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THE MOTIONS OF THE SATELLITES OF MARS

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SUMMARY

In the first part of the paper, analytical expressions for the secular and periodic variations in the orbital elements of the satellites of Mars due to the combined action of the oblateness of Mars and the attraction of the Sun are developed. The periodic variations are kept as small as possible by referring the elements to the appropriate Laplacian planes. In the second part of the paper, improved values of the arbitrary parameters of the theory are determined by an analysis of all available observations of the positions of the satellites during the period 1877–1969. It is concluded that the observational data are not sufficiently accurate to determine the secular accelerations, if any, of the mean longitudes of the satellites.

1. INTRODUCTION

The aim of this investigation is to develop a new theory of the orbital motions of Phobos and Deimos, the satellites of Mars, and to make a new analysis of all available observations of their positions in order to provide:

(a) an improved basis for ephemerides of the satellites for use in, for example, future space-probe missions to Mars; and
(b) more reliable data about Mars and about the controversial secular accelerations of the satellites.

The development of the theory is given in Part I of this paper (Sections 2–8), while the method and results of the analysis are given in Part II (Sections 9–15).

Earlier work on the orbits of these satellites has been reviewed by Wilkins (1967, 1968, 1969, 1970), who made a preliminary analysis of the observational data for the period 1877–1929 to re-determine the orbital elements and who then used later data in an attempt to determine the secular acceleration of Phobos. He concluded that the secular acceleration was much smaller than the value that had been deduced by Sharpless (1945). The character of the residuals that he obtained suggested, however, that neglected periodic perturbations might be significant. Further, he made independent solutions for each satellite even though it was realized that this introduced more unknowns than was necessary, and it did not permit the direct use of relative measures of the satellites. The elements obtained did not, in fact, satisfy the consistency relations deduced by Woolard (1944). The theory developed here does not suffer from these disadvantages.

Phobos and Deimos move in close, nearly circular orbits that are inclined at small angles to the equator of Mars. The principal perturbations are due to the oblateness of Mars, but it is necessary to take into account the smaller perturbations due to the Sun. The masses of the satellites are believed to be very small, and so their mutual perturbations have been ignored. Non-gravitational forces are not considered except in so far as the expressions for the mean longitudes may contain arbitrary secular terms. Perturbations from other planets are considered to be negligible.

The sizes of the largest periodic perturbations are estimated in Section 3. It is apparent that these terms are likely to be very small, but perhaps just detectable in the geocentric observations. Hence these terms were calculated to remove a source of possible error. In fact they were subsequently found to be somewhat smaller than the estimates, and their inclusion in the theory of the motion of the satellites had an insignificant effect on the mean residuals and standard errors of the solution.

At first, we used the equator of Mars as the reference plane so that we could use Brouwer's Earth-satellite solution (1959) for computing the perturbations due to the oblateness of Mars. We then found, however, large periodic perturbations due to the Sun in the inclinations and longitudes of the nodes of the satellites' orbits. (The amplitudes are about $0^\circ\cdot 5$ in inclination and 30° in longitude of the node, and the periods are about 54 years, for Deimos; these perturbations are considerably smaller for Phobos since it is much closer to Mars.) We therefore found that it was desirable to follow the more usual procedure and refer the orbital elements to the so-called Laplacian planes (Tisserand 1896), since by so doing large periodic perturbations are avoided—the orbital planes precess almost uniformly on the Laplacian planes since the positions of the latter are chosen so that the periodic variations due to the oblateness and the Sun just cancel each other. The precession of Mars itself under the action of the Sun causes the Laplacian planes to precess; the corresponding additional perturbations have been computed. It was found that it is sufficiently accurate to treat the orbit of the Sun with respect to Mars as an ellipse.

The development of the theory has been carried out in quite a general fashion except in so far as we have normally limited the expansion of the disturbing function to terms that are larger than about 0·0001, since this is of ample accuracy. The equations for the variations of the elements were derived from the expansion of the disturbing function and then integrated to first order to give analytical series for the perturbations of the elements.

The resulting expressions then formed the basis of a computer program for the evaluation of the apparent positions of the satellites for any given time of observation from initial values of the orbital elements and of the parameters defining the gravitational potential of Mars. These parameters depended on the orientation of the equator of Mars with respect to the plane of the mean orbit of Mars, the mass of Mars relative to the Sun, and the coefficients J_2, J_3, J_4 in the usual notation in the expression for the potential. (Other terms in the potential are ignored.)

The program also includes the evaluation of the derivatives of the positions with respect to the arbitrary parameters and the least-squares solution for the corrections to any chosen set of parameters from the observational data. The program was used in an iterative fashion for various selections of data in order that the reliability of the results could be established.

Part I—Construction of Theory of Motion of Satellites

2. Formulation of the Problem

The theory of the motion of each satellite is to be developed with respect to an arbitrary reference plane that passes through the intersection of the planes of the equator and orbit of Mars. Later, as explained in the introduction, we shall choose this plane in such a way as to minimize the periodic perturbations in the orbital elements. We refer to the reference plane as the Laplacian plane. The relative positions of the various planes on the celestial sphere at some time t are shown in Fig. 1. The quantities shown there, and some other quantities which will subsequently be required, are defined as follows:

h is the longitude of the node of the satellite orbit on the Laplacian plane, from an arbitrary origin;
g is the argument of the pericentre of the satellite orbit;
f is the true anomaly of the satellite;
I is the inclination of the satellite orbit to the Laplacian plane;
a is the semi-major axis of the satellite orbit;
e is the eccentricity of the satellite orbit;
l is the mean anomaly of the satellite;
n is the mean motion of the satellite about Mars.

h', g', f', I', a', e', l' and n' are similar quantities for the orbit of the Sun about Mars.

x is the angle subtended at Mars between the satellite and the Sun;
y is the latitude of the satellite above the equator of Mars;
g^* is the argument of the pericentre of the satellite orbit measured from the ascending node of the satellite orbit on the equator;
i is the inclination of the Laplacian plane to the equatorial plane of Mars.

α, β, γ, δ, η and η' are auxiliary angles which will be used only in the derivation of the expansion of the disturbing function. (After the expansion has been obtained some of these symbols will be used to represent other quantities.) We see that

$$\gamma = g-\beta; \quad \delta = g'-\alpha.$$

We shall also use the following quantities:

$\lambda = l+g+h$, the mean longitude of the satellite;
$\tilde{\omega} = g+h$, the longitude of the pericentre of the satellite;
$\theta = \sin I$;
$Q = i+I'$, the inclination of the equator of Mars to the orbital plane of Mars.

The inclination Q is about $25°$, but its precise value is to be determined from the observations of the satellites. The position of the Laplacian plane is determined by the theoretical relationship between i and I' that is deduced later, subject to the restraint $i+I' = Q$. The angles i are small, approximately $0°{\cdot}01$ for Phobos and $1°$ for Deimos.

We assume Mars to be moving in a fixed elliptical orbit. An investigation showed that the neglected variations in the orbit of Mars have no significant effects on the motions of the satellites. The Sun moves in a similar elliptical orbit relative to Mars.

We assume the equator of Mars to be at a constant inclination Q to the orbit of Mars, and to be precessing around it at a constant rate. Let the longitude (measured from some suitable origin) of the ascending node of the equator of Mars on the orbital plane of Mars be P, so that $P = P_0+\dot{P}t$, where P_0 and $\dot{P}$ are constants to be determined from the satellite observations.

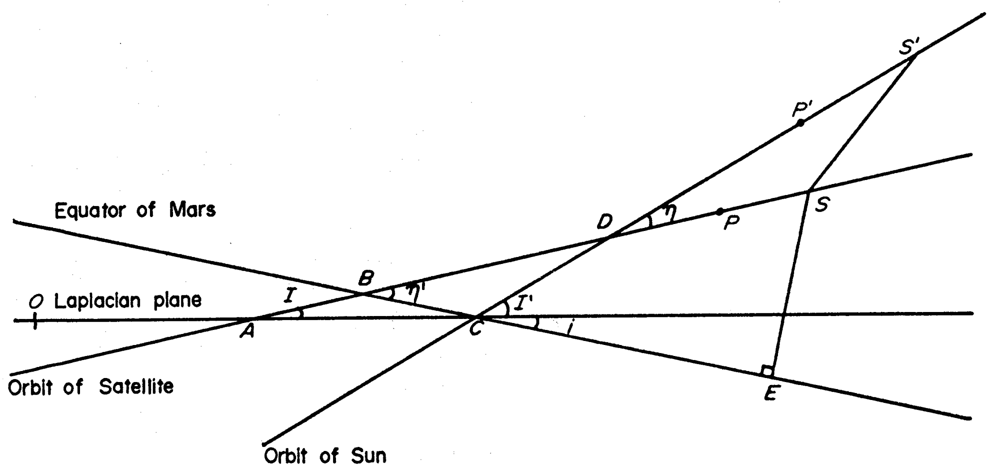

FIG. 1. *Planetocentric view of reference planes.*
S and S' are the positions of the satellite and the Sun. P and P' are the pericentres of the satellite and the Sun. O is the origin of longitude.

$OA = h$	$OC = h'$	$SS' = x$	$CD = \alpha$	$DP' = \delta$
$AP = g$	$CP' = g'$	$SE = y$	$AD = \beta$	$DP = \gamma$
$PS = f$	$P'S' = f'$			
$BP = g^*$				

The disturbing function for perturbations on the satellite due to the Sun, the figure of Mars, and the effects of the precession of the equator of Mars is given by

$$R = R_S + R_F + R_P$$

where

$$R_S = k^2 S(1/\Delta - \mathbf{r}.\mathbf{r}'/r'^3)$$

$$R_F = -\frac{k^2 M}{r} \sum_{n=2}^{\infty} J_n (r_0/r)^n P_n(\sin y)$$

$$R_P = na^2(1-e^2)^{1/2}(-\sin I \cos h \sin I' + \cos I \cos I')\dot{P}$$

and k^2 is the constant of gravitation;

S is the mass of the Sun;
M is the mass of Mars;
Δ is the distance between the satellite and the Sun;
$\mathbf{r}$ is the position vector of the satellite relative to Mars;
$\mathbf{r}'$ is the position vector of the Sun relative to Mars;
r_0 is the equatorial radius of Mars;
J_n ($n = 2, \infty$) are the zonal coefficients in the harmonic expansion of the potential of Mars;
P_n ($n = 2, \infty$) are the Legendre polynomials.

The mean motions of the satellite and the Sun about Mars, n and n' respectively, are defined by

$$n^2 a^3 = k^2 M, \quad n'^2 a'^3 = k^2 S.$$

The term R_P is a correction to R to allow for the fact that our reference plane, the Laplacian plane, is not inertial, due to the precession of the equator of Mars. The expression for R_P is obtained from expressions given by Goldreich (1965).

3. EXPANSION OF THE DISTURBING FUNCTION

We now expand the disturbing function in powers of e, $\theta = \sin I$, and e'. For Phobos and Deimos e and θ are $O(10^{-2})$ or smaller, and $e' = 0{\cdot}09$. The observed distances of the satellites from Mars are always less than 100″, and the observations are given at most to an accuracy of 0″·01, so we aim to obtain an accuracy in the computed positions of the satellites of 1 part in 10^4. This is equivalent to about 20″ in the orbital longitude of the satellites.

The part of the disturbing function due to solar perturbations has a factor $(n'/n)^2$. As n'/n is of order 10^{-3}, we see that the only significant perturbations due to the Sun are those of long period (i.e. those whose arguments are independent of l), as these terms appear with small divisors. Their arguments depend on g, h, l', g' and h', and their periods are typically of the order of the period of Mars. Hence their small divisors will be of order n'/n, and so the perturbations in the elements due to these terms will be of order n'/n, or 10^{-3}. So to obtain an accuracy of 1 part in 10^4, it will be sufficient to calculate the long-period perturbations to the first power of e, θ and e', and to achieve this it will be necessary to expand the solar part of the disturbing function to the second power of these quantities, although terms in e'^2 can be neglected as no differentiation with respect to e' occurs.

For the perturbations due to the oblateness of Mars, we shall calculate terms due to the first three harmonics, J_2, J_3 and J_4. The perturbations due to J_2 appear with a factor $J_2(r_0/a)^2$, which is of order 10^{-4} for Phobos and smaller for Deimos. Hence, for these perturbations, we only need calculate terms independent of e, θ and e', for which we shall require an expansion of the oblateness part of the disturbing function to the first power of e and θ.

For the Earth, J_3 and J_4 are of the order of $J_2{}^2$, and we shall assume that the values for Mars are similar, so the only terms in J_3 and J_4 we need consider are the long-period ones, which will have a small divisor of order J_2.

The above remarks apply for the perturbations in a, e, θ and λ. The quantities $\tilde{\omega}$ and h occur in the expressions for the positions of the satellites with factors of e and θ respectively, and so perturbations in these quantities can be evaluated to one power less.

We shall evaluate the secular rates of change of the elements to higher accuracy than that for the periodic terms; to the second power of e, θ and e' for λ, and to the first power for $\tilde{\omega}$ and h.

We propose only to make a solution to first order in J_2. Brouwer (1959) has shown that long-period terms arise in the second-order solution, and these terms have a small divisor of order J_2, thus becoming terms of first order. However, it can be seen from Brouwer's solution that the largest of these terms, in l, has a factor θ^2, and is thus negligible for our purposes.

4. THE SOLAR PART OF THE DISTURBING FUNCTION

The solar part of R is

$$R_S = k^2 S(1/\Delta - \mathbf{r}.\mathbf{r}'/r'^3)$$

where

$$\Delta^2 = r^2 + r'^2 - 2rr'\cos x \quad \text{and} \quad \mathbf{r}.\mathbf{r}' = rr'\cos x.$$

Hence by expanding r'/Δ we obtain

$$R_S = \frac{k^2 S}{r'}\left[1 + \frac{r^2}{r'^2}\left(-\tfrac{1}{2} + \tfrac{3}{2}\cos^2 x\right) + O\left(\frac{r}{r'}\right)^3\right].$$

Now $r/r' = O(10^{-7})$, so the terms of order $(r/r')^3$ are negligible. The term k^2S/r' is independent of the coordinates of the satellite and hence can be ignored. Thus, putting $k^2S = n'^2a'^3$, we have

$$R_S = n'^2a^2\left(\frac{r}{a}\right)^2\left(\frac{a'}{r'}\right)^3(-\tfrac{1}{2}+\tfrac{3}{2}\cos^2 x).$$

From the spherical triangle DSS' in Fig. 1 we obtain

$$\cos^2 x = \cos^2(f'+\delta)\cos^2(f+\gamma)+\sin^2(f'+\delta)\sin^2(f+\gamma)\cos^2\eta$$
$$+2\cos(f'+\delta)\cos(f+\gamma)\sin(f'+\delta)\sin(f+\gamma)\cos\eta,$$

where

$$\gamma = g-\beta,\quad \delta = g'-\alpha,$$

and we have the following expansions in elliptic motion (obtained from Cayley's tables (1861)):

$$\left(\frac{a'}{r'}\right)^3 = c+3e'\cos l'+O(e'^2)$$
$$\left(\frac{a'}{r'}\right)^3\cos 2f' = -\tfrac{1}{2}e'\cos l'+\cos 2l'+\tfrac{7}{2}e'\cos 3l'+O(e'^2)$$
$$\left(\frac{a'}{r'}\right)^3\sin 2f' = -\tfrac{1}{2}e'\sin l'+\sin 2l'+\tfrac{7}{2}e'\sin 3l'+O(e'^2)$$
$$\left(\frac{r}{a}\right)^2 = 1+\tfrac{3}{2}e^2+\text{short-period terms}$$
$$\left(\frac{r}{a}\right)^2\cos 2f = \tfrac{5}{2}e^2+\text{short-period terms}$$
$$\left(\frac{r}{a}\right)^2\sin 2f = 0+\text{short-period terms}$$

where $c = (1-e'^2)^{-3/2}$. This quantity is not expanded in powers of e' so that a greater accuracy can be obtained in the secular terms. The neglected short-period terms have arguments depending on l.

Hence,

$$R_S = \tfrac{3}{8}n'^2a^2[(c+3e'\cos l')(-\tfrac{1}{3}-\tfrac{1}{2}e^2+(1+\tfrac{3}{2}e^2)\cos^2\eta+\tfrac{5}{2}e^2\cos 2\gamma\sin^2\eta)$$
$$+\{-\tfrac{1}{2}e'\cos(l'+2\delta)+\cos(2l'+2\delta)+\tfrac{7}{2}e'\cos(3l'+2\delta)\}$$
$$\times\{(1+\tfrac{3}{2}e^2)\sin^2\eta+\tfrac{5}{2}e^2\cos 2\gamma(1+\cos^2\eta)\}+5e^2\sin 2\gamma\cos\eta$$
$$\times\{-\tfrac{1}{2}e'\sin(l'+2\delta)+\sin(2l'+2\delta)+\tfrac{7}{2}e'\sin(3l'+2\delta)\}].$$

From the spherical triangle ACD in Fig. 1 we obtain

$$\sin\eta\sin\beta = \sin I'\sin(h'-h)$$
$$\sin\eta\cos\beta = -\cos I'\sin I+\sin I'\cos I\cos(h'-h)$$
$$\cos\eta = \cos I'\cos I+\sin I'\sin I\cos(h'-h)$$
$$\sin\eta\cos\alpha = \sin I'\cos I-\cos I'\sin I\cos(h'-h)$$
$$\sin\eta\sin\alpha = \sin I\sin(h'-h).$$

Putting $\sin I = \theta$ and $\cos I = (1-\theta^2)^{1/2}$ and expanding in powers of θ, we obtain:

$$\begin{aligned}
\cos 2(g-\beta) &= \cos(2g+2h-2h')+O(\theta)\\
\sin 2(g-\beta) &= \sin(2g+2h-2h')+O(\theta)\\
\cos 2(g'-\alpha) &= \cos 2g'+O(\theta)\\
\sin 2(g'-\alpha) &= \sin 2g'+O(\theta)\\
\cos\eta &= \cos I + O'(\theta)\\
\cos^2\eta &= \cos^2 I'+\theta\sin 2I'\cos(h-h')\\
&\quad+\theta^2[-1+\tfrac{3}{2}\sin^2 I'+\tfrac{1}{2}\sin^2 I'\cos(2h-2h')]+O(\theta^3)\\
\cos 2(g-\beta)\sin^2\eta &= \sin^2 I'\cos(2g+2h-2h')+O(\theta)\\
\cos 2(g'-\alpha)\sin^2\eta &= \sin^2 I'\cos 2g'-\theta\sin I'(1+\cos I')\cos(2g'-h+h')\\
&\quad+\theta\sin I'(1-\cos I')\cos(2g'+h-h')\\
&\quad+\theta^2[-\tfrac{3}{2}\sin^2 I'\cos 2g'+\tfrac{1}{4}(1+\cos I')^2\cos(2g'-2h+2h')\\
&\qquad+\tfrac{1}{4}(1-\cos I')^2\cos(2g'+2h-2h')]+O(\theta^3)\\
\sin 2(g'-\alpha)\sin^2\eta &= \sin^2 I'\sin 2g'-\theta\sin I'(1+\cos I')\sin(2g'-h+h')\\
&\quad+\theta\sin I'(1-\cos I')\sin(2g'+h-h')\\
&\quad+\theta^2[-\tfrac{3}{2}\sin^2 I'\sin 2g'+\tfrac{1}{4}(1+\cos I')^2\sin(2g'-2h+2h')\\
&\qquad+\tfrac{1}{4}(1-\cos I')^2\sin(2g'+2h-2h')]+O(\theta^3).
\end{aligned}$$

The neglected terms in the expansions of $\sin 2(g'-\alpha)$ and $\cos 2(g'-\alpha)$ are in fact of order $\theta/\sin I'$. These expansions are only valid because I' is fairly large, with $\sin I' \doteqdot 0{\cdot}4$.

Hence we obtain the following expression for the solar part of the disturbing function.

$$\begin{aligned}
R_S = {}& \tfrac{3}{8}n'^2a^2c[-\tfrac{1}{3}-\tfrac{1}{2}e^2+(1+\tfrac{3}{2}e^2)(\cos^2 I'+\tfrac{3}{2}\theta^2\sin^2 I'-\theta^2)]\\
&+\tfrac{3}{8}n'^2a^2[c(1+\tfrac{3}{2}e^2)\,\theta\sin 2I'\cos(h-h')+\tfrac{1}{2}c\theta^2\sin^2 I'\cos(2h-2h')\\
&\quad+\tfrac{5}{2}e^2\sin^2 I'\cos(2g+2h-2h')-e'\cos l'+3e'\cos^2 I'\cos l'\\
&\quad+\tfrac{3}{2}e'\theta\sin 2I'(\cos(l'-h+h')+\cos(l'+h-h'))\\
&\quad-\tfrac{1}{2}e'\sin^2 I'\cos(l'+2g')\\
&\quad+\tfrac{1}{2}e'\theta\sin I'(1+\cos I')\cos(l'+2g'-h+h')\\
&\quad-\tfrac{1}{2}e'\theta\sin I'(1-\cos I')\cos(l'+2g'+h-h')\\
&\quad+(1+\tfrac{3}{2}e^2-\tfrac{3}{2}\theta^2)\sin^2 I'\cos(2l'+2g')\\
&\quad-\theta\sin I'(1+\cos I')\cos(2l'+2g'-h+h')\\
&\quad+\theta\sin I'(1-\cos I')\cos(2l'+2g'+h-h')+\tfrac{7}{2}e'\sin^2 I'\cos(3l'+2g')\\
&\quad-\tfrac{7}{2}e'\theta\sin I'(1+\cos I')\cos(3l'+2g'-h+h')\\
&\quad+\tfrac{7}{2}e'\theta\sin I'(1-\cos I')\cos(3l'+2g'+h-h')\\
&\quad+\tfrac{1}{4}\theta^2(1+\cos I')^2\cos(2l'+2g'-2h+2h')\\
&\quad+\tfrac{1}{4}\theta^2(1-\cos I')^2\cos(2l'+2g'+2h-2h')\\
&\quad+\tfrac{5}{4}e^2(1+\cos I')^2\cos(2l'+2g'-2g-2h+2h')\\
&\quad+\tfrac{5}{4}e^2(1-\cos I')^2\cos(2l'+2g'+2g+2h-2h')].
\end{aligned}$$

The first part of R_S gives rise to the secular changes in the elements, and is given exactly.

5. THE PART OF THE DISTURBING FUNCTION DUE TO THE OBLATENESS OF MARS

We shall only consider perturbations due to the J_2, J_3 and J_4 harmonics in the potential of Mars. Substituting the expressions for the Legendre polynomials, and putting $k^2M = n^2a^3$, we obtain the following expression for the oblateness part of the disturbing function.

$$R_F = -\frac{n^2a^3}{r}\left[\tfrac{1}{2}J_2\left(\frac{r_0}{r}\right)^2(3\sin^2 y-1)+\tfrac{1}{2}J_3\left(\frac{r_0}{r}\right)^3(5\sin^3 y-3\sin y)\right.$$
$$\left.+\tfrac{1}{8}J_4\left(\frac{r_0}{r}\right)^4(35\sin^4 y-30\sin^2 y+3)\right].$$

From the spherical triangle *BES* in Fig. 1 we see that $\sin y = \sin\eta' \sin(f+g^*)$.

As was stated earlier, we need to calculate periodic terms in R_F to the first power of e and θ, and secular terms to the second power. Also we can neglect any short-period terms with a factor of J_3 or J_4 (i.e. terms whose arguments depend on l). We shall subsequently see that η' is a small angle of the same order as I, and as we are neglecting terms of order $\sin^3 I$ we may neglect the terms in $\sin^3 y$ and $\sin^4 y$.

Hence we require expressions for

$$\left(\frac{a}{r}\right)^3\sin^2 y,\quad \left(\frac{a}{r}\right)^3,\quad \left(\frac{a}{r}\right)^4\sin y,\quad \left(\frac{a}{r}\right)^5\sin^2 y\quad\text{and}\quad \left(\frac{a}{r}\right)^5.$$

Now

$$\sin y = \sin\eta'(\sin f\cos g^*+\cos f\sin g^*)$$

and

$$\sin^2 y = \tfrac{1}{2}\sin^2\eta'(1-\cos 2f\cos 2g^*+\sin 2f\sin 2g^*)$$

and we obtain the following expansions from Cayley's tables (1861):

$$\left(\frac{a}{r}\right)^3 = (1-e^2)^{-3/2}+3e\cos l+\text{periodic terms of order } e^2$$

$$\left(\frac{a}{r}\right)^4\cos f = e+O(e^3)+\text{short-period terms}$$

$$\left(\frac{a}{r}\right)^5 = 1+5e^2+O(e^4)+\text{short-period terms}$$

$$\left(\frac{a}{r}\right)^5\cos 2f = \tfrac{3}{4}e^2+O(e^4)+\text{short-period terms}$$

$$\left(\frac{a}{r}\right)^3\cos 2f,\quad \left(\frac{a}{r}\right)^3\sin 2f,\quad \left(\frac{a}{r}\right)^4\sin f\quad\text{and}\quad \left(\frac{a}{r}\right)^5\sin 2f$$

contain only short-period terms.

Hence we obtain

$$R_F = -\tfrac{1}{2}n^2J_2r_0^2[(1+\tfrac{3}{2}e^2)(\tfrac{3}{2}\sin^2\eta'-1)-3e\cos l]+\frac{3}{2a}n^2J_3r_0^3\,e\sin\eta'\sin g^*$$
$$-\frac{n^2J_4r_0^4}{8a^2}[3+15e^2-15\sin^2\eta'].$$

From the spherical triangle ABC in Fig. 1 we obtain

$$\cos\eta' = \cos i \cos I - \sin i \sin I \cos(h'-h)$$
$$\sin\eta' \cos(g-g^*) = \cos i \sin I + \sin i \cos I \cos(h'-h)$$
$$\sin\eta' \sin(g-g^*) = \sin i \sin(h'-h).$$

Hence putting $\sin I = \theta$ and $\cos I = (1-\theta^2)^{1/2}$ and expanding in powers of θ we obtain

$$\sin\eta' \sin g^* = \theta \sin g + \sin i \sin(g+h-h') + O(\theta^2, \sin^2 i)$$
$$\sin^2\eta' = \theta^2 + \sin^2 i + \theta \sin 2i \cos(h-h') + O(\theta^3, \sin^3 i)$$

giving

$$R_F = \tfrac{1}{4}n^2 J_2 r_0^2[2+3e^2-3\theta^2-3\sin^2 i - 3(1+\tfrac{3}{2}e^2)\theta \sin 2i \cos(h-h') + 6e\cos l]$$
$$+\frac{3}{2a}n^2 J^3 r_0^3[e\theta \sin g + e \sin i \sin(g+h-h')]$$
$$-\frac{\eta^2 J_4 r_0^4}{8a^2}[3+15e^2-15\theta^2-15\sin^2 i].$$

The largest periodic terms in J_4 have a factor $e^2\theta^2 J_4$ and have been neglected. The terms in J_3 are very small but have been included as they are the largest ones occurring.

6. THE PART OF THE DISTURBING FUNCTION DUE TO THE PRECESSION OF THE EQUATOR OF MARS

We shall put $\dot{P}$, the precession rate of the equator of Mars, equal to pn', so that p is a dimensionless number. Then expanding R_P in powers of e and θ we obtain

$$R_P = pn'na^2(-\theta \cos h \sin I' + \cos I' - \tfrac{1}{2}e^2 \cos I' - \tfrac{1}{2}\theta^2 \cos I').$$

Ultimately we shall attempt to determine p from the satellite observations. However, an estimate of its value can be obtained as follows.

The precessional rate is given in terms of the principal moments of inertia of Mars by

$$pn' = -\frac{3n'^2(C-A)\cos Q}{2\omega C(1-e'^2)^{3/2}}$$

where ω is the angular speed of rotation of Mars.

The moments of inertia can be calculated from

$$\frac{C-A}{Mr_0^2} = J_2$$

$$\frac{C}{Mr_0^2} = \tfrac{2}{3}\left[1-\tfrac{2}{5}\left(\frac{5\sigma}{2f}-1\right)^{1/2}\right]$$

where σ is the ratio of the centrifugal acceleration to the apparent acceleration at the equator of Mars, and f is the flattening of Mars. The flattening is calculated from

$$f = \tfrac{3}{2}J_2 + \tfrac{1}{2}\sigma.$$

These equations can be found in Kaula (1968).

Taking the values of J_2 and f given by Wilkins (1967) these equations give $p = -1{\cdot}08 \times 10^{-5}$, corresponding to a period of precession of 173 000 years. The effects are therefore unlikely to be significant.

7. CHOICE OF REFERENCE PLANE

The above expansion of the disturbing function is for an arbitrary small value of i. By appropriate choice of i we can almost completely eliminate the largest terms in the disturbing function. The corresponding reference plane is the Laplacian plane.

In R_S we have the term

$$\tfrac{3}{8}n'^2a^2(1-e'^2)^{-3/2}(1+\tfrac{3}{2}e^2)\,\theta \sin 2I' \cos(h-h')$$

and in R_F we have the term

$$-\tfrac{3}{4}n^2J_2r_0^2(1+\tfrac{3}{2}e^2)\,\theta \sin 2i \cos(h-h').$$

These terms are of long period and produce large periodic perturbations in the orbital elements of the satellites (considerably larger, in fact, than any of the other perturbations). However, they can be almost completely eliminated if we choose i so that

$$2n_0^2J_2\left(\frac{r_0}{a_0}\right)^2 \sin 2i = n'^2(1-e'^2)^{-3/2}\sin 2(Q-i)$$

using

$$I' = Q-i.$$

Q is the inclination of the orbit of Mars to the equator of Mars, and a_0 and n_0 are the constant parts of a and n in the final solution. Then the above two terms will cancel, apart from very small parts due to the perturbations in a and n which we shall neglect.

8. SOLUTION OF THE EQUATION OF MOTION

The equations of motion are the usual differential equations for the osculating elements a, e, I, λ, $\tilde{\omega}$ and h (see Brouwer & Clemence (1961), p. 284). We change the equation for I into one for θ, and expand the right-hand sides of the equations in powers of e and θ, retaining only sufficient terms to give the desired accuracy in the solution. The equations become

$$\frac{da}{dt} = \frac{2}{na}\frac{\partial R}{\partial \lambda} \tag{1}$$

$$\frac{de}{dt} = -\frac{1}{ena^2}\frac{\partial R}{\partial \tilde{\omega}} \tag{2}$$

$$\frac{d\theta}{dt} = -\frac{1}{\theta na^2}\frac{\partial R}{\partial h} \tag{3}$$

$$\frac{d\lambda}{dt} = n - \frac{2}{na}\frac{\partial R}{\partial a} + \frac{1}{2na^2}\left(e\frac{\partial R}{\partial e} + \theta\frac{\partial R}{\partial \theta}\right) \tag{4}$$

$$\frac{d\tilde{\omega}}{dt} = \frac{1}{ena^2}\frac{\partial R}{\partial e} \tag{5}$$

$$\frac{dh}{dt} = \frac{1}{\theta na^2}\frac{\partial R}{\partial \theta} \tag{6}$$

where R is to be considered a function of a, e, θ, λ, $\tilde{\omega}$ and h, and n is given by

$$n^2a^3 = k^2M.$$

To obtain a first-order solution of these equations we first consider only the secular part of R (i.e. the part independent of λ, $\tilde{\omega}$ and h). The right-hand sides of equations (1), (2) and (3) are then zero, giving a solution of the equation of the form $a = a_0$, $e = e_0$, $\theta = \theta_0$, where a_0, e_0 and θ_0 are constants. We shall define n_0 by the equation $n_0{}^2a_0{}^3 = k^2M$. Substituting this solution in equations (4), (5) and (6), the right-hand sides become constants, and the equations can be integrated to give a solution of the form

$$\bar{\lambda} = \lambda_0 + (n_0 + \alpha n')\, t$$
$$\bar{\tilde{\omega}} = \tilde{\omega}_0 + \beta n' t$$
$$\bar{h} = h_0 + \gamma n' t$$

where λ_0, $\tilde{\omega}_0$ and h_0 are constants, and the dimensionless coefficients α, β and γ are given by

$$\alpha = F_2N_0(3 - \tfrac{9}{2}\sin^2 i + \tfrac{21}{4}e_0{}^2 - \tfrac{21}{4}\theta_0{}^2) + \frac{c}{N_0}(-1 + \tfrac{3}{2}\sin^2 I') - \tfrac{15}{4}F_4N_0 - p\cos I' \tag{7}$$

$$\beta = \tfrac{3}{2}F_2N_0 + \frac{c}{N_0}(\tfrac{3}{4} - \tfrac{9}{8}\sin^2 I') - \tfrac{15}{4}F_4N_0 - p\cos I' \tag{8}$$

$$\gamma = -\tfrac{3}{2}F_2N_0 - \frac{c}{N_0}(\tfrac{3}{4} - \tfrac{9}{8}\sin^2 I') + \tfrac{15}{4}F_4N_0 - p\cos I' \tag{9}$$

where $F_2 = J_2(r_0/a_0)^2$, $F_3 = J_3(r_0/a_0)^3$, $F_4 = J_4(r_0/a_0)^4$, and $N_0 = n_0/n'$. It is found that for Phobos $\alpha \doteqdot 1{\cdot}7$, and for Deimos $\alpha \doteqdot 0{\cdot}066$. For both satellites, $\beta \doteqdot \alpha/2$ and $\gamma \doteqdot -\beta$.

We now consider the remaining part of R. Inserting the above solution in the right-hand sides of equations (1), (2) and (3) the equations become the differential equations for the periodic perturbations in a, e and θ. They can be easily integrated to give the following expressions for a, e and θ (N.B., we have put $\bar{l} = \bar{\lambda} - \bar{\tilde{\omega}}$, $\bar{g} = \bar{\tilde{\omega}} - \bar{h}$).

$$a = a_0(1 + 3F_2e_0\cos\bar{l}) \tag{10}$$

$$\begin{aligned} e = e_0 &+ \tfrac{3}{2}F_2\cos\bar{l} - \tfrac{3}{2}F_3N_0\left(\frac{\theta_0}{\beta-\gamma}\sin\bar{g} + \frac{\sin i}{\beta}\sin(\bar{g}+\bar{h}-h')\right) \\ &+ \frac{3}{8N_0}\left[-\frac{5e_0\sin^2 I'}{2\beta}\cos(2\bar{g}+2\bar{h}-2h') \right. \\ &\quad + \frac{5e_0(1+\cos I')^2}{4-4\beta}\cos(2l'+2g'-2\bar{g}-2\bar{h}+2h') \\ &\quad \left. - \frac{5e_0(1-\cos I')^2}{4+4\beta}\cos(2l'+2g'+2\bar{g}+2\bar{h}-2h')\right] \end{aligned} \tag{11}$$

$$\theta = \theta_0 + \frac{3F_3N_0e_0}{2(\beta-\gamma)}\sin\bar{g} + \frac{p\sin I'}{\gamma}\cos\bar{h} + \frac{3}{8N_0}$$
$$\times\left[-\frac{\theta_0\sin^2 I'}{2\gamma}\cos(2\bar{h}-2h') + \frac{3e'\sin 2I'}{2-2\gamma}\cos(l'-\bar{h}+h')\right.$$
$$-\frac{3e'\sin 2I'}{2+2\gamma}\cos(l'+\bar{h}-h') + \frac{e'\sin I'(1+\cos I')}{2-2\gamma}\cos(l'+2g'-\bar{h}+h')$$
$$+\frac{e'\sin I'(1-\cos I')}{2+2\gamma}\cos(l'+2g'+\bar{h}-h')$$
$$-\frac{\sin I'(1+\cos I')}{2-\gamma}\cos(2l'+2g'-\bar{h}+h')$$
$$-\frac{\sin I'(1-\cos I')}{2+\gamma}\cos(2l'+2g'+\bar{h}-h')$$
$$-\frac{7e'\sin I'(1+\cos I')}{6-2\gamma}\cos(3l'+2g'-\bar{h}+h')$$
$$-\frac{7e'\sin I'(1-\cos I')}{6+2\gamma}\cos(3l'+2g'+\bar{h}-h')$$
$$+\frac{\theta_0(1+\cos I')^2}{4-4\gamma}\cos(2l'+2g'-2\bar{h}+2h')$$
$$\left.-\frac{\theta_0(1-\cos I')^2}{4+4\gamma}\cos(2l'+2g'+2\bar{h}-2h')\right]. \qquad (12)$$

The perturbation in a is below the rejection limit decided upon, but it has been retained as it is the largest term occurring in a. The periodic terms in equations (4), (5) and (6) are integrated in a similar manner to give the periodic perturbations in λ, $\tilde{\omega}$ and h. The periodic term in a causes a perturbation $-4{\cdot}5n_0F_2e_0\cos\bar{l}$ in n. This must be taken into account on the right-hand side of equation (4) but it produces a perturbation of order e_0J_2 in λ which is below our rejection limit. The expressions for λ, $\tilde{\omega}$ and h are found to be

$$\lambda = \lambda_0 + (n_0+\alpha n')\,t$$
$$-\frac{3}{16N_0}\left[-8e'\sin l' + 24e'\cos^2 I'\sin l' + \frac{3\theta_0^2\sin^2 I'}{2\gamma}\sin(2\bar{h}-2h')\right.$$
$$+\frac{15e_0^2\sin^2 I'}{2\beta}\sin(2\bar{g}+2\bar{h}-2h')$$
$$-4e'\sin^2 I'\sin(l'+2g') + 4\sin^2 I'\sin(2l'+2g')$$
$$-\frac{7\theta_0\sin I'(1+\cos I')}{2-\gamma}\sin(2l'+2g'-\bar{h}+h')$$
$$+\frac{7\theta_0\sin I'(1-\cos I')}{2+\gamma}\sin(2l'+2g'+\bar{h}-h')$$
$$\left.+\frac{28}{3}e'\sin^2 I'\sin(3l'+2g')\right]. \qquad (13)$$

$$\tilde{\omega} = \tilde{\omega}_0+\beta n't+\frac{3F_2}{2e_0}\sin\bar{l}-\frac{3F_3}{2e_0}N_0\left(\frac{\theta_0}{\beta-\gamma}\cos\bar{g}+\frac{\sin i}{\beta}\cos(\bar{g}+\bar{h}-h')\right)$$

$$+\frac{3}{8N_0}\left[\frac{5\sin^2 I'}{2\beta}\sin(2\bar{g}+2\bar{h}-2h')+\frac{3\sin^2 I'}{2}\sin(2l'+2g')\right.$$

$$+\frac{5(1+\cos I')^2}{4-4\beta}\sin(2l'+2g'-2\bar{g}-2\bar{h}+2h')$$

$$\left.+\frac{5(1-\cos I')^2}{4+4\beta}\sin(2l'+2g'+2\bar{g}+2\bar{h}-2h')\right]. \qquad (14)$$

$$h = h_0+\gamma n't-\frac{3e_0F_3N_0}{2\theta_0(\beta-\gamma)}\cos\bar{g}-\frac{p\sin I'}{\theta_0\gamma}\sin\bar{h}$$

$$+\frac{3}{8N_0}\left[\frac{\sin^2 I'}{2\gamma}\sin(2\bar{h}-2h')-\tfrac{3}{2}\sin^2 I'\sin(2l'+2g')\right.$$

$$+\frac{3e'\sin 2I'}{2\theta_0(1-\gamma)}\sin(l'-\bar{h}+h')+\frac{3e'\sin 2I'}{2\theta_0(1+\gamma)}\sin(l'+\bar{h}-h')$$

$$+\frac{e'\sin I'(1+\cos I')}{2\theta_0(1-\gamma)}\sin(l'+2g'-\bar{h}+h')$$

$$-\frac{e'\sin I'(1-\cos I')}{2\theta_0(1+\gamma)}\sin(l'+2g'+\bar{h}-h')$$

$$-\frac{\sin I'(1+\cos I')}{\theta_0(2-\gamma)}\sin(2l'+2g'-\bar{h}+h')$$

$$+\frac{\sin I'(1-\cos I')}{\theta_0(2+\gamma)}\sin(2l'+2g'+\bar{h}-h')$$

$$-\frac{7e'\sin I'(1+\cos I')}{2\theta_0(3-\gamma)}\sin(3l'+2g'-\bar{h}+h')$$

$$+\frac{7e'\sin I'(1-\cos I')}{2\theta_0(3+\gamma)}\sin(3l'+2g'+\bar{h}-h')$$

$$+\frac{(1+\cos I')^2}{4-4\gamma}\sin(2l'+2g'-2\bar{h}+2h')$$

$$\left.+\frac{(1-\cos I')^2}{4+4\gamma}\sin(2l'+2g'+2\bar{h}-2h')\right]. \qquad (15)$$

Part II—Comparison of Theory with Observations

9. Reference system adopted

In Part I we specified the orbit of Mars by a set of orbital elements. We now take these elements to be Newcomb's mean elements of Mars at the epoch 1950·0 referred to the ecliptic and equinox of 1950·0. A test solution for the elements of the satellites using Newcomb's elements of Mars at the epoch 1900·0 gave results insignificantly different from those obtained using the epoch 1950·0. The adopted elements of Mars are:

Mean anomaly	$l' = 204°\cdot5069 + n't$
Argument of perihelion	$\omega' = 285°\cdot9667$
Longitude of ascending node	$\Omega' = 49°\cdot1719$
Inclination to ecliptic	$i' = 1°\cdot8500$
Eccentricity	$e' = 0°\cdot09336$
Mean motion	$n' = 0°\cdot52402\,07666$/day
Mean distance	$a' = 1\cdot5237151$ *A.U.*

n' and a' have been given to a large number of figures as in the numerical work n and a were measured in terms of these quantities. They satisfy the equation $n'^2a'^3 = k^2S$. The time t is measured in days from *JED* 241 4800·5. We take this date as our osculating epoch as it is that used by Wilkins (1968) in his solution for Deimos, and it is close to the centre of the time span covered by the observations.

We have not yet specified our origin of longitude in the Laplacian plane, nor our origin in the orbital plane of Mars from which the angle P is measured. We shall take as our origin of longitude in the Laplacian plane the ascending node of this plane on the orbital plane of Mars (as defined by the elements above). We shall measure P from the ascending node of the orbital plane of Mars on the ecliptic of 1950·0. The positions of these planes are shown in Fig. 2.

As a consequence of these definitions we find

$$g' = \omega' - P, \quad h' = 180°.$$

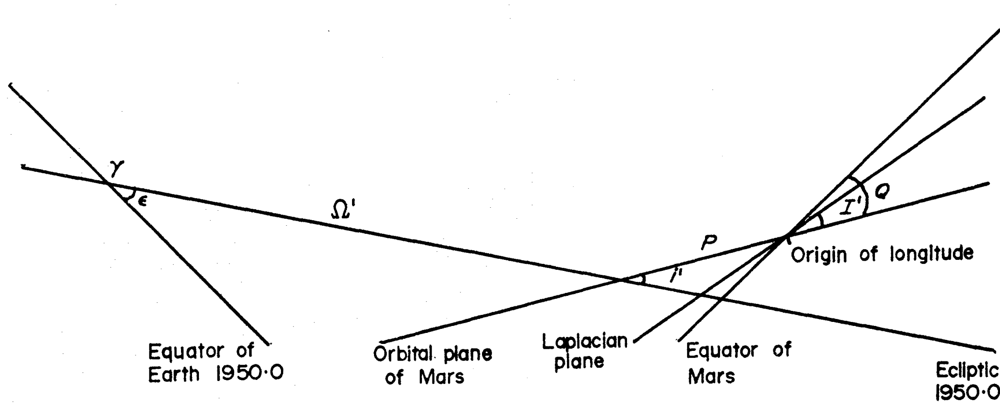

FIG. 2. *Reference planes referred to the ecliptic and the equator of the Earth.*

10. EXPRESSIONS FOR THE COORDINATES OF THE SATELLITES

In order to compare the theory of Part I with the observational data it is first of all necessary to calculate for each time of observation the apparent position of the satellite relative to Mars as seen from the Earth. It is therefore necessary to calculate the perturbed values of the elements, then to calculate rectangular coordinates of the satellite in the reference frame used in the theory. These are transformed to a geocentric equatorial reference frame, and then the apparent position of the satellite relative to Mars on the geocentric celestial sphere is calculated.

As e and θ are small for both satellites, $\tilde{\omega}$ and h will be poorly determined, so we shall use the quantities u, v, r and s instead, where $u = e\cos\tilde{\omega}$, $v = e\sin\tilde{\omega}$,

$r = \theta \cos h$, $s = \theta \sin h$. Also we shall put $u_0 = e_0 \cos \tilde{\omega}_0$, $v_0 = e_0 \sin \tilde{\omega}_0$, $r_0 = \theta_0 \cos h_0$, $s_0 = \theta_0 \sin h_0$.

Now in Part I we put $N_0 = n_0/n'$. We shall also put $A_0 = a_0/a'$, and $A = a/a'$. We shall make no use of the quantity n/n'. If we measure the mass of Mars M in units of the Sun's mass, we have $N_0^2 A_0^3 = M$. Now N_0 can be determined from the observations to a considerably greater accuracy than A_0, so we shall solve for N_0 and M from the observations, and calculate A_0 from the above equation.

We shall put $K_2 = J_2(r_0/a')^2$, $K_3 = J_3(r_0/a')^3$, $K_4 = J_4(r_0/a')^4$. These quantities are independent of the orbital elements of the satellites, and can thus be determined from the observations of both satellites.

In view of the interest in the possible secular accelerations of the satellites, we add an empirical term mt^2 to the mean longitude of each satellite.

We add a reminder that the angle P in Fig. 2 is given by $P = P_0 + pn't$, and Q is the inclination of the equator of Mars to the orbital plane of Mars.

So we take as our fundamental constants to be determined from the observations the following 21 quantities:

$$N_0, \lambda_0, u_0, v_0, r_0, s_0, m \quad \text{for each satellite,}$$
$$M, P_0, p, Q, K_2, K_3, K_4.$$

We now give a series of formulae to be used to compute positions of the satellites corresponding to the observed positions, assuming that a set of values of the above constants is known at the epoch *JED* 241 4800·5. We assume that all angular quantities are in radians, so $n't$ and mt^2 must be in radians also.

$n't = 0{\cdot}0091458\ 87716\ (JD - 241\ 4800{\cdot}5)$, where JD is the Julian ephemeris date of the observation, antedated for light time (see Section 10, page 267).

$$P = P_0 + pn't$$
$$l' = 3{\cdot}5693 + n't, \quad g' = 4{\cdot}9911 - P, \quad e' = 0{\cdot}09336$$
$$A_0 = M^{1/3} N_0^{-2/3}$$
$$e_0^2 = u_0^2 + v_0^2, \quad \theta_0^2 = r_0^2 + s_0^2, \quad \tan \tilde{\omega}_0 = v_0/u_0, \quad \tan h_0 = s_0/r_0$$
$$F_2 = K_2/A_0^2, \quad F_3 = K_3/A_0^3, \quad F_4 = K_4/A_0^4$$
$$c = (1 - e'^2)^{-3/2}$$
$$2N_0^2 F_2 \sin 2i = c \sin 2(Q - i) \quad \text{(to be solved iteratively for } i)$$
$$I' = Q - i$$

α, β and γ are now obtained from equations (7), (8) and (9) of Part I.

$$\bar{\lambda} = \lambda_0 + (N_0 + \alpha)\, n't + mt^2$$
$$\bar{\tilde{\omega}} = \tilde{\omega}_0 + \beta n't, \quad \bar{h} = h_0 + \gamma n't$$
$$l = \bar{\lambda} + \bar{\tilde{\omega}}, \quad \bar{g} = \bar{\tilde{\omega}} - \bar{h}$$
$$A = A_0(1 + 3F_2 e_0 \cos l)$$
$$\lambda = \bar{\lambda} - \frac{3}{16N_0}\Bigg[-8e' \sin l + 24e' \cos^2 I' \sin l' + \frac{3\theta_0^2 \sin^2 I'}{2\gamma} \sin 2\bar{h}$$
$$+ \frac{15e_0^2 \sin I'}{2\beta} \sin(2\bar{g} + 2\bar{h}) - 4e' \sin^2 I' \sin(l' + 2g')$$
$$+ 4 \sin^2 I' \sin(2l' + 2g')$$
$$+ \frac{7\theta_0 \sin I'(1 + \cos I')}{2 - \gamma} \sin(2l' + 2g' - \bar{h})$$

$$-\frac{7\theta_0 \sin I'(1-\cos I')}{2+\gamma}\sin(2l'+2g'+\bar{h})$$
$$+\tfrac{28}{3}e'\sin^2 I'\sin(3l'+2g')\Big].$$

We convert the expressions for e, θ, $\tilde{\omega}$ and h given by equations (11), (12), (14) and (15) to expressions for u, v, r and s, giving

$$u = u_0\cos\beta n't - v_0\sin\beta n't + \tfrac{3}{2}F_2\cos\bar{\lambda} + \frac{3F_3N_0\theta_0}{2(\beta-\gamma)}\sin\bar{h}$$
$$+\frac{3}{8N_0}\Bigg[-\frac{5e_0\sin^2 I'}{2\beta}\cos\bar{\tilde{\omega}}$$
$$+\left(\frac{5e_0(1+\cos I')^2}{4-4\beta}-\tfrac{3}{4}e_0\sin^2 I'\right)\cos(2l'+2\bar{g}-\bar{\tilde{\omega}})$$
$$+\left(-\frac{5e_0(1-\cos I')^2}{4+4\beta}+\tfrac{3}{4}e_0\sin^2 I'\right)\cos(2l'+2g'+\bar{\tilde{\omega}})\Bigg].$$

$$v = v_0\cos\beta n't + u_0\sin\beta n't + \tfrac{3}{2}F_2\sin\bar{\lambda} - \tfrac{3}{2}F_3N_0\left(\frac{\theta_0}{\beta-\gamma}\cos\bar{h}-\frac{\sin i}{\beta}\right)$$
$$+\frac{3}{8N_0}\Bigg[\frac{5e_0\sin^2 I'}{2\beta}\sin\bar{\tilde{\omega}}$$
$$+\left(\frac{5e_0(1+\cos I')^2}{4-4\beta}+\tfrac{3}{4}e_0\sin^2 I'\right)\sin(2l'+2g'-\bar{\tilde{\omega}})$$
$$+\left(\frac{5e_0(1-\cos I')^2}{4+4\beta}+\tfrac{3}{4}e_0\sin^2 I'\right)\sin(2l'+2g'+\bar{\tilde{\omega}})\Bigg].$$

$$r = r_0\cos\gamma n't - s_0\sin\gamma n't + \frac{3F_3N_0e_0}{2\beta-2\gamma}\sin\bar{\tilde{\omega}} + \frac{p\sin I'}{\gamma}$$
$$+\frac{3}{8N_0}\Bigg[-\frac{\theta_0\sin^2 I'}{2\gamma}\cos\bar{h}-\frac{3\gamma e'\sin 2I'}{1-\gamma^2}\cos l'$$
$$-\frac{e'\sin I'(1+\gamma\cos I')}{1-\gamma^2}\cos(l'+2g')$$
$$+\frac{2\sin I'(2+\gamma\cos I')}{4-\gamma^2}\cos(2l'+2g')$$
$$+\frac{7e'\sin I'(3+\gamma\cos I')}{9-\gamma^2}\cos(3l'+2g')$$
$$+\left(\frac{\theta_0(1+\cos I')^2}{4-4\gamma}+\tfrac{3}{4}\theta_0\sin^2 I'\right)\cos(2l'+2g'-\bar{h})$$
$$-\left(\frac{\theta_0(1-\cos I')^2}{4+4\gamma}+\tfrac{3}{4}\theta_0\sin^2 I'\right)\cos(2l'+2g'+\bar{h})\Bigg].$$

$$s = s_0\cos\gamma n't + r_0\sin\gamma n't - \frac{3F_3N_0e_0}{2\beta-2\gamma}\cos\bar{\tilde{\omega}}$$
$$+\frac{3}{8N_0}\Bigg[\frac{\theta_0\sin^2 I'}{2\gamma}\sin\bar{h}-\frac{3e'\sin 2I'}{1-\gamma^2}\sin l'$$

$$-\frac{e'\sin I'(\gamma+\cos I')}{1-\gamma^2}\sin(l'+2g')$$
$$+\frac{2\sin I'(\gamma+2\cos I')}{4-\gamma^2}\sin(2l'+2g')$$
$$+\frac{7e'\sin I'(\gamma+3\cos I')}{9-\gamma^2}\sin(3l'+2g')$$
$$+\left(\frac{\theta_0(1+\cos I')^2}{4-4\gamma}-\tfrac{3}{4}\theta_0\sin^2 I'\right)\sin(2l'+2g'-\bar{h})$$
$$+\left(\frac{\theta_0(1-\cos I')^2}{4+4\gamma}-\tfrac{3}{4}\theta_0\sin^2 I'\right)\sin(2l'+2g'+\bar{h})\Bigg].$$

Let (X, Y, Z) be the rectangular coordinates of the satellite relative to Mars in a system in which the XY-plane lies in the Laplacian plane, and the X-axis points towards the ascending node of this plane on the orbit of Mars. Let R be the distance of the satellite from the centre of Mars. Then

$$X/R = \cos(f+g)\cos h-\sin(f+g)\sin h\cos I$$
$$Y/R = \cos(f+g)\sin h+\sin(f+g)\cos h\cos I$$
$$Z/R = \sin(f+g)\sin I.$$

These formulae are not suitable for the calculation of X, Y and Z as they involve the quantities g and h which are not well determined. If we expand R and f in terms of the elliptic elements, and ignore powers of e and θ greater than the second, we obtain the following formulae used to calculate X^*, Y^* and Z^*, where $X^* = X/a$, $Y^* = Y/a$, $Z^* = Z/a$.

$$X^* = \cos\lambda-\tfrac{3}{2}u+\tfrac{1}{2}u\cos 2\lambda+\tfrac{1}{2}v\sin 2\lambda-(\tfrac{3}{8}u^2+\tfrac{5}{8}v^2+\tfrac{1}{2}s^2)\cos\lambda$$
$$+(\tfrac{1}{4}uv+\tfrac{1}{2}rs)\sin\lambda+\tfrac{3}{8}(u^2-v^2)\cos 3\lambda+\tfrac{3}{4}uv\sin 3\lambda.$$

$$Y^* = \sin\lambda-\tfrac{3}{2}v+\tfrac{1}{2}u\sin 2\lambda-\tfrac{1}{2}v\cos 2\lambda+(\tfrac{1}{4}uv+\tfrac{1}{2}rs)\cos\lambda$$
$$-(\tfrac{5}{8}u^2+\tfrac{3}{8}v^2+\tfrac{1}{2}r^2)\sin\lambda-\tfrac{3}{4}uv\cos 3\lambda+\tfrac{3}{8}(u^2-v^2)\sin 3\lambda.$$

$$Z^* = r\sin\lambda-s\cos\lambda-\tfrac{3}{2}vr+\tfrac{3}{2}us-\tfrac{1}{2}(vr+us)\cos 2\lambda+\tfrac{1}{2}(ur-vs)\sin 2\lambda.$$

Let (X_1, Y_1, Z_1) be the transformation of (X^*, Y^*, Z^*) to a system in which the X_1Y_1-plane lies in the plane of the orbit of Mars and the X_1-axis points towards the ascending node of this plane on the plane of the ecliptic of 1950·0.

Then

$$X_1 = X^*\cos P-(Y^*\cos I'-Z^*\sin I')\sin P$$
$$Y_1 = X^*\sin P+(Y^*\cos I'-Z^*\sin I')\cos P$$
$$Z_1 = Y^*\sin I'+Z^*\cos I'.$$

Let (ξ_1, η_1, ζ_1) be the components of (X_1, Y_1, Z_1) in a system in which the $\xi_1\eta_1$-plane lies in the plane of the Earth's equator of 1950·0, and the ξ_1-axis points towards the equinox of 1950·0. So we have

$$\xi_1 = CN.X_1-SN.CI.Y_1+SN.SI.Z_1$$
$$\eta_1 = CE.SN.X_1+(CE.CN.CI-SE.SI).Y_1+(-CE.CN.SI-SE.CI).Z_1$$
$$\zeta_1 = SE.SN.X_1+(SE.CN.CI+CE.SI).Y_1+(-SE.CN.SI+CE.CI).Z_1,$$

where $CE = \cos\epsilon$, $SE = \sin\epsilon$, $CN = \cos\Omega'$, $SN = \sin\Omega'$, $CI = \cos i'$, $SI = \sin i'$ and $\epsilon = 23^\circ{\cdot}44579$, the obliquity of the ecliptic at 1950·0.

Most of the observations of the satellites are referred to the true equator and equinox of date, so for these we apply precession (from 1950·0 to the date of the observation) and nutation to (ξ_1, η_1, ζ_1) to obtain (ξ, η, ζ).

Some of the photographic observations are referred to the mean equator and equinox of 1900·0, so for these we apply precession from 1950·0 to 1900·0 to obtain (ξ, η, ζ).

Other photographic observations are referred to the mean equator and equinox of 1950·0, so for these we can use (ξ_1, η_1, ζ_1) directly.

The transformation from (X_1, Y_1, Z_1) to (ξ, η, ζ) depends only on known quantities, so we can compute the transformation matrix for the time of each observation and store it on a magnetic tape with the details of the observation.

The observed quantities are either the position angle p and distance s of the satellite relative to the centre of Mars, or $x = \Delta\alpha\cos\delta$ and $y = \Delta\delta$, where

$\Delta\alpha$ = difference between right ascensions of satellite and Mars

$\Delta\delta$ = difference between declinations of satellite and Mars

δ = declination of Mars.

We put

α = right ascension of Mars,

ρ = distance of Mars from the Earth in *A.U.*

Then

$$x = A(a'/\rho)(\eta\cos\alpha - \xi\sin\alpha)$$
$$y = A(a'/\rho)(\zeta\cos\delta - \xi\sin\delta\cos\alpha - \eta\sin\delta\sin\alpha)$$

where

$$a' = 1{\cdot}5237151\ A.U.$$

and x, y are measured in radians.

These expressions neglect terms of order $(a\xi/r)^2$, etc. Now $(a\xi/r)$ is about 4×10^{-4} at most, so these expressions give at least four-figure accuracy in x and y which is sufficient for our purposes.

Then p and s are calculated from

$$\tan p = x/y, \quad s^2 = x^2 + y^2.$$

We convert x, y and s to seconds of arc and p to degrees to correspond to the observational data. However, when computing the root-mean-square residual of the observations compared with the theory, we convert the residuals in position angle to seconds of arc by multiplying by $s/57{\cdot}29578$.

The observations of Deimos relative to Phobos are all in terms of x and y. Let (x_{DP}, y_{DP}) be the coordinates of Deimos relative to Phobos, and (x_D, y_D) and (x_P, y_P) be the coordinates of Deimos and Phobos respectively relative to Mars. Then

$$x_{DP} = x_D - x_P, \quad y_{DP} = y_D - y_P.$$

The apparent coordinates of the satellites and Mars relative to the Earth at time t are given approximately by their geometric positions at time $t-\tau$ relative to the position of the Earth at time $t-\tau$, where τ is the light-time from Mars to the Earth. This approximation assumes the motion of the Earth to be rectilinear and uniform during the light-time. Hence we antedate all observation times by the light-time for that instant, and evaluate the position of the satellite at this antedated time. We obtain the apparent position of Mars at this time by interpolating a geometric geocentric ephemeris to the antedated time.

We make no parallax corrections for the position of the observer, as the effect of parallax on the apparent position of the satellites relative to Mars is negligible.

11. OBSERVATIONAL DATA

G. A. Wilkins has collected all available observations of the satellites of Mars, and has had most of them punched on cards, with the time of observation antedated for light-time and converted to ephemeris time. The remaining observations were punched in a similar manner, and all of them were copied on to a magnetic tape, together with the transformation matrix mentioned in Section 10 and the coordinates of Mars. It is hoped to publish a list of these observations in some form in the near future.

The total number of observations on the magnetic tape is 3143, covering a time span from 1877 to 1969. The observations were regarded as being of equal weight, and those with residuals greater than 2″ were rejected, so in fact 3107 were used, made up as follows:

Deimos relative to Phobos	106
Deimos relative to Mars	1508
Phobos relative to Mars	1493.

Table IV gives the root-mean-square residual for the observations of each observer at each opposition.

12. METHOD OF CALCULATION

A computer program was written which, for a given set of approximate values of the 21 arbitrary constants, will compute the position of the satellite from the formulae in Section 10 at the time of each observation. Hence the residuals *O–C* are calculated. The derivatives of x, y, p and s with respect to the arbitrary constants were calculated by differentiating analytically the expressions in Section 10. The differentiation was done in the following stages.

Derivatives of N_0, λ, u, v, r, s, P, Q, K_2, M w.r.t. N_0, λ_0, u_0, v_0, r_0, s_0, P_0, p, Q, K_2, K_3, K_4, M, m.
Derivatives of X^*, Y^*, Z^*, P, I' w.r.t. N_0, λ, u, v, r, s, P, Q, K_2, M.
Derivatives of ξ, η, ζ w.r.t. X^*, Y^*, Z^*, P, I'.
Derivatives of x, y w.r.t. ξ, η, ζ and w.r.t. N_0, M (the latter due to the dependence of x, y on A).
Derivatives of p, s w.r.t. x, y (if necessary).

Each set of derivatives was expressed as the elements of a matrix, and the derivatives of x, y (or p, s) with respect to the arbitrary constants were calculated by multiplying together these matrices. The derivatives are not required to great accuracy so most of the periodic perturbations were neglected in calculating them.

The computer program uses these derivatives to make a least-squares solution for corrections to the arbitrary constants. It can deal with observations of Phobos relative to Mars, Deimos relative to Mars, and Deimos relative to Phobos, and it makes a simultaneous solution for the corrections to the elements of both satellites.

Having obtained a corrected set of constants, the program was run again using this set as the initial values. This process was repeated until successive sets of constants agreed to within their standard errors, and no further reduction of the root-mean-square residual was obtained. This convergence generally occurred in one or two iterations.

Initial values of the constants were obtained from the values of the elements given by Wilkins (1968). His elements, converted to the reference system used in this paper, are given in Table III.

13. RESULTS

It was found that the J_3 and J_4 coefficients of the potential of Mars, and p, the precession rate of Mars, could not be determined significantly from the observations. So in the final solution J_3 and J_4 were taken as zero, and p was given the value -1.08×10^{-5}, corresponding to a period of precession of 173 000 years.

TABLE I(a)

Data from solution using all available observations

Epoch *JED* 241 4800·5 = 1899 May 25·5

		Phobos	Deimos
N_0	(in units of n')	2152.5365 ± 0.0009	544.11497 ± 0.00004
λ_0	(in radians)	3.242 ± 0.002	4.2540 ± 0.0010
u_0		-0.0163 ± 0.0007	-0.0014 ± 0.0003
v_0		-0.0086 ± 0.0007	-0.0014 ± 0.0003
r_0		0.0113 ± 0.0010	0.0224 ± 0.0004
s_0		-0.0132 ± 0.0010	-0.0221 ± 0.0004
m	(in radians day^{-2})	$(0.13 \pm 0.02) \times 10^{-9}$	$(-0.8 \pm 0.6) \times 10^{-11}$
P_0	(in radians)	0.6155 ± 0.0010	
Q	(in radians)	0.4401 ± 0.0004	
$J_2 (r_0/a')^2$		$(0.4356 \pm 0.0003) \times 10^{-12}$	
M	(in units of solar mass)	$(0.3228 \pm 0.0003) \times 10^{-6}$	

TABLE I(b)

Data from solution omitting observations of 1877, 1879 *and* 1881

Epoch *JED* 241 4800·5 = 1899 May 25·5

		Phobos	Deimos
N_0	(in units of n')	2152.5373 ± 0.0009	544.11509 ± 0.00004
λ_0	(in radians)	3.237 ± 0.002	4.2510 ± 0.0010
u_0		-0.0161 ± 0.0007	-0.0009 ± 0.0003
v_0		-0.0064 ± 0.0007	-0.0013 ± 0.0003
r_0		0.0106 ± 0.0010	0.0229 ± 0.0004
s_0		-0.0159 ± 0.0010	-0.0229 ± 0.0004
m	(in radians day^{-2})	$(-0.11 \pm 0.02) \times 10^{-9}$	$(-5.2 \pm 0.7) \times 10^{-11}$
P_0	(in radians)	0.6168 ± 0.0010	
Q	(in radians)	0.4399 ± 0.0004	
$J_2 (r_0/a')^2$		$(0.4354 \pm 0.0003) \times 10^{-12}$	
M	(in units of solar mass)	$(0.3227 \pm 0.0003) \times 10^{-6}$	

TABLE II

Quantities derived from solution given in Table I(a)

Epoch *JED* 2414800·5

	Phobos	Deimos
Semi-major axis (in *A.U.*)	$(0\cdot6270 \pm 0\cdot0002) \times 10^{-4}$	$(1\cdot5683 \pm 0\cdot0004) \times 10^{-4}$
Eccentricity	$0\cdot0184 \pm 0\cdot0007$	$0\cdot0020 \pm 0\cdot0003$
Inclination to Laplacian plane (I)	$1^\circ\cdot00 \pm 0^\circ\cdot07$	$1^\circ\cdot80 \pm 0^\circ\cdot03$
Mean longitude at epoch (λ_0)	$185^\circ\cdot7 \pm 0^\circ\cdot1$	$243^\circ\cdot74 \pm 0^\circ\cdot06$
Longitude of pericentre at epoch ($\tilde{\omega}_0$)	$208^\circ \pm 2^\circ$	$226^\circ \pm 8^\circ$
Longitude of node at epoch (h_0)	$311^\circ \pm 4^\circ$	$315^\circ\cdot4 \pm 0^\circ\cdot8$
Inclination of Laplacian plane to equatorial plane of Mars (i)	$0^\circ\cdot00938 \pm 0^\circ\cdot00001$	$0^\circ\cdot895 \pm 0^\circ\cdot001$
Daily mean motion in longitude ($n_0 + \alpha n'$)	$1128^\circ\cdot8442 \pm 0^\circ\cdot0010$	$285^\circ\cdot16192 \pm 0^\circ\cdot00004$
Daily mean motion of pericentre ($\beta n'$)	$0^\circ\cdot4345 \pm 0^\circ\cdot0004$	$0^\circ\cdot01814 \pm 0^\circ\cdot00002$
Daily mean motion of node ($\gamma n'$)	$-0^\circ\cdot4354 \pm 0\cdot0004$	$-0^\circ\cdot01813 \pm 0^\circ\cdot00002$
Longitude of node of equator of Mars on orbital plane of Mars (P_0)	$35^\circ\cdot27 \pm 0^\circ\cdot06$	
Inclination of equator to Mars to orbital plane of Mars (Q)	$25^\circ\cdot22 \pm 0^\circ\cdot03$	
Ratio of solar mass to mass of Mars	3097000 ± 3000	
$J_2 r_0^2$ (in $A.U.^2$)	$(1\cdot0114 \pm 0\cdot0008) \times 10^{-12}$	

TABLE III

Values of the elements given by Wilkins (1968) *converted to the reference system used in this paper*

Epoch *JED* 2414800·5

	Phobos	Deimos
Semi-major axis (in *A.U.*)	$(0\cdot6259 \pm 0\cdot0005) \times 10^{-4}$	$(1\cdot5689 \pm 0\cdot0005) \times 10^{-4}$
Eccentricity	$0\cdot018 \pm 0\cdot001$	$0\cdot0 \pm 0\cdot0003$
Inclination to Laplacian plane (I)	$0^\circ\cdot9 \pm 0^\circ\cdot1$	$1^\circ\cdot80 \pm 0^\circ\cdot02$
Mean longitude at epoch	$186^\circ\cdot6 \pm 0^\circ\cdot1$	$243^\circ\cdot66 \pm 0^\circ\cdot05$
Longitude of pericentre at epoch	$209^\circ \pm 3^\circ$	$210^\circ \pm 20^\circ$
Longitude of node at epoch	$305^\circ \pm 5^\circ$	$311^\circ\cdot9 \pm 0^\circ\cdot9$
Daily mean motion in longitude	$1128^\circ\cdot8443 \pm 0^\circ\cdot0001$	$285^\circ\cdot16192 \pm 0^\circ\cdot00001$

This value was calculated from the formulae of Section 6, using values given by Wilkins (1967) for J_2 and f. In fact, if the values of J_2 and f obtained from our final solution are used then p is only changed slightly, to $-1\cdot07 \times 10^{-5}$.

The solution finally adopted is given in Table I(a). It was obtained using all observations with residuals less than 2″, and gave a root-mean-square residual of 0″·550. Formal standard errors are given throughout. The values of certain quantities derived from those in Table I(a) are given in Table II, together with some of the quantities in Table I(a) expressed in more conventional units. It can be seen that the values obtained for the elements of the satellites are in substantial agreement with those obtained by Wilkins (1968), which are given in Table III converted to our reference system. In the present solution we have obtained a significant value for the eccentricity of the orbit of Deimos.

The solution given in Table I(b) was obtained by omitting the observations of 1877, 1879 and 1881, as it will be shown that these observations give residuals in longitude that stand out from the general trend. An ephemeris computed using the values of Table I(b) could well be more accurate at the present time than one using Table I(a).

A solution was tried in which observations with residuals greater than 1″ were omitted. This caused about 9 per cent of the observations to be omitted, and gave a root-mean-square residual of 0″·426. However, it produced no significant changes in the values of the constants.

A solution was tried using the value of the mass of Mars (1/3098600) obtained from the Mariner IV spacecraft observations (Null, Gordon & Tito 1967). It gave a very small increase in the root-mean-square residual but gave no significant changes in the values of the constants.

The periodic terms calculated in Part I were found to be very small, and omitting them had an insignificant effect on the root-mean-square residual and the values of the constants.

To calculate J_2 and f, the flattening of Mars, from the data in Table I(a), we need to know values for r_0, the equatorial radius of Mars, and k^2M. The Mariner IV observations gave $k^2M = 42830 \pm 8\ \mathrm{km^3\,s^{-2}}$, and O'Handley *et al.* (1970) give $r_0 = 3393 \pm 2$ km, obtained from Earth-based radar ranging. Then, using the formulae

$$f = \tfrac{3}{2}J_2 + \tfrac{1}{2}\sigma - 5{\cdot}8 \times 10^{-6}$$

(the constant being a correction for second-order terms in J_2 and σ)

$$\sigma = \frac{r_0\omega^2}{k^2Mr_0^{-2}(1+3J_2/2) - r_0\omega^2}$$

where σ is the ratio of the centrifugal acceleration to the apparent acceleration at the equator of Mars, and ω is the angular speed of rotation of Mars, we obtain

$$J_2 = 0{\cdot}001966 \pm 0{\cdot}000003$$
$$f = 0{\cdot}005238 \pm 0{\cdot}000009$$
$$\sigma = 0{\cdot}004590 \pm 0{\cdot}000008.$$

Also we calculate the moment of inertia of Mars from the formula in Section 6, giving

$$C/Mr_0^2 = 0{\cdot}376.$$

Values for the right ascension and declination of the pole of Mars relative to the mean equator and equinox of date have been calculated from the values in Table I(a). A period of precession of 173 000 years was taken for the equator of Mars. The standard errors given are calculated from the standard errors of P_0 and Q.

At the epoch t,

$$\alpha = 317^\circ{\cdot}31 \pm 0^\circ{\cdot}05 + 0^\circ{\cdot}0068\,(t - 1950{\cdot}0)$$
$$\delta = 52^\circ{\cdot}65 \pm 0^\circ{\cdot}03 + 0^\circ{\cdot}0035\,(t - 1950{\cdot}0).$$

The values of these quantities in current use in the Astronomical Ephemeris are obtained from de Vaucouleurs (1964), and are

$$\alpha = 316^\circ{\cdot}55 + 0^\circ{\cdot}006750\ (t - 1905{\cdot}0)$$

$$\delta = 52^\circ{\cdot}85 + 0^\circ{\cdot}003479\ (t - 1905{\cdot}0).$$

Table VI gives the number of observations, their root-mean-square residual, and the telescope used, for each observer at each opposition at which he has observed the satellites.

TABLE IV

Analysis of observation residuals

Opposition	Observer	Instrument	No. of observations used	R.M.S. residual (″ of arc)
1877	A. Hall (Snr.)	Washington 26″	171	0·78
1879	A. Hall (Snr.)	Washington 26″	168	0·53
1881	A. Hall (Snr.)	Washington 26″	32	0·56
1888	J. E. Keeler	Lick 36″	166	0·51
1890	A. Hall (Snr.)	Washington 26″	19	0·81
	J. E. Keeler	Lick 36″	73	0·53
1892	A. Hall (Snr.)	Washington 26″	70	0·95
	W. W. Campbell	Lick 36″	179	0·51
1894	W. W. Campbell	Lick 36″	734	0·42
	S. J. Brown	Washington 26″	112	0·65
1896	S. Kostinsky	Pulkovo 12″	4	0·47
	S. J. Brown	Washington 26″	58	0·56
	W. J. Hussey	Lick 36″	80	0·68
	J. M. Schaeberle	Lick 36″	116	0·59
1907	H. L. Rice	Washington 26″	52	0·57
1909	S. Kostinsky	Pulkovo 13″	48	0·61
	A. Hall (Jnr.)	Washington 26″	220	0·53
1922	A. Hall (Jnr.)	Washington 26″	28	0·65
1924	A. Hall (Jnr.)	Washington 26″	137	0·41
	E. C. Bower	Washington 26″	85	0·63
1926	A. Hall (Jnr.)	Washington 26″	75	0·37
	H. E. Burton	Washington 26″	107	0·30
1928	H. E. Burton	Washington 26″	16	0·29
1941	H. E. Burton	Washington 26″	48	0·59
	G. M. Raynsford	Washington 26″	26	0·62
1956	R. S. Richardson	Mt Wilson 60″	21	0·65
	G. P. Kuiper	McDonald 82″	44	0·38
		Pulkovo 9″	33	0·84
1963		Kiev 16″	22	0·77
1967		Pulkovo 9″	52	0·66
		Pulkovo 26″	26	0·25
		Kiev 16″	36	0·54
		Kazan 16″	22	0·71
		Kiev Astrograph	7	0·93
1969	D. Pascu	Flagstaff 61″	20	0·21

14. SECULAR ACCELERATIONS

The values given for the secular accelerations of Phobos and Deimos in Table I(a) are greater than their standard errors. (N.B. The acceleration of Deimos is negative.) However, further investigation shows that these values cannot be afforded much confidence.

A solution for the arbitrary constants was made in which the secular accelerations of Phobos and Deimos were given zero values. Then, using this set of constants as initial values, for each opposition a solution was made using only the observations at that opposition, and the only quantities solved for were corrections to the mean longitudes at epoch of Phobos and Deimos. The corrections obtained are interpreted as corrections to be added to the computed longitude to give the observed longitude, and are given in Figs 3 and 4 for Phobos and Deimos. (At the opposition of 1963 all the observations were of Deimos, so no correction was obtained for Phobos.) Any secular acceleration in longitude should show up as a parabola in these residuals.

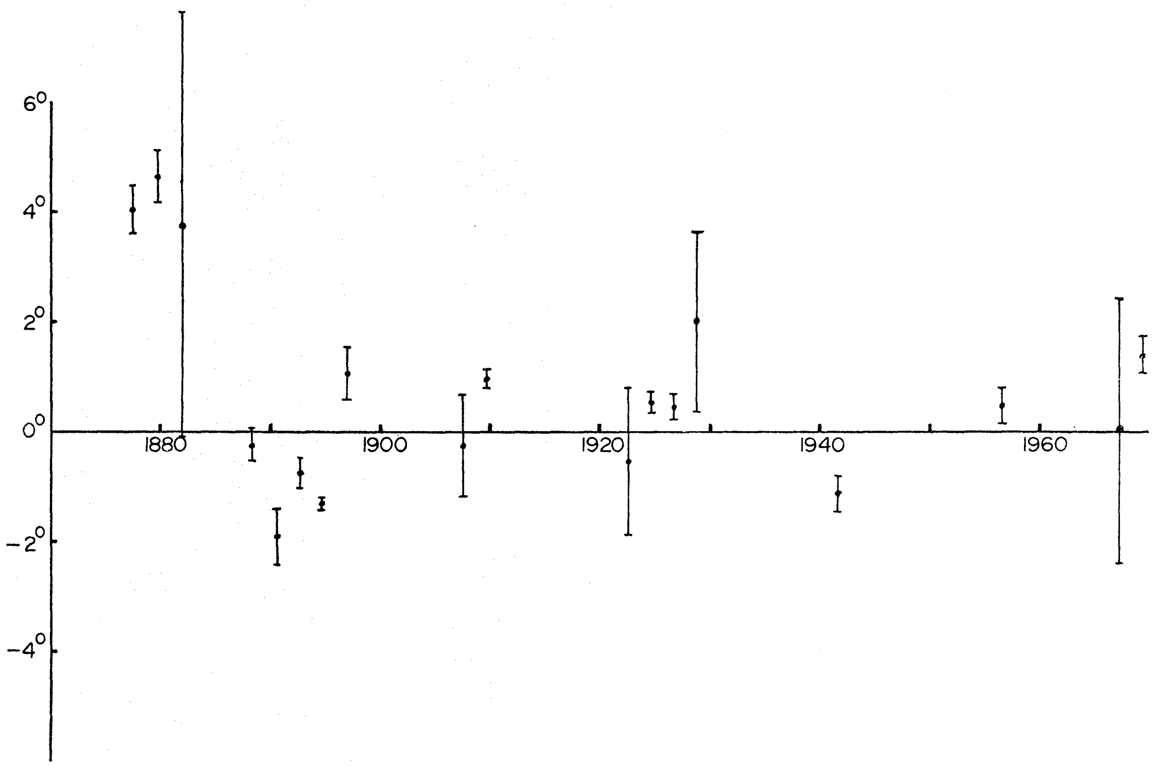

FIG. 3. *Residuals in longitude for Phobos.*

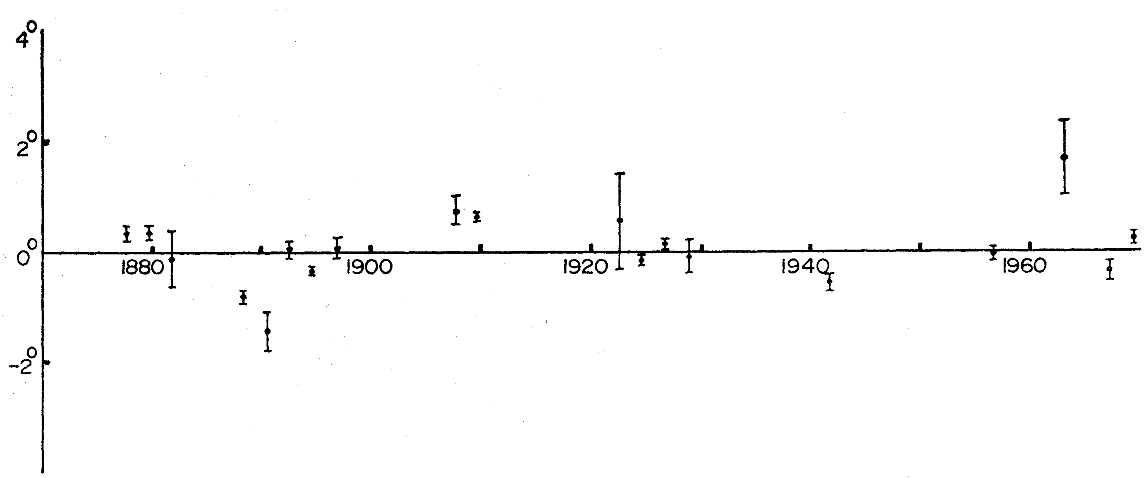

FIG. 4. *Residuals in longitude for Deimos.*

It can be seen from Fig. 3 that the positive secular acceleration for Phobos given in Table I(a) is due to the three large positive residuals at 1877, 1879 and 1881. Accordingly a solution was made omitting these observations and is given in Table I(b). A negative acceleration is obtained for Phobos. Sharpless (1945) obtained a positive acceleration for Phobos from an analysis of observations from 1877 to 1941. Accordingly a solution was made omitting observations later than 1941. The value obtained for the acceleration of Phobos was close to that of Sharpless. In all these solutions a negative acceleration was obtained for Deimos, but its magnitude varied considerably. The results of all these solutions are given below, with the secluar accelerations expressed in degrees year^{-2}.

	Phobos	Deimos
Solution using all observations	$+0{\cdot}00096 \pm 0{\cdot}00016$	$-0{\cdot}000063 \pm 0{\cdot}000044$
Solution omitting 1877, 1879, 1881 observations	$-0{\cdot}00083 \pm 0{\cdot}00019$	$-0{\cdot}000396 \pm 0{\cdot}000056$
Solution omitting observations after 1941	$+0{\cdot}00170 \pm 0{\cdot}00029$	$-0{\cdot}000182 \pm 0{\cdot}000094$
Values obtained by Sharpless	$+0{\cdot}00188 \pm 0{\cdot}00025$	$-0{\cdot}000266 \pm 0{\cdot}000243$

From these results it would seem that the observations of the satellites are not sufficiently accurate to show conclusively if the longitudes of the satellites are affected by secular accelerations but a small negative acceleration is indicated for Deimos.

Even if the large residuals in longitude of the early observations are believed, the general trend of the residuals does not correspond particularly well to a parabola that would be caused by a secular acceleration. The general trends of the residuals are remarkably similar, which suggests that they could perhaps be due to systematic timing errors. An error of 1 minute in time would cause a residual in longitude of $0^\circ{\cdot}78$ for Phobos and $0^\circ{\cdot}20$ for Deimos. However, we cannot see how systematic timing errors of the order of a few minutes necessary to explain the observed residuals could arise.

In a recent paper (Shor, Glebova & Sorokina 1971) the observations of the satellites are re-examined by a method similar to that used by Sharpless. Observations up to 1956 are used, and a value for the secular acceleration of Phobos of $(0{\cdot}101 \pm 0{\cdot}011) \times 10^{-7}$ deg day^{-2} [or $0{\cdot}00134 \pm 0{\cdot}00015$ deg year^{-2}] is obtained.

15. CONCLUSIONS

It is found that the periodic perturbations in the motions of the satellites are very small, and have little noticeable effect on their positions as seen from the Earth. Also, perturbations due to the J_3 and J_4 coefficients in the potential of Mars, and due to the precession of the equator of Mars, are not large enough for the determination of these quantities. The observations are not sufficiently accurate to show conclusively if the longitudes of the satellites are affected by secular accelerations. The values obtained for the orbital elements of the satellites are basically in agreement with those obtained by previous investigators. However, the values obtained, and in particular those determining the position of the equator of Mars, should be more accurate than previous values due to the longer time interval covered and the use of a more complete theoretical model of the motions of the satellites. Also the values derived for J_2 and f should be superior to previous values due to the use of more accurate values of r_0 and k^2M.

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