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The Multi-way Relay Channel

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***Abstract*—The multi-user communication channel, in which multiple users exchange information with the help of a single relay terminal, called the *multi-way relay channel*, is considered. In this model, multiple interfering clusters of users communicate simultaneously, where the users within the same cluster wish to exchange messages among themselves. It is assumed that the users cannot receive each other's signals directly, and hence the relay terminal is the enabler of communication. A relevant metric to study in this scenario is the symmetric rate achievable by all users, which we identify for amplify-and-forward (AF), decode-and-forward (DF) and compress-and-forward (CF) protocols. We also present an upper bound for comparison. The two extreme cases, namely full data exchange, in which every user wants to receive messages of all other users, and pairwise data exchange, consisting of multiple two-way relay channels, are investigated and presented in detail.**

I. INTRODUCTION

Relaying in wireless networks can provide robustness, extended coverage, and energy efficiency. The relay channel was studied in [1] in detail as a building block for wireless networks that employ relaying strategies. Recently, it has been recognized that effective relaying protocols can be devised to facilitate cooperation between two users when they want to exchange information simultaneously over a single relay terminal. This channel model, called the *two-way relay channel (TRC)*, has been studied in detail; see [2], [3], [4], [5] and the references therein. In the TRC, unlike the classical relay channel, we can exploit the structure of the network to design more efficient protocols and harvest the benefits of network coding in the physical layer.

Here, we extend the TRC model studied in previous work in two directions: First, we consider clusters of multiple nodes that want to exchange information among themselves. Second, we consider multiple such clusters communicating simultaneously over a single relay terminal. This would model, for example, multiple sensor networks in the same environment served by a single access point, where nodes in each network want to exchange some control information among themselves. We term this model the *multi-way relay channel (mRC)*, and consider a total of N users grouped into $L \geq 1$ clusters of $K \geq 2$ distinct users each, i.e., $N = KL$. In the special case of $L = 1$ and $K = 2$, this model reduces to the TRC.

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We note that the symmetric rate performance is a relevant metric in this setting, and derive the achievable symmetric rate with the corresponding multi-way extensions of decode-and-forward (DF), amplify-and-forward (AF) and compress-and-forward (CF). We provide a comparison of these rates for a symmetric Gaussian network scenario. It is shown in [5] that the CF scheme achieves within a half bit of the capacity for the symmetric TRC, while DF achieves the capacity when the additional sum-rate constraint is not the bottleneck. Here, we explore the behavior of these protocols for a large network. We show that CF achieves a symmetric rate within a constant bit offset from the capacity, where this gap diminishes as the number of users in the system increases.

We also investigate the special case of two users per cluster, i.e., $K = 2, L > 1$, and provide a generalization of the lattice coding scheme proposed in [3] and [4]. While for TRC lattice coding also achieves within a half bit of the capacity [4] and performs close to the upper bound for a large range of power constraints, we show here that CF outperforms lattice coding as the number of clusters increases.

II. SYSTEM MODEL

We consider a Gaussian mRC in which multiple users exchange messages with the help of a single relay terminal. In this model users do not receive each other's transmissions, hence the relay is essential for communication. We consider full-duplex communication, that is, all terminals including the relay can receive and transmit simultaneously. There are $L \geq 1$ clusters of nodes in the network, where each cluster has $K \geq 2$ users. Users in cluster j, $j \in \mathcal{I}_L \triangleq \{1, \ldots, L\}$ are denoted by $T_{j1}, \ldots, T_{jK}$ while the relay terminal is denoted by R (see Fig. 1). $W_{ji} \in \mathcal{W}_{ji}$ is the message of user T_{ji}. User T_{ji} wants to decode messages $(W_{j1}, \ldots, W_{jK})$.

The Gaussian mRC channel is modeled as

$$Y_r = \sum_{j=1}^{L}\sum_{i=1}^{K} X_{ji} + Z_r \quad (1)$$

$$Y_{ji} = X_r + Z_{ji}, j \in \mathcal{I}_L \text{ and } i \in \mathcal{I}_L \quad (2)$$

where Z_r is zero-mean Gaussian noise at the relay with variance N_r, and Z_{ij} is zero-mean Gaussian noise at user T_{ji} with variance N_{ji}. These noise variables are independent of

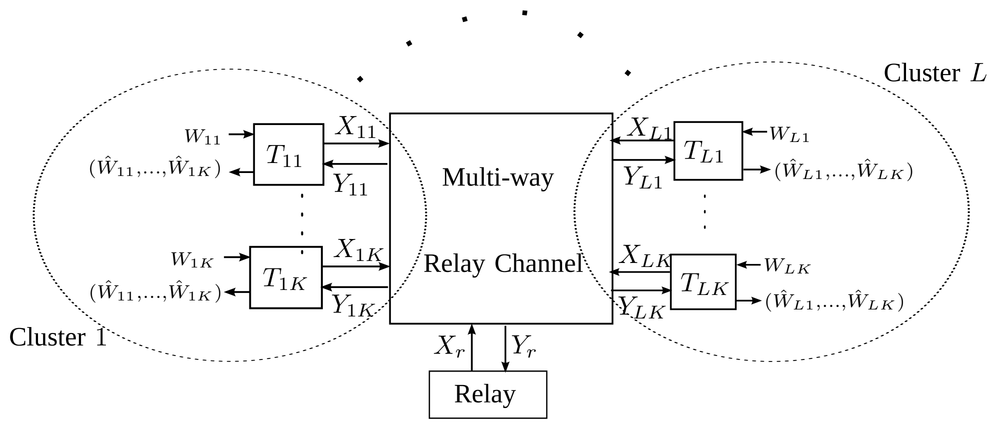

Fig. 1: The mRC with L clusters, each of which has K distinct terminals. All terminals in a cluster want to receive the messages of all the other terminals in the same cluster. The relay terminal facilitates the data exchange between the terminals.

each other and the channel inputs. Average power constraints apply on the transmitted signals at the relay and at users T_{ji} for all $j \in \mathcal{I}_L$ and $i \in \mathcal{I}_L$:

$$\frac{1}{n} E\left[\sum_{t=1}^{n} X_{r,t}^2\right] \leq P_r \text{ and } \frac{1}{n} E\left[\sum_{t=1}^{n} X_{ji,t}^2\right] \leq P_{ji}. \quad (3)$$

Note that, although we have a full-duplex operation, the effect of the transmitted signal of each user on its received signal will be ignored since it is known at the transmitter, and hence can be subtracted.

A $(2^{nR_{11}}, \dots, 2^{nR_{1K}}, \dots, 2^{nR_{L1}}, \dots, 2^{nR_{LK}}, n)$ code for the mRC consists of $N = LK$ sets of integers $\mathcal{W}_{ji} = \{1, 2, \dots, 2^{nR_{ji}}\}$ for $j \in \mathcal{I}_L$ and $i \in \mathcal{I}_L$ as the message sets, N encoding functions f_{ji} at the users such that $x_{ji}^n = f_{ji}(W_{ji})$, a set of encoding functions $\{f_{r,t}\}_{t=1}^n$ at the relay such that $x_{r,t} = f_{r,t}(Y_{r,1}, \dots, Y_{r,t-1}), \quad 1 \leq t \leq n$, and N decoding functions $g_{ji} : \mathcal{Y}_{ji}^n \times \mathcal{W}_{ji} \to (\mathcal{W}_{j1}, \dots, \mathcal{W}_{jK})$.

Note that we consider "restricted encoders", that is f_{ji} depend only on messages W_{ji} and not on the received signals. The average probability of error for this system is defined as

$$P_e^n = \Pr \bigcup_{j \in \mathcal{I}_L, i \in \mathcal{I}_L} \left\{ g_{ji}(W_{ji}, Y_{ji}^n) \neq (W_{j1}, \dots, W_{jK}) \right\}.$$

Observe that the condition $P_e^n \to 0$ implies that individual average error probabilities also go to zero. We assume that the messages W_{ji}, $j \in \mathcal{I}_L$, $i \in \mathcal{I}_L$, are chosen independently and uniformly over the message sets $\mathcal{W}_{ji}$.

Definition 1: A rate tuple $(R_{11}, \dots, R_{1K}, \dots, R_{L1}, \dots, R_{LK})$ is said to be *achievable* for an mRC with L clusters of users with K users each if there exists a sequence of $(2^{nR_{11}}, \dots, 2^{nR_{1K}}, \dots, 2^{nR_{L1}}, \dots, 2^{nR_{LK}}, n)$ codes such that $P_e^n \to 0$ as $n \to \infty$. The corresponding *capacity region* is the convex closure of all achievable rate tuples.

We focus on the equal rate points of the capacity region, i.e., $R_{ji} = R$, $j \in \mathcal{I}_L$ and $i \in \mathcal{I}_L$. We define the symmetric capacity with L clusters and K users in each cluster as

$$C_{sym}^{L,K} \triangleq \sup\{R : (R, \dots, R) \text{ is achievable}\}.$$

Our goal is to find lower and upper bounds on the symmetric capacity of the network. The symmetric capacity is relevant in applications in which the messages correspond to some control information that needs to be shared by the nodes in the network, and the system performance is dominated by the minimum rate. To simplify the notation and to focus on the fundamental behavior of the analyzed schemes, we consider a symmetric network, that is, $P_{ji} = P$ and $N_{ji} = 1$ for all $j \in \mathcal{I}_L$, $i \in \mathcal{I}_L$. We use the notation $C(x) \triangleq \frac{1}{2}\log(1 + x)$.

III. Bounds on the Symmetric Capacity

In this section, we provide upper and lower bounds on the symmetric capacity of the symmetric Gaussian mRC. The following proposition presents an upper bound.

Proposition 1: For a symmetric Gaussian mRC with L clusters of K users each, the symmetric capacity is upper bounded by

$$R_{UB}^{L,K} = \min\left\{\frac{C(L(K-1)P)}{L(K-1)}, \frac{C(P_r)}{L(K-1)}\right\}. \quad (4)$$

Proof: To prove this upper bound, consider an equivalent network in which one user from each cluster does not have a message to transmit. Moreover, assume that only the users without messages want to decode the messages of the other users, that is, users with messages are the source terminals while the users without messages are the sink terminals. The symmetric capacity for this network with $L(K-1)$ messages constitutes an upper bound for the original mRC. Observe that this remaining network is a multiple access relay network, in which L multiple access relay channels operate simultaneously over a single relay terminal.

In this network, consider the cuts around the source terminals and the sink terminals. The cut around the source terminals forms a symmetric multiple access channel (MAC) with $L(K-1)$ users, and the achievable symmetric rates are bounded by $\frac{C(L(K-1)P)}{L(K-1)}$. The cut around the sink terminals is a symmetric Gaussian broadcast channel with L messages of rate $(K-1)R$ each, where each message is destined for a single receiver. Since this is a degraded broadcast channel, the total rate can be bounded by $C(P_r)$. ■

Next we identify symmetric rates achievable with various relaying schemes. We consider AF, DF and CF schemes, and find the corresponding symmetric rates. A symmetric rate achievable with AF relaying is characterized in the next proposition.

Proposition 2: For a symmetric Gaussian mRC with L clusters of K users each, the following symmetric rate is achievable with AF relaying:

$$R_{AF}^{L,K} = \frac{1}{L(K-1)} C\left(\frac{PP_r}{1+P_r+KP}\right). \quad (5)$$

Proof: In the case of the AF protocol, we consider time division among the clusters. Due to the symmetry of the network and the equal number of users within each cluster, equal time allocation maximizes the achievable symmetric rate. Within the timeslot of each cluster, all the users in that cluster transmit, and the relay scales its received signal and broadcasts to the users. Within the timeslot for cluster j, the relay's transmit signal is given by $X_r = \sqrt{\frac{P_r}{KP+1}}(X_{j1} + \cdots + X_{jK} + Z_r)$. Each user subtracts its own transmit signal from the received signal of the relay, and decodes the messages of the other users in its own cluster. For each receiver, this is equivalent to a MAC with $K-1$ users, and the maximum achievable symmetric rate for this MAC is given by (5). ■

Next we consider DF relaying, in which the relay decodes messages from all the users, and broadcasts each message to its recipients. DF consists of two transmission phases: the first phase is the MAC from the users to the relay, and the second phase is the broadcast channel from the relay to the users. In the broadcast phase, we consider time division transmission among the clusters, that is, the relay divides the channel block into L timeslots, and for $j \in \mathcal{I}_L$, broadcasts the messages $W_{j,1}, \ldots, W_{j,k}$ to users $T_{j,1}, \ldots, T_{j,K}$ within the j-th timeslot. For broadcasting within the j-th timeslot, the relay uses the transmission scheme introduced in [6], where we consider $W_{j,1}, \ldots, W_{j,k}$ as the source message and $W_{j,i}$ as the correlated side information at user $T_{j,i}$. The symmetric rate achievable with DF is then found as given in the following proposition.

Proposition 3: For the symmetric Gaussian mRC with L clusters of K users each, the following symmetric rate is achievable with DF relaying:

$$R_{DF}^{L,K} = \min\left\{\frac{C(LKP)}{LK}, \frac{C(P_r)}{L(K-1)}\right\}. \quad (6)$$

Remark 1: Comparing (6) and (4), we can show that DF achieves the symmetric capacity if $P_r \leq (1+LKP)^{1-\frac{1}{K}} - 1$. This corresponds to the case in which the relay power is the bottleneck, i.e., the symmetric capacity is limited by the rate that the relay can broadcast to the users. The range of P_r for which DF is optimal increases as the number of clusters, the number of users within each cluster or the power constraint P of the users increases.

Next, we consider CF relaying, in which the relay terminal quantizes its received signal and broadcasts this quantized channel output to the users, again using the coding scheme that we employed with DF to exploit the side information at the users. Similar to AF, we consider time division among the user clusters in the multiple access phase as well as in the broadcast phase. This will prevent multiple user clusters from interfering with each other's signals, which would decrease the quality of the quantized signal broadcast by the relay. Within the timeslot for each cluster, the transmission from the relay can be considered as broadcasting the relay's received signal to the users with minimum distortion [7].

Proposition 4: For a symmetric Gaussian mRC with L clusters of K users each, the following symmetric rate is achievable with CF relaying:

$$R_{CF}^{L,K} = \frac{1}{L(K-1)} C\left(\frac{(K-1)PP_r}{1+(K-1)P+P_r}\right). \quad (7)$$

Proof: We use Gaussian codebooks for quantization without claiming optimality. Consider transmission over timeslot j, $j \in \mathcal{I}_L$. We have

$$\hat{Y}_r = X_{j,1} + \cdots + X_{k,K} + Z_r + Q, \quad (8)$$

where Q is a zero mean Gaussian random variable with variance N_Q. For $\hat{Y}_r$ to be decoded at all receivers, we need

$$I(Y_r; \hat{Y}_r | X_{j,i}) \leq I(X_r; Y_{j,i}), \quad (9)$$

or equivalently, in the symmetric case, $N_Q \geq \frac{(K-1)P+1}{P_r}$. The achievable rate hence satisfies

$$(K-1)R_{CF}^{L,K} = C\left(\frac{(K-1)P}{1+N_Q}\right). \quad (10)$$

Using the minimum allowable N_Q, we obtain (7). ■

Remark 2: Comparing (5) and (7), we observe that, for an arbitrary number of clusters and terminals within each cluster $(L \geq 1, K \geq 2)$, CF achieves a *higher symmetric rate* than AF. Yet, in some implementations, the lowre complexity of AF might be more compelling than the better performance of CF.

In the next theorem, we prove that the CF protocol achieves rates within a constant number of bits of the symmetric capacity for an arbitrary number of clusters and users.

Theorem 1: For a symmetric Gaussian mRC with L clusters of K users each, the CF protocol achieves rates within $\frac{\log(L+1)}{2L(K-1)}$ bits of the symmetric capacity.

Proof: First, assume that $P_r \geq L(K-1)P$. Then we have the following chain of inequalities:

$$R_{CF}^{L,K} = \frac{1}{L(K-1)} C\left(\frac{(K-1)PP_r}{1+(K-1)P+P_r}\right) \quad (11)$$

$$= \frac{1}{2L(K-1)}\Big[\log(L+L(K-1)P) + \log\left\{\frac{1+P_r}{L(1+(K-1)P+P_r)}\right\}\Big] \quad (12)$$

$$\geq R_{UB}^{L,K} + \frac{1}{2L(K-1)} \log\left\{\frac{1+P_r}{L(1+P_r)+P_r}\right\} \quad (13)$$

$$\geq R_{UB}^{L,K} - \frac{\log(L+1)}{2L(K-1)}, \quad (14)$$

where (13) follow from the assumption that $P_r \geq L(K-1)P$.

Next, assuming $P_r < L(K-1)P$, we have

$$R_{CF}^{L,K} = \frac{1}{2L(K-1)}\Big[\log(1+P_r) + \log\left\{\frac{1+(K-1)P}{1+(K-1)P+P_r}\right\}\Big] \quad (15)$$

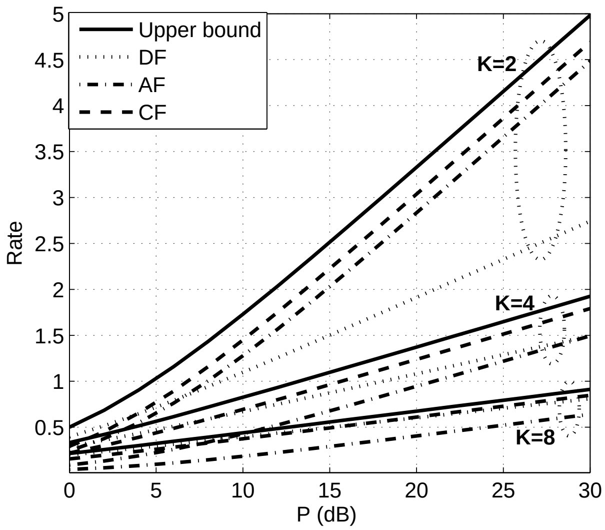

Fig. 2: Achievable symmetric rate versus the user power, P. The relay power is equal to the total user power, i.e., $P_r = KP$. We illustrate rates for $K = 2, 4$ and 8 users.

$$\geq R_{UB}^{L,K} + \frac{1}{2L(K-1)} \log \left\{ \frac{1+(K-1)P}{1+(L+1)(K-1)P} \right\}$$
$$\geq R_{UB}^{L,K} - \frac{\log(L+1)}{2L(K-1)}. \quad (16)$$

■

Remark 3: It is noteworthy that the constant gap to the capacity is a function only of L and K, and is independent of the power constraints of the users and the relay. Moreover, the gap goes to zero as either K or L goes to infinity, independent of how the power constraints scale with the number of users. Hence, we conclude that for a large system of many clusters and/or many users within each cluster, the CF protocol is nearly optimal in terms of the symmetric capacity.

IV. Special Cases

A. Multi-way Relay Channel with Full Data Exchange

In this section, we consider a special mRC with a single cluster $L = 1$, that is, each user wants to decode all the messages in the system. We term this model the *mRC with full data exchange*.

Assume that the relay's power scales with the number of users, i.e., $P_r = KP$. In this case we have $R_{UB}^{1,K} = \frac{C((K-1)P)}{K-1}$ and $R_{DF}^{1,K} = \frac{C(KP)}{K}$. We can see that, with increasing power, the gap between the two increases and can be arbitrarily large when P is very high. In Fig. 2, we plot the upper bound and achievable symmetric rates for this setup. Achievable rates and the upper bound converge as the number of users increases. We have a finite gap between the symmetric rate achievable with the CF scheme and the upper bound at all power values; and especially for a small number of users, the rate of CF dominates the rate of DF for a wide range of power values. We can also see that the symmetric rate achievable by AF follows that of CF with a constant gap as well. Although not

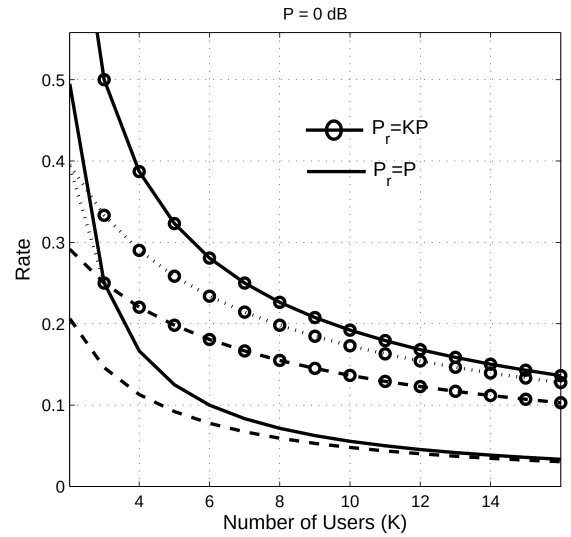

Fig. 3: Achievable symmetric rate versus the number of users with $P = 0$ dB. In the figure, the straight line is the upper bound, while the dotted and dashed lines correspond to the DF and CF rates, respectively. We illustrate both $P_r = P$ and $P_r = KP$.

included here due to space limitations, similar observations are made when the relay power does not scale with the number of users, i.e., $P_r = P$.

In Fig. 3 we plot the upper bound and the achievable rates versus the number of users for the mRC with full data exchange. As expected, the rate per user diminishes as the number of users increases in the system. With the number of users increasing, both DF and CF get very close to the upper bound. The DF scheme achieves the upper bound with a smaller number of users when the relay power does not scale with the number of users in the system.

B. Multi-way Relay Channel with Pairwise Data Exchange

In the previous subsection we focused on full-data exchange, in which case each user wants to learn the messages of all other users. This constitutes one extreme in the mRC model. Another extreme would be to assume that users are paired, and each user is interested only in the data of its partner, i.e., $L \geq 1$ and $K = 2$. This model is equivalent to having multiple two-way relay channels served simultaneously by a single relay terminal [8]. We term this model the *mRC with pairwise data exchange*.

In the case of the pairwise data exchange model, another achievability scheme is obtained by structured codes. In particular, nested lattice codes are used for the Gaussian TRC [3], [4], which allows the relay to decode only the modulo sum of the messages rather than decoding the individual messages. Then the relay can broadcast the modulo sum to both users, each of which can decode the other user's message by subtracting its own message. Unfortunately this structured coding scheme does not scale with an increasing number of users within each cluster, that is, by knowing the modulo sum

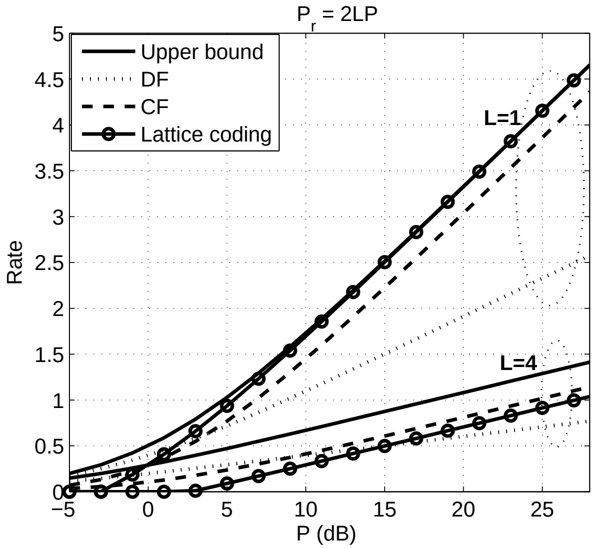

Fig. 4: Symmetric capacity upper bound and achievable rates versus power P for the pairwise data exchange model.

of more than two messages and only one of the messages, the users cannot decode the remaining messages.

In pairwise data exchange with $L > 1$ clusters, we will have the relay first decode the modulo sums of all the message pairs, and then broadcast each pair's sum only to the users in that pair by time-division among the pairs. For the multiple access phase, the relay employs successive decoding to decode the modulo sums of the pairs. We consider time division for the multiple access phase as well; however, this is not among the users but among the decoding orders. In each time slot the decoding order of the pairs at the relay is shifted. This way, each pair experiences each decoding order once. Using nested lattices as in [3], when no other transmission occurs, the modulo sum of two messages can be decoded at the relay at a rate $\frac{1}{2}\log\left(\frac{1}{2}+P\right)$. Hence, by time division and shifted decoding order at the relay, each pair's modulo sum can be decoded at the relay at a rate

$$\frac{1}{L}\sum_{j=1}^{L}\frac{1}{2}\log\left(\frac{1}{2}+\frac{P}{1+2(j-1)P}\right) \tag{17}$$

$$=\frac{1}{2L}\log\prod_{j=1}^{L}\left(\frac{1}{2}+\frac{P}{1+2(j-1)P}\right) \tag{18}$$

$$=\frac{1}{2L}\log 2^{-L}\prod_{j=1}^{L}\left(\frac{1+2jP}{1+2(j-1)P}\right) \tag{19}$$

$$=\frac{1}{2L}\log\left(1+2LP\right)-\frac{1}{2}. \tag{20}$$

For the broadcasting of the modulo sums from the relay to the pairs, the rate is bounded by the rate that can be transmitted to each user: $\frac{1}{L}C(P_r)$. Hence, the following symmetric rate can be achieved by nested lattice codes:

$$R_{lattice}^{L,2}=\min\left\{\frac{C(2LP)}{L}-\frac{1}{2},\frac{C(P_r)}{L}\right\}. \tag{21}$$

Remark 4: It is easy to see that lattice coding achieves rates within $1/2$ bit of the symmetric capacity. This constant bit gap decays to $\frac{L-1}{2L}$ in the high SNR limit. For $L > 2$, the gap for lattice coding is larger than the gap for CF even in the infinite SNR limit; however, this does not directly lead to a claim of higher symmetric rates with CF.

In Fig. 4 we illustrate the upper bound and the achievable rates for the pairwise data exchange model as functions of P, while $P_r = 2LP$. Similar observations as in Section IV-A apply for DF and CF schemes. The lattice coding performs within a constant bit offset from the symmetric capacity as well. As seen in the figure, for $L = 1$, lattice coding outperforms CF and its gap with the upper bound decays to zero. However, this is not the case when the number of clusters increases. For $L = 4$, we see that CF outperforms lattice coding for all power values. It is also noteworthy that DF achieves the highest rate in the low power regime.

V. Conclusion

We have considered the multi-way relay channel in which multiple clusters of users communicate simultaneously over a single relay terminal (no cross-reception between the users), and the users in each cluster want to exchange information among themselves. We have shown that the CF scheme achieves a symmetric rate within a constant bit offset from the capacity, while this constant gap decays to zero with increasing number of users in the system independent of the scaling behavior of the power constraints. We have also investigated symmetric rate achievable by nested lattice codes for the case of multiple clusters with two users each. We have shown that lattice coding outperforms other schemes for a single cluster, but falls short of the CF performance as the number of clusters increases. Our results provide insights into various design tradeoffs associated with relaying between clusters of communicating nodes.

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