THE MULTI-ATTRIBUTE GROUP DECISION MAKING METHOD BASED ON THE INTERVAL GREY LINGUISTIC VARIABLES WEIGHTED HARMONIC AGGREGATION OPERATORS

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Abstract. With respect to the characteristics of fuzziness, complexity and uncertainty for many group-decision making problems in real world, the paper proposes a novel method based on the interval grey linguistic variables hybrid weighted harmonic aggregation operators to solve the multiple attribute group decision making problems in which the attribute values and the weights take the form of the interval grey linguistic variables. In the approach, the relative concepts and the operation rules of interval grey linguistic variables are defined, and some operators (such as interval grey linguistic weighted harmonic aggregation (IGLWHA) operator, interval grey linguistic ordered weighted harmonic aggregation (IGLOWHA) operator, and interval grey linguistic hybrid weighted harmonic aggregation (IGLHWHA) operator) are proposed to solve the group decision making problems. The computational results from an illustrative example have shown that the proposed approach is feasible and effective for the group-decision making problems.

Keywords: group decision-making; fuzzy numbers; grey fuzzy number; interval grey linguistic variables; the interval grey linguistic variables hybrid weighted harmonic aggregation operator.

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Introduction

Multiple attribute decision making (MADM) has been extensively applied to various areas such as society, economics, management, military and engineering technology, etc. Since the objective things are complex, uncertainty and human thinking is ambiguous, the majority of multi-attribute decision-making problems are uncertain and fuzzy, so fuzziness is the major factor which should be considered in the process of decision making. On the other hand, decision-making problems have the greyness in the process of dealing with the incomplete information. "Greyness" means amount of information is smaller' and inadequate. For example, in agriculture planting, even if the sown area, seed, fertilizer, irrigation and other information are completely clear, it is still difficult to accurately predict the productive output. The productive output is grey. Another example, in 2050, China's total population will be controlled between 15 and 16 billion. This "between 15 and 16 billion" is a concept of grey, we cannot know the accurate value. So, the "greyness" is a concept about "quantity". However, "fuzziness" means a concept is not clear. For example, about "young people", it is very difficult to designate an exact range in which they are young people and out which they are not, so it is fuzzy. Other examples, such as "hot water", "wet" etc. are fuzzy. So "fuzziness" is a concept about "quality". Obviously, "greyness" and "fuzziness" don't mean that some information is "grey information" and the other part of the information is "fuzzy information" for the same problem because they don't describe the same concept (Bu, Zhang 2002). In general, we can simply interpret "greyness" and "fuzziness" as width and depth of an evaluation object. In reality, the decision making problems have not only the fuzziness, but also the greyness, which are called the grey fuzzy multi-attribute decision making problems. For example, about ability of innovation management of enterprises, it has the fuzziness and greyness simultaneously, because the concept of ability of innovation management is unclear, i.e., it has the fuzziness; at the same time, we cannot get all information about ability of innovation management of enterprises, so it has the greyness. There are similar examples, such as moral evaluation, working ability assessment, evaluation of a person's level of knowledge, etc.

About fuzzy theory, Zadeh (1965) firstly proposed the theory of Fuzzy Sets, The core idea is to extend membership function to any value in the closed interval [0,1]. Then fuzzy sets had been extended to interval numbers, triangular fuzzy numbers, trapezoidal fuzzy numbers, linguistic variables, intuitionistic fuzzy numbers, etc. and widely used in the field of decision-making (Herrera *et al.* 1996; Liu 2011; Liu, Su 2010; Liu, Zhang 2010; Zhang, Liu 2010; Yu 2013; Razavi Hajiagha *et al.* 2013). About grey theory, Deng (1982) firstly proposed the theory of Grey Systems, then grey theory has been the rapid development, and a series of grey decision-making methods were proposed (Deng 2002; Liu, W. L., Liu, P. D. 2010). The grey fuzzy theory, which combined the fuzzy theory and the grey theory, takes into account the greyness and fuzzynes of decision making problems, and is more in line with the objective reality of things. At present, there have been consistent efforts to the research on grey fuzzy decision making problems. Bu and Zhang (2002), Choobineh and Li (1993a, b), Jin and Lou (2003, 2004), Luo and Liu (2004) studied the ranking method of grey fuzzy number. Bu and Zhang (2002) transformed the grey fuzzy number into the interval number, and then utilized the ranking method of interval number to rank the order

of alternatives. For the grey fuzzy multiple attribute decision making problems which both the fuzzy part and the grey part took the form of real number, Jin and Lou (2003) proposed the decision making model which utilized the difference between the alternatives and the fuzzy positive ideal solution, and between the alternatives and the negative ideal solution to rank the orders based on the Hamming distance. Jin and Lou (2004) utilized the distance between each alternative and the grey fuzzy ideal solution to rank the orders of alternatives. In order to solve the grey fuzzy decision making problems, Luo and Liu (2004) utilized the maximum entropy formula to determine attribute weights, then ranked the orders of alternatives based on the linear combination of fuzzy information and grey information. Zhu *et al.* (2006) constructed the evaluation model in which the fuzzy part and the grey part took the form of interval number and the real number respectively. Meng *et al.* (2007) proposed the interval numbers to present greyness and fuzziness of grey fuzzy decision making problems, and the mathematical model of interval valued grey fuzzy comprehensive evaluation is established, and the application to the selection of the preferred project is given. Wang and Wang (2008) extended the fuzzy part and the grey part of grey fuzzy decision making problems to interval numbers, and ranked the order of alternatives based on the ordered weight aggregation (OWA) operator. Zhang (2013) proposed the interval grey linguistic variables ordered weighted aggregation (IGLOWA) operator, and then use the Choquet integral to develop the interval grey linguistic correlated ordered arithmetic aggregation (IGLCOA) operator and the interval grey linguistic correlated ordered geometric aggregation (IGLCOGA) operator.

Because the linguistic variables are easier to express fuzzy information, and the research on multi-attribute decision making based on the linguistic variables has made great achievements (Alonso *et al.* 2009; Cabrerizo *et al.* 2010a, b; Herrera *et al.* 2009; Herrera-Viedma *et al.* 2003; Kim, Ahn 1999; Martínez *et al.* 2009; Xu 2004, 2007, 2008). So, this paper proposes the concept of interval grey linguistic variables in which the fuzzy part and the grey part adopt linguistic variables and interval numbers respectively, and then studies the operation rules and the multiple attribute decision making method based on interval grey linguistic variables.

The remainder of the paper is organized as follows: Section 1 introduces some relative knowledge; Section 2 defines the interval grey linguistic variables and proposes some weighted harmonic aggregation operators; Section 3 gives a method based on the interval grey linguistic variables hybrid weighted harmonic aggregation operators to solve the multiple attribute group decision making problems; Section 4 presents an illustrative example to verify effectiveness of this method and to illustrate its decision making steps; Finally, conclusions are given in the final section.

1. Preliminaries

1.1. Grey number (Deng 2002; Luo 2005; Lu 2009)

Grey number is the basic unit to express the greyness. We can call only knowing the ranges roughly and not knowing the exact value as grey number. In the application, the grey number

generally refers to a range or an uncertain number, and it can be expressed by "$\otimes$". Grey number can be divided into the following categories:

1) The grey number only with a lower bound

   The grey number in this type can be expressed as $\mu_A(x) \to [0,1]$, where $\underline{a}$ is the lower bound of the grey number $\otimes$ and it's also a certain number.

2) The grey number only with a upper bound

   The grey number in this type can be expressed as $\otimes \in (-\infty, \overline{a}]$, where $\overline{a}$ is the upper bound of the grey number $\otimes$ and it's also a certain number.

3) The grey number with interval number

   The grey number in this type can be expressed as $\otimes \in [\underline{a}, \overline{a}]$, where $\underline{a}$ and $\overline{a}$ are a certain number, and $\underline{a}$ is the lower bound, $\overline{a}$ is the upper bound of the grey number $\otimes$.

4) The grey number with three-point interval number

   The grey number in this type can be expressed as $\otimes \in [\underline{a}, a, \overline{a}]$, where $\underline{a}$, a and $\overline{a}$ are a certain number; $\underline{a}$ and $\triangledown i$ are the lower bound and upper bound of the grey number $\otimes$ respectively, and a is the center of gravity which can be get most likely.

5) The black number and white number

   When grey number $\otimes \in (-\infty, +\infty)$, we can call $\otimes$ as a black number. It shows that information is completely unknown; when grey number $\otimes \in [\underline{a}, \overline{a}]$ and $\underline{a} = \overline{a}$, we can call $\otimes$ as a white number. It shows information is completely known.

1.2. Grey fuzzy math
(Chen 1994; Li and Wang 1994; Wang and Song 1988; Wang 1996)

Definition 1: Let $\underset{\otimes}{\tilde{A}}$ be the fuzzy subset in the space $X = \{x\}$, if the membership degree $\mu_A(x)$ of x to $\underset{\otimes}{\tilde{A}}$ has the greyness $\nu_A(x)$ in the interval [0, 1], then $\underset{\otimes}{\tilde{A}}$ is called the grey fuzzy set in space X:

$$\underset{\otimes}{\tilde{A}} = \left\{ \left(x, \mu_A(x), \nu_A(x) \right) \mid x \in X \right\}. \tag{1}$$

The set pair mode is $\underset{\otimes}{\tilde{A}} = \left(\tilde{A}, \underset{\otimes}{A} \right)$, where $\tilde{A} = \left\{ (x, \mu_A(x)) \mid x \in X \right\}$ is called the fuzzy part of $\underset{\otimes}{\tilde{A}}$, and $\underset{\otimes}{A} = \left\{ (x, \nu_A(x)) \mid x \in X \right\}$ is called the grey part of $\underset{\otimes}{\tilde{A}}$.

So the grey fuzzy set is regarded as the generalization of the fuzzy set and the grey set.

Definition 2: Let $X = \{x\}$ and $Y = \{y\}$ be the given space, if $\nu_R(x, y)$ is the greyness of the membership function $\mu_R(x, y)$ of $\tilde{R}$ which is the fuzzy relationship between x and y, then grey fuzzy set $\underset{\otimes}{\tilde{R}} = \left\{ ((x, y), \mu_R(x, y), \nu_R(x, y)) \mid x \in X, y \in Y \right\}$ is called the grey fuzzy relationship in direct product space $X \times Y$, which is represented as the grey fuzzy matrix mode:

$$\underset{\otimes}{\tilde{R}} = \begin{bmatrix} (\mu_{11}, \nu_{11}) & (\mu_{12}, \nu_{12}) & \cdots & (\mu_{1n}, \nu_{1n}) \\ (\mu_{21}, \nu_{21}) & (\mu_{22}, \nu_{22}) & \cdots & (\mu_{2n}, \nu_{2n}) \\ \vdots & \vdots & \vdots & \vdots \\ (\mu_{m1}, \nu_{m1}) & (\mu_{m2}, \nu_{m2}) & \cdots & (\mu_{mn}, \nu_{mn}) \end{bmatrix}. \tag{2}$$

And $\tilde{\underset{\otimes}{R}} = \left(\tilde{R}, \underset{\otimes}{R} \right)$ represents the grey fuzzy relationship in direct product space $X \times Y$, where $\tilde{R} = \{((x,y), \mu_A(x,y)) \mid x \in X, y \in Y\}$ represents the fuzzy relationship in direct product space $X \times Y$, and $\underset{\otimes}{R} = \{((x,y), \nu_A(x,y)) \mid x \in X, y \in Y\}$ represents the grey relationship in direct product space $X \times Y$.

1.3. Linguistic evaluation set and its extension

Suppose that $S = (s_1, s_2, \cdots, s_l)$ is a finite and totally ordered discrete term set, where l is the odd number. In real situation, l is equal to 3, 5, 7, 9 etc. In this paper, $l = 7$. For example, a set S could be given as follows:

$S = (s_1, s_2, s_3, s_4, s_5, s_6, s_7) = \{$very poor, poor, slightly poor, fair, slightly good, good, very good$\}$.

Usually, in these cases, it requires that s_i and s_j must satisfy the following additional characteristics (Herrera, Herrera-Viedma 2000):

1) The set is ordered: $s_i \prec s_j$, if and only if $i < j$;
2) There is the negation operator: $neg(s_i) = s_{l-i}$;
3) Maximum operator: $\max(s_i, s_j) = s_i$, if $i \geq j$;
4) Minimum operator: $\min(s_i, s_j) = s_i$, if $i \leq j$.

For any linguistic labels $S = (s_1, s_2, \cdots, s_l)$, the relationship between the element s_i and its subscript i is strictly monotone increasing (Herrera *et al.* 1996; Xu 2006a), so the function can be defined as follows:

$$f : s_i = f(i) \cdot$$

Obviously, the function $f(i)$ is the strictly monotone increasing function about subscript i. To preserve all the given information, the discrete linguistic label $S = (s_1, s_2, \cdots, s_l)$ is extended to a continuous linguistic label $\overline{S} = \{s_\alpha \mid \alpha \in R\}$ which satisfied the above characteristics. If $s_\alpha \in S$, then s_α is called an original linguistic label, otherwise, s_α is called a virtual linguistic label. In general, the decision maker uses the original label to evaluate attributes and alternatives, and the virtual labels can only appear in the course of operation.

Let $s_i, s_j \in \overline{S}$ and $\lambda_1, \lambda_2 \in [0,1]$, n is a positive integer, then operational laws of linguistic variables are given as follows (Xu 2006b):

1) $\beta s_i = s_{\beta \times i}$;   (3)

2) $s_i \oplus s_j = s_{i+j}$;   (4)

3) $s_i / s_j = s_{i/j}$, if $j \neq 0$;   (5)

4) $\left(s_i \right)^n = s_{i^n}$;   (6)

5) $\lambda_1(s_i \oplus s_j) = \lambda_1 s_i \oplus \lambda_1 s_j$, $(\lambda_1 + \lambda_2)s_i = \lambda_1 s_i \oplus \lambda_2 s_i$.   (7)

Definition 3: Let s_α, s_β be the two linguistic variables, then the distance between s_α and s_β is defined as follows:

$$d(s_\alpha, s_\beta) = |\alpha - \beta|/l \,. \tag{8}$$

2. Interval grey linguistic variables

2.1. The definition of interval grey linguistic variables

Definition 4: Let $\underset{\otimes}{\tilde{A}} = \left(\tilde{A}, \underset{\otimes}{A} \right)$ be the grey fuzzy number, if its fuzzy part $\tilde{A}$ is a linguistic variable $s_\alpha \in \bar{S}$, and its grey part $\underset{\otimes}{A}$ is a closed interval $\left[g_A^L, g_A^U \right] \subseteq [0,1]$, where g_A^L is the interval lower limit, g_A^U is the interval upper limit, and $g_A^L \le g_A^U$, then $\underset{\otimes}{\tilde{A}}$ is called the interval grey linguistic variables.

Because the linguistic variables are easier to express fuzzy information, it is more reasonable to utilize the linguistic variables to represent the fuzzy part, and for the grey part which indicates the amount of information obtained, it is more accurate to reflect the information obtained by decision maker using the interval numbers. The larger the greyness of the grey part is, the less information obtained and the lower credibility of the obtained information is. The lower the credibility of the obtained value is, the lower the usage value of the information is. When the greyness rises, the obtained information becomes useless. On the other hand, the smaller the greyness is, the more information obtained is, which causes higher credibility of the obtained value. That finally leads to higher usage value of the obtained information.

2.2. The operation of the interval grey linguistic variables

Supposed that $\underset{\otimes}{\tilde{A}} = \left(s_\alpha, \left[g_A^L, g_A^U \right] \right)$, $\underset{\otimes}{\tilde{B}} = \left(s_\beta, \left[g_B^L, g_B^U \right] \right)$ and $\underset{\otimes}{\tilde{C}} = \left(s_\lambda, \left[g_C^L, g_C^U \right] \right)$ are the three interval grey linguistic variables. Based on the concept of the interval grey linguistic variables, the linguistic operational rules and extension principle, the operation rules of interval grey linguistic variables are defined as follows:

1) $\underset{\otimes}{\tilde{A}} + \underset{\otimes}{\tilde{B}} = \left(s_{\alpha+\beta}, \left[\max\left(g_A^L, g_B^L \right), \max\left(g_A^U, g_B^U \right) \right] \right)$; $\tag{9}$

2) $\underset{\otimes}{\tilde{A}} - \underset{\otimes}{\tilde{B}} = \left(s_{\alpha-\beta}, \left[\max\left(g_A^L, g_B^L \right), \max\left(g_A^U, g_B^U \right) \right] \right)$; $\tag{10}$

3) $\underset{\otimes}{\tilde{A}} \times \underset{\otimes}{\tilde{B}} = \left(s_{\alpha\times\beta}, \left[\max\left(g_A^L, g_B^L \right), \max\left(g_A^U, g_B^U \right) \right] \right)$; $\tag{11}$

4) $\underset{\otimes}{\tilde{A}} / \underset{\otimes}{\tilde{B}} = \left(s_{\alpha/\beta}, \left[\max\left(g_A^L, g_B^L \right), \max\left(g_A^U, g_B^U \right) \right] \right)$; $\tag{12}$

5) $k \underset{\otimes}{\tilde{A}} = \left(s_{k\times\alpha}, \left[g_A^L, g_A^U \right] \right)$; $\tag{13}$

6) $\left(\underset{\otimes}{\tilde{A}} \right)^k = \left(s_{\alpha^k}, \left[g_A^L, g_A^U \right] \right)$. $\tag{14}$

2.3. The distance between the two interval grey linguistic variables

Definition 5: Let $\tilde{A}_{\otimes}$, $\tilde{B}_{\otimes}$, $\tilde{C}_{\otimes}$ be the interval grey linguistic variables, $\tilde{Z}_{\otimes}$ be the set of the interval grey linguistic variables, f be the mapping, $f: \tilde{Z}_{\otimes} \times \tilde{Z}_{\otimes} \to \tilde{R}_{\otimes}$ .If $d\left(\tilde{A}_{\otimes}, \tilde{B}_{\otimes}\right)$ satisfies the following equation:

1) $0 \leq d\left(\tilde{A}_{\otimes}, \tilde{B}_{\otimes}\right) \leq 1$, $d\left(\tilde{A}_{\otimes}, \tilde{A}_{\otimes}\right) = 0$;

2) $d\left(\tilde{A}_{\otimes}, \tilde{B}_{\otimes}\right) = d\left(\tilde{B}_{\otimes}, \tilde{A}_{\otimes}\right)$;

3) $d\left(\tilde{A}_{\otimes}, \tilde{B}_{\otimes}\right) + d\left(\tilde{B}_{\otimes}, \tilde{C}_{\otimes}\right) \geq d\left(\tilde{A}_{\otimes}, \tilde{C}_{\otimes}\right)$.

Then $d\left(\tilde{A}_{\otimes}, \tilde{B}_{\otimes}\right)$ is called the distance between the interval grey linguistic variable $\tilde{A}_{\otimes}$ and $\tilde{B}_{\otimes}$.

Definition 6: Let $\tilde{A}_{\otimes} = \left(s_\alpha, \left[g_A^L, g_A^U\right]\right)$ and $\tilde{B}_{\otimes} = \left(s_\beta, \left[g_B^L, g_B^U\right]\right)$ be the interval grey linguistic variables, then the Hamming distance $d\left(\tilde{A}_{\otimes}, \tilde{B}_{\otimes}\right)$ between the interval grey linguistic variable $\tilde{A}_{\otimes}$ and $\tilde{B}_{\otimes}$ is defined as follows:

$$d\left(\tilde{A}_{\otimes}, \tilde{B}_{\otimes}\right) = \frac{1}{2(l-1)}\left(\left|\alpha\left(1 - g_A^L\right) - \beta\left(1 - g_B^L\right)\right| + \left|\alpha\left(1 - g_A^U\right) - \beta\left(1 - g_B^U\right)\right|\right). \tag{15}$$

It is easy to verify that the equation (15) satisfies the three equation of definition 5.

Specially, if $g_A^L = g_A^U = g_B^L = g_B^U = 0$, then the interval grey linguistic variable is reduced to linguistic variable, and the equation (15) is transformed into equation (8). That is, the equation (8) is the special case of equation (15).

2.4. The comparing method of interval grey linguistic variables
(1) C-OWA aggregate operator

Definition 7 (Yager 2004)**:** function $\rho: [0,1] \to [0,1]$ satisfied that:

1) $\rho(0) = 0$;

2) $\rho(1) = 1$;

3) if $x > y$, then $\rho(x) > \rho(y)$,

then ρ is called the basic unit-interval monotonic (BUM) function.

Definition 8 (Yager 2004): Let $[a, b]$ be the interval number, and

$$f_\rho([a,b]) = \int_0^1 \frac{d\rho(y)}{dy}(b - y(b - a))dy, \tag{16}$$

then f is called the continuous interval number OWA (C-OWA)operator.

If $\rho(y) = y^\delta (\delta \geq 0)$, then $f_\rho([a,b]) = \dfrac{b + \delta a}{\delta + 1}$.

(2) The expectation value and the rank method of interval grey linguistic variables

Let $\tilde{A}_{\otimes} = \left(s_\alpha, \left[g_A^L, g_A^U \right] \right)$ be the interval linguistic variable, the expectation value of interval grey linguistic variable $\tilde{A}_{\otimes}$ is defined as follows:

$$I(\tilde{A}_{\otimes}) = s_\alpha \times f_\rho \left(\left[\left(1 - g_A^U \right), \left(1 - g_A^L \right) \right] \right). \tag{17}$$

Suppose $\tilde{A}_{\otimes} = \left(s_\alpha, \left[g_A^L, g_A^U \right] \right)$ and $\tilde{B}_{\otimes} = \left(s_\beta, \left[g_B^L, g_B^U \right] \right)$ are two interval linguistic variables, if $I(\tilde{A}_{\otimes}) > I(\tilde{B}_{\otimes})$, then $\tilde{A}_{\otimes} > \tilde{B}_{\otimes}$, and If $I(\tilde{A}_{\otimes}) = I(\tilde{B}_{\otimes})$ and $s_\alpha \succ s_\beta$, then $\tilde{A}_{\otimes} > \tilde{B}_{\otimes}$.

2.5. Interval grey linguistic variables hybrid weighted harmonic aggregation (*IGLHWHA*) operator

Definition 9: Let $\tilde{A}_{\otimes j} = \left(s_{\alpha_j}, \left[g_j^L, g_j^U \right] \right)$ be the group of interval grey linguistic variables, and $IGLWHA : \Omega^n \to \Omega$, if

$$IGLWHA_W \left(\tilde{A}_{\otimes 1}, \tilde{A}_{\otimes 2}, \cdots, \tilde{A}_{\otimes n} \right) = \left(\sum_{j=1}^n \frac{w_j}{\tilde{A}_{\otimes j}} \right)^{-1}, \tag{18}$$

where Ω is the set of all the interval grey linguistic variables, $W = \left(w_1, w_2, \cdots, w_n \right)$ is the weight vector of $\tilde{A}_{\otimes j}$ $(j = 1, 2, \cdots, n)$, and $\sum_{j=1}^n w_j = 1$, then *IGLWHA* is called the interval grey linguistic weighted harmonic aggregation (*IGLWHA*)operator.

Based on the operation rules of interval grey linguistic variables, the equation (18) is deduced to:

$$IGLWHA_W \left(\tilde{A}_{\otimes 1}, \tilde{A}_{\otimes 2}, \cdots, \tilde{A}_{\otimes n} \right) = \left(\sum_{j=1}^n \frac{w_j}{\tilde{A}_{\otimes j}} \right)^{-1} = \left(s_{\left(\sum_{j=1}^n \frac{w_j}{\alpha_j} \right)^{-1}}, \left[\max_j \left(g_j^L \right), \max_j \left(g_j^U \right) \right] \right). \tag{19}$$

Example 1. Assume $W = (0.2, 0.3, 0.1, 0.4)$ and

$$\tilde{A}_{\otimes 1} = \left(s_2, [0.2, 0.4] \right), \tilde{A}_{\otimes 2} = \left(s_1, [0.1, 0.3] \right), \tilde{A}_{\otimes 3} = \left(s_5, [0.4, 0.5] \right), \tilde{A}_{\otimes 4} = \left(s_4, [0.3, 0.6] \right).$$

By Definition 9, we have

$$IGLWHA_W\left(\tilde{A}_{\otimes 1},\tilde{A}_{\otimes 2},\tilde{A}_{\otimes 3},\tilde{A}_{\otimes 4}\right)=\left(s_{\left(\sum_{j=1}^{4}\frac{w_j}{\alpha_j}\right)^{-1}},\left[\max_j\left(g_j^L\right),\max_j\left(g_j^U\right)\right]\right)=$$

$$\left(s_{\left(\frac{w_1}{\alpha_1}+\frac{w_2}{\alpha_2}+\frac{w_3}{\alpha_3}+\frac{w_4}{\alpha_4}\right)^{-1}},\left[\max\left(g_1^L,g_2^L,g_3^L,g_4^L\right),\max\left(g_1^U,g_2^U,g_3^U,g_4^U\right)\right]\right)=$$

$$\left(s_{\left(\frac{0.2}{2}+\frac{0.3}{1}+\frac{0.1}{5}+\frac{0.4}{4}\right)^{-1}},\left[\max\left(0.2,0.1,0.4,0.3\right),\max\left(0.4,0.3,0.5,0.6\right)\right]\right)=$$

$$\left(s_{\left(\frac{0.2}{2}+\frac{0.3}{1}+\frac{0.1}{5}+\frac{0.4}{4}\right)^{-1}},\left[0.4,0.6\right]\right)=\left(s_{(0.52)^{-1}},\left[0.4,0.6\right]\right)=\left(s_{1.92},\left[0.4,0.6\right]\right).$$

It's easy to prove that the *IGLWHA* operator has the following properties.

1) **Theorem 1** (Commutativity)

   If $\left(\tilde{A}_{\otimes 1}',\tilde{A}_{\otimes 2}',\cdots,\tilde{A}_{\otimes n}'\right)$ is any permutation of $\left(\tilde{A}_{\otimes 1},\tilde{A}_{\otimes 2},\cdots,\tilde{A}_{\otimes n}\right)$, then

$$IGLWHA_W\left(\tilde{A}_{\otimes 1}',\tilde{A}_{\otimes 2}',\cdots,\tilde{A}_{\otimes n}'\right)=IGLWHA_W\left(\tilde{A}_{\otimes 1},\tilde{A}_{\otimes 2},\cdots,\tilde{A}_{\otimes n}\right).$$

2) **Theorem 2** (Idempotency)

   If $\tilde{A}_{\otimes j}=\tilde{A}_{\otimes}$ for all j, then

$$IGLWHA_W\left(\tilde{A}_{\otimes 1},\tilde{A}_{\otimes 2},\cdots,\tilde{A}_{\otimes n}\right)=\tilde{A}_{\otimes}.$$

3) **Theorem 3** (Monotonicity)

   If $\tilde{A}_{\otimes j}\le\tilde{A}_{\otimes j}'$ for all j, then

$$IGLWHA_W\left(\tilde{A}_{\otimes 1},\tilde{A}_{\otimes 2},\cdots,\tilde{A}_{\otimes n}\right)\le IGLWHA_W\left(\tilde{A}_{\otimes 1}',\tilde{A}_{\otimes 2}',\cdots,\tilde{A}_{\otimes n}'\right).$$

Definition 10: Let $\tilde{A}_{\otimes j}=\left(s_{\alpha_j},\left[g_j^L,g_j^U\right]\right)$ be the group of interval grey linguistic variables, and $IGLOWHA:\Omega^n\to\Omega$, if

$$IGLOWHA_\omega\left(\tilde{A}_{\otimes 1},\tilde{A}_{\otimes 2},\cdots,\tilde{A}_{\otimes n}\right)=\left(\sum_{j=1}^{n}\frac{\omega_j}{\tilde{A}_{\otimes\sigma_j}}\right)^{-1}, \tag{20}$$

where Ω is the set of all the interval grey linguistic variables, $\omega=(\omega_1,\omega_2,\cdots,\omega_n)$ is the weight vector associated with the function *IGLOWHA*, and $\sum_{j=1}^{n}\omega_j=1$. Supposed that the

possible permutation of $(1,2,\cdots,n)$ is $(\sigma_1,\sigma_2,\cdots,\sigma_n)$, and for any j, we can get $\tilde{A}_{\otimes\sigma_{j-1}} \geq \tilde{A}_{\otimes\sigma_j}$,
then *IGLOWHA* is called the interval grey linguistic ordered weighted harmonic aggregation

(*IGLOWHA*)operator.

Based on the operation rules of interval grey linguistic variables, the equation (20) is
deduced to:

$$IGLOWHA_\omega\left(\tilde{A}_{\otimes 1},\tilde{A}_{\otimes 2},\cdots,\tilde{A}_{\otimes n}\right) = \left(\sum_{j=1}^n \frac{\omega_j}{\tilde{A}_{\otimes\sigma_j}}\right)^{-1} =$$

$$\left(s_{\left(\sum_{j=1}^n \frac{w_j}{\alpha_{\sigma_j}}\right)^{-1}}, \left[\max_j\left(g_j^L\right),\max_j\left(g_j^U\right)\right]\right). \tag{21}$$

The characteristic of *IGLOWHA* operator is that interval grey linguistic variables
$\tilde{A}_{\otimes j}$ $(j=1,2,\cdots,n)$ is ranked in descending order and aggregated with weights. ω_j is associated
with the jth position of the aggregation process, and $\tilde{A}_{\otimes j}$ isn't associated with ω_j. So ω is
called the position weighted vector.

According to the actual situation, the position weight vector $\omega=(\omega_1,\omega_2,\cdots,\omega_n)$ is de-
termined by the following method:

1) The method by Wang and Xu (2008):

$$\omega_1 = \frac{1-\alpha}{n}+\alpha, \omega_j = \frac{1-\alpha}{n}, j\neq 1, \alpha\in[0,1]. \tag{22}$$

2) The position weighted vector ω is determined by the method which proposed by Wang
and Xu (2008). The equation is shown as follows:

$$\omega_{i+1} = \frac{C_{n-1}^i}{2^{n-1}} \qquad i=0,1,\cdots,n-1. \tag{23}$$

Example 2. Assume $\omega=\left(\frac{1}{8},\frac{3}{8},\frac{3}{8},\frac{1}{8}\right)$ (calculated by Eq. 23) and

$\tilde{A}_{\otimes 1} = \left(s_2,[0.2,0.4]\right)$, $\tilde{A}_{\otimes 2} = \left(s_1,[0.1,0.3]\right)$, $\tilde{A}_{\otimes 3} = \left(s_5,[0.4,0.5]\right)$, $\tilde{A}_{\otimes 4} = \left(s_4,[0.3,0.6]\right)$.

Firstly, calculate the expectation value of interval grey linguistic variables $\tilde{A}_{\otimes 1},\tilde{A}_{\otimes 2},\tilde{A}_{\otimes 3}$ and $\tilde{A}_{\otimes 4}$
according to Eq. (17).

Suppose $\rho(y)=y^2$, then $f_\rho([a,b])=\frac{b+2a}{3}$. So,

$$I(\tilde{A}_{\otimes 1}) = s_{\alpha_1}\times f_\rho\left(\left[\left(1-g_1^U\right),\left(1-g_1^L\right)\right]\right) = s_2\times f_\rho\left(\left[\left(1-0.4\right),\left(1-0.2\right)\right]\right) =$$

$$s_2\times f_\rho\left(\left[0.6,0.8\right]\right) = s_2\times\frac{0.8+2\times 0.6}{3} = s_2\times 0.67 = s_{1.34}.$$

Similarly,

$$I(\tilde{A}_{\otimes_2}) = s_{\alpha_2} \times f_\rho\left(\left[\left(1-g_2^U\right),\left(1-g_2^L\right)\right]\right) = s_1 \times f_\rho\left(\left[\left(1-0.3\right),\left(1-0.1\right)\right]\right) = s_1 \times \frac{0.9+2\times0.7}{3} = s_{0.77};$$

$$I(\tilde{A}_{\otimes_3}) = s_{\alpha_3} \times f_\rho\left(\left[\left(1-g_3^U\right),\left(1-g_3^L\right)\right]\right) = s_5 \times f_\rho\left(\left[\left(1-0.5\right),\left(1-0.4\right)\right]\right) = s_5 \times \frac{0.6+2\times0.5}{3} = s_{2.65};$$

$$I(\tilde{A}_{\otimes_4}) = s_{\alpha_4} \times f_\rho\left(\left[\left(1-g_4^U\right),\left(1-g_4^L\right)\right]\right) = s_4 \times f_\rho\left(\left[\left(1-0.6\right),\left(1-0.3\right)\right]\right) = s_4 \times \frac{0.7+2\times0.4}{3} = s_{2.00}.$$

Secondly, we rank the $\tilde{A}_{\otimes_1}, \tilde{A}_{\otimes_2}, \tilde{A}_{\otimes_3}$ and $\tilde{A}_{\otimes_4}$, the results are shown as follows:

$$\sigma_1 = 3, \ \sigma_2 = 4, \ \sigma_3 = 1, \ \sigma_4 = 2.$$

Finally, by definition 10, we have:

$$IGLOWHA_\omega\left(\tilde{A}_{\otimes_1}, \tilde{A}_{\otimes_2}, \tilde{A}_{\otimes_3}, \tilde{A}_{\otimes_4}\right) = \left(s_{\left(\sum_{j=1}^4 \frac{\omega_j}{\alpha_{\sigma_j}}\right)^{-1}}, \left[\max_j\left(g_j^L\right), \max_j\left(g_j^U\right)\right]\right) =$$

$$\left(s_{\left(\frac{\omega_1}{\alpha_3}+\frac{\omega_2}{\alpha_4}+\frac{\omega_3}{\alpha_1}+\frac{\omega_4}{\alpha_2}\right)^{-1}}, \left[\max\left(g_1^L,g_2^L,g_3^L,g_4^L\right), \max\left(g_1^U,g_2^U,g_3^U,g_4^U\right)\right]\right) =$$

$$\left(s_{\left(\frac{0.125}{5}+\frac{0.375}{4}+\frac{0.375}{2}+\frac{0.125}{1}\right)^{-1}}, \left[\max\left(0.2,0.1,0.4,0.3\right), \max\left(0.4,0.3,0.5,0.6\right)\right]\right) =$$

$$\left(s_{(0.43)^{-1}}, \left[0.4,0.6\right]\right) = \left(s_{2.32}, \left[0.4,0.6\right]\right).$$

It's easy to prove that the *IGLOWHA* operator has the following properties.

1) **Theorem 1** (Commutativity).

If $\left(\tilde{A}_{\otimes_1}', \tilde{A}_{\otimes_2}', \cdots, \tilde{A}_{\otimes_n}'\right)$ is any permutation of $\left(\tilde{A}_{\otimes_1}, \tilde{A}_{\otimes_2}, \cdots, \tilde{A}_{\otimes_n}\right)$, then

$$IGLOWHA_\omega\left(\tilde{A}_{\otimes_1}', \tilde{A}_{\otimes_2}', \cdots, \tilde{A}_{\otimes_n}'\right) = IGLOWHA_\omega\left(\tilde{A}_{\otimes_1}, \tilde{A}_{\otimes_2}, \cdots, \tilde{A}_{\otimes_n}\right).$$

2) **Theorem 2** (Idempotency)

If $\tilde{A}_{\otimes_j} = \tilde{A}_{\otimes}$ for all j, then

$$IGLOWHA_\omega\left(\tilde{A}_{\otimes_1}, \tilde{A}_{\otimes_2}, \cdots, \tilde{A}_{\otimes_n}\right) = \tilde{A}_{\otimes}.$$

3) **Theorem 3** (Monotonicity).

If $\tilde{A}_{\otimes j} \leq \tilde{A}_{\otimes j}'$ for all j, then

$$IGLOWHA_\omega\left(\tilde{A}_{\otimes 1},\tilde{A}_{\otimes 2},\cdots,\tilde{A}_{\otimes n}\right) \leq IGLOWHA_\omega\left(\tilde{A}_{\otimes 1}',\tilde{A}_{\otimes 2}',\cdots,\tilde{A}_{\otimes n}'\right).$$

The *IGLWHA* operator only focuses on the importance of each interval grey linguistic variable itself, and *IGLOWHA* operator only weights the position of each interval grey linguistic variable, therefore, both of them have certain one-sidedness. In order to overcome the above weaknesses, the interval grey linguistic variable hybrid weighted harmonic aggregation operator is defined as follows:

Definition 11: Let $\tilde{A}_{\otimes j} = \left(s_{\alpha_j},\left[g_j^L,g_j^U\right]\right)$ be the group of interval grey linguistic variables, and $IGLHWHA:\Omega^n \to \Omega$, if

$$IGLHWHA_{\omega,\tilde{W}_\otimes}\left(\tilde{A}_{\otimes 1},\tilde{A}_{\otimes 2},\cdots,\tilde{A}_{\otimes n}\right) = \left(\sum_{j=1}^n \frac{\omega_j}{\tilde{B}_{\otimes\sigma_j}}\right)^{-1}, \tag{24}$$

where Ω is the set of all the interval grey linguistic variables, $\omega=(\omega_1,\omega_2,\cdots,\omega_n)$ is the weight vector associated with the function *IGLHWHA*, and $\sum_{j=1}^n \omega_j = 1$. $\tilde{W}_\otimes = \left(\tilde{W}_{\otimes 1},\tilde{W}_{\otimes 2},\cdots,\tilde{W}_{\otimes n}\right)$ is the weight vector of $\tilde{A}_{\otimes j}$ $(j=1,2,\cdots,n)$, and $\tilde{W}_{\otimes j}$ is the interval grey linguistic variable represented as $\tilde{W}_{\otimes j} = \left(s_{w_j},\left[w_j^L,w_j^U\right]\right)$. $(\sigma_1,\sigma_2,\cdots,\sigma_n)$ is a permutation of $(1,2,\cdots,n)$, and $\tilde{B}_{\otimes j} = \frac{\tilde{A}_{\otimes j}}{\tilde{W}_{\otimes j}} = \left(s_{\beta_j},\left[g_{B_j}^L,g_{B_j}^U\right]\right)$, for any j, we can get $\tilde{B}_{\otimes\sigma_{j-1}} \geq \tilde{B}_{\otimes\sigma_j}$, then *IGLHWHA* is called the interval grey linguistic hybrid weighted harmonic aggregation (*IGLHWHA*)operator.

Example 3. Assume $\omega=\left(\frac{1}{8},\frac{3}{8},\frac{3}{8},\frac{1}{8}\right)$ (calculated by Eq. 23) and

$$\tilde{A}_{\otimes 1} = \left(s_2,\left[0.2,0.4\right]\right), \quad \tilde{A}_{\otimes 2} = \left(s_1,\left[0.1,0.3\right]\right), \quad \tilde{A}_{\otimes 3} = \left(s_5,\left[0.4,0.5\right]\right), \quad \tilde{A}_{\otimes 4} = \left(s_4,\left[0.3,0.6\right]\right);$$

$$\tilde{W}_{\otimes 1} = \left(s_3,\left[0.3,0.4\right]\right), \quad \tilde{W}_{\otimes 2} = \left(s_4,\left[0.4,0.4\right]\right), \quad \tilde{W}_{\otimes 3} = \left(s_2,\left[0.5,0.7\right]\right), \quad \tilde{W}_{\otimes 4} = \left(s_4,\left[0.2,0.6\right]\right).$$

Firstly, calculate the $\tilde{B}_{\otimes j} = \frac{\tilde{A}_{\otimes j}}{\tilde{W}_{\otimes j}}$.

$$\tilde{B}_{\otimes 1} = \frac{\tilde{A}_{\otimes 1}}{\tilde{W}_{\otimes 1}} = \frac{\left(s_2,\left[0.2,0.4\right]\right)}{\left(s_3,\left[0.3,0.4\right]\right)} = \left(s_{2/3},\left[\max(0.2,0.3),\max(0.4,0.4)\right]\right) = \left(s_{0.67},\left[0.3,0.4\right]\right);$$

$$\tilde{B}_{\otimes 2} = \frac{\tilde{A}_{\otimes 2}}{\tilde{W}_{\otimes 2}} = \frac{\left(s_1,[0.1,0.3]\right)}{\left(s_4,[0.4,0.4]\right)} = \left(s_{1/4},[\max(0.1,0.4),\max(0.3,0.4)]\right) = \left(s_{0.25},[0.4,0.4]\right);$$

$$\tilde{B}_{\otimes 3} = \frac{\tilde{A}_{\otimes 3}}{\tilde{W}_{\otimes 3}} = \frac{\left(s_5,[0.4,0.5]\right)}{\left(s_2,[0.5,0.7]\right)} = \left(s_{5/2},[\max(0.4,0.5),\max(0.5,0.7)]\right) = \left(s_{2.50},[0.5,0.7]\right);$$

$$\tilde{B}_{\otimes 4} = \frac{\tilde{A}_{\otimes 4}}{\tilde{W}_{\otimes 4}} = \frac{\left(s_4,[0.3,0.6]\right)}{\left(s_4,[0.2,0.6]\right)} = \left(s_{4/4},[\max(0.3,0.2),\max(0.6,0.6)]\right) = \left(s_1,[0.3,0.6]\right).$$

Secondly, calculate the expectation value of interval grey linguistic variables $\tilde{B}_{\otimes 1}, \tilde{B}_{\otimes 2}, \tilde{B}_{\otimes 3}$ and $\tilde{B}_{\otimes 4}$ according to Eq. (17).

Suppose $\rho(y) = y^2$, then $f_\rho([a,b]) = \dfrac{b+2a}{3}$. So,

$$I(\tilde{B}_{\otimes 1}) = s_{0.67} \times f_\rho\left(\left[(1-0.4),(1-0.3)\right]\right) = s_{0.67} \times \frac{0.7+2\times 0.6}{3} = s_{0.42};$$

$$I(\tilde{B}_{\otimes 2}) = s_{0.25} \times f_\rho\left(\left[(1-0.4),(1-0.4)\right]\right) = s_{0.25} \times \frac{0.6+2\times 0.6}{3} = s_{0.15};$$

$$I(\tilde{B}_{\otimes 3}) = s_{2.5} \times f_\rho\left(\left[(1-0.7),(1-0.5)\right]\right) = s_{2.5} \times \frac{0.5+2\times 0.3}{3} = s_{0.92};$$

$$I(\tilde{B}_{\otimes 4}) = s_1 \times f_\rho\left(\left[(1-0.6),(1-0.3)\right]\right) = s_1 \times \frac{0.7+2\times 0.4}{3} = s_{0.5}.$$

Thirdly, we rank the $\tilde{B}_{\otimes 1}, \tilde{B}_{\otimes 2}, \tilde{B}_{\otimes 3}$ and $\tilde{B}_{\otimes 4}$, the results are shown as follows:

$$\sigma_1 = 3, \sigma_2 = 4, \sigma_3 = 1, \sigma_4 = 2.$$

Finally, By Definition 9, we have

$$IGLHWHA_{\omega,\tilde{W}_\otimes}\left(\tilde{A}_{\otimes 1}, \tilde{A}_{\otimes 2}, \tilde{A}_{\otimes 3}, \tilde{A}_{\otimes 4}\right) = \left(\sum_{j=1}^{4} \frac{\omega_j}{\tilde{B}_{\otimes \sigma_j}}\right)^{-1} = \left(s_{\left(\sum_{j=1}^{4} \frac{\omega_j}{\beta_{\sigma_j}}\right)^{-1}}, \left[\max_j\left(g_{B_j}^L\right), \max_j\left(g_{B_j}^U\right)\right]\right) =$$

$$\left(s_{\left(\frac{\omega_1}{\beta_3}+\frac{\omega_2}{\beta_4}+\frac{\omega_3}{\beta_1}+\frac{\omega_4}{\beta_2}\right)^{-1}}, \left[\max\left(g_{B_1}^L, g_{B_2}^L, g_{B_3}^L, g_{B_4}^L\right), \max\left(g_{B_1}^U, g_{B_2}^U, g_{B_3}^U, g_{B_4}^U\right)\right]\right) =$$

$$\left(s_{\left(\frac{0.125}{2.5}+\frac{0.375}{1}+\frac{0.375}{0.67}+\frac{0.125}{0.25}\right)^{-1}}, \left[\max\left(0.3,0.4,0.5,0.3\right), \max\left(0.4,0.4,0.7,0.6\right)\right]\right) =$$

$$\left(s_{(1.48)^{-1}}, [0.5,0.7]\right) = \left(s_{0.67}, [0.5,0.7]\right).$$

Obviously, *IGLWHA* operator and *IGLOWHA* operator are the special cases of the *IGLHWHA* operator, which not only shows the importance of interval grey linguistic variables themselves, but also shows the importance of the position of the interval grey linguistic variables.

3. The multi-attribute group decision making method based on the *IGLHWHA* operator

3.1. The description of multiple attribute group decision making problem based on the interval grey linguistic variables

Let $E = \{e_1, e_2, \cdots, e_p\}$ be the experts set in the group decision making, $A = \{A_1, A_2, \cdots, A_m\}$ be the set of alternatives, and $C = \{C_1, C_2, \cdots, C_n\}$ be the attribute set with respect to the alternatives. Supposed that $\tilde{A}^k_{\otimes ij} = \left(t^k_{ij}, \left[g^L_{ijk}, g^U_{ijk} \right] \right)$ is the attribute value in the attribute C_j with respect to the alternative A_i, given by expert e_k, and $\tilde{A}^k_{\otimes} = \left[\tilde{A}^k_{\otimes ij} \right]_{m \times n}$ is the decision making matrix given by the expert e_k, and $\tilde{W}^k_{\otimes} = \left(\tilde{W}^k_{\otimes 1}, \tilde{W}^k_{\otimes 2}, \cdots, \tilde{W}^k_{\otimes n} \right)$, $\tilde{W}^k_{\otimes j} = \left(\eta_{jk}, \left[w^L_{jk}, w^U_{jk} \right] \right)$ is the attribute weight, where $t^k_{ij}, \eta_{jk} \in S, S$ is the linguistic label. Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_p)$ be the experts weight, and $\sum_{k=1}^{p} \lambda_k = 1$. The attribute weight is unknown. We can rank the order of the alternatives based on the given information.

3.2. Decision making steps

1) Aggregate the evaluation information of each expert

According to the different attributes' attribute values and weights which were given by different experts under different alternatives, we can aggregate the attribute values and weights into group decision making information. Based on the $IGLWHA_\lambda$ operator, we can get the group decision making matrix $\tilde{X}_{\otimes} = \left[\tilde{X}_{\otimes ij} \right]_{m \times n}$ and the group decision making weight vector $\tilde{W}_{\otimes} = \left(\tilde{W}_{\otimes 1}, \tilde{W}_{\otimes 2}, \cdots, \tilde{W}_{\otimes n} \right)$, where $\tilde{X}_{\otimes ij} = \left(t_{ij}, \left[g^L_{ij}, g^U_{ij} \right] \right)$ and $\tilde{W}_{\otimes j} = \left(\eta_j, \left[w^L_j, w^U_j \right] \right)$. Then

$$\tilde{X}_{\otimes ij} = IGLWHA_\lambda \left(\tilde{A}^1_{\otimes ij}, \tilde{A}^2_{\otimes ij}, \cdots, \tilde{A}^p_{\otimes ij} \right), \tilde{W}_{\otimes j} = IGLWHA_\lambda \left(\tilde{W}^1_{\otimes j}, \tilde{W}^2_{\otimes j}, \cdots, \tilde{W}^p_{\otimes j} \right).$$

2) Calculate the comprehensive evaluation value of each alternative

We utilize the *IGLHWHA* operator to calculate the comprehensive evaluation value of each alternative $\tilde{Z}_{\otimes i} = \left(c_i, \left[z_i^L, z_i^U \right]\right) = IGLHWHA_{\omega, \tilde{W}_{\otimes}}\left(\tilde{X}_{\otimes i1}, \tilde{X}_{\otimes i2}, \cdots, \tilde{X}_{\otimes in} \right) = \left(\sum_{j=1}^n \frac{\omega_j}{\tilde{B}_{\otimes i\sigma_j}} \right)^{-1}$,

where $\omega = (\omega_1, \omega_2, \cdots, \omega_n)$ is the weight vector associated with the function, and $\sum_{j=1}^n \omega_j = 1$;

$\tilde{W}_{\otimes} = \left(\tilde{W}_{\otimes 1}, \tilde{W}_{\otimes 2}, \cdots, \tilde{W}_{\otimes n} \right)$ is the weight vector of $\tilde{X}_{\otimes ij}$ $(j = 1, 2, \cdots, n)$, $(\sigma_1, \sigma_2, \cdots, \sigma_n)$ is a permutation of $(1, 2, \cdots, n)$, and $\tilde{B}_{\otimes j} = \dfrac{\tilde{A}_{\otimes j}}{\tilde{W}_{\otimes j}}$, for any j, we can get $\tilde{B}_{\otimes \sigma_{j-1}} \geq \tilde{B}_{\otimes \sigma_j}$.

3) Rank the alternatives

Because $\tilde{Z}_{\otimes i}$ is the interval grey linguistic variables, we can get the ranking alternatives by the expectation value $I(\tilde{Z}_{\otimes i})$. The larger the value $I(\tilde{Z}_{\otimes i})$ is, the better the alternative is.

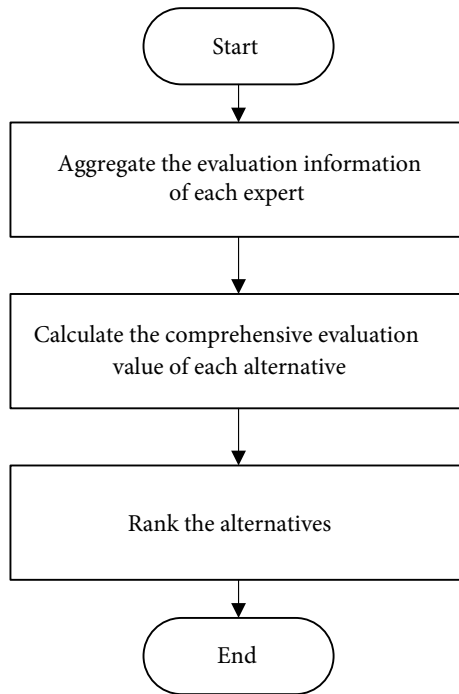The flowchart of this method is shown in Fig. 1.

Fig. 1. The flowchart of the proposed method

4. Practical examples

To evaluate the performance of the proposed method, practical examples are presented as follows: now there are four enterprises $\{A_1, A_2, A_3, A_4\}$, the object is to evaluate the technological innovation ability of the enterprises.

The first step is to develop the evaluation criterion for the project. The criterion is shown as follows: the ability of innovative resources investment (C_1), the ability of innovation management (C_2), the ability of innovation tendency (C_3) and the ability of research and development (C_4). Based on the four criterions, the three experts $\{e_1, e_2, e_3\}$ are invited to evaluate the technological innovation ability of the four enterprises. Supposed that $\lambda = (0.4, 0.32, 0.28)$ be the weight vector about the three experts, and the evaluating values given by the experts adopting interval grey linguistic variables are shown in Tables 1, 2 and 3, and the criterion weight values are shown in Table 4. Let $S = (s_1, s_2, s_3, s_4, s_5, s_6, s_7)$ be the linguistic label. The problem is to rank the four enterprises based on their technological innovation ability.

Table 1. The attribute values of each attribute with respect to four enterprises given by expert e_1

Enterprises	Attribute (C_1)	Attribute (C_2)	Attribute (C_3)	Attribute (C_4)
A1	$(s_5, [0.2, 0.3])$	$(s_2, [0.4, 0.4])$	$(s_5, [0.5, 0.5])$	$(s_3, [0.2, 0.4])$
A2	$(s_4, [0.4, 0.4])$	$(s_5, [0.4, 0.5])$	$(s_3, [0.1, 0.2])$	$(s_4, [0.5, 0.5])$
A3	$(s_3, [0.2, 0.3])$	$(s_4, [0.2, 0.3])$	$(s_4, [0.3, 0.3])$	$(s_5, [0.2, 0.3])$
A4	$(s_6, [0.5, 0.6])$	$(s_2, [0.2, 0.2])$	$(s_3, [0.2, 0.4])$	$(s_3, [0.3, 0.4])$

Table 2. The attribute values of each attribute with respect to four enterprises given by expert e_2

Enterprises	Attribute (C_1)	Attribute (C_2)	Attribute (C_3)	Attribute (C_4)
A1	$(s_4, [0.1, 0.3])$	$(s_3, [0.2, 0.3])$	$(s_3, [0.2, 0.2])$	$(s_6, [0.4, 0.5])$
A2	$(s_5, [0.4, 0.5])$	$(s_3, [0.3, 0.4])$	$(s_4, [0.2, 0.4])$	$(s_3, [0.2, 0.3])$
A3	$(s_4, [0.2, 0.4])$	$(s_4, [0.2, 0.3])$	$(s_2, [0.4, 0.4])$	$(s_3, [0.3, 0.3])$
A4	$(s_5, [0.3, 0.4])$	$(s_4, [0.4, 0.5])$	$(s_2, [0.3, 0.4])$	$(s_4, [0.2, 0.4])$

Table 3. The attribute values of each attribute with respect to four enterprises given by expert e_3

Enterprises	Attribute (C_1)	Attribute (C_2)	Attribute (C_3)	Attribute (C_4)
A1	$(s_5, [0.2, 0.4])$	$(s_3, [0.3, 0.3])$	$(s_4, [0.4, 0.5])$	$(s_4, [0.2, 0.3])$
A2	$(s_4, [0.3, 0.3])$	$(s_5, [0.3, 0.4])$	$(s_2, [0.1, 0.2])$	$(s_3, [0.1, 0.2])$
A3	$(s_4, [0.2, 0.3])$	$(s_5, [0.3, 0.4])$	$(s_1, [0.1, 0.2])$	$(s_4, [0.2, 0.3])$
A4	$(s_3, [0.2, 0.3])$	$(s_3, [0.1, 0.3])$	$(s_4, [0.3, 0.4])$	$(s_5, [0.4, 0.5])$

Table 4. The attribute weight value given by experts

Experts	Attribute (C_1)	Attribute (C_2)	Attribute (C_3)	Attribute (C_4)
e_1	$(s_5, [0.2, 0.3])$	$(s_3, [0.1, 0.2])$	$(s_2, [0.3, 0.4])$	$(s_3, [0.2, 0.3])$
e_2	$(s_3, [0.2, 0.3])$	$(s_4, [0.2, 0.4])$	$(s_3, [0.1, 0.2])$	$(s_3, [0.1, 0.2])$
e_3	$(s_4, [0.2, 0.2])$	$(s_3, [0.1, 0.2])$	$(s_2, [0.1, 0.2])$	$(s_3, [0.2, 0.3])$

The evaluation steps used in this paper are proposed as follows:

(1) Based on the Eq. (19), to aggregate the evaluation information (shown in Tables 1, 2, 3 and 4) about the experts $\{e_1, e_2, e_3\}$, then we can get the group decision making matrix $\tilde{X}_\otimes$ and attribute weight vector $\tilde{W}_\otimes$:

$$
\tilde{X}_\otimes = \begin{bmatrix}
(s_{4.63}, [0.20, 0.40]) & (s_{2.50}, [0.40, 0.40]) & (s_{3.90}, [0.50, 0.50]) & (s_{3.90}, [0.40, 0.50]) \\
(s_{4.27}, [0.40, 0.50]) & (s_{4.12}, [0.40, 0.50]) & (s_{2.83}, [0.20, 0.40]) & (s_{3.33}, [0.50, 0.50]) \\
(s_{3.53}, [0.20, 0.40]) & (s_{4.24}, [0.30, 0.40]) & (s_{1.85}, [0.40, 0.40]) & (s_{3.90}, [0.30, 0.30]) \\
(s_{4.46}, [0.50, 0.60]) & (s_{2.68}, [0.40, 0.50]) & (s_{2.75}, [0.30, 0.40]) & (s_{3.71}, [0.40, 0.50])
\end{bmatrix} ;
$$

$$
\tilde{W}_\otimes = \left((s_{3.90}, [0.20, 0.30]), (s_{3.26}, [0.20, 0.40]), (s_{2.24}, [0.30, 0.40]), (s_{3.00}, [0.20, 0.30]) \right).
$$

(2) Calculate $\tilde{B}_{\otimes ij} = \dfrac{\tilde{X}_{\otimes ij}}{\tilde{W}_{\otimes j}}$;

$$
\tilde{B}_\otimes = \begin{bmatrix}
(s_{1.188}, [0.20, 0.40]), (s_{0.767}, [0.40, 0.40]), (s_{1.740}, [0.50, 0.50]), (s_{1.299}, [0.40, 0.50]) \\
(s_{1.097}, [0.40, 0.50]), (s_{1.264}, [0.40, 0.50]), (s_{1.264}, [0.30, 0.40]), (s_{1.111}, [0.50, 0.50]) \\
(s_{0.906}, [0.20, 0.40]), (s_{1.299}, [0.30, 0.40]), (s_{0.827}, [0.40, 0.40]), (s_{1.299}, [0.30, 0.30]) \\
(s_{1.146}, [0.50, 0.60]), (s_{0.821}, [0.40, 0.50]), (s_{1.229}, [0.30, 0.40]), (s_{1.238}, [0.40, 0.50])
\end{bmatrix} .
$$

3) Calculate the comprehensive evaluation values of each alternative

If the BUM function is $\rho(y) = y^2$, then $f_\rho([a, b]) = \dfrac{b + 2a}{3}$. The position vector is determined by the equation (23), and $\omega = \left(\dfrac{1}{8}, \dfrac{3}{8}, \dfrac{3}{8}, \dfrac{1}{8} \right)$. According to the equation (24), we can get the comprehensive evaluation values of each alternative:

$$
\tilde{Z}_\otimes = \begin{pmatrix}
s_{1.19}, [0.50, 0.50] \\
s_{1.18}, [0.50, 0.50] \\
s_{1.05}, [0.40, 0.40] \\
s_{1.13}, [0.50, 0.60]
\end{pmatrix} .
$$

4) Rank the alternatives.

We select the BUM function $\rho(y) = y^2$; the expectation value $I(\tilde{Z}_{\otimes i})$ is calculated by Eq. (17).

$$I(\tilde{Z}_{\otimes 1}) = s_{0.596}, \; I(\tilde{Z}_{\otimes 2}) = s_{0.588}, \; I(\tilde{Z}_{\otimes 3}) = s_{0.632}, \; I(\tilde{Z}_{\otimes 4}) = s_{0.490}.$$

So the orders of technological innovation ability of the four enterprises $\{A_1, A_2, A_3, A_4\}$ are shown as follows:

$$A_3 \succ A_1 \succ A_2 \succ A_4.$$

5) Discuss.

In order to illustrate the effect of the position weight vector ω on decision making of this example, we use the different value ω to rank the alternatives, and the ranking results are shown as follows:

1) For $\omega = \left(\dfrac{1}{4}, \dfrac{1}{4}, \dfrac{1}{4}, \dfrac{1}{4} \right)$, the orders are $A_3 \succ A_2 \succ A_1 \succ A_4$.

2) For $\omega = \left(\dfrac{2}{5}, \dfrac{1}{5}, \dfrac{1}{5}, \dfrac{1}{5} \right)$ which can be calculated by Eq. (22) in $\alpha = 0.2$, the orders are $A_3 \succ A_1 \succ A_2 \succ A_4$.

3) For $\omega = \left(\dfrac{3}{8}, \dfrac{1}{8}, \dfrac{1}{8}, \dfrac{3}{8} \right)$, the orders are $A_3 \succ A_2 \succ A_1 \succ A_4$.

These show that the position weight vector ω has a certain impact on ranking.

In addition, in order to verify the validity of the method proposed in this paper, we use the method proposed by Meng *et al.* (2007) to sort this example. Because the method proposed by Meng *et al.* (2007) is for grey fuzzy decision making problems in which the fuzzy part and the grey part of grey fuzzy numbers take the form of the interval numbers, and in this paper, the fuzzy part of grey fuzzy numbers takes the form of the linguistic variables, and the grey part takes the form of the interval numbers. So in order to use the method proposed by Meng *et al.* (2007), we change the linguistic variables of fuzzy part of grey fuzzy numbers to interval numbers according to Liu and Meng (2009) firstly. The ranking result is shown as follows:

$$A_3 \succ A_2 \succ A_1 \succ A_4.$$

Obviously, two methods have the same ranking results; this verifies the validity of the method in this paper.

In order to further illustrate the validity of the proposed method, we use the evaluation data presented by Meng *et al.* (2007). Firstly, we convert the interval numbers to the linguistic variables by the methods proposed by Liu (2009), and then we can use the methods in this paper. By calculating, we get the ranking of 4 alternatives. It is shown as follows:

$$y_2 \succ y_4 \succ y_1 \succ y_3.$$

It is the same as the ranking result produced in Meng *et al.* (2007).

Conclusions

The traditional grey fuzzy decision making methods are generally suitable for decision making the information taking the form of crisp numbers, or interval numbers in the fuzzy part and the grey part of grey fuzzy numbers, and yet they will fail in dealing with the linguistic information of grey fuzzy numbers. In this paper, with respect to multiple attribute group decision making (MAGDM) problems in which the attribute values and the attribute weights take the form of the interval grey linguistic variables, some new group decision making analysis methods are developed. Firstly, the concept of interval grey linguistic variables is proposed, in which the fuzzy part takes the form of the linguistic variables, and the grey part takes the form of the interval numbers. Then, the operation rules of interval grey linguistic variables are defined, and some operators (such as interval grey linguistic weighted harmonic aggregation (IGLWHA) operator, interval grey linguistic ordered weighted harmonic aggregation (IGLOWHA) operator, and interval grey linguistic hybrid weighted harmonic aggregation (IGLHWHA) operator) are proposed to solve the group decision making problems. Finally, the computational results from an illustrative example have shown that the proposed approach is feasible and effective for the group-decision making problems, and it is easier to understand and implement. This study promotes the development of the theory and method of grey fuzzy multiple attribute decision making and provides a new idea to solve the grey fuzzy multiple attribute decision making. Because interval grey linguistic variables take into account the greyness and fuzziness of the same problem, they can get more rational decision-making results. However, in order to do this, we must get more data, and sometimes, it is very difficult to obtain these data. In the future, we will research data acquisition methods, and continue working in the extension and application of the developed operators and methods to other domains.

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