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The motion of a non-isolated vortex on the beta-plane

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The trajectory of a non-isolated monopole on the beta-plane is calculated as an asymptotic expansion in the ratio of the strength of the vortex to the beta-effect. The method of matched asymptotic expansions is used to solve the equations of motion in two regions of the flow: a near field where the beta-effect enters as a first-order forcing in relative vorticity, and a wave field in which the dominant balance is a linear one between the beta-effect and the rate of change of relative vorticity. The resulting trajectory is computed for Gaussian and Rankine vortices.

1. Introduction

The Earth's atmosphere and oceans contain an array of strongly swirling coherent structures. The winter polar vortex, tropical cyclones, and tornadoes are examples of vortices in the atmosphere. The large-scale ocean circulation contains a large number of eddies: Gulf Stream rings, Kuroshio rings, meddies (Mediterranean eddies), and many others (Wunsch 1981). Many of these structures are generated in frontal regions, and are potentially of great importance in the horizontal transport of quantities such as heat and momentum, as well as biota.

One strand of the meteorological literature has naturally concentrated on the formation of tropical cyclones and their subsequent motion. Starting with Rossby (1949) and Adem (1956), this line of work has concentrated in particular on the division between the cyclone and the environment (see e.g. Kasahara & Platzman 1963). More recently, Chan & Williams (1987) showed very clearly how an intense vortex in a quiescent environment on the beta-plane will decay into Rossby waves in the absence of nonlinearity, whereas it will propagate coherently and to the northwest because of nonlinearity. Willoughby (1988), Smith & Ulrich (1990), Ross & Kurihara (1992) and Smith & Weber (1993) have also examined the partition of the flow into background and cyclone contributions. One feature that has emerged from this work is the importance of the difference in magnitude between the vorticity gradient across a cyclone and the background vorticity gradient beta. The ratio of the latter to the former is small, and expansions in this parameter ϵ are considered in the last two of these papers.

The study of these structures in oceanography dates back to the MODE (Mid-Ocean Dynamics Experiment) programme (MODE Group 1978). Theoretical work since has followed a great number of directions. An interesting overview of some of these lines of research is the review of Flierl (1987), which concentrates on isolated

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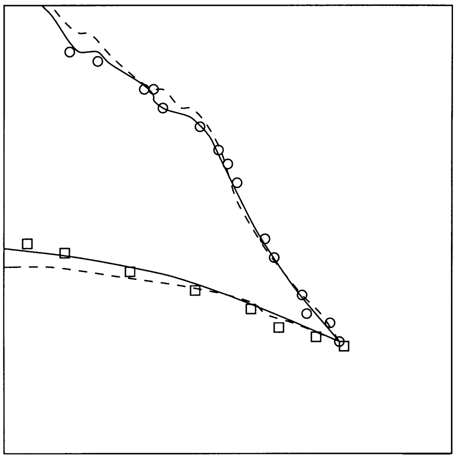

FIGURE 1. Trajectories of sink and stirred vortices on an inclined plane. Symbols represent the results of laboratory experiments: stirred vortices are indicated by squares, sink vortices by circles. The solid (dashed) lines correspond to viscous (inviscid) one-layer, quasi-geostrophic, numerical simulations. Adapted from Carnevale *et al.* (1991).

structures with weak far fields. Previously, Flierl, Stern & Whitehead (1993) had shown that steadily propagating isolated structures on the beta-plane cannot be simple, monopolar vortices with non-zero angular momentum. The evolution of non-isolated structures, they argued, generates long barotropic Rossby waves that alter the momentum balance.

The motion of a monopolar vortex on a beta-plane has been studied in rotating tank experiments since Firing & Beardsley (1976). Carnevale, Kloosterziel & van Heijst (1991) provide a good overview of such work. Figure 1, adapted from the latter paper, shows the difference between the trajectory of a sink vortex and the trajectory of a stirred vortex. The former has non-zero circulation; the latter has zero circulation. The trajectories appear noticeably different.

More recently, Sutyrin & Flierl (1994) examined the evolution of small disturbances to localized step-profile vorticity distributions in the presence of a weak beta-effect. The far field is not treated separately, although it is recognized that in the non-divergent case the solution will break down in the far field when the basic-state circulation is non-zero. Thus the Rankine vortex falls outside the remit of their paper. Reznik & Dewar (1994, hereafter referred to as RD94) looked at the evolution of isolated vortices in strictly two-dimensional flows. A variety of profiles were studied, all having zero circulation and some with non-localized vorticity. The asymptotic expansion was recognized to become invalid in the far field at second order, and a non-rigorous patching was outlined. The evolution of a vortex with Gaussian streamfunction (and hence zero circulation) was considered by Korotaev & Fedotov (1994) who sought a quasi-equilibrium regime. While an asymptotic expansion was used, the scalings used were not very clearly motivated. Indeed, as the authors admitted, '...this mathematical formulation does not provide an exact (even in a strict asymptotic sense) solution of the initial-value problem [...]; some fine-scale features of the solution are ignored'.

Given the differences in trajectory presented in figure 1, which are presumably due to strong far-field effects, the breakdown of the asymptotic theories of Smith & Weber (1993), Sutyrin & Flierl (1994) and RD94 in the far field, and the presumed importance of the mechanism of barotropic wave radiation to adjust to isolated conditions (Flierl *et al.* 1983), this paper attempts to address these issues by examining the evolution of non-isolated vortices on the beta-plane, concentrating on the far-field response and its influence on the vortex motion. This is carried out via an expansion in the parameter ϵ of the barotropic vorticity equation, which results in separate near- and far-field solutions. The presence of this non-dimensional parameter is a major difference from the point vortex case which was studied by Reznik (1992). The present approach has obvious limitations, not least in the simplified equations adopted, but should be capable of giving some insight into the physical mechanisms underlying the evolution of strong vortical structures in the presence of an anisotropic dispersion mechanism such as the beta-effect.

The mathematical problem is posed in §2, where the expansions adopted and the assumptions made are presented. The linear beta-plane equation governing the far-field evolution is solved in §3, and the asymptotic properties of the relevant Green's function are calculated. These results are necessary for §4, where the zeroth-order equations are solved in the near and far fields. The first-order solution is calculated in §5. Some ways of following the motion of the vortex are discussed in §6, and two examples are treated in §7: the Gaussian and Rankine vortices. Finally, the results are discussed in §8.

2. Statement of the problem

For the kind of phenomena under consideration, the barotropic vorticity equation on the beta-plane is appropriate (as may be shown by scale analysis for rapidly rotating shallow stuctures, for example a hurricane of radius 500 km and 10 km in depth; cf. Pedlosky 1987). Then the governing dynamic principle is the conservation of absolute vorticity:

$$\frac{\mathrm{D}q}{\mathrm{D}t} = 0, \tag{2.1}$$

where $q = \zeta + \beta y$, ζ being the relative vorticity of the flow, and y the meridional coordinate. This equation is relatively amenable to analytical treatment, and should help provide insight into the dynamical processes in operation. In particular, it conserves absolute vorticity and also supports Rossby waves.

The equation of motion (2.1) will be non-dimensionalized using the physical scales appropriate to the vortex: length L (corresponding to the radius of the vortex, say) and velocity V (for example the velocity at $r = L$).† The result is

$$\frac{\partial}{\partial t}\nabla^2\psi + J(\psi, \nabla^2\psi) + \epsilon\frac{\partial \psi}{\partial x} = 0, \tag{2.2}$$

where $\psi(x, y)$ is the streamfunction. The relative vorticity is given by $\zeta = \nabla^2\psi$. The parameter

$$\epsilon \equiv \frac{\beta L^2}{V} \tag{2.3}$$

is a measure of the weakness of the planetary vorticity gradient compared to the relative vorticity gradient across the vortex, or equivalently of the strength of the vortex. It is a small parameter for strong vortices, and will serve as the expansion parameter in the asymptotic scheme used to solve the equation of motion.

† The presence of two independent physical scales shows that the present analysis must differ from that for the point vortex case.

The equation of motion and streamfunction will be expanded as asymptotic series in ϵ, yielding

$$\psi = \psi_0 + \epsilon\psi_1 + \cdots, \tag{2.4}$$

and the hierarchy of equations

$$\frac{\partial}{\partial t}\nabla^2\psi_0 + J(\psi_0, \nabla^2\psi_0) = 0, \tag{2.5a}$$

$$\frac{\partial}{\partial t}\nabla^2\psi_1 + J(\psi_0, \nabla^2\psi_1) + J(\psi_1, \nabla^2\psi_0) + \frac{\partial\psi_0}{\partial x} = 0, \tag{2.5b}$$

and so forth. The first of these equations is the Euler equation on an f-plane. The second is the linearized Euler equation (or Rayleigh equation) for the perturbation streamfunction ψ_1 about a swirling basic state ψ_0, forced by the beta-effect.

However, the size of the terms in (2.2) will change at very large distances from the origin, where L is no longer an appropriate length scale. A search for a distinguished scaling gives a new variable $R = \epsilon r$, with the resulting governing equation for $\phi(\boldsymbol{R}, t; \epsilon) \equiv \psi(\boldsymbol{r} = \boldsymbol{R}/\epsilon, t; \epsilon)$

$$\frac{\partial}{\partial t}\nabla_R^2\phi + \epsilon^2 J_R(\phi, \nabla_R^2\phi) + \frac{\partial\phi}{\partial X} = 0. \tag{2.6}$$

This equation holds in the far field of the vortex. Again, the streamfunction ϕ is expanded in ϵ, and the resulting equations solved to give an asymptotic representation of ϕ. The governing equation at zeroth order is linear, and corresponds to the problem of linear wave propagation on the beta-plane. RD94 mentions the change of balance in the far field, but does not carry out a matching procedure between the expansions in the two regions. The two equations (2.2) and (2.6) are valid in different parts of the flow domain, and so their solutions must be matched onto each other in an appropriate manner.

The initial condition will be a radially symmetric vortex with streamfunction $\Psi(r)$, monotonic vorticity $Q \equiv \nabla^2\Psi$ and monotonic angular velocity $\Omega \equiv \Psi'/r$. The vortex will be taken to be localized, i.e. its vorticity will decay faster than any power of r at infinity, but non-isolated, so that its circulation Γ is non-zero. These two conditions imply the following results for large r:

$$Q = O\left(\frac{1}{r^\infty}\right), \tag{2.7a}$$

$$\Psi = \frac{\Gamma}{2\pi}\ln r + O\left(\frac{1}{r^\infty}\right). \tag{2.7b}$$

The order-infinity notation represents a contribution that decays faster than any power of r. A derivation of these results is given by Llewellyn Smith (1995). The vorticity must be bounded everywhere, but may have discontinuities, as in the case of the Rankine vortex. Without loss of generality, Γ will be taken positive, which corresponds to a cyclone, and Q' will be taken to be zero at the origin. Then the angular velocity near the origin becomes

$$\Omega(r) = \Omega_0 + \tfrac{1}{2}r^2\Omega_0'' + \cdots. \tag{2.8}$$

The two asymptotic expansions in the near and far fields will be matched by the method of matched asymptotic expansions. Van Dyke's (1975) rule gives

$$\psi^{(n,m)} = \phi^{(m,n)}, \tag{2.9}$$

where $\psi^{(n,m)}$ is the inner solution truncated to order n in the inner variable, subsequently re-expressed in the outer variable and then truncated to order m in that variable. For this rule to give correct results, terms of logarithmic order must be included in the expansion truncated at algebraic order.

The motion of the vortex will be calculated in a moving coordinate frame, where the centre of the frame is fixed by some prescription corresponding to a possible definition of the centre of the vortex, for example the vorticity maximum.

The expansion in ϵ will not in general be uniform in time. For long times, the structure of the velocity field becomes very complicated with the formation of a wake behind the vortex (e.g. Sutyrin *et al.* 1994). Formally, the presence of terms such as ϵt will upset the relative order of terms in the expansion. The asymptotic solution can be expected to hold for times up to and including $O(1)$, and also for larger times up to some unknown breakdown order to be determined later from the form of the solution.

3. Linear solution

The solution to the linear initial value problem

$$\frac{\partial}{\partial t}\nabla^2\psi + \epsilon\frac{\partial\psi}{\partial x} = 0, \tag{3.1}$$

with $\psi = \Psi$ at time $t = 0$, may be obtained using a Green's function approach. The Green's function for this problem is derived in Kamenkovich (1989). It is the solution to

$$\frac{\partial}{\partial t}\nabla^2 L + \epsilon\frac{\partial L}{\partial x} = \delta(\boldsymbol{r})\delta(t). \tag{3.2}$$

There is no elementary expression for L, but its Laplace transform $\bar{L}(p)$ is found to be

$$\bar{L} = -\frac{1}{2\pi p}\exp\left(-\frac{\epsilon x}{2p}\right)K_0\left(\frac{\epsilon r}{2p}\right), \tag{3.3}$$

where r is the usual plane polar distance.

The large-$|p|$ behaviour of $\bar{L}\exp(pt)$ governs the causal behaviour of the Green's function through the inverse Laplace transform. For large $|p|$, the appropriate behaviour is

$$\bar{L}e^{pt} = -\frac{1}{2\pi p}\exp\left(-\frac{\epsilon x}{2p} + pt\right)[\ln p + O(1)], \tag{3.4}$$

where γ is Euler's constant. Writing $p = p_r + ip_i$ gives

$$\mathrm{Re}\left(-\frac{\epsilon x}{2p} + pt\right) = p_r\left(t - \frac{\epsilon x}{2|p|^2}\right). \tag{3.5}$$

For $t < 0$, the contour may be closed in the right half-plane and L is zero. Accordingly L has the appropriate causal behaviour. For $t > 0$, the contour must be closed in the left half-plane, and L is not zero. Hence there is an instantaneous response over all space. This is due to the fact that the velocity of Rossby waves increases without bound with wavelength.

A variety of representations exists for L in the original time variable. One integral representation for K_0 leads to the formula

$$L = -\frac{1}{\pi}\int_0^\infty \frac{J_0(2[\epsilon rt(u^2 + c^2)]^{1/2})}{(u^2 + 1)^{1/2}}\,\mathrm{d}u, \tag{3.6}$$

where $c = \cos(\theta/2)$, θ being the usual polar angle. Use of the convolution theorem leads to other expressions for L; explicit results may also be derived for L on the x-axis (see Appendix A).

The behaviour of the Green's function near the origin is needed for later use. Expanding $\bar{L}$ in r gives

$$\bar{L} = -\frac{1}{2\pi p}\left[\ln p - \ln\frac{\epsilon r}{4} - \gamma\right] + \frac{\epsilon x}{4\pi p^2}\left[\ln p - \ln\frac{\epsilon r}{4} - \gamma\right] \\ -\frac{\epsilon^2 x^2}{16\pi p^3}\left[\ln p - \ln\frac{\epsilon r}{4} - \gamma\right] - \frac{\epsilon^2 r^2}{32\pi p^3}\left[\ln p - \ln\frac{\epsilon r}{4} - \gamma + 1\right] + O(\epsilon^3 r^3/p^4). \quad (3.7)$$

This series may be inverted term by term to give

$$L = \frac{1}{2\pi}\left[\ln\frac{\epsilon r t}{4} + 2\gamma\right] - \frac{\epsilon x t}{4\pi}\left[\ln\frac{\epsilon r t}{4} + 2\gamma - 1\right] \\ + \frac{\epsilon^2 t^2}{128\pi}\left[(2x^2 + r^2)\left(2\ln\frac{\epsilon r t}{4} + 4\gamma - 3\right) - 2r^2\right] + O(\epsilon^3 r^3 t^3). \quad (3.8)$$

It is clear that the natural expansion variables in the above expressions are $\epsilon \boldsymbol{r}/p$ and $\epsilon \boldsymbol{r}t$. The second expression is clearly non-uniform in time and space. When $\epsilon r t = O(1)$, the asymptotic ordering breaks down.

Laplace transforming the governing equation (3.1) in time gives

$$p\nabla^2\bar{\psi} + \epsilon\frac{\partial\bar{\psi}}{\partial x} = Q_i, \quad (3.9)$$

where $Q_i = \nabla^2\Psi$ is the initial vorticity. This may be solved using the previously derived Green's function, leading to

$$\bar{\psi} = \bar{L} * Q_i, \quad (3.10)$$

where $*$ is the spatial convolution operator. The inverse Laplace transform of this equation gives

$$\psi = L * Q_i \quad (3.11)$$

as the solution to the linear initial value problem.

4. Zeroth-order solution

4.1. *Inner solution*

4.1.1. *Derivation*

The zeroth-order equation in the near field is

$$\frac{\partial}{\partial t}\nabla^2\psi_0 + J(\psi_0, \nabla^2\psi_0) = 0, \quad (4.1)$$

with initial condition $\psi_0 = \Psi$ at $t = 0$. The boundary conditions at infinity come from the matching with the outer solution.

However, $\psi_0 = \Psi$ is clearly a steady solution to the equation, ignoring the boundary condition. The eigenfunctions of the Laplace operator are also solutions of the equation which do not necessarily satisfy any boundary conditions; they correspond to irrotational flow. A simple form for the inner field, regular at $r = 0$, is then

$$\psi_0 = \Psi(r) + \sum_{n=0}^{\infty} a_n(t) r^n \mathrm{e}^{\mathrm{i}n\theta}, \quad (4.2)$$

where the a_n are all initially zero for $n > 0$. The constant function of space $a_0(t)$ is physically irrelevant but will be included for completeness. Complex notation will be used for this type of trigonometric sum; the real part is to be understood. This sum includes all the necessary terms for a complete zeroth-order solution to the inner problem, as will be shown by the matching.

4.1.2. *Far-field behaviour*

The form (2.7) holds for the far field of the initial streamfunction, and so the inner solution may be rewritten as

$$\psi^{(0,.)} = \frac{\Gamma}{2\pi} \ln\left(\frac{R}{\epsilon}\right) + \sum_{n=0}^{\infty} a_n \left(\frac{R}{\epsilon}\right)^n \mathrm{e}^{\mathrm{i}n\theta} + O\left(\frac{\epsilon^\infty}{R^\infty}\right) \tag{4.3}$$

in the far-field coordinate.

The order term cannot appear at any stage in the matching procedure. In addition, there can be no terms in the above sum with n greater than zero, since these would have to match onto terms of the far-field solution containing negative powers of ϵ, whereas the perturbation expansion in the outer field cannot be large as ϵ becomes very small. Hence all the a_n are zero except for $n = 0$, and the appropriate truncation for Van Dyke's rule is

$$\psi^{(0,0)} = \frac{\Gamma}{2\pi}(\ln R - \ln \epsilon) + a_0(t). \tag{4.4}$$

This shows that there must be terms of the form $\psi_0'(r) \ln \epsilon$ in the inner expansion. However, such terms cannot be dynamically significant since the dominant motion is at zeroth-order (i.e. smaller than $\ln \epsilon$), and hence the only possible term is constant in space. This corresponds to a streamfunction of the form

$$\psi_0' = a'(t). \tag{4.5}$$

Therefore the correct truncated expansion to use, which includes logarithmic terms, is

$$\psi^{(0,0)} = \frac{\Gamma}{2\pi}(\ln R - \ln \epsilon) + a_0(t) + a'(t) \ln \epsilon. \tag{4.6}$$

Primes will be used to denote functions of logarithmic order in ϵ.

4.2. *Outer solution*

4.2.1. *Derivation*

The zeroth-order equation in the far field is

$$\frac{\partial}{\partial t} \nabla_R^2 \phi_0 + \frac{\partial \phi_0}{\partial X} = 0. \tag{4.7}$$

The boundary condition at the origin comes from the matching. The initial condition may be written as $\phi_0 = \Phi(R)$, where $\Phi(R)$ is the zeroth-order term in the expansion of $\Psi(R/\epsilon)$. However, the vorticity $q(r)$ is assumed to be localized, and hence decays faster than any power of $1/r$ in the far field. Expanding its counterpart $Q(R)$ in ϵ gives nothing, since all the terms decay too fast. Hence the initial condition for the vorticity is $Q_0(R) = 0$, and the solution to (4.7) can be taken from §3 as

$$\phi_0 = \sum_{n=0}^{\infty} A_{i\dots j} \frac{\partial^n}{\partial X_i \dots \partial X_j} L(\boldsymbol{R}, t). \tag{4.8}$$

Tensorial notation is used for the multipole sum, and the number of indices on A is equal to n. This sum corresponds to a homogeneous solution of (4.7) with unknown forcing at the origin.

4.2.2. *Near-field behaviour*

The behaviour of the Green's function and its derivatives near the origin can be obtained from (3.8), using the appropriate far-field variable:

$$L = \frac{1}{2\pi}\left[\ln\frac{Rt}{4} + 2\gamma\right] - \frac{Xt}{4\pi}\left[\ln\frac{Rt}{4} + 2\gamma - 1\right] + O(R^2t^2). \tag{4.9}$$

Hence

$$\nabla_R L = \frac{1}{2\pi}\frac{\boldsymbol{R}}{R^2} - \frac{t}{4\pi}\left[\ln\frac{Rt}{4} + 2\gamma - 1\right]\boldsymbol{i} - \frac{Xt}{4\pi}\frac{\boldsymbol{R}}{R^2} + O(Rt). \tag{4.10}$$

As expected, each differentiation raises the degree of singularity of L. Changing to the inner variable by

$$\frac{\partial}{\partial X_i} = \frac{1}{\epsilon}\frac{\partial}{\partial x_i} \tag{4.11}$$

leads to a new expression for the sum:

$$\sum_{n=0}^{\infty} A_{i\ldots j}\frac{\partial^n}{\partial X_i \ldots \partial X_j}L(\boldsymbol{R},t) = A_0\left\{\frac{1}{2\pi}\left[\ln\frac{\epsilon rt}{4} + 2\gamma\right] - \frac{\epsilon xt}{4\pi}\left[\ln\frac{\epsilon rt}{4} + 2\gamma - 1\right]\right.$$
$$\left. + O(\epsilon^2)\right\} + \boldsymbol{A}_1 O(\epsilon^{-1}) + \cdots, \tag{4.12}$$

where $\boldsymbol{A}_1$ is a vector representation of A_i. This shows that all the $A_{i\ldots j}$ must be zero for $n \geqslant 1$, since any higher A would have to match onto terms of negative order in ϵ in the inner expansion, and the leading-order behaviour of the inner solution is of order zero.

4.3. *Matching*

The truncated expansions for the inner and outer solutions are

$$\psi^{(0,0)} = \frac{\Gamma}{2\pi}\ln R + a_0 + \ln\epsilon\left(a' - \frac{\Gamma}{2\pi}\right) \tag{4.13}$$

and

$$\phi^{(0,0)} = \frac{A_0}{2\pi}\left[\ln\frac{\epsilon rt}{4} + 2\gamma\right] \tag{4.14}$$

respectively. Van Dyke's rule then leads to the the following three relations:

$$A_0 = \Gamma, \quad a_0 = \frac{\Gamma}{2\pi}\left[\ln\frac{t}{4} + 2\gamma\right], \quad a' = \frac{\Gamma}{2\pi}. \tag{4.15a–c}$$

If there had been a logarithmic term in the far field (ϕ_0' say), it would have had to satisfy (4.7), and hence been proportional to L. The three equations of (4.15a–c) could then have been satisfied only with ϕ_0' identically zero.

The complete zeroth-order solution, including logarithmic terms, is

$$\psi_0 = \Psi(r) + \frac{\Gamma}{2\pi}\left[\ln\frac{t}{4} + 2\gamma\right] \tag{4.16a}$$

and

$$\psi_0' = \frac{\Gamma}{2\pi} \tag{4.16b}$$

for the near field, and

$$\phi_0 = \Gamma\, L(\boldsymbol{R}, t) \tag{4.16c}$$

for the far field. In the Laplace coordinate, these expressions become

$$\overline{\psi_0} = \frac{\Psi(r)}{p} - \frac{\Gamma}{2\pi}\left[\ln p - \ln \tfrac{1}{4} - \gamma\right] \tag{4.17a}$$

and

$$\overline{\psi_0'} = \frac{\Gamma}{2\pi p}, \tag{4.17b}$$

and

$$\overline{\phi_0} = \Gamma\, \bar{L}(\boldsymbol{R}, p) \tag{4.17c}$$

for the two regions respectively.

The only dynamically significant part of the inner solution is the initial streamfunction. The other terms are just functions of time that match onto the outer solution. The Green's function term corresponds to the response to a vortex of circulation Γ at the origin. To the far field, the only 'visible' property of the vortex, at zeroth order, is its circulation. This illustrates the important difference between isolated and non-isolated vortices.

5. First-order solution

5.1. *Inner solution*

5.1.1. *Derivation*

The governing equation for the first-order solution in the inner field is the linearized inhomogeneous Euler equation

$$\left(\frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \theta}\right)\nabla^2 \psi_1 - \frac{Q'}{r}\frac{\partial \psi_1}{\partial \theta} + \Omega r \cos\theta = 0, \tag{5.1}$$

with zero initial condition. The forcing term comes from the beta-effect acting on the radial order-zero streamfunction. Laplace transforming in time, and decomposing into radial modes, leads to

$$(p + \mathrm{i}l\Omega)\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} r \frac{\mathrm{d}}{\mathrm{d}r} - \frac{l^2}{r^2}\right]\overline{\psi_1^l} - \frac{Q'}{r}\mathrm{i}l\overline{\psi_1^l} = -\frac{\Omega r \delta_{1l}}{p}, \tag{5.2}$$

where the real part of this equation is to be understood, and where

$$\psi_1 = \sum_l \psi_1^l \mathrm{e}^{\mathrm{i}l\theta}. \tag{5.3}$$

Each mode ψ_1^l has zero initial condition, and (5.2) is homogeneous for all modes except $l = 1$, which is therefore the only non-zero solution. Dropping the subscript l, the governing equation for mode one may be rewritten as

$$-\frac{\mathrm{d}}{\mathrm{d}r}\left(r \frac{\mathrm{d}\overline{\psi_1}}{\mathrm{d}r}\right) + \left[\frac{1}{r} + \frac{\mathrm{i}Q'}{p + \mathrm{i}\Omega}\right]\overline{\psi_1} = \frac{1}{p + \mathrm{i}\Omega}\frac{\Omega r^2}{p}. \tag{5.4}$$

The change of variable $\overline{\psi_1} = r(p + \mathrm{i}\Omega)f$ leads to the equation

$$-\frac{1}{r(p+\mathrm{i}\Omega)}\frac{\mathrm{d}}{\mathrm{d}r}\left[r^3(p+\mathrm{i}\Omega)^2\frac{\mathrm{d}f}{\mathrm{d}r}\right] = \frac{1}{p+\mathrm{i}\Omega}\frac{\Omega r^2}{p}. \tag{5.5}$$

The solution to this equation is

$$\overline{\psi_1} = -\frac{r(p+\mathrm{i}\Omega(r))}{p}\int_{B(p)}^{r}\frac{h(v)-h(A(p))}{v^3(p+\mathrm{i}\Omega(v))^2}\,\mathrm{d}v, \tag{5.6}$$

where h is defined by

$$h(v) = \int_0^v \Omega(u)u^3\,\mathrm{d}u \tag{5.7}$$

and A and B are undetermined functions of p. Changing A and B corresponds to adding multiples of the homogeneous solutions. This is essentially the solution presented in RD94, although the derivation has followed Smith & Rosenbluth (1990). However, the far field behaviour of (5.6) is quite different here.

The function $h(v)$ is the basic-state relative angular momentum within a disc of radius v. The behaviour of the function $h(v)$ is given by

$$h(v) = \tfrac{1}{4}\Omega_0 v^4 + O(v^6) \tag{5.8}$$

for small v, and by

$$h(v) = \frac{\Gamma v^2}{4\pi} + H + O(v^{-\infty}) \tag{5.9}$$

for large v, where

$$H = \int_0^\infty\left[\Omega(u) - \frac{\Gamma}{4\pi u^2}\right]u^3\,\mathrm{d}u. \tag{5.10}$$

This quantity will in general be non-zero, even for vortices with zero circulation, although there will be vortices for which it vanishes. For vortices with zero circulation, it has been called the Relative Angular Momentum (RAM) in the tropical cyclone literature (Willoughby 1988).

Denoting by $\overline{f_l}$ the mode-l solution to (5.2) that is well-behaved at the origin, a complete solution to the inner problem is given by

$$\overline{\psi_1} = -\frac{r(p+\mathrm{i}\Omega(r))}{p}\mathrm{e}^{\mathrm{i}\theta}\int_0^r\frac{h(v)}{v^3(p+\mathrm{i}\Omega(v))^2}\,\mathrm{d}v + \sum_l \overline{b_l}(p)\overline{f_l}(r,p)\mathrm{e}^{\mathrm{i}l\theta}. \tag{5.11}$$

The homogeneous solutions are multiplied by functions of p which remain to be determined. The functions A and B have been fixed to ensure convergence of the integral. This freedom is due to the presence of the unspecified function $\overline{b_1}$. The solutions for the lowest two modes can be written down explicitly as

$$\overline{f_0} = 1 \tag{5.12a}$$

and

$$\overline{f_1} = r(p+\mathrm{i}\Omega). \tag{5.12b}$$

5.1.2. *Far-field behaviour*

Truncating the complete inner solution to first order gives

$$\overline{\psi^{(1,)}} = \overline{\psi_0} - \frac{\epsilon r(p+\mathrm{i}\Omega(r))}{p}\mathrm{e}^{\mathrm{i}\theta}\int_0^r\frac{h(v)}{v^3(p+\mathrm{i}\Omega(v))^2}\,\mathrm{d}v + \sum_l \overline{b_l}(p)\overline{f_l}(r,p)\mathrm{e}^{\mathrm{i}l\theta}. \tag{5.13}$$

The leading-order behaviour of the integral for small ϵ and $R = O(1)$ is given by $\Gamma \ln(R/\epsilon)/4\pi p^2$. The order-zero and order-one portions of the integral are calculated in Appendix B. The contribution of the integral term in the far-field variable is

$$-\frac{\Gamma X}{4\pi p^2}\ln\frac{R}{\epsilon}-\frac{Re^{i\theta}}{p^2}\int_0^\infty \frac{h(v)\Omega(v)}{v^3(p+i\Omega(v))^2}(\Omega(v)-2ip)\,dv+\frac{\Gamma X}{8\pi p^2}+\frac{X\Psi_0}{2p^2}+O(\epsilon^2), \quad (5.14)$$

using (B 9) and the far-field behaviour of Ω. The expression $Re^{i\theta}$ can be replaced by X when it is multiplying a real quantity.

The asymptotic form of the integral term suggests that mode-zero and mode-one homogeneous terms will be needed in the solution. In the far-field variable, the appropriate mode-zero solution is

$$\overline{f_0} = 1, \quad (5.15)$$

while the mode-one solution is

$$\overline{f_1} = r(p + i\Omega) = \frac{Rp}{\epsilon} + O(\epsilon). \quad (5.16)$$

Owing to the factor ϵ multiplying it in (5.13), the mode-zero term is of first order in the far-field variable expansion. Discarding all modes other than zero and one gives the contribution

$$\overline{b_1}Re^{i\theta}p + \epsilon\overline{b_0} + O(\epsilon^2) \quad (5.17)$$

to (5.13) from the homogeneous terms.

5.1.3. *The $O(\epsilon \ln \epsilon)$ terms*

The form of the integral term shows that a logarithmic term will also be required in the expansion. The inner solution at $O(\ln \epsilon)$ was found in (4.5) and is dynamically insignificant. The governing equation for the $O(\epsilon \ln \epsilon)$ term is

$$\left(\frac{\partial}{\partial t}+\Omega\frac{\partial}{\partial \theta}\right)\nabla^2\psi_1' - \frac{Q'}{r}\frac{\partial \psi_1'}{\partial \theta} = 0, \quad (5.18)$$

which is the homogeneous counterpart of (5.1). This has the solution

$$\psi_1' = \sum_l \overline{b_l'}(p)\overline{f_l}(r,p)e^{i\theta}, \quad (5.19)$$

analogous to the homogeneous term of (5.11). The form of the logarithmic term in (5.14) shows that the only term actually required is the mode-one response. Its far-field behaviour is given by (5.16).

Putting these results together leads to

$$\begin{aligned}\overline{\psi^{(1,)}} = \overline{\psi^{(0,1)}} &- \frac{\Gamma X}{4\pi p^2}\ln\frac{R}{\epsilon}-\frac{Re^{i\theta}}{p^2}\int_0^\infty \frac{h(v)\Omega(v)}{v^3(p+i\Omega(v))^2}(\Omega(v)-2ip)\,dv \\ &+\frac{\Gamma X}{8\pi p^2}+\frac{X\Psi_0}{2p^2}+\overline{b_1}Re^{i\theta}p+\epsilon\overline{b_0}+\overline{b_1'}Re^{i\theta}p\ln\epsilon+O(\epsilon^2). \quad (5.20)\end{aligned}$$

The zeroth- and first-order truncations may now be easily computed.

5.2. *Outer solution*

5.2.1. *Derivation*

The governing equation is again (4.7), since nonlinearity only enters at second order in the far field. The solution with zero initial condition, and unspecified behaviour at the origin, is

$$\phi_1 = \sum_{n=0}^{\infty} B_{i\ldots j} \frac{\partial^n}{\partial X_i \ldots \partial X_j} L(\boldsymbol{R}, t). \tag{5.21}$$

It is not necessary to compute terms of logarithmic order in the far field at this order. If this were done, they would be identically zero through the matching process. At higher orders, however, such terms will be required.

5.2.2. *Near-field behaviour*

The limiting behaviour of such a sum has already been investigated in §4.2.2. Working in the Laplace coordinate, the first-order truncation of the outer solution is

$$\begin{aligned}
\overline{\phi_1^{(1,\cdot)}} &= \Gamma\,\bar{L}(\boldsymbol{R}, p) + \epsilon \sum_n B_{i\ldots j} \frac{\partial^n}{\partial X_i \ldots \partial X_j} \bar{L}(\boldsymbol{R}, p) \\
&= \Gamma \left\{ -\frac{1}{2\pi p} \left[\ln p - \ln \frac{\epsilon r}{4} - \gamma \right] + \frac{\epsilon x}{4\pi p^2} \left[\ln p - \ln \frac{\epsilon r}{4} - \gamma \right] + O(\epsilon^2) \right\} \\
&\quad + \epsilon \sum_n B_{i\ldots j} \frac{\partial^n}{\epsilon^n \partial x_i \ldots \partial x_j} \left\{ -\frac{1}{2\pi p} \left[\ln p - \ln \frac{\epsilon r}{4} - \gamma \right] \right. \\
&\quad \left. + \frac{\epsilon x}{4\pi p^2} \left[\ln p - \ln \frac{\epsilon r}{4} - \gamma \right] + O(\epsilon^2) \right\},
\end{aligned} \tag{5.22}$$

when rewritten in the inner coordinate. As before, the B must be zero, except for B_0 and B_i, since there is no ϵ^{-1} term in the inner expansion. Hence, truncating to the appropriate order gives

$$\begin{aligned}
\overline{\phi_1^{(1,1)}} &= -\frac{\Gamma}{2\pi p} \left[\ln p - \ln \frac{\epsilon r}{4} - \gamma \right] + \frac{\boldsymbol{B}_1}{2\pi p} \cdot \frac{\boldsymbol{r}}{r^2} \\
&\quad + \epsilon \left\{ \frac{1}{\pi} \left(\frac{\Gamma x}{4p^2} - \frac{B_0}{2p} \right) \left[\ln p - \ln \frac{\epsilon r}{4} - \gamma \right] + \frac{\boldsymbol{B}_1}{4\pi p^2} \cdot \left(\boldsymbol{i} \left[\ln p - \ln \frac{\epsilon r}{4} - \gamma \right] - x \frac{\boldsymbol{r}}{r^2} \right) \right\},
\end{aligned} \tag{5.23}$$

where $\boldsymbol{B}_1$ is the vector form of B_i. The truncation of $\overline{\phi^{(1,\cdot)}}$ to zeroth order may be written down immediately from the previous line as

$$\overline{\phi_1^{(1,0)}} = -\frac{\Gamma}{2\pi p} \left[\ln p - \ln \frac{\epsilon r}{4} - \gamma \right] + \frac{\boldsymbol{B}_1}{2\pi p} \cdot \frac{\boldsymbol{r}}{r^2}. \tag{5.24}$$

The 'off-diagonal' element $\psi^{(1,0)} = \phi^{(0,1)}$ of Van Dyke's rule requires the first-order truncation of the zeroth-order outer field. This is

$$\overline{\phi^{(0,1)}} = \overline{\phi^{(0,0)}} + \frac{\epsilon \Gamma x}{4\pi p^2} \left[\ln p - \ln \frac{\epsilon r}{4} - \gamma \right]. \tag{5.25}$$

5.3. *Matching*

5.3.1. $\overline{\psi^{(0,1)}} = \overline{\phi^{(1,0)}}$

All Van Dyke truncations of the zeroth-order inner solution are the same, since the only powers of ϵ present when expressed in the far field coordinate are zero (including the logarithmic term) and infinity. Hence

$$\overline{\psi^{(0,1)}} = \overline{\psi^{(0,0)}} \tag{5.26}$$

and no further calculation is required. The off-diagonal element of Van Dyke's rule gives

$$\overline{\phi^{(1,0)}} = \overline{\phi^{(0,0)}} + \frac{\boldsymbol{B}_1}{2\pi p} \cdot \frac{\boldsymbol{R}}{\epsilon R^2} = \overline{\psi^{(0,1)}}. \tag{5.27}$$

Using the result of the zeroth-order matching gives $\boldsymbol{B}_1 = \boldsymbol{0}$.

5.3.2. $\overline{\psi^{(1,0)}} = \overline{\phi^{(0,1)}}$

Truncating (5.20) at zeroth order, and equating the result with (5.25) leads to the equation

$$\begin{aligned}\overline{\psi^{(0,1)}} - \frac{\Gamma X}{4\pi p^2} \ln \frac{R}{\epsilon} - \frac{R\mathrm{e}^{\mathrm{i}\theta}}{p^2} \int_0^\infty \frac{h(v)\Omega(v)}{v^3(p+\mathrm{i}\Omega(v))^2} (\Omega(v) - 2\mathrm{i}p)\, \mathrm{d}v \\ + \frac{\Gamma X}{8\pi p^2} + \frac{X\Psi_0}{2p^2} + \overline{b_1} R\mathrm{e}^{\mathrm{i}\theta} p + \overline{b_1'}(p) R\mathrm{e}^{\mathrm{i}\theta} p \ln \epsilon = \overline{\phi^{(0,0)}} + \frac{\Gamma X}{4\pi p^2} \left[\ln p - \ln \frac{R}{4} - \gamma \right],\end{aligned} \tag{5.28}$$

written in the far-field variable. Again, the two terms formally truncated at zeroth order cancel from the zeroth-order matching (using the preceding paragraph). The logarithmic terms in R cancel also, and the equation decouples into two, since the $\ln \epsilon$ terms must be taken into account separately. The two resulting equations are

$$\begin{aligned}-\frac{R\mathrm{e}^{\mathrm{i}\theta}}{p^2} \int_0^\infty \frac{h(v)\Omega(v)}{v^3(p+\mathrm{i}\Omega(v))^2} (\Omega(v) - 2\mathrm{i}p)\, \mathrm{d}v + \frac{\Gamma X}{8\pi p^2} + \frac{X\Psi_0}{2p^2} + \overline{b_1} R\mathrm{e}^{\mathrm{i}\theta} p \\ = \frac{\Gamma X}{4\pi p^2} \left[\ln p - \ln \tfrac{1}{4} - \gamma \right],\end{aligned} \tag{5.29a}$$

and

$$\frac{\Gamma X}{4\pi p^2} + \overline{b_1'}(p) R\mathrm{e}^{\mathrm{i}\theta} p = 0. \tag{5.29b}$$

The real part of both equations is understood. However, both decouple into an equation in X and an equation in Y, and solving these is equivalent to solving the original complex equations for $\overline{b_1}$ and $\overline{b_1'}$, replacing X by its complex counterpart $R\mathrm{e}^{\mathrm{i}\theta}$. This leads to

$$\overline{b_1} = \frac{\Gamma}{4\pi p^3} \left[\ln p - \ln \tfrac{1}{4} - \gamma - \tfrac{1}{2} \right] - \frac{\Psi_0}{2p^3} + \frac{1}{p^3} \int_0^\infty \frac{h(v)\Omega(v)}{v^3(p+\mathrm{i}\Omega(v))^2} (\Omega(v) - 2\mathrm{i}p)\, \mathrm{d}v \tag{5.30a}$$

and

$$\overline{b_1'} = -\frac{\Gamma}{4\pi p^3}. \tag{5.30b}$$

5.3.3. $\overline{\psi^{(1,1)}} = \overline{\phi^{(1,1)}}$

Truncating (5.20) at first order now gives

$$\overline{\psi^{(1,1)}} = \overline{\psi^{(1,0)}} + \epsilon \overline{b_0}(p), \tag{5.31}$$

since the integral does not contribute an order-one term to the expansion. The necessary truncation of the outer expansion has already been calculated, and may be rewritten as

$$\overline{\phi^{(1,1)}} = \overline{\phi^{(0,1)}} - \frac{\epsilon B_0}{2p} \left[\ln p - \ln \frac{R}{4} + \gamma \right]. \tag{5.32}$$

Van Dyke's rule gives

$$\overline{b_0} = -\frac{B_0}{2p}\left[\ln p - \ln\frac{R}{4} + \gamma\right]. \tag{5.33}$$

The only possible solution to this equation is $\overline{b_0} = B_0 = 0$; any other choice leaves a logarithmic term that cannot be matched. Hence there is no first-order disturbance in the far field.

The complete solution to the first-order problem is thus given by the outer solution

$$\phi_1 = 0 \tag{5.34a}$$

and the inner solution

$$\begin{aligned}\overline{\psi_1} = &-\frac{r(p+\mathrm{i}\Omega(r))\mathrm{e}^{\mathrm{i}\theta}}{p}\int_0^r \frac{h(v)}{v^3(p+\mathrm{i}\Omega(v))^2}\,\mathrm{d}v\\ &+\frac{r(p+\mathrm{i}\Omega(r))}{p^3}\mathrm{e}^{\mathrm{i}\theta}\left\{\frac{\Gamma}{4\pi}\left[\ln p - \ln\tfrac{1}{4} - \gamma - \tfrac{1}{2}\right] - \frac{\Psi_0}{2}\right.\\ &\left.+\int_0^\infty \frac{h(v)\Omega(v)}{v^3(p+\mathrm{i}\Omega(v))^2}(\Omega(v)-2\mathrm{i}p)\,\mathrm{d}v\right\},\end{aligned} \tag{5.34b}$$

with the logarithmic term

$$\overline{\psi_1'} = -\frac{\Gamma\, r(p+\mathrm{i}\Omega(r))\mathrm{e}^{\mathrm{i}\theta}}{4\pi p^3}. \tag{5.34c}$$

An equivalent expression for the $O(\epsilon)$ solution, which highlights its behaviour in the matching region, is

$$\begin{aligned}\overline{\psi_1} = \frac{r(p+\mathrm{i}\Omega(r))\mathrm{e}^{\mathrm{i}\theta}}{p^3}&\left\{-\frac{\Psi(r)}{2} + \frac{h(r)}{2r^2} + \frac{\Gamma}{4\pi}\left[\ln p - \ln\tfrac{1}{4} - \gamma - \tfrac{1}{2}\right]\right.\\ &\left.+\int_r^\infty \frac{h(v)\Omega(v)}{v^3}\frac{\Omega(v)-2\mathrm{i}p}{(p+\mathrm{i}\Omega(v))^2}\,\mathrm{d}v.\right\}.\end{aligned} \tag{5.35}$$

In the time variable, the inner solution is

$$\begin{aligned}\psi_1 = &-r\mathrm{e}^{\mathrm{i}\theta}\left(\frac{\partial}{\partial t} + \mathrm{i}\Omega(r)\right)\int_0^r \frac{h(v)}{v^3}\left[\frac{\mathrm{i}t\mathrm{e}^{-\mathrm{i}\Omega(v)t}}{\Omega(v)} + \frac{1-\mathrm{e}^{-\mathrm{i}\Omega(v)t}}{\Omega(v)^2}\right]\mathrm{d}v\\ &+r\mathrm{e}^{\mathrm{i}\theta}\left(\frac{\partial}{\partial t} + \mathrm{i}\Omega(r)\right)\left\{\frac{\Gamma t^2}{8\pi}\left[-\ln\frac{t}{4} - 2\gamma + 1\right] - \frac{\Psi_0 t^2}{4}\right.\\ &\left.+\int_0^\infty \frac{h(v)}{v^3}\left[-\frac{t^2}{2} + \frac{\mathrm{i}t\mathrm{e}^{-\mathrm{i}\Omega(v)t}}{\Omega(v)} + \frac{\mathrm{e}^{-\mathrm{i}\Omega(v)t}-1}{\Omega(v)^2}\right]\mathrm{d}v\right\},\end{aligned} \tag{5.36a}$$

$$\psi_1' = -\frac{\Gamma r\mathrm{e}^{\mathrm{i}\theta}}{4\pi}\left[t + \mathrm{i}\Omega(r)\frac{t^2}{2}\right]. \tag{5.36b}$$

While the last expression seems to suggest that the asymptotic expansion must lose validity for $t = O(\epsilon^{-1/2})$, when ψ_1' will be of order ϵ^{-1} and hence become a zeroth-order term (actually an $O(\ln\epsilon)$ term), this is erroneous. In fact, the spatial dependence of the solution must be taken into account as well. This will be returned to later.

6. Vortex trajectory

6.1. *Moving coordinate system*

Working in a coordinate system moving with the vortex will be advantageous in this asymptotic framework. Assuming that the velocity of this new frame with respect to the old is (U, V), the equation of motion becomes

$$\frac{\partial}{\partial t}\nabla^2\psi + J(\psi, \nabla^2\psi) + \epsilon\frac{\partial \psi}{\partial x} - \left(U\frac{\partial}{\partial x} + V\frac{\partial}{\partial y}\right)\nabla^2\psi = 0, \tag{6.1}$$

in the inner region, and

$$\frac{\partial}{\partial t}\nabla_R^2\phi + \frac{\partial \phi}{\partial X} + \epsilon^2 J_R(\phi, \nabla_R^2\phi) - \epsilon\left(U\frac{\partial}{\partial X} + V\frac{\partial}{\partial Y}\right)\nabla_R^2\phi = 0 \tag{6.2}$$

in the outer region. However, the relative velocity between the frames is of order ϵ (or $\epsilon \ln \epsilon$), since it cannot appear in the absence of the beta-effect. Hence the previous analysis is valid to order zero in the near field, and order one in the far field. The full new equations have to be solved at higher orders to find a solution to the equation of motion in the new frame.

In the inner region, the first-order equation for ψ_1 becomes

$$\frac{\partial}{\partial t}\nabla^2\psi_1 + J(\psi_0, \nabla^2\psi_1) + J(\psi_1, \nabla^2\psi_0) = -\frac{\partial \psi_0}{\partial x} + \left(U_1\frac{\partial}{\partial x} + V_1\frac{\partial}{\partial y}\right)\nabla^2\psi_0, \tag{6.3}$$

where $U = \epsilon U_1 + \epsilon \ln \epsilon U_1' + \cdots$, and similarly for V. This is just equation (5.1) with an extra forcing term due to the relative motion of the frame. Since this is a linear equation, the solution can be decomposed into a part due to the beta-effect, which has already been calculated, and a part due to the new forcing term, provided both satisfy appropriate boundary and initial conditions.

The part of the streamfunction ψ_1^f due to the change of frame may be written as

$$\psi_1^f = \int_0^t \left(U_1(\tau)\frac{\partial \Psi}{\partial x} + V_1(\tau)\frac{\partial \Psi}{\partial y}\right) \mathrm{d}\tau \tag{6.4}$$

$$= X_1(t)\frac{\partial \Psi}{\partial x} + Y_1(t)\frac{\partial \Psi}{\partial y} \tag{6.5}$$

$$= (X_1(t)\cos\theta + Y_1(t)\sin\theta)\Psi', \tag{6.6}$$

where $\Psi(r)$ is the original streamfunction and $\boldsymbol{X}(t) = \epsilon \boldsymbol{X}_1 + \cdots$ is the location of the origin in the new frame, as seen from the old one. This solution may be written in complex form as

$$\psi_1^f = Z_1^*(t)\mathrm{e}^{\mathrm{i}\theta} r\Omega, \tag{6.7}$$

where the asterisk denotes complex conjugation, and the real part is again to be understood. The function Ψ may be taken instead of ψ_0, since the two differ only by a constant function of space. The function $r\Omega\mathrm{e}^{\mathrm{i}\theta}$ is actually a steady mode-one solution to the linearized Euler equation in the absence of beta (cf. Michalke & Timme 1967; Llewellyn Smith 1995). The corresponding vorticity is easily calculated to be

$$\nabla^2\psi_1^f = X_1\frac{\partial Q}{\partial x} + Y_1\frac{\partial Q}{\partial y} = Z_1^*\mathrm{e}^{\mathrm{i}\theta}Q'. \tag{6.8}$$

Hence (6.3) is satisfied. This part of the streamfunction decays at infinity, and clearly vanishes at $t = 0$. Its Laplace transform is simply obtained by replacing Z_1 by $\overline{Z_1}$. Expressed in terms of the far-field space variable and of the Laplace transform variable, ψ_1^f takes the form

$$\overline{\psi^{f(1,)}} = \epsilon \overline{Z_1^*} e^{i\theta} \left[\frac{\Gamma \epsilon}{2\pi R} + O\left(\frac{\epsilon^\infty}{R^\infty} \right) \right]. \tag{6.9}$$

This is formally a second-order quantity, so the contribution to the streamfunction from the change of frame does not affect the far-field expansion to zeroth and first order through matching.

The governing equation at $O(\epsilon \ln \epsilon)$ is

$$\frac{\partial}{\partial t}\nabla^2\psi_1' + J(\psi_0, \nabla^2\psi_1') + J(\psi_1', \nabla^2\psi_0) = -\left(U_1' \frac{\partial}{\partial x} + V_1' \frac{\partial}{\partial y} \right) \nabla^2\psi_0. \tag{6.10}$$

There is no forcing due to the beta-effect; nor is there advection of $O(\ln \epsilon)$ vorticity. The solution to the homogeneous problem was found in (5.34c), while the solution to the drift problem is clearly formally the same as above, i.e.

$$\psi_1^{f\prime} = Z_1^{*\prime}(t) e^{i\theta} r \Omega. \tag{6.11}$$

The analysis of the previous paragraph shows that this term does not require matching in the far field until second order in the logarithmic terms.

6.2. *Origin of the coordinate system*

It is logical to centre the moving coordinate system in the middle of the vortex. This condition will determine X_1 and Y_1 and hence close the set of equations. This is easier than solving an implicit set of equations in the original frame. However, the centre of the vortex must be specified somehow. A number of possibilities for identifying the centre of an initially monopolar distribution exist. Four are presented here: the vorticity maximum, the streamfunction maximum, and the location of the particle initially at the origin, all of which are described in RD94, and also a pseudo-secularity condition.

6.2.1. *Relative vorticity maximum*

One way of defining the centre of the vortex is to look at the maximum in relative vorticity. Presumably, this approach will not work if there is no such point (e.g. for the Rankine vortex). The position of the origin is then determined by the condition

$$\nabla\left(\nabla^2\psi\right)|_O = \mathbf{0}. \tag{6.12}$$

Clearly, the inner solution is the relevant one to employ here. Working in plane polars and expanding in ϵ, this corresponds to

$$\left.\frac{\partial}{\partial r}\nabla^2(\psi_0 + \ln\epsilon\psi_0' + \epsilon\psi_1 + \epsilon\psi_1^f + \epsilon\ln\epsilon\psi_1' + \epsilon\ln\epsilon\psi_1^{f\prime} + \cdots)\right|_{r=0} = 0. \tag{6.13}$$

The $O(1)$ term of (6.13) vanishes at the origin, since Ψ has a maximum there, and the $O(\ln \epsilon)$ term is dynamically insignificant. As for the other terms, it may be seen that

$$\nabla^2 r(p + i\Omega(r)) = i(3\Omega'(r) + r\Omega''(r)), \tag{6.14}$$

with the obvious special case $p = 0$. In addition

$$
\begin{aligned}
&- \frac{r(p+\mathrm{i}\Omega(r))}{p^3}\mathrm{e}^{\mathrm{i}\theta}\int_0^r \frac{h(v)}{v^3(p+\mathrm{i}\Omega(v))^2}\,\mathrm{d}v \\
&\quad = \left[-\frac{r(p+\mathrm{i}\Omega_0)}{p} + O(r^2)\right]\int_0^r \left[\frac{\Omega_0 v}{4} + O(1)\right]\left[\frac{1}{(p+\mathrm{i}\Omega_0)^2} + O(v)\right]\mathrm{d}v \\
&\quad = -\frac{\Omega_0 r^3 \mathrm{e}^{\mathrm{i}\theta}}{8p(p+\mathrm{i}\Omega_0)} + O(r^4\mathrm{e}^{\mathrm{i}\theta}),
\end{aligned}
\tag{6.15}
$$

for small r, which leads to

$$
\nabla^2\left[-\frac{r(p+\mathrm{i}\Omega(r))}{p^3}\mathrm{e}^{\mathrm{i}\theta}\int_0^r \frac{h(v)}{v^3(p+\mathrm{i}\Omega(v))^2}\,\mathrm{d}v\right] = -\frac{\Omega_0 r \mathrm{e}^{\mathrm{i}\theta}}{p(p+\mathrm{i}\Omega_0)} + O(r^2\mathrm{e}^{\mathrm{i}\theta}). \tag{6.16}
$$

Hence (6.13) becomes the two equations

$$
4\Omega_0''(\overline{Z_1^*} + \mathrm{i}\overline{b_1}) - \frac{\Omega_0}{p(p+\mathrm{i}\Omega_0)} = 0 \tag{6.17}
$$

and

$$
4\Omega_0''(\overline{Z_1^*} + \mathrm{i}\overline{b_1'}) = 0. \tag{6.18}
$$

Clearly, if $\Omega_0'' = 0$, the first equation is inconsistent and the second one useless. This happens for vorticity profiles with an inflection point at the origin. In this special case, taking the r-derivative of (6.12) at the origin will lead to an answer. A higher number of derivatives may need to be taken for profiles that are very flat at the origin. However, if the vorticity profile is actually constant about the origin, as is the case for the Rankine vortex, the technique will not work at all.

When $\Omega_0'' \neq 0$, the resulting equations for the motion of the vortex are

$$
\overline{Z_1^*} = -\frac{\mathrm{i}}{p^3}\left\{\frac{\Gamma}{4\pi}\left[\ln p - \ln\tfrac{1}{4} - \gamma - \tfrac{1}{2}\right] - \frac{\Psi_0}{2} + \int_0^\infty \frac{h(v)\Omega(v)}{v^3(p+\mathrm{i}\Omega(v))^2}(\Omega(v) - 2\mathrm{i}p)\,\mathrm{d}v\right\} + \frac{\Omega_0}{4\Omega_0'' p(p+\mathrm{i}\Omega_0)}, \tag{6.19}
$$

and

$$
\overline{Z_1^{*\prime}} = \frac{\mathrm{i}\Gamma}{4\pi p^3}. \tag{6.20}
$$

Inverse Laplace transforming these expressions and taking their complex conjugates gives

$$
\begin{aligned}
Z_1^{rv} = {} & \frac{\mathrm{i}\Gamma t^2}{8\pi}\left[-\ln\frac{t}{4} - 2\gamma + 1\right] - \frac{\mathrm{i}\Psi_0 t^2}{4} \\
& + \mathrm{i}\int_0^\infty \frac{h(v)}{v^3}\left[-\frac{t^2}{2} - \frac{\mathrm{i}t\mathrm{e}^{\mathrm{i}\Omega(v)t}}{\Omega(v)} + \frac{\mathrm{e}^{\mathrm{i}\Omega(v)t} - 1}{\Omega(v)^2}\right]\mathrm{d}v + \frac{\mathrm{i}}{4\Omega_0''}(1 - \mathrm{e}^{\mathrm{i}\Omega_0 t})
\end{aligned}
\tag{6.21}
$$

and

$$
Z_1^{rv\prime} = -\frac{\mathrm{i}\Gamma t^2}{8\pi}. \tag{6.22}
$$

It is interesting to note that the first-order trajectory has an oscillatory component, forced at what might be termed the vortex frequency Ω_0. This oscillatory term is governed by the local behaviour of the basic state near the origin, while the other terms depend on the circulation and other global properties of the vortex.

The small-time behaviour of the displacement of the centre is given by

$$Z_1^{rv} = \frac{\Omega_0 t}{4\Omega_0''} + \frac{\mathrm{i}\Gamma t^2}{8\pi}\left[-\ln\frac{t}{4} - 2\gamma + 1\right] - \frac{\mathrm{i}\Psi_0 t^2}{4} + O(t^3) \tag{6.23}$$

and by (6.22). The initial displacement of the vortex is zonal and to the west, since $\Omega_0 > 0$ and $\Omega_0'' < 0$. It comes from the oscillatory component of the displacement, and hence depends on the local behaviour of the basic-state vorticity near the origin. The next-order term, which depends on the global properties of the vortex, is meridional and to the north; it contains a logarithmic contribution. These results agree qualitatively with experimental observations of the motion of cyclones on the beta-plane (cf. Carnevale *et al.* 1991).

The large-time behaviour of the trajectory can also be derived. For large t, the result is, from Appendix C,

$$Z_1^{rv} = \frac{\mathrm{i}\Gamma t^2}{8\pi}\left[-\ln\frac{t}{4} - 2\gamma + 1\right] + \frac{\mathrm{i}\Gamma t^2}{16\pi}\left[-\ln\frac{\Gamma t}{2\pi} - \gamma + \frac{3}{2} + \frac{\mathrm{i}\pi}{2}\right] + O(t). \tag{6.24}$$

This result is unphysical, as can be seen by comparing it to computed vortex trajectories. It predicts a displacement to the south, which is clearly erroneous. However, the asymptotic expansion used breaks down for asymptotically large time. Either the current expansion must match onto some other expansion valid for these larger times, or there is no expansion possible for large times.

6.2.2. *Streamfunction maximum*

The position of the streamfunction maximum is determined by the condition

$$\nabla\psi|_O = \mathbf{0}. \tag{6.25}$$

The working is as above, replacing $\nabla^2\psi$ by ψ. Rewriting the two first-order equations leads to

$$\overline{b_1}(p + \mathrm{i}\Omega_0)\mathrm{e}^{\mathrm{i}\theta} + \overline{Z_1^*}\mathrm{e}^{\mathrm{i}\theta}\Omega_0 = 0 \tag{6.26a}$$

and

$$\overline{b_1'}(p + \mathrm{i}\Omega_0)\mathrm{e}^{\mathrm{i}\theta} + \overline{Z_1^{*\prime}}\mathrm{e}^{\mathrm{i}\theta}\Omega_0 = 0. \tag{6.26b}$$

Substituting the appropriate expressions gives

$$\overline{Z_1^*} = -\frac{p + \mathrm{i}\Omega_0}{\Omega_0 p^3}\left\{\frac{\Gamma}{4\pi}\left[\ln p - \ln\tfrac{1}{4} - \gamma - \tfrac{1}{2}\right] - \frac{\Psi_0}{2} + \int_0^\infty \frac{h(v)\Omega(v)}{v^3(p + \mathrm{i}\Omega(v))^2}(\Omega(v) - 2\mathrm{i}p)\,\mathrm{d}v\right\}, \tag{6.27a}$$

and

$$\overline{Z_1^{*\prime}} = \frac{\Gamma(p + \mathrm{i}\Omega_0)}{4\pi\Omega_0 p^3}. \tag{6.27b}$$

In the time variable, these expressions become

$$Z_1^{sf} = \left(1 + \frac{\mathrm{i}}{\Omega_0}\frac{\mathrm{d}}{\mathrm{d}t}\right)\left[Z_1^{rv} - \frac{\mathrm{i}}{4\Omega_0''}(1 - \mathrm{e}^{\mathrm{i}\Omega_0 t})\right] \tag{6.28a}$$

or

$$Z_1^{sf} = \left(\mathrm{i}t^2 - \frac{2t}{\Omega_0}\right)\left\{\frac{\Gamma}{8\pi}\left[\ln\frac{t}{4} - 2\gamma + 1\right] - \frac{\Psi_0}{4}\right\} + \frac{\Gamma t}{8\pi\Omega_0} - \frac{\Psi_0 t^2}{4}$$
$$+ \int_0^\infty \frac{h(v)}{v^3}\left[-\frac{\mathrm{i}t^2}{2} + \left(\frac{t}{\Omega_0} - \frac{\mathrm{i}}{\Omega(v)^2}\right)\left(1 - \mathrm{e}^{\mathrm{i}\Omega(v)t}\right) + \frac{t\mathrm{e}^{\mathrm{i}\Omega(v)t}}{\Omega(v)}\right] \mathrm{d}v, \qquad (6.28b)$$

and

$$Z_1^{sf\prime} = \left(1 + \frac{\mathrm{i}}{\Omega_0}\frac{\mathrm{d}}{\mathrm{d}t}\right) Z_1^{rv\prime} = \frac{\Gamma}{4\pi}\left[\frac{t}{\Omega_0} - \frac{\mathrm{i}t^2}{2}\right]. \qquad (6.28c)$$

There is now no dependence on Ω_0''.

The small-time behaviour of the trajectory can be obtained from (6.23):

$$Z_1^{sf} = -\frac{\Gamma t}{4\pi\Omega_0}\left[\ln t/4 - 2\gamma + \tfrac{1}{2}\right] + \frac{\Psi_0 t}{2\Omega_0} + \frac{\mathrm{i}\Gamma t^2}{8\pi}\left[-\ln t - \ln\tfrac{1}{4} - 2\gamma + 1\right]$$
$$-\frac{\mathrm{i}\Psi_0 t^2}{4} - \frac{\mathrm{i}t^2}{\Omega_0}\int_0^\infty \frac{h(v)\Omega(v)^2}{v^3}\mathrm{d}v + O(t^3). \qquad (6.29)$$

The initial displacement is again to the west thanks to the $t\ln t$ term. The large time behaviour can similarly be obtained, and is given by

$$Z_1^{sf} = \frac{\mathrm{i}\Gamma t^2}{8\pi}\left[-\ln\frac{t}{4} - 2\gamma + 1\right] + \frac{\mathrm{i}\Gamma t^2}{16\pi}\left[-\ln\frac{\Gamma t}{2\pi} - \gamma + \tfrac{3}{2} + \frac{\mathrm{i}\pi}{2}\right] + O(t\ln t). \qquad (6.30)$$

This is almost the same as (6.24).

6.2.3. *Motion of the origin*

The particle initially at the origin of space moves with the flow, and its motion is described by the equations

$$\left.\frac{\partial\psi}{\partial y}\right|_{r=0} = -U \quad \text{and} \quad \left.\frac{\partial\psi}{\partial x}\right|_{r=0} = V. \qquad (6.31a,b)$$

Transforming to the Laplace variable, and working in polar coordinates, these two equations together correspond to

$$\left.\frac{\partial\overline{\psi}}{\partial r}\right|_{r=0} = \mathrm{i}\mathrm{e}^{\mathrm{i}\theta}(\overline{U} - \mathrm{i}\overline{V}) = \mathrm{i}p\mathrm{e}^{\mathrm{i}\theta}\overline{Z^*}. \qquad (6.32)$$

This expression may be expanded in powers of ϵ, which corresponds to adding the appropriate truncations of the right-hand side of (6.32) to (6.26*a*) and (6.26*b*). This leads to

$$\overline{b_1}(p + \mathrm{i}\Omega_0)\mathrm{e}^{\mathrm{i}\theta} + \overline{Z_1^*}\mathrm{e}^{\mathrm{i}\theta}\Omega_0 = \mathrm{i}p\mathrm{e}^{\mathrm{i}\theta}\overline{Z_1^*}. \qquad (6.33a)$$

and

$$\overline{b_1'}(p + \mathrm{i}\Omega_0)\mathrm{e}^{\mathrm{i}\theta} + \overline{Z_1^{*\prime}}\mathrm{e}^{\mathrm{i}\theta}\Omega_0 = \mathrm{i}p\mathrm{e}^{\mathrm{i}\theta}\overline{Z_1^{*\prime}}. \qquad (6.33b)$$

These equations are almost the same as for the relative vorticity maximum technique. The only difference is the term $\mathrm{i}(1 - \mathrm{e}^{-\mathrm{i}\Omega_0 t})/4\Omega_0''$. The large-time behaviour of this technique will thus be given by (6.24), while the small-time behaviour will be given by

$$Z_1^o = \frac{\mathrm{i}\Gamma t^2}{8\pi}\left[-\ln\frac{t}{4} - 2\gamma + 1\right] - \frac{\mathrm{i}\Psi_0 t^2}{4} - \frac{t^3}{3}\int_0^\infty \frac{h(v)\Omega(v)^2}{v^3}\mathrm{d}v + O(t^4). \qquad (6.34)$$

With this technique, the vortex initially moves to the north, which is different from the previous methods. It also moves more slowly initially.

6.2.4. *Pseudo-secularity condition*

The three conditions derived above make crucial use of the fact that the extremum of a mode-one quantity has a natural expression in terms of the derivative of that quantity near the origin. This was used to cancel off the mode-one component $Z_1^* r\Omega$ which corresponds, to first order, to a change of frame. In fact, higher-mode terms in the inner solution ψ_1 do not play any part in the matching.

The effect of a change in origin on a function of space, however, suggests another way of picking the centre of the frame, namely to choose Z such that the streamfunction has no $r\Omega$ term.† This is a well-defined criterion, and corresponds to picking the reference frame in which the dynamical contribution to the streamfunction does not contain any contribution from the special change-of-frame solution.

This condition is easy to apply in the Laplace variable; it leads to

$$\overline{Z_1^*} + \mathrm{i}\overline{b_1} = 0 \tag{6.35}$$

and

$$\overline{Z_1^{*\prime}} + \mathrm{i}\overline{b_1'} = 0. \tag{6.36}$$

Thus

$$\overline{Z_1^*} = -\frac{\mathrm{i}}{p^3}\left\{\frac{\Gamma}{4\pi}\left[\ln p - \ln\tfrac{1}{4} - \gamma - \tfrac{1}{2}\right] - \frac{\Psi_0}{2} + \int_0^\infty \frac{h(v)\Omega(v)}{v^3(p+\mathrm{i}\Omega(v))^2}(\Omega(v) - 2\mathrm{i}p)\,\mathrm{d}v\right\}, \tag{6.37}$$

which becomes

$$Z_1^s = \frac{\mathrm{i}\Gamma t^2}{8\pi}\left[-\ln\frac{t}{4} - 2\gamma + 1\right] - \frac{\mathrm{i}\Psi_0 t^2}{4} + \mathrm{i}\int_0^\infty \frac{h(v)}{v^3}\left[-\frac{t^2}{2} + \frac{\mathrm{i}t\mathrm{e}^{\mathrm{i}\Omega(v)t}}{\Omega(v)} + \frac{\mathrm{e}^{\mathrm{i}\Omega(v)t} - 1}{\Omega(v)^2}\right]\mathrm{d}v. \tag{6.38}$$

The expression for the logarithmic displacement is also very simple and takes the form

$$\overline{Z_1^{*\prime}} = \frac{\mathrm{i}\Gamma}{4\pi p^3}, \tag{6.39}$$

that is

$$Z_1^{s\prime} = -\frac{\mathrm{i}\Gamma t^2}{8\pi}. \tag{6.40}$$

Interestingly enough, this recovers the result of the displacement of the origin. In a sense, this is because this technique is concerned with minimizing the first-order difference between frames, and the Lagrangian technique of tracking the origin corresponds very naturally to picking a frame. However, this technique will behave very differently for higher orders. It may be called a ‘pseudo-secularity’ condition, since it does not really remove the ultimate breakdown in the asymptotic expansion. Nevertheless, it fully removes all extraneous order-one contributions that may be ascribed merely to a change of frame. In effect, it corresponds to the transformation that most fully keeps the vortex symmetric. The overall physical effect is not just a change of frame though, since such a transformation would have higher-order terms too. For higher-order terms, this technique may also be used, although it cannot remove asymmetries in modes other than mode one.

† I am grateful to Peter Haynes for suggesting this approach.

The behaviour of the trajectory is already known from previous results. It is clear that while this technique removes unwanted changes of frame, the predicted trajectory will eventually reverse to the south. Hence the expansion cannot remain valid longer than with any other technique.

7. Examples

7.1. *Rankine vortex*

The Rankine profile is a distribution with vorticity $2\Omega_0$ within a radius d say, and zero vorticity outside. Then

$$\Omega(r) = \Omega_0 \tag{7.1}$$

inside the disc, and

$$\Omega(r) = \frac{\Omega_0 d^2}{r^2} \tag{7.2}$$

outside it. The corresponding streamfunction is

$$\Psi(r) = \tfrac{1}{2}\Omega_0(r^2 - d^2) + \Omega_0 d^2 \ln d \tag{7.3}$$

inside the disc, and

$$\Psi(r) = \Omega_0 d^2 \ln r \tag{7.4}$$

outside it. This corresponds to a circulation of $2\pi\Omega_0 d^2$, and a value of the streamfunction of $\Psi_0 = \Omega_0 d^2 \left(\ln d - \frac{1}{2}\right)$ at the origin. For the numerical calculations of this section, the values $\Omega_0 = \frac{1}{2}$ and $d = 1$ have been chosen, leading to $\Gamma = \pi$.

The integral in (6.21) can be evaluated analytically. The position of the centre of the vortex is given by

$$\begin{aligned} Z_1^{rv} = {} & \frac{\mathrm{i}\Omega_0 t^2}{4}\left[-\ln t - \ln \tfrac{1}{4} - 2\gamma + 1\right] - \frac{\mathrm{i}\Omega_0 d^2 t^2(\ln d - 12)}{4} \\ & + \frac{\mathrm{i}\Omega_0 d^2 t^2}{8}\left[-\ln \Omega_0 t - \gamma + \frac{\mathrm{i}\pi}{2} + \tfrac{1}{2} - E_1(\mathrm{i}\Omega_0 t)\right] + \frac{d^2 t}{8} + \frac{\mathrm{i}d^2}{8\Omega_0}\left(\mathrm{e}^{\mathrm{i}\Omega_0 t} - 1\right). \end{aligned} \tag{7.5}$$

However, the large-time asymptotic behaviour of the vortex is not given by (6.24). This is due to the assumption about the strict monotonicity of Ω used in Appendix C to derive the large-time behaviour, which is not appropriate in the case of the Rankine vortex. The oscillatory behaviour of the trajectory comes from the exponential integral term, which leads to a leading-order oscillatory term $t\mathrm{e}^{\mathrm{i}\Omega_0 t}/8$ for large times. These oscillations are hence of lower order than the actual displacement of the vortex, which behaves like t^2.

The trajectory of the streamfunction maximum may also be calculated. This results in

$$\begin{aligned} Z_1^{sf} = {} & \left(t^2 + \frac{2\mathrm{i}t}{\Omega_0}\right)\left\{\frac{\mathrm{i}\Omega_0 d^2}{4}\left[-\ln\frac{t}{4} - 2\gamma + 1\right] - \frac{\mathrm{i}\Omega_0 d^2(\ln d - \frac{1}{2})}{4}\right\} \\ & + \left(t^2 + \frac{2\mathrm{i}t}{\Omega_0}\right)\frac{\mathrm{i}\Omega_0 d^2}{8}\left[-\ln \Omega_0 t - \gamma + \frac{\mathrm{i}\pi}{2} - E_1(-\mathrm{i}\Omega_0 t) + \tfrac{1}{2}\right] \\ & - \frac{d^2 t}{8}\left(\mathrm{e}^{\mathrm{i}\Omega_0 t} - 4\right) \end{aligned} \tag{7.6}$$

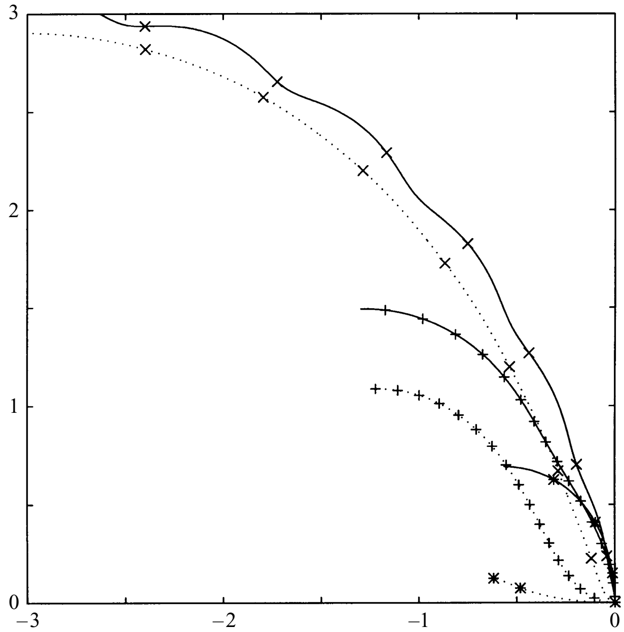

FIGURE 2. Trajectory of a Rankine vortex stopping at the maximum y-position. Full lines correspond to the position of the origin (v), dotted lines to the streamfunction maximum (s). For each technique, there is a lower line, corresponding to $\epsilon = 0.5$, with stars at unit time intervals and a maximum time of 3.75 for v and 2 for s, a middle line with $\epsilon = 0.05$, crosses at unit time intervals and maximum time 16.625 for v and 16 for s, and an upper line, with $\epsilon = 0.005$, crosses every 10 time units and maximum time 78.75 for v and 79.5 for s.

and

$$Z_1^{sf\prime} = \frac{d^2 t}{2} - \frac{\mathrm{i}\Omega_0 d^2 t^2}{4}. \tag{7.7}$$

Figure 2 shows the trajectory of the vortex, identified by the two different techniques, until the time at which the trajectory starts to curve back down to the south. The motion of the centre of the vortex is oscillatory, but for the streamfunction maximum, the oscillations are suppressed for this trajectory, since the $-d^2 t e^{\mathrm{i}\Omega_0 t}/8$ term cancels with the leading-order contribution from the exponential integral. There are oscillations, but they are at a higher order, and hence cannot be seen on the plot. The difference between the two methods is most marked for large ϵ.

The solution is expected to break down for large time (see the discussion in §8) and the time at which the motion reversed towards the south is a natural choice for truncating the trajectory (in fact, if the solution is not truncated at some time, the displacement to the south becomes enormous and the initial behaviour is completely hidden). In addition, previous numerical and laboratory experiments have shown motion to the northwest, while the large-time displacement Z_1 grows without bound. The time of validity of the expansion may be evaluated from the maximum y-position of the trajectory, for the particle-tracking and for the streamfunction-maximum techniques. Figure 3 shows logarithmic plots of time against ϵ for both methods. A straight line fit to this curve, using values of ϵ between 10^{-6} and 0.1 gives a slope of −0.6642 for the particle technique, and −0.7092 for the streamfunction method. This suggests that the expansion breaks down for times of order $\epsilon^{-2/3}$. These values become −0.6064 and −0.6956, respectively, when ϵ is restricted to the

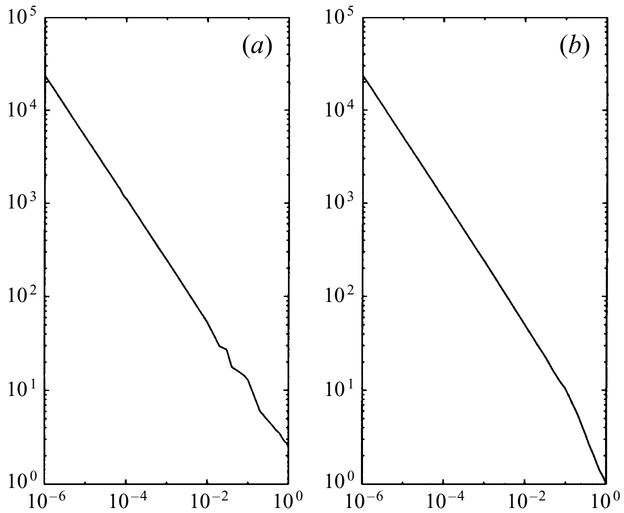

FIGURE 3. Log-log plot of approximate time of breakdown of expansion versus ϵ for the Rankine vortex; (*a*) corresponds to the motion-of-the-origin technique, (*b*) to the streamfunction maximum. The slope of the line gives the power of ϵ for breakdown.

range 0.005 to 0.1, or slightly under two decades. This dependence on $\epsilon^{-2/3}$ may be understood as follows. The maximum value of y must occur for large time (or else the expansion is useless), and hence the dominant contribution to the time-derivative of the imaginary part of $Z_1^{sf} + \ln \epsilon Z_1^{sf\,\prime}$ is

$$-\frac{\Gamma t \ln t}{4\pi} - \frac{\Gamma t \ln t}{8\pi} - \frac{\Gamma t \ln \epsilon}{4\pi}. \tag{7.8}$$

This vanishes asymptotically for $t = O(\epsilon^{-N})$ when $N + N/2 - 1 = 0$, i.e. for $t = O(\epsilon^{-2/3})$. The same argument holds for the streamfunction method, since the two behave identically for large time.

7.2. *Gaussian vortex*

The vorticity profile for the Gaussian vortex will be taken to be

$$Q(r) = \mathrm{e}^{-r^2}. \tag{7.9}$$

This corresponds to angular velocity

$$\Omega(r) = \frac{1 - \mathrm{e}^{-r^2}}{2r^2} \tag{7.10}$$

and streamfunction

$$\Psi(r) = \tfrac{1}{2} \ln r + \tfrac{1}{4} E_1(r^2). \tag{7.11}$$

The values of the streamfunction and angular velocity at the origin are given by $\Psi_0 = -\gamma/4$ and $\Omega_0 = \frac{1}{2}$. The circulation of the vortex is equal to π. The integrals in the preceding section cannot be evaluated in closed form, but may be calculated numerically. This was done using the NAG routine D01AMF.

Figure 4 shows the trajectory of the vortex, using the path of the origin and the streamfunction-maximum techniques. Again, the trajectory is truncated when it turns south. The path of the maximum in relative vorticity could be plotted, but it is almost identical to figure 4 except for large ϵ, where high resolution is needed for small time.

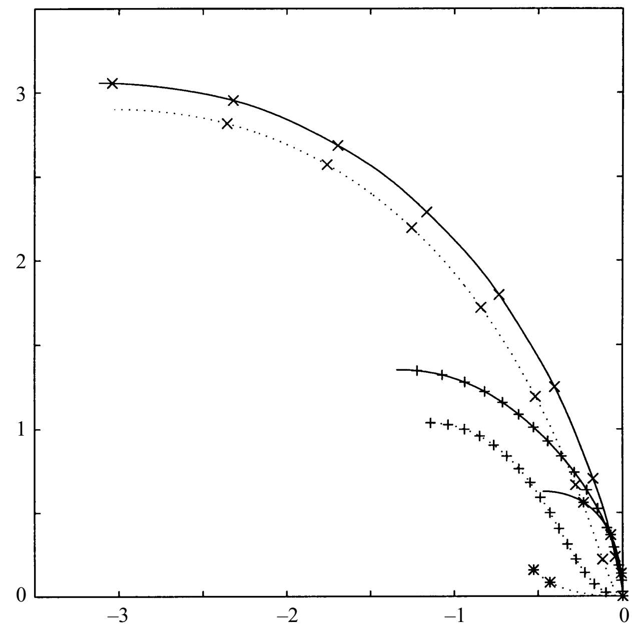

FIGURE 4. Trajectory of a Gaussian vortex stopping at the maximum y-position. Full lines correspond to the particle at the origin (v), dotted lines to the streamfunction maximum (s). For each technique, there is a lower line, corresponding to $\epsilon = 0.5$, with stars at unit time intervals and a maximum time of 3.875 for v and 2.25 for s, a middle line with $\epsilon = 0.05$, crosses at unit time intervals and maximum time 17.75 for v and 16.125 for s, and an upper line, with $\epsilon = 0.005$, crosses every 10 time units and maximum time 81 for v and 79.75 for s.

Again, the time of validity of the expansion may be estimated by considering the y-position reached. Figure 5 shows the maximum time values as functions of ϵ. The slopes of the lines are now −0.688 and −0.679. The same argument as in the preceding section explains the $-\frac{2}{3}$ slope for both methods.

8. Conclusions

The initial-value problem for the evolution of a circular vortex on the beta-plane has been solved to first order in an expansion in ϵ, the non-dimensional beta-effect. To zeroth order, the near-field response is just the initial condition: an intense vortex is unaffected by the beta-induced perturbation. The far field zeroth-order response is the Green's function of the linear Rossby wave equation, with amplitude equal to the circulation of the initial vortex. This response only exists therefore for non-isolated vortices.

The first-order response in the near field is a time-dependent dipole, which corresponds to the beta-gyres (Sutyrin & Flierl 1994). There is an $O(\epsilon \ln \epsilon)$ response proportional to the circulation; this means that the trajectory is different for different values of ϵ. The first-order far-field response is identically zero. The solution (5.36a) reduces to that of RD94 in the case of zero circulation, in which case the logarithmic response vanishes.

The equation of motion may be solved in a reference frame centred in the vortex, and the steady mode-one solution $r\Omega(r)$ to the radial Rayleigh equation may be generalized by convolution to enable the problem to be solved in the new frame. The

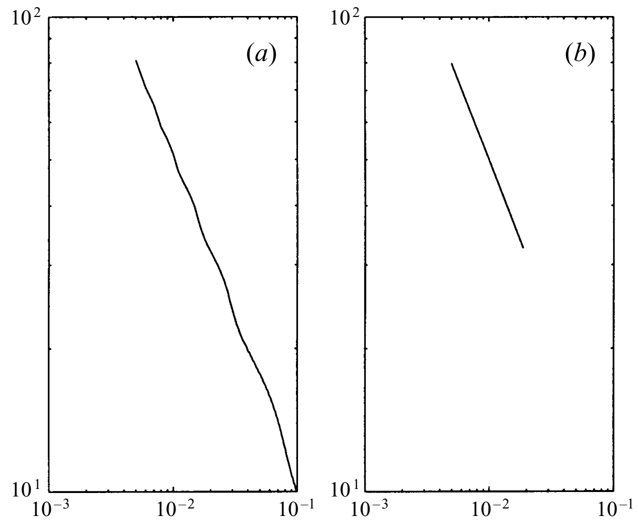

FIGURE 5. Log-log plot of approximate time of breakdown of expansion versus ϵ for the Gaussian vortex; (*a*) corresponds to the motion-of-the-origin technique, (*b*) to the streamfunction maximum. The slope of the line gives the power of ϵ for breakdown.

fundamental issue then becomes locating the centre of the vortex. Four possibilities are set out (the first three following RD94): the maximum in relative vorticity, the maximum in streamfunction, the position of the particle initially at the origin of space, and a pseudo-secularity condition. The first of these techniques fails when the basic-state vorticity is constant at the origin, as in the case of the Rankine vortex. The second is not Galilean invariant. The third turns out to be almost identical with the first. This is to be expected, since the difference between the relative vorticity of the point initially at the origin and its relative vorticity at a later is essentially ϵy, y being the meridional displacement, which is an asymptotically small quantity.

The final technique is probably the most interesting. Mathematically, it leads to the same result as the Lagrangian origin-following approach to first order. However, it is obtained by removing all terms of the form $r\Omega$, or convolutions of the trajectory with the steady mode-one solution.

The large-time behaviour of all these methods gives motion to the south, which disagrees with experiments for non-isolated vortices (cf. Carnevale *et al.* 1991). In addition, the time at which the paths curve back down to the south is $O(\epsilon^{-2/3})$, which suggests that naive estimation of the breakdown of the expansion as occurring at $O(\epsilon^{-1})$ is simplistic. This may be explained in a general way by considering a rescaling of time and space given by $r = \epsilon^{\beta} r_*$ and $t = t^{\alpha} t_*$. The governing equation of motion is then

$$\frac{\partial}{\partial t_*}\nabla_*^2\psi + \epsilon^{\alpha-2\beta} J_*(\psi, \nabla_*^2\psi) + \epsilon^{1+\alpha+\beta}\frac{\partial\psi}{\partial x_*} = 0. \tag{8.1}$$

The streamfunction scales like $\epsilon^{-\alpha+2\beta}$ implicity. The dynamical evolution of the system must be along the distinguished scalings $\alpha = 2\beta$, which corresponds to vorticity advection, and $1+\alpha+\beta = 0$, where the rate-of-change and beta terms balance. In the former case, the expansion parameter is $\epsilon^{1+3\alpha/2}$, while in the latter, it is $\epsilon^{3\alpha+2}$. When $\alpha = -\frac{2}{3}$, the expansion becomes disordered, and all terms are formally of the same size.

The profiles used here correspond to localized vortices, for which the far-field angular velocity and vorticity have simple behaviour. The question of the relevance of non-isolated vortices is worth addressing at this point. Numerical experiments, such as two-dimensional turbulence, and rotating-tank experiments tend to create vortices which appear to have non-zero circulation, although of course boundary effects guarantee zero overall circulation. The infinite beta-plane approximation is just an idealization in any case to try and understand the properties of these structures without having to consider boundary effects.

In addition, as mentioned by RD94, isolated vortices will also generate a non-zero response in the far field at second order (if they have non-zero RAM – cf. Llewellyn Smith 1996). Hence it is important to understand the coupling processes between the near and far fields. Using non-isolated vortices leads to results at $O(1)$ which are analytically simpler than the $O(\epsilon^2)$ analysis required in the isolated case.

The question of how well these trajectory predictions correspond to experiment and observation has not been addressed in this work. However, it is interesting to examine the present results in the light of the work of Smith, Weber & Krause (1995), who examined the evolution of the symmetric circulation of a hurricane using numerical simulation and the theory of Smith & Weber (1993). In the present theory, the symmetric circulation vanishes to $O(\epsilon^2)$ for isolated vortices which corresponds to the case of Smith *et al.* (1995), and the spontaneous appearance of circulation which is a problem with the theory of Smith & Weber (1993) does not occur. For non-isolated vortices, the $O(1)$ symmetric circulation is unbounded for large R as a consequence of the form of the far-field impulsive response, and hence it is hard to assign meaning to its evolution.

The current state of the theory of this paper is not yet at a stage to compare with observations, which are in any case difficult to make. Previous rotating-tank experiments give qualitative agreement, but detailed measurements are difficult, and the experiments of course take place in bounded domains. Numerical calculations would also seem to support the theory's basic predictions, but exact comparisons are difficult, due to the limitations of the calculations. In particular, the vast majority of previous numerical simulations have enforced zero circulation by subtracting out a constant value of background vorticity in doubly periodic domains. The resulting initial condition has zero circulation but the relative vorticity does not decay towards the boundaries. Consequently, any comparison must carefully account for the fact that the numerical simulation effectively has zero circulation. Alternatively numerical techniques capable of simulating flows on unbounded domains or of incorporating flows with circulation in doubly periodic domains (and with the shear layers that would tend to develop near the boundaries as a result) need to be developed. At present, finite-difference simulations using radiation conditions show sensitivity to resolution and initial conditions (G. F. Carnevale 1996, private communication). One aim of the current work is to try and develop ways of validating such numerical codes in the future, since the problem considered is one of the simplest possible which combines vorticity conservation and wave radiation.

This research was supported by NERC award GT4/93/125/P. Conversations with Professor D. G. Crighton, Dr G. F. Carnevale and Dr M. T. Montgomery were useful. The author is grateful to the last of these and to Dr P. H. Haynes for pointing out the paper by Smith & Rosenbluth. The comments of an anonymous referee led to significant improvements in the manuscript.

Appendix A. Representations of L and values on the x-axis

The expression for $\bar{L}$ may be written in the two following ways:

$$\bar{L} = -\frac{1}{2\pi}\left(\frac{1}{p^{1/2}}\exp\left(\frac{\epsilon r}{2p}\right)K_0\left(\frac{\epsilon r}{2p}\right)\right)\left(\frac{1}{p^{1/2}}\exp\left(-\frac{\epsilon rc^2}{p}\right)\right) \tag{A 1}$$

$$= -\frac{1}{2\pi}\left\{\left[\exp\left(-\frac{\epsilon rc^2}{p}\right)-1\right]+1\right\}\frac{1}{p}\exp\left(\frac{\epsilon r}{2p}\right)K_0\left(\frac{\epsilon r}{2p}\right). \tag{A 2}$$

Use of the convolution theorem then gives

$$L = -\frac{2}{\pi^2}\int_0^{\pi/2} K_0\left(2[\epsilon rt]^{1/2}\sin\phi\right)\cos\left(2c[\epsilon rt]^{1/2}\cos\phi\right)\mathrm{d}\phi \tag{A 3}$$

and

$$L = \frac{2c}{\pi}(\epsilon rt)^{1/2}\int_0^{\pi/2} J_1\left(2c[\epsilon rt]^{1/2}\cos\phi\right)\ I_0\left([\epsilon rt]^{1/2}\cos\phi\right)K_0\left([\epsilon rt]^{1/2}\cos\phi\right)\sin\phi\,\mathrm{d}\phi \\ -\frac{1}{\pi}I_0\left([\epsilon rt]^{1/2}\right)K_0\left([\epsilon rt]^{1/2}\right) \tag{A 4}$$

respectively, where $c = \cos\theta/2$ is positive. These expressions are potentially computationally more efficient than (3.6) since they are integrals over a finite range.

On the positive x-axis,

$$L = \tfrac{1}{2}J_0\left([\epsilon rt]^{1/2}\right)Y_0\left([\epsilon rt]^{1/2}\right), \tag{A 5}$$

while on the negative x-axis,

$$L = -\frac{1}{\pi}I_0\left([\epsilon rt]^{1/2}\right)K_0\left([\epsilon rt]^{1/2}\right). \tag{A 6}$$

Using (A 3) leads to the same result on the positive x-axis, while the expression on the negative x-axis may be simplified to give

$$\int_0^{\pi/2} K_0(2z\sin\theta)\,\mathrm{d}\theta = \frac{\pi}{2}I_0(z)K_0(z). \tag{A 7}$$

This is 6.681.4 of Gradshteyn & Ryzhik (1980). Proceeding analogously with (A 4) leads to two definite integrals:

$$\int_0^{\pi/2} K_0(2z\sin\theta)\cos(2z\cos\theta)\,\mathrm{d}\theta = -\frac{\pi^2}{4}J_0(z)Y_0(z) \tag{A 8}$$

and

$$\int_0^{\pi/2} J_1(2z\cos\theta)I_0(z\sin\theta)K_0(\sin\theta)\sin\theta\,\mathrm{d}\theta = \frac{\pi}{2z}\left(\tfrac{1}{2}J_0(z)Y_0(z)+\frac{1}{\pi}I_0(z)K_0(z)\right). \tag{A 9}$$

These two integrals do not seem to appear in Gradshteyn & Ryzhik (1980) or in Luke (1962).

Appendix B. Small-ϵ behaviour of the integral in (5.13)

The integral in (5.13) is

$$I = \int_0^{R/\epsilon}\frac{h(v)}{v^3(p+\mathrm{i}\Omega(v))^2}\,\mathrm{d}v. \tag{B 1}$$

It clearly behaves logarithmically for small ϵ, but it is the order-zero and order-one terms that are of interest. The singular part of the integral may be subtracted off by writing

$$I = \int_0^{R/\epsilon} \frac{h(v)}{v^3}\left[\frac{1}{(p+\mathrm{i}\Omega(v))^2} - \frac{1}{p^2}\right] \mathrm{d}v + \int_0^{R/\epsilon} \frac{h(v)}{p^2 v^3}\,\mathrm{d}v. \tag{B 2}$$

The first integral in (B 2) exists for $\epsilon = 0$ and may be expressed as a Taylor series in ϵ. The far-field behaviour of the integrand is given by

$$-\frac{\mathrm{i}\Gamma^2\epsilon^3}{4\pi p^3 R^3} + O\left(\frac{\epsilon^5}{R^5}\right) \tag{B 3}$$

for large R/ϵ. Using the chain rule,

$$\frac{\mathrm{d}I_1}{\mathrm{d}\epsilon} = -\frac{R}{\epsilon^2}\left[-\frac{\mathrm{i}\Gamma^2\epsilon^3}{4\pi p^3 R^3} + O\left(\frac{\epsilon^5}{R^5}\right)\right]. \tag{B 4}$$

This leads to

$$I_1 = \frac{1}{p^2}\int_0^\infty \frac{h(v)\Omega(v)}{v^3(p+\mathrm{i}\Omega(v))^2}(\Omega(v) - 2\mathrm{i}p)\,\mathrm{d}v + O\left(\frac{\epsilon^2}{R^2}\right). \tag{B 5}$$

The second integral in (B 2) may be rewritten as

$$I_2 = \frac{1}{p^2}\int_0^{R/\epsilon}\int_0^v \frac{u^3\Omega(u)}{v^3}\,\mathrm{d}u\,\mathrm{d}v, \tag{B 6}$$

which may be transformed into

$$I_2 = \frac{\Psi(R/\epsilon)}{2p^2} - \frac{\Psi_0}{2p^2} - \frac{\epsilon^2 h(R/\epsilon)}{2p^2R^2}. \tag{B 7}$$

The behaviour of this expression for small ϵ is

$$I_2 = \frac{\Gamma}{4\pi p^2}\ln\frac{R}{\epsilon} - \frac{\Psi_0}{2p^2} - \frac{\Gamma}{8\pi p^2} + O\left(\frac{\epsilon^2}{R^2}\right). \tag{B 8}$$

The first term is exactly the logarithmic term required by the matching, while the second and third are constants which may not be discarded. Their appearance is due to the boundary condition at infinity which ensures that no constant term appears in the expansion of Ψ for small ϵ.

Combining the two above expressions gives

$$I = \frac{\Gamma}{4\pi p^2}\ln\frac{R}{\epsilon} + \frac{1}{p^2}\int_0^\infty \frac{h(v)\Omega(v)}{v^3(p+\mathrm{i}\Omega(v))^2}(\Omega(v) - 2\mathrm{i}p)\,\mathrm{d}v - \frac{\Psi_0}{2p^2} - \frac{\Gamma}{8\pi p^2} + O(\epsilon^2). \tag{B 9}$$

Appendix C. Large-time behaviour for the vortex motion

The integral

$$I = \int_0^\infty \frac{h(v)}{v^3}\left[-\frac{t^2}{2} - \frac{\mathrm{i}t\mathrm{e}^{\mathrm{i}\Omega(v)t}}{\Omega(v)} + \frac{\mathrm{e}^{\mathrm{i}\Omega(v)t} - 1}{\Omega(v)^2}\right]\mathrm{d}v \tag{C 1}$$

governs the behaviour of the motion of the vortex. Differentiating in time gives

$$\frac{\mathrm{d}I}{\mathrm{d}t} = tJ(t) = t\int_0^\infty \frac{h(v)}{v^3}\left(\mathrm{e}^{\mathrm{i}\Omega(v)t} - 1\right)\,\mathrm{d}v. \tag{C 2}$$

The change of variable $u = \Omega(v)$ (a single-valued transformation since the angular velocity is monotonic) leads to

$$J(t) = \int_0^{\Omega_0} g(u)\left(\mathrm{e}^{\mathrm{i}ut} - 1\right)\,\mathrm{d}u, \tag{C 3}$$

where

$$g(u) = -\frac{h(v)}{v^3\Omega'(v)}. \tag{C 4}$$

The behaviour of g at the points 0 and Ω_0 can be deduced from the behaviour of $\Omega(v)$ and $h(v)$ near $v = 0$ and $v \to \infty$, and is given by

$$g(u) = \frac{\Gamma}{8\pi u} + O(1), \quad g(u) = -\frac{\Omega_0}{4\Omega_0''} + O\left((\Omega_0 - u)^{1/2}\right), \tag{C 5a,b}$$

for u near 0 and Ω_0 respectively.

Applying the method of stationary phase to J shows that the dominant contribution to the integral comes from the origin. This suggests subtracting off the singular behaviour of g from the integrand, and hence rewriting J as

$$J = \int_0^{\Omega_0}\left[g(u) - \frac{\Gamma}{8\pi u}\right]\left(\mathrm{e}^{\mathrm{i}ut} - 1\right)\,\mathrm{d}u + \frac{\Gamma}{8\pi}\int_0^{\Omega_0}\frac{\mathrm{e}^{\mathrm{i}ut} - 1}{u}\,\mathrm{d}u \tag{C 6}$$

$$= J_1(t) + \frac{\Gamma}{8\pi}\left[-\ln\Omega_0 t - \gamma + \frac{\mathrm{i}\pi}{2} - E_1(-\mathrm{i}\Omega_0 t)\right]. \tag{C 7}$$

The behaviour of the derivative of J_1 can be obtained by integration by parts, thanks to the earlier change of variable. The result

$$\frac{\mathrm{d}J_1}{\mathrm{d}t} = \mathrm{i}\int_0^{\Omega_0} u\left[g(u) - \frac{\Gamma}{8\pi u}\right]\mathrm{e}^{\mathrm{i}ut}\,\mathrm{d}u \tag{C 8}$$

leads to

$$\frac{\mathrm{d}J_1}{\mathrm{d}t} = \left[\Omega_0 g(\Omega_0) - \frac{\Gamma}{8\pi}\right]\frac{\mathrm{e}^{\mathrm{i}\Omega_0 t}}{t} + O\left(\frac{1}{t^2}\right). \tag{C 9}$$

If J_1 is rewritten in the form

$$J_1 = \int_0^{\Omega_0}\left[g(u) - \frac{\Gamma}{8\pi u}\right]\mathrm{e}^{\mathrm{i}ut}\,\mathrm{d}u - \int_0^{\Omega_0}\left[g(u) - \frac{\Gamma}{8\pi u}\right]\,\mathrm{d}u, \tag{C 10}$$

the Riemann–Lebesgue lemma shows that the first integral must be $o(1)$ for large t. The actual behaviour of J_1 is known up to a constant term (by integrating (C 9)), and hence this term must come entirely from the second integral in (C 10). This integral may be rewritten in the original variable as

$$-\lim_{R\to\infty}\int_0^R\left[\frac{h(v)}{v^3} + \frac{\Gamma}{8\pi}(\ln\Omega(v))'\right]\,\mathrm{d}v; \tag{C 11}$$

this is equal to

$$
\begin{aligned}
-\lim_{R\to\infty} & \left\{ \frac{\Psi(R)}{2} - \frac{\Psi_0}{2} - \frac{h(R)}{2R^2} + \frac{\Gamma}{8\pi}\ln\frac{\Omega(R)}{\Omega_0} \right\} \\
&= -\lim_{R\to\infty} \left\{ \frac{\Gamma}{4\pi}\ln R - \frac{\Psi_0}{2} - \frac{\Gamma}{8\pi} + O\left(\frac{1}{R^2}\right) + \frac{\Gamma}{8\pi}\ln\left[\frac{\Gamma}{2\pi\Omega_0 R^2} + O\left(\frac{1}{R^\infty}\right)\right]\right\} \\
&= \frac{\Psi_0}{2} + \frac{\Gamma}{8\pi} - \frac{\Gamma}{8\pi}\ln\frac{\Gamma}{2\pi\Omega_0} \qquad \text{(C 12)}
\end{aligned}
$$

(see Appendix B).

Combining the above results leads to

$$
\begin{aligned}
\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\Gamma t}{8\pi}\left[-\ln\frac{\Gamma t}{2\pi} - \gamma + 1 + \frac{\mathrm{i}\pi}{2} - E_1(-\mathrm{i}\Omega_0 t)\right] + \frac{\Psi_0 t}{2} \\
+\mathrm{i}\left[\frac{\Gamma}{8\pi} - \Omega_0 g(\Omega_0)\right]\mathrm{e}^{\mathrm{i}\Omega_0 t} + O\left(\frac{1}{t}\right). \qquad \text{(C 13)}
\end{aligned}
$$

Integrating once and using appropriate order relations gives

$$
I = \frac{\Gamma t^2}{16\pi}\left[-\ln\frac{\Gamma t}{2\pi} - \gamma + \frac{3}{2} + \frac{\mathrm{i}\pi}{2}\right] + \frac{\Psi_0 t^2}{4} + O(t). \qquad \text{(C 14)}
$$

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