 4.1. A transformation of the form

$$\zeta \longmapsto z_0 \circ \delta_r(\zeta)\,, \quad r > 0, z_0 \in \mathbb{R}^{N+1}\,, \tag{4.1}$$

preserves the class of differential equations considered. More precisely, if u is a weak solution of (1.1) in the cylinder $R_r(z_0)$ then the function

$$v(\zeta) = u(z_0 \circ \delta_r(\zeta))$$

is a solution to the equation

$$\text{div}\ (\tilde{A}Dv)(\zeta) + Yv(\zeta) = 0\,, \quad \zeta \in R_1\,,$$

where $\tilde{A}(\zeta) = A(z_0 \circ \delta_r(\zeta))$ satisfies hypothesis [H.1] with the same constant μ as A.

Lemma 4.2. *There exists a positive constant $\bar{c}$ such that, for every ϱ, r, with $\frac{1}{2} \le \varrho < r \le 1$ and $z_0 \in \mathbb{R}^{N+1}$, it holds*

$$R_{\bar{c}(r-\varrho)}(z) \subseteq R_r(z_0)\,, \quad \forall\ z \in R_\varrho(z_0)\,. \tag{4.2}$$

Proof. By the change of variables $z = z_0 \circ \delta_r(\zeta)$, it suffices to prove (4.2) for $z_0 = 0$ and $r = 1$. By expression (2.5) of the dilations (δ_λ), it is clear that

$$R_\varrho \subseteq \{(x,t) \in \mathbb{R}^{N+1} | |x| < \varrho, |t| < \varrho^2\}\,, \quad \forall\ \varrho < 1\,.$$

Then the thesis is a consequence of the following inclusion

$$R_\varepsilon(z) \subseteq \{(\xi,\tau) | |x-\xi| < c\varepsilon, |t-\tau| < (c\varepsilon)^2\}, \quad \forall\, z \in R_\varrho, \varepsilon < 1\,, \tag{4.3}$$

for some positive constant c. Indeed, if we choose $\varepsilon \le \frac{1-\varrho}{c}$, we get

$$R_\varepsilon(z) \subseteq R_1\,, \quad \forall\, z \in R_\varrho\,,$$

and this shows (4.2) with $\bar{c} = c^{-1}$.

We are left with the proof of (4.3). If $\zeta = (\xi,\tau) \in R_\varepsilon(z)$ then

$$\zeta = z \circ \bar{z} = (\bar{x} + E(\bar{t})x, t + \bar{t})\,,$$

for some $\bar{z} \in R_\varepsilon$. Hence

$$|\xi - x| = |\bar{x} + (E(\bar{t}) - E(0))x| \le \bar{x}| + |\bar{t}| \max_{|s|\le|1} \|E'(s)\| \le c\varepsilon\,, \quad \tau - t| = |\bar{t}| < \varepsilon^2\,,$$

where $c = 1 + |\max_{|s|\le 1} \|E'(s)\|$. □

We are now in position to prove Theorem 1.2.

Proof of Theorem 1.2. It suffices to give the proof in the case $z_0 = 0$, $\varrho = \frac{1}{2}$ and $r = 1$. Then, by a transformation of the form (4.1), we get

$$\sup_{R_{\frac{\theta}{2}}(z)} u^p \le \frac{c}{\theta^{Q+2}} \int_{R_\theta(z)} u^p \tag{4.4}$$

for every $z \in R_\varrho(z_0)$ and $\theta > 0$ suitably small, with $c = c(Q,\mu)$. Keeping in mind Lemma 4.2, we set $\theta = \bar{c}(r-\varrho)$ in (4.4), and we obtain

$$\sup_{R_{\frac{\bar{c}(r-\varrho)}{2}}(z)} u^p \le \frac{\tilde{c}}{(r-\varrho)^{Q+2}} \int_{R_r(z_0)} u^p\,, \quad \forall\, z \in R_\varrho(z_0)\,,$$

which yields (1.7).

We are left with the proof of (1.7) for $z_0 = 0$, $\varrho = 1/2$ and $r = 1$. We first consider the case $p > 0$ which is technically more complicated. Combining Theorems 3.1 and 3.3, we obtain the following estimate: if q, $\delta > 0$ verify the condition

$$|q - 1/2| \ge \delta\,,$$

then there exists a positive constant $c_\delta = c(\delta, Q, \mu)$, such that

$$\|u^q\|_{L^{2\kappa}(R_\varrho)} \le \frac{c_\delta}{(r-\varrho)^2} \|u^q\|_{L^2(R_r)}\,, \tag{4.5}$$

for every ϱ, r, $\frac{1}{2} \le \varrho < r \le 1$, where $\kappa = 1 + \frac{2}{Q}$.

Fixed a suitable $\delta > 0$ as we shall specify later and $p > 0$, we iterate inequality (4.5) by choosing

$$\varrho_n = \frac{1}{2}\left(1 + \frac{1}{2^n}\right), \quad p_n = \frac{p\kappa^n}{2}, \quad n \in \mathbb{N} \cup \{0\}\,.$$

We set $v = u^{\frac{p}{2}}$. If $p > 0$ is such that

$$|p\kappa^n - 1| \geq 2\delta\,, \quad \forall\, n \in \mathbb{N} \cup \{0\}\,, \tag{4.6}$$

by (4.5), we obtain

$$\|v^{\kappa^n}\|_{L^{2\kappa}(R_{\varrho_{n+1}})} \leq \frac{c_\delta}{(\varrho_n - \varrho_{n+1})^2} \|v^{\kappa^n}\|_{L^2(R_{r_n})}\,, \quad \forall\, n \in \mathbb{N} \cup \{0\}\,. \tag{4.7}$$

Since

$$\|v^{\kappa^n}\|_{L^{2\kappa}} = (\|v\|_{L^{2\kappa^{n+1}}})^{\kappa^n} \quad \text{and} \quad \|v^{\kappa^n}\|_{L^2} = (\|v\|_{L^{2\kappa^n}})^{\kappa^n}\,,$$

we can rewrite (4.7) in the form

$$\|v\|_{L^{2\kappa^{n+1}}(R_{\varrho_{n+1}})} \leq \left(\frac{c_\delta}{(\varrho_n - \varrho_{n+1})^2}\right)^{\frac{1}{\kappa^n}} \|v\|_{L^{2\kappa^n}(R_{\varrho_n})}\,.$$

Iterating this inequality, we obtain

$$\|v\|_{L^{2\kappa^{n+1}}(R_{\varrho_{n+1}})} \leq \prod_{j=0}^{n} \left(\frac{c_\delta}{(\varrho_j - \varrho_{j+1})^2}\right)^{\frac{1}{\kappa^j}} \|v\|_{L^2(R_1)}\,,$$

and letting n go to infinity, we get

$$\sup_{R_{\frac{1}{2}}} v \leq c\|v\|_{L^2(R_1)}\,,$$

where

$$c = \prod_{j=0}^{\infty} \left(\frac{c_\delta}{(\varrho_j - \varrho_{j+1})^2}\right)^{\frac{1}{\kappa^j}}\,,$$

is a finite constant, dependent on δ. Thus, we have proved that

$$\sup_{R_{\frac{1}{2}}} u^p \leq c^2 \int_{R_1} u^p\,, \tag{4.8}$$

for every $p > 0$ which verifies condition (4.6).

We now make a suitable choice of $\delta > 0$, only dependent on the homogeneous dimension Q, in order to show that (4.8) holds for every positive p. We remark that, if p is a number of the form

$$p_m = \frac{\kappa^m(\kappa+1)}{2}\,, \quad m \in \mathbb{Z}\,,$$

then (4.6) is satisfied with $\delta = (2Q+4)^{-1}$, for every $m \in \mathbb{Z}$. Therefore (4.8) holds for such a choice of p, with c only dependent on Q, μ. On the other hand, if p is an arbitrary positive number, we consider $m \in \mathbb{Z}$ such that

$$p_m = \frac{\kappa^m(\kappa+1)}{2} \leq p < p_{m+1}\,. \tag{4.9}$$

Hence, by (4.8), we have

$$\sup_{R_{\frac{1}{2}}} u \leq \left(c^2 \int_{R_1} u^{p_m}\right)^{\frac{1}{p_m}} \leq c^{\frac{2}{p_m}} \left(\int_{R_1} u^p\right)^{\frac{1}{p}}\,,$$

so that, by (4.9), we obtain

$$\sup_{R_{\frac{1}{2}}} u^p \le c^{\frac{2p}{pm}} \int_{R_1} u^p \le c^{2\kappa} \int_{R_1} u^p \,.$$

This concludes the proof of (1.7) for $p > 0$.

We next consider $p < 0$. In this case, assuming that $u \ge u_0$ for some positive constant u_0, estimate (1.7) can be proved as in the case $p > 0$ or even more easily since condition (4.5) is satisfied for every $p < 0$. On the other hand, if u is a non-negative solution, it suffices to apply (1.7) to $u + \frac{1}{n}$, $n \in \mathbb{N}$, and to let n go to infinity, by the monotone convergence theorem. □

Proceeding as in the proof of Theorem 1.2, we obtain the following

Corollary 4.3. *Let u be a non-negative weak sub-solution of* (1.1) *in* Ω. *Let* $z_0 \in \Omega$ *and* r, ϱ, $\frac{1}{2} \le \varrho < r \le 1$, *such that* $R_r(z_0) \subseteq \Omega$. *Then we have*

$$\sup_{R_\varrho(z_0)} u \le \left(\frac{c}{(r-\varrho)^{Q+2}} \int_{R_r(z_0)} u^p \right)^{\frac{1}{p}} , \quad \forall\, p \ge 1 \,, \tag{4.10}$$

$$\inf_{R_\varrho(z_0)} u \ge \left(\frac{c}{(r-\varrho)^{Q+2}} \int_{R_r(z_0)} u^p \right)^{\frac{1}{p}} , \quad \forall\, p < 0 \,, \tag{4.11}$$

where $c = c(Q, \mu)$. *Estimate* (4.11) *is meaningful only when* $u^p \in L^1(R_r(z_0))$.

We close this section by proving the local boundedness of weak solutions to (1.1).

Proof of Corollary 1.4. We consider a sequence $(g_n)_{n\in\mathbb{N}}$ in $C^\infty(\mathbb{R}, [0, +\infty[)$ with the following properties:

$$g_n(s) \downarrow \max(0, s) \,, \quad s \in \mathbb{R} \,, \quad \text{as } n \to \infty \,,$$

and, for every $n \in \mathbb{N}$, g_n is a monotone increasing, convex function which is linear out of a fixed compact set. By Lemma 2.6, $(g_n(u))$ and $(g_n(-u))$ are sequences of non-negative sub-solutions of L, which converge to $u^+ = \max(0, u)$ and $u^- = \max(0, -u)$ respectively. Thus, the thesis follows applying (4.10) of corollary (4.3) to $g_n(u)$, $g_n(-u)$ and passing at limit as n goes to infinity. □

5. Bounds on the Set $R_r((x_0, t_0)) \cap \{t < t_0\}$

In this section we prove that Theorem 1.2 also holds in the sets

$$R_r^-((x_0, t_0)) \equiv R_r((x_0, t_0)) \cap \{t < t_0\} \,, \tag{5.1}$$

in the case of negative exponents p. This result is analogous to [1, Theorem 3] (see also inequality (6^-) in the statement of [2, Lemma 1] and states that, in some sense, every point of the set $\overline{R_\varrho^-(z_0)}$ can be considered as an interior point of $R_r^-(z_0)$, when $\varrho < r$, even if it belongs to its topological boundary.

Proposition 5.1. *Let u be a non-negative weak sub-solution of* (1.1) *in* Ω. *Let $z_0 \in \Omega$ and r, ϱ, $\frac{1}{2} \le \varrho < r \le 1$, such that $\overline{R_r^-(z_0)} \subseteq \Omega$ and let $p < 0$. Then there exists a positive constant c which depends on μ and on the homogeneous dimension Q such that*

$$\sup_{R_\varrho^-(z_0)} u^p \le \frac{c}{(r-\varrho)^{Q+2}} \int_{R_r^-(z_0)} u^p \,, \tag{5.2}$$

provided that the last integral converges.

Proof. We proceed exactly as in the proof of Theorem 1.2, by using the following two estimates:

$$\|D_{m_0} u^p\|_{L^2(R_\varrho^-)} \le \frac{c\sqrt{\mu(\mu+\varepsilon)}}{\varepsilon(r-\varrho)} \|u^p\|_{L^2(R_r^-)} \,, \quad \text{where} \quad \varepsilon = \frac{|2p-1|}{4p} \,, \tag{5.3}$$

and

$$\|u^p\|_{L^{2\kappa}(R_\varrho^-)} \le \frac{c}{r-\varrho} (\|u^p\|_{L^2(R_r^-)} + \|D_{m_0} u^p\|_{L^2(R_r^-)}) \,, \tag{5.4}$$

for every negative p and for any ϱ, r with $\frac{1}{2} \le \varrho < r \le 1$.

In order to prove the Caccioppoli type inequality (5.3) we introduce a function $\chi_n(t)$ defined as

$$\chi_n(s) = \begin{cases} 1\,, & \text{if } s \le 0\,, \\ 1 - ns\,, & \text{if } 0 \le s \le 1/n\,, \\ 0\,, & \text{if } s \ge 1/n\,, \end{cases}$$

for every $n \in \mathbb{N}$. We proceed as in the proof of Theorem 3.1: we let $v = u^p$ and, for every $\psi \in C_0^\infty(R_r)$ we consider the function $\phi = u^{2p-1}\psi^2$. It is not restrictive to assume $z_0 = 0$, $r = 1$ and $u \ge u_0$, for some positive constant u_0. Since χ_n is Lipschitz continuous and v, $D_{m_0} v$, $Yv \in L^2(R_1)$, we can use $\chi_n(t)\phi(x,t)$ as a test function in (2.15). We find

$$\begin{aligned} 0 &\le \frac{p}{2} \int_{R_1} \langle ADu, D(\chi_n \phi)\rangle - \chi_n \phi Y u \\ &= \int_{R_1} \left(1 - \frac{1}{2p}\right) \chi_n \psi^2 \langle ADv, Dv\rangle + \chi_n v \psi \langle ADv, D\psi\rangle - \frac{\chi_n \psi^2}{4} Y(v^2) \end{aligned}$$

(using the identity

$$\chi_n \psi^2 Y(v^2) = Y(\chi_n \psi^2 v^2) - 2v^2 Y(\chi_n \psi^2)$$

and applying the divergence theorem)

$$= \int_{R_1} \chi_n \left[\left(1 - \frac{1}{2p}\right) \psi^2 \langle ADv, Dv\rangle + v\psi\langle ADv, D\psi\rangle + \frac{v^2\psi}{2} Y\psi \right]$$

$$+ n \int_{R_1 \cap \{0 \le t \le 1/n\}} \frac{v^2\psi^2}{4} .$$

Note that the last integral above is non-negative, then, by letting $n \to \infty$ we find

$$\int_{R_1^-} \left(1 - \frac{1}{2p}\right) \psi^2 \langle ADv, Dv\rangle + \psi\langle ADv, D\psi\rangle + \frac{v^2\psi}{2} Y\psi \le 0 .$$

After that, we follow the same line used in the proof of Theorem 3.1 and we obtain (5.3).

The Sobolev type inequality (5.4) can be proved exactly as Theorem 3.3: it is sufficient to note that the fundamental solution $\Gamma_0(x, t, \xi, \tau)$ vanishes in the set $\{\tau > t\}$. Then, when representing $u^p(x, t)$, we actually have

$$\begin{aligned} u^p(x,t) &= u^p\psi(x,t) \\ &\le \int_{R_r \cap \{\tau \le t\}} [\langle A_0 D(u^p\psi), D\Gamma_0(x,t,\cdot)\rangle - \Gamma_0(x,t,\cdot) Y(u^p\psi)](\xi,\tau) d\xi d\tau , \end{aligned}$$

for every $(x, t) \in R_\varrho^-$. This means that we integrate the functions u^p and $D_{m_0} u^p$ only in the set R_r^-, thus we can use their $L^2(R_r^-)$ norm in the estimates (2.11) and (2.12) of Corollary 2.2. We then get (5.4) as in the proof of Theorem 3.3. This completes the proof. □

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