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THE MULTIDIMENSIONAL STATIONARY-PHASE METHOD FOR AN ASYMPTOTIC ESTIMATE OF EDGE EFFECTS IN THE MULTIPLE KIRCHHOFF DIFFRACTION BY CURVED SURFACES

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Abstract. An analytical approach is proposed to study the contribution of edge effects in the multiple high-frequency diffraction, according to guidelines of classical Kirchhoff theory in (scalar) wave propagation. We start from a suitable asymptotic analysis of the Kirchhoff diffraction integral, here set up in a generalized (iterated) form to describe the multiple reflections from an arbitrary sequence of curved reflecting (smooth) surfaces. The explicit formula obtained for a concrete example of double reflection is compared with the results from direct numerical simulation.

1. Introduction. Mathematical diffraction theories have many important practical applications in acoustics, optics and electromagnetism [1]–[4]. Two different viewpoints, resulting in geometrical or analytical approaches, did appear in these theories from the very beginning, being applied to the same physical phenomena. The first geometrical approach was originated from the classical Gaussian wave beam ideas [5], both in optics and acoustics. Its further development led to the modern Ray theory [6], [7], and finally to the powerful instrument of the Geometrical Diffraction Theory [8], [9] (see also [3]). In common, the basic concepts of this approach to diffraction phenomena are founded upon a certain high-frequency asymptotic analysis of the wave equation, aimed to find purely geometrical quantities: phase, wave beam spreading, amplitude change on the act of diffraction and its further decrease with distance, etc. It should be noted that such ideas have been tested in a number of special cases for which exact analytical solutions exist (e.g., diffraction by half-planes and wedges). Typically, the latter ones can be attained

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either by Sommerfeld's exact diffraction theory [10], or by the Biot-Tolstoy theory [11], or by some others close to them.

The classical Kirchhoff Physical Diffraction Theory represents an alternative analytical approach which moved forward from absolutely different ideas. Starting from the Kirchhoff-Helmholtz integral representation for the exact general solution of the Helmholtz wave equation [4], [10], where one can find two unknown functions in the integrand (namely, the scalar wave field and its normal derivative on the boundary surface), Kirchhoff theory is founded on some reliable physical hypotheses which in the case of rigid scatterer lead to an explicit mathematical expression for the amplitude of the wave field over the scatterer's surface [3], [12], [13]. When estimating its precision, it should be noted that for a long time there were numerous heated debates as to whether this theory is asymptotic at $k \to \infty$ (k is the wave number). Recently, it was proved that the leading asymptotic term of the exact solution coincides with Kirchhoff's prediction if the solution is constructed in the "light" zone and the boundary surface of the obstacle is convex. A strict mathematical proof of that is given for soft obstacles in [12] and for rigid ones in [13]. Therefore, despite the fact that in some particular cases Kirchhoff theory is numerically less precise when compared with experimental data [14], in the regimes where it works well this theory provides the correct high-frequency asymptotic representation.

If under Kirchhoff's assumption all quantities in the integrand become known, it is evident that - in frames of Kirchhoff theory - the problem is reduced to the evaluation of a certain surface integral. Generally, such an integral can be calculated by an appropriate direct numerical method. However, when working in the high-frequency regime, the more natural procedure is to construct its asymptotic representation. This can be attained analytically by the use of the multidimensional *stationary phase method*, which has been developed for smooth reflecting surfaces by many authors independently; see for example [15]–[18]. While, in its basic form, Kirchhoff's diffraction theory was proposed for single reflection only, in [19] we extended it to the case of multiple reflection by an iterated procedure accounting for an arbitrary sequence of smooth reflecting surfaces; in such a paper, the problem was reduced to a $2N$-fold diffraction integral, where N is the number of the surfaces, and then partially estimated by the quoted method.

When comparing the two classes of methods - geometrical vs. analytical - the early experience of their application showed that if a certain diffraction problem can be studied by a method from the first class, then, as a rule, this can also be solved by an analytical approach, perhaps with a longer mathematical treatment but still successful (or vice versa). However, some time later it was recognized that this rule is not universal. In the case of multiple reflections, a two-dimensional extension of the Kirchhoff theory permitted us to analytically solve many new problems with curved reflectors [20], and we could not find in literature analogous solutions (for most of them) constructed by any alternative method. Moreover, when the approach of [20] has been extended to a three-dimensional context in [19], some new analytical solutions, again for curved surfaces, could also be constructed in explicit form (see in particular the treatment given there for the double diffraction by two different spheres). One thus should conclude that in some cases the analytical approach looks more powerful than the geometrical one.

The comment in the previous paragraph is related to the leading high-frequency asymptotic term. Coming back to the question of the precision of Kirchhoff's theory, it is shown in [19], [20] that there is a significant disagreement between direct numerical and analytical asymptotic predictions, either for single or multiple reflections. This difference is more significant in the three-dimensional case. As discovered in those papers, the difference occurs because, in the asymptotic estimate of the multiple Kirchhoff diffraction integral, only the leading term is usually taken into account. In [21] we improve the precision of Kirchhoff theory by extracting the subsequent (second) term in the high-frequency asymptotic expansion, in the case of single reflection only: it is proved that such a second term is related to the so-called *edge effects*, namely to the wave contribution coming from the boundary edge line of the given reflecting surface. From the mathematical procedure, it turns out that the leading asymptotic term (which is physically related to the simple *specular* reflection) is of the order $O(1)$, while the second asymptotic term has the order $O(1/\sqrt{k})$, $k \to \infty$. In our recent works [22], [23], we extended this result to the case of multiple diffraction, but only for flat reflecting surfaces. Therefore, the real goal of the present study is to extend these results to multiple reflections from curved surfaces, by again using the multidimensional stationary-phase method applied to a multifold diffraction integral of Kirchhoff type. As it will be seen, this study requires very lengthy mathematical transformations, which, to our knowledge, have never been previously performed by any other existing (non-numerical) method. We will illustrate the analytical procedure by a concrete example of double diffraction, and compare the explicit formula obtained with the results coming from a direct numerical treatment of the relevant integrals.

2. Basics and analysis of the proposed method. In a given high-frequency (harmonic) regime, let us consider the sequence of reflections of a scalar wave, radiated from the source point x_0, from a set of thin rigid surfaces $S_1, S_2, \ldots, S_N$, as shown in Figure 1. The extension of the classical Kirchhoff diffraction theory, from single to multiple reflection [19], [22], [23], reduces the amplitude of the re-reflected (*scattered*) wave at the receiving point x, $p^{sc}(x)$, to the following repeated integral taken over all surfaces $S_1, S_2, \ldots, S_N$:

$$p^{sc}(x) \sim \left(\frac{ik}{2\pi}\right)^N \int_{S_1} \cdots \int_{S_N} \frac{e^{ikg}}{r_{01}} \prod_{\nu=1}^{N} \frac{\cos\left(\vec{r}_{\nu,\nu+1} \widehat{\ } \vec{n}_\nu\right)}{r_{\nu,\nu+1}} \, dS_1 \ldots dS_N, \qquad k \to \infty\,. \tag{2.1}$$

Here k is the wave number; $\vec{r}_{\nu,\nu+1} = y_\nu - y_{\nu+1}$ $(y_\nu \in S_\nu)$; $r_{\nu,\nu+1} = |\vec{r}_{\nu,\nu+1}|$ $(\nu = 1, \ldots, N)$; $y_{N+1} \equiv x$; $\vec{r}_{N,N+1} \equiv \vec{r}_{Nx}$; $\vec{r}_{01} = y_1 - x_0$, $r_{01} = |\vec{r}_{01}|$. $\vec{n}_\nu$ are the unit outer normals to S_ν. The *phase function* g in (2.1) has the form

$$g = g(y_1, \ldots, y_N) = r_{01} + \sum_{\nu=1}^{N} r_{\nu,\nu+1} = |y_1 - x_0| + \cdots + |y_N - x|\,. \tag{2.2}$$

It is assumed that all reflecting surfaces are convex and all of them are illuminated; moreover, both source and receiving points are placed in the light zone.

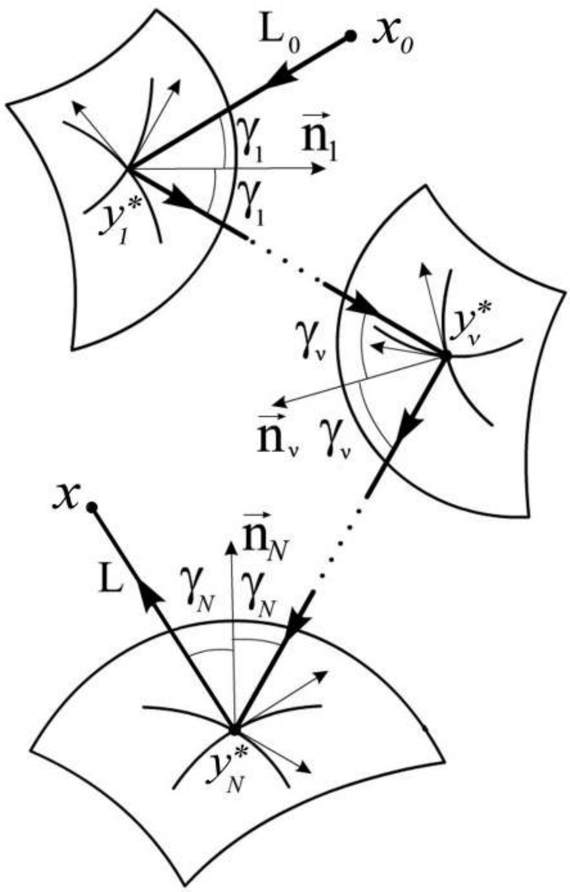

FIG. 1. Multiple reflections from a set of smooth curved surfaces: x_0 - source, x - receiver, y^* - specular reflection points, γ - incidence and reflection angles

The integral in (2.1) can be further estimated asymptotically as $k \to \infty$. From classical results on the asymptotic expansions of integrals [1]–[18], it follows that, if $\nabla g \neq 0$, then this integral has the order $o\,(k^{-N})$; hence, in such a case $p^{sc}(x) \to 0,\ k \to \infty$. Therefore, a not trivial asymptotic value of the scattered field is possible only when there is at least one *stationary point* y^*, defined by $\nabla g(y^*) = 0,\ y^* = (y_1^*, \cdots, y_N^*)$. Let us restrict our consideration to the case when such a special point is isolated; then, integral (2.1) can be estimated by the multidimensional stationary-phase method. To this aim, let us write out the first two asymptotic terms for an integral - over some domain $\Omega \subset E^M$ - having the following general form [24]:

$$I_M(k) = \int_\Omega f(y) e^{ikg(y)}\, dy$$
$$\sim \left(\frac{2\pi}{k}\right)^{M/2} \exp\left[ikg(y^*) + \frac{\pi i}{4}\,\mathrm{sign}\,(g''_*)\right] \frac{f(y^*)}{|\det g''_*|^{1/2}} \left[1+O\left(\frac{1}{k}\right)\right] + F_{\partial\Omega}(k)\left[1+O\left(\frac{1}{k}\right)\right], \tag{2.3a}$$

$$F_{\partial\Omega}(k) \sim \frac{1}{ik} \int_{\partial\Omega} \frac{f(y)}{|\nabla g(y)|^2} \frac{\partial g}{\partial n_y} e^{ikg(y)}\, d\sigma_y\,, \quad k \to \infty\,, \tag{2.3b}$$

where, as indicated above, y^* is a (M-dimensional) stationary point for function g:

$$\frac{\partial g(y^*)}{\partial y_1} = \cdots = \frac{\partial g(y^*)}{\partial y_M} = 0\,. \tag{2.4}$$

Moreover, $g''_* = \left\{\partial^2 g/\partial y_m \partial y_\mu\right\}$, $m, \mu = 1, \ldots, M$, denotes the symmetric Hessian matrix of function g, calculated at the stationary point, and $\mathrm{sign}(g''_*)$ is its *sign*, namely, the difference between the numbers of its positive and negative eigenvalues. Note that the contribution $F_{\partial\Omega}(k)$ given by the boundary of domain Ω into $I_M(k)$ is expressed by an integral over the hyper-surface $\partial\Omega$, of dimension $M-1$; it is proved that this term has a greater vanishing order, as $k \to \infty$, than the first (*leading*) term in Eq. (2.3a).

Formula (2.3), when applied to integral (2.1), attains a great amount of physical meaning. The first leading asymptotic term in (2.3a) corresponds to the "specular" multiple reflections [19] (the residue term, of the order $O(1/k^{M/2+1})$, can be neglected). Let us pass to the asymptotic analysis of the boundary contribution $F_{\partial\Omega}(k)$ in more detail, addressed to its physical interpretation. In this connection, the following two special cases for stationary points may occur, which are discussed here in the order of their decreasing asymptotic importance [24]:

(i) *Boundary stationary point of the first kind.* This is a usual isolated stationary point of the full M-dimensional phase function $g(y)$ (see Eq. (2.4)) falling on the boundary $\partial\Omega$: $y^* \in \partial\Omega$. It is proved [24] that if the boundary hyper-surface is smooth in a vicinity of such a boundary stationary point, then its asymptotic contribution to integral $I_M(k)$ in (2.3) is simply one half of the contribution from the interior stationary points of the phase function. So, it is obvious from Eq. (2.3) that the asymptotic contribution of 1^{st} kind boundary stationary points has the order $O(1/k^{M/2})$, the same as the leading asymptotic term given by the explicit expression in Eq. (2.3a). We assume that no stationary point y^* of function $g(y)$ falls on the boundary $\partial\Omega$; with such an assumption, the denominator present in $F_{\partial\Omega}$ never vanishes. For the example considered below, this assumption is valid, hence we do not further study this kind of boundary stationary point.

(ii) *Boundary stationary point of the second kind.* This is a stationary point of the phase function appearing in the boundary integral of Eq. (2.3b). Of course, the phase function here is the same full phase function but restricted to the boundary hyper-surface $\partial\Omega$: $\tilde{g}(y) = g(y)$, $y \in \partial\Omega$.

In order to estimate the boundary contribution of such boundary stationary points, through quantity $F_{\partial\Omega}(k)$, let us notice that this quantity itself is expressed in the form of a certain integral like $I_M(k)$ in Eq. (2.3a). It is sufficient to take into account only its principal contribution, given by Eq. (2.3a) with $\partial\Omega$ ($M-1$- dim) in place of Ω (M

- dim) and a suitable re-definition of the integrand. Let us assume that the $(M-1)$-dimensional phase function $\tilde{g}(y)$, for $F_{\partial\Omega}(k)$, has itself at least one stationary point. Thus, in this case the asymptotic behaviour of the boundary integral is defined by the factor $(2\pi/k)^{(M-1)/2}$. When formula (2.3) is applied to integral (2.1), one has $M = 2N$. Therefore, the leading contribution given by the full-phase stationary point (physically by the specular reflection) to integral (2.1) has the asymptotic order $O(1/k^N)$, while the contribution coming from the boundary is given by the factor $(1/ik)\cdot(2\pi/k)^{(M-1)/2}$, i.e., has the order $O(1/k^{N+1/2})$. The latter contribution is physically interpreted as the contribution due to the edge effects, and thus this contribution (when there is at least one boundary stationary point) turns out to be asymptotically more significant than all other residue terms in Eq. (2.3), so becoming the second leading term in the asymptotic representation of p^{sc} as $k\to\infty$ (of course, the final asymptotic estimate of p^{sc} must take into account the factor $(ik/2\pi)^N$; see Eq. (2.1)).

Clearly, when there are several stationary points for any phase function, the total contribution is given by the sum of the contributions from all stationary points. The reader can easily imagine what happens when either full phase function $g(y),\ y\in\Omega$, or/and boundary phase function $\tilde{g}(y),\ y\in\partial\Omega$, have no stationary point in their domain. In this case, the corresponding terms will be absent in the (two-leading-terms) asymptotic expansion.

Several examples of an application of these asymptotic ideas, in the case of single reflection, can be found in our recent work [21]. Here below we will study one example showing the application of the proposed approach to the case of double reflection from spherical reflecting surfaces.

3. Double reflection from a pair of spherical domes. The geometry of this example is shown in Figure 2. One can see two reflectors ($N = 2$) in the form of spherical domes S_1, S_2 of the same size. Axes η_1, ζ_1, η_2, ζ_2 all lie in the same plane (that of the sheet) which arranges a symmetric cross-section; axes ξ_1, ξ_2 are thus orthogonal to this plane. Due to evident symmetry, the double specular reflection $x_0 \to y_1^* \to y_2^* \to x$ takes place with both incident angles $\gamma_1 = \gamma_2 = 45^0$. Axes ζ_1 and ζ_2 are directed so that they pass through specular reflection points y_1^* and y_2^*, respectively, which are the apical points of the domes.

Any point P with Cartesian coordinates $(\xi_1^P,\ \eta_1^P,\ \zeta_1^P)$ in the first system has different coordinates in the second system given by

$$\xi_2^P = \xi_1^P, \qquad \eta_2^P = \sqrt{2}a - \eta_1^P, \qquad \zeta_2^P = (\sqrt{2}+2)a - \zeta_1^P, \tag{3.1}$$

where a is the common radius of the spherical domes.

It follows from (2.1) that, with $k\to\infty$,

$$p^{sc}(x) \sim \left(\frac{ik}{2\pi}\right)^2 \int\limits_{S_1}\int\limits_{S_2} \frac{f_1(y)f_2(y)e^{ikg(y)}}{\varphi(y)\,\mu(y)\,\psi(y)}\, dS_1 dS_2\,, \tag{3.2a}$$

$$g(y) = \varphi(y) + \mu(y) + \psi(y), \quad y = y_1\times y_2 \ \ (y_1\in S_1,\ \ y_2\in S_2), \tag{3.2b}$$

$$\varphi(y) = r_{01} = |y_1 - x_0|,\ \mu(y) = r_{1,2} = |y_1 - y_2|,\ \psi(y) = r_{2x} = |y_2 - x|, \tag{3.2c}$$

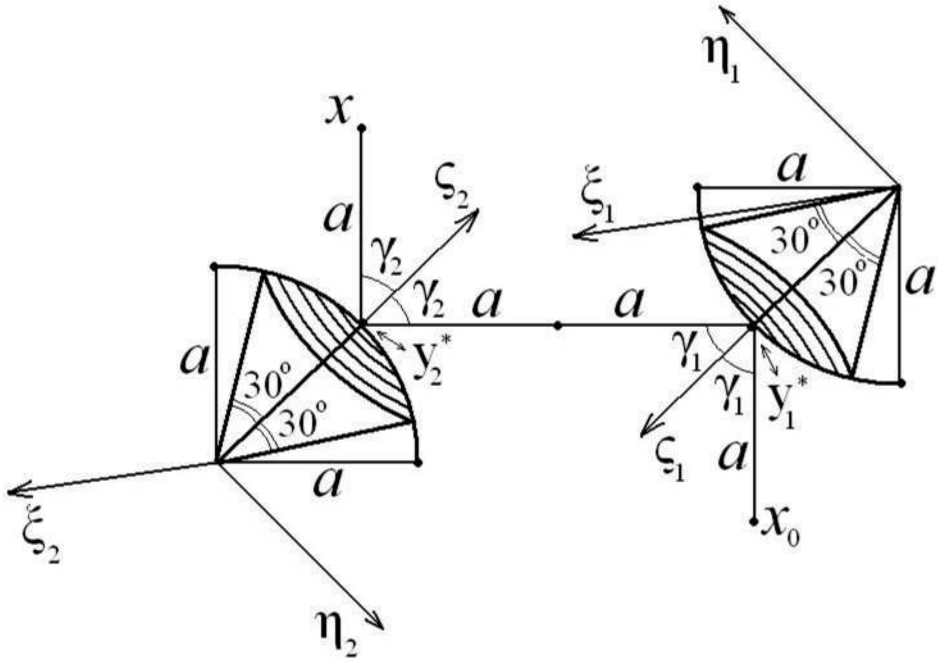

FIG. 2. Double reflection from two spherical domes

$$f_1(y) = \cos\left(\vec{r}_{1,2} \overset{\wedge}{} \vec{n}_1\right) = \frac{(\vec{r}_{1,2} \cdot \vec{n}_1)}{\mu}\,, \qquad f_2(y) = \cos\left(\vec{r}_{2x} \overset{\wedge}{} \vec{n}_2\right) = \frac{(\vec{r}_{2x} \cdot \vec{n}_2)}{\psi}\,. \tag{3.2d}$$

For further analysis, it is convenient to introduce the local polar coordinate systems

$$\begin{cases} \xi_1 = \rho_1 \cos\theta_1 \\ \eta_1 = \rho_1 \sin\theta_1 \end{cases}, \qquad \begin{cases} \xi_2 = \rho_2 \cos\theta_2 \\ \eta_2 = \rho_2 \sin\theta_2 \end{cases}, \tag{3.3}$$

for the points on the spherical domes:

$$y_1 = \left(\xi_1,\, \eta_1,\, \zeta_1 = \sqrt{a^2 - \rho_1^2}\right), \qquad y_2 = \left(\xi_2,\, \eta_2,\, \zeta_2 = \sqrt{a^2 - \rho_2^2}\right), \tag{3.4}$$

with $-\pi \le \theta_1, \theta_2 \le \pi,\ \ 0 \le \rho_1, \rho_2 \le d\ (< a)$. It holds that

$$dS_1 = \frac{a}{\zeta_1}\, d\xi_1\, d\eta_1 = \frac{a\rho_1 d\rho_1 d\theta_1}{\sqrt{a^2 - \rho_1^2}}\,, \qquad dS_2 = \frac{a}{\zeta_2}\, d\xi_2\, d\eta_2 = \frac{a\rho_2 d\rho_2 d\theta_2}{\sqrt{a^2 - \rho_2^2}}\,. \tag{3.5}$$

By choosing $d = a/2$, the outer radius of the domes forms an angle of 30° with respect to the corresponding axis of symmetry, ζ_1 or ζ_2. Moreover, the source and receiver points have the following cartesian coordinates:

$$\begin{aligned} x_0 &= (\xi_0 = 0,\, \eta_0 = -a/\sqrt{2},\, \zeta_0 = a + a/\sqrt{2}\,), \\ x &= (\xi = 0,\, \eta = -a/\sqrt{2},\, \zeta = a + a/\sqrt{2}\,), \end{aligned} \tag{3.6}$$

while the unit normals to the domes are

$$\vec{n}_1 = (\,\xi_1/a,\ \eta_1/a,\ \sqrt{a^2 - \rho_1^2}\,/a\,), \quad \vec{n}_2 = (\,\xi_2/a,\ \eta_2/a,\ \sqrt{a^2 - \rho_2^2}\,/a\,), \tag{3.7}$$

in the respective coordinate systems.

The leading asymptotic term in Eq. (3.2), corresponding physically to the double specular reflection, has been studied in detail in the authors' recent work [19, Section 5], hence there is no need to repeat the development. The symmetric geometry considered here can be directly extracted from the general asymmetric case there treated, if one puts in all relevant formulas of that section the following particular values:

$$\begin{gathered} h\cos\alpha = a\sqrt{2}, \qquad h\sin\alpha = (2+\sqrt{2})a\ , \\ \alpha+\beta=\pi, \qquad b=a, \qquad \gamma_1=\gamma_2=\pi/4\,, \\ L_0=L=a,\ L_1=2a, \qquad g(y^*)=L_0+L+L_1=4a\ . \end{gathered} \tag{3.8}$$

After that, the quoted leading term can be taken from Eq. (47) of [19]. By adding to (3.2) the full application of Eq. (2.3a, 2.3b), we further get

$$p^{sc}(x) \sim p^{sc}_{spec}(x)\left[1+O\left(\frac{1}{k}\right)\right] + F_\Gamma(k)\left[1+O\left(\frac{1}{k}\right)\right], \quad k\to\infty\,, \tag{3.9a}$$

where

$$p^{sc}_{spec}(x) = \frac{\cos\gamma_1\ \cos\gamma_2}{L_0L_1L\sqrt{\det g''_*}}\, e^{ik(L_0+L_1+L)} = \frac{e^{4ika}}{4a^3\sqrt{\det g''_*}} \tag{3.9b}$$

denotes the scattered field coming from the specular reflections, and it holds that

$$\begin{aligned} \det g''_* = &\left[\left(\frac{2\cos\gamma_1}{a}+\frac{1}{L_0}+\frac{1}{L_1}\right)\left(\frac{2\cos\gamma_2}{b}+\frac{1}{L}+\frac{1}{L_1}\right)-\frac{1}{L_1^2}\right] \\ &\times\left\{\left[\frac{2\cos\gamma_1}{a}+\left(\frac{1}{L_0}+\frac{1}{L_1}\right)\cos^2\gamma_1\right]\left[\frac{2\cos\gamma_2}{b}+\left(\frac{1}{L}+\frac{1}{L_1}\right)\cos^2\gamma_2\right]\right. \\ &\left.-\left(\frac{\cos\gamma_1\cos\gamma_2}{L_1}\right)^2\right\} = \frac{(3+2\sqrt{2})(6+5\sqrt{2})}{2a^4}\,. \end{aligned} \tag{3.9c}$$

Moreover, recalling Eqs. (3.5), it is clear that

$$F_\Gamma(k) = \frac{ika^2}{4\pi^2}\int\limits_\Gamma \frac{f_1f_2}{\varphi\,\mu\,\psi}\ \frac{(\nabla g\cdot\vec{n})}{|\nabla g|^2}\ \frac{\rho_1\rho_2\,e^{ikg}d\sigma_{\rho\theta}}{\sqrt{a^2-\rho_1^2}\sqrt{a^2-\rho_2^2}}\,, \tag{3.10}$$

where Γ denotes a union of the boundary hyper-planes for domain $S_1\times S_2$ when surfaces S_1 and S_2 are represented in the chosen polar coordinate systems; the symbol $d\sigma_{\rho\theta}$ stands for an elementary area of such (three-dimensional) boundary hyper-planes, to which $\vec{n}$ is a unit normal. Our aim is now to asymptotically estimate this integral by using Eq. (2.3a) itself.

It can be easily shown that all quantities involved in Eq. (3.10) can be expressed in polar coordinates, as follows:

$$\varphi = \varphi(\rho_1, \theta_1) = \left[\sqrt{2}a\rho_1 \sin\theta_1 + (3+\sqrt{2})a^2 - (2+\sqrt{2})a\sqrt{a^2-\rho_1^2}\right]^{1/2},$$

$$\psi = \psi(\rho_2, \theta_2) = \left[\sqrt{2}a\rho_2 \sin\theta_2 + (3+\sqrt{2})a^2 - (2+\sqrt{2})a\sqrt{a^2-\rho_2^2}\right]^{1/2},$$

$$\mu = \mu(\rho_1,\theta_1,\rho_2,\theta_2) = \Big[(10+4\sqrt{2})a^2 - 2(2+\sqrt{2})a\left(\sqrt{a^2-\rho_1^2}+\sqrt{a^2-\rho_2^2}\right) \tag{3.11a}$$

$$+2\sqrt{a^2-\rho_1^2}\sqrt{a^2-\rho_2^2} - 2\sqrt{2}a(\rho_1\sin\theta_1+\rho_2\sin\theta_2) - 2\rho_1\rho_2\cos(\theta_1+\theta_2)\Big]^{1/2},$$

$$g = \varphi + \mu + \psi = g(\rho_1,\theta_1,\rho_2,\theta_2),$$

while functions f_1 and f_2 are (from Eqs. (3.2d); see also Eqs. (3.1), (3.6), (3.7)):

$$f_1 = \frac{1}{a\mu}[a^2 - \xi_1\xi_2 + \eta_1\eta_2 + \zeta_1\zeta_2 - \sqrt{2}a\eta_1 - (2+\sqrt{2})a\zeta_1]$$

$$= \frac{a^2-\rho_1\rho_2\cos(\theta_1+\theta_2)+\sqrt{a^2-\rho_1^2}\sqrt{a^2-\rho_2^2}-\sqrt{2}a\,\rho_1\sin\theta_1-(2+\sqrt{2})a\sqrt{a^2-\rho_1^2}}{a\mu},$$

$$f_2 = \frac{a^2-\eta\eta_2-\zeta\zeta_2}{a\psi} = \frac{a+(1/\sqrt{2})\rho_2\,\sin\theta_2-(1+1/\sqrt{2})\sqrt{a^2-\rho_2^2}}{\psi}. \tag{3.11b}$$

In the polar coordinate system $(\rho_1, \theta_1, \rho_2, \theta_2)$, the domain $S_1 \times S_2$ is a four-dimensional rectangular parallelepiped, hence its boundary Γ is given by the union of some three-dimensional rectangular parallelepipeds. Let us list all such three-dimensional boundary hyper-planes for $S_1 \times S_2$. They are: (i) $\rho_1 = d$; (ii) $\rho_2 = d$; (iii) $\rho_1 = 0$; (iv) $\rho_2 = 0$; (v) $\theta_1 = -\pi$; (vi) $\theta_1 = \pi$; (vii) $\theta_2 = -\pi$; (viii) $\theta_2 = \pi$.

Obviously, cases (iii) and (iv) give no contribution since in these cases the trivial values $\rho_1 = 0$ or $\rho_2 = 0$ are involved as factors in the integrand in (3.10). The pair of cases (v) and (vi) gives contributions which cancel out each other, since (in view of the evident periodicity) all terms included in the integrand are the same, except terms $(\nabla g \cdot \vec{n})$ which are of the same value but with opposite signs, due to the opposite direction of the unit normals for $\theta_1 = -\pi$ and $\theta_1 = \pi$. By analogy, cases (vii) and (viii) give contributions canceling out each other too. Hence, the only non-trivial cases that remain are the first two, which are studied in detail below.

First of all, it is convenient to write the components of gradient $\nabla g(\rho_1, \theta_1, \rho_2, \theta_2)$:

$$\frac{\partial g}{\partial \rho_1} = \frac{\zeta_0\rho_1/\sqrt{a^2-\rho_1^2} - \eta_0\sin\theta_1}{\varphi}$$
$$+\frac{[(2+\sqrt{2})a-\sqrt{a^2-\rho_2^2}]\,\rho_1/\sqrt{a^2-\rho_1^2}-\sqrt{2}a\sin\theta_1-\rho_2\cos(\theta_1+\theta_2)}{\mu}, \tag{3.12a}$$

$$\frac{\partial g}{\partial \theta_1} = -\frac{\eta_0\rho_1\cos\theta_1}{\varphi} + \frac{\rho_1\rho_2\sin(\theta_1+\theta_2) - \sqrt{2}a\rho_1\cos\theta_1}{\mu}, \tag{3.12b}$$

$$\frac{\partial g}{\partial \rho_2} = \frac{\zeta\rho_2/\sqrt{a^2-\rho_2^2} - \eta\sin\theta_2}{\psi} + \frac{[(2+\sqrt{2})a-\sqrt{a^2-\rho_1^2}]\,\rho_2/\sqrt{a^2-\rho_2^2}-\sqrt{2}a\sin\theta_2-\rho_1\cos(\theta_1+\theta_2)}{\mu}\,, \tag{3.12c}$$

$$\frac{\partial g}{\partial \theta_2} = -\frac{\eta\rho_2\cos\theta_2}{\psi} + \frac{\rho_1\rho_2\sin(\theta_1+\theta_2) - \sqrt{2}a\rho_2\cos\theta_2}{\mu}\,. \tag{3.12d}$$

After that, let us begin to treat the case:

(i) $\rho_1 = d$. Here the phase function is three-dimensional: $\tilde{g}_1(\theta_1, \rho_2, \theta_2) = g(d, \theta_1, \rho_2, \theta_2)$; moreover, in Eq. (3.10) it holds that $d\sigma_{\rho\theta} = d\theta_1\,d\rho_2\,d\theta_2$ and $(\nabla g \cdot \vec{n}) = \partial g/\partial\rho_1$. The stationary points of function $\tilde{g}_1(\theta_1, \rho_2, \theta_2)$ are defined by Eqs. (3.12, b-d) set equal to zero. The first two of them are easily calculated as follows.

(i_1) The values $\theta_1^* = \pi/2$, $\theta_2^* = -\pi/2$ automatically provide vanishing expressions (3.12b) and (3.12d). Thus, Eq. (3.12c) gives an equation to find the stationary value for quantity ρ_2; to this aim, let us introduce the new function

$$F(\rho_2) = (\partial\tilde{g}_1/\partial\rho_2)(\pi/2, \rho_2, -\pi/2).$$

Obviously, it is a continuous function on the interval $\rho_2 \in [0, d]$ (recall that $d = a/2$). Its values at the ends of this interval are

$$F(0) = -\frac{a}{\sqrt{2}\,\psi(0,-\pi/2)} + \frac{\sqrt{2}a - a/2}{\mu(d,\pi/2,0,-\pi/2)} = -\frac{1}{\sqrt{2}} + \frac{\sqrt{2}-1/2}{\sqrt{6+\sqrt{2}-\sqrt{3}-\sqrt{6}}} < 0, \tag{3.13a}$$

since $\psi(0,-\pi/2) = a$ and $\mu(d,\pi/2,0,-\pi/2) = \sqrt{6+\sqrt{2}-\sqrt{3}-\sqrt{6}}a$, along with

$$F(d) = \frac{(1+\sqrt{2}-\sqrt{3})a}{\sqrt{6}\;\psi(d,-\pi/2)} + \frac{(2+\sqrt{2}-\sqrt{3}+\sqrt{6})a}{\sqrt{3}\;\mu(d,\pi/2,d,-\pi/2)} > 0. \tag{3.13b}$$

Due to opposite signs at the ends of the interval, there is at least one point ρ_2^* such that $F(\rho_2^*) = 0$; a detailed investigation shows that such a point is actually unique and holds: $\rho_2^* = 0.07792a$.

The contribution of this boundary stationary point to the value of integral (3.10) can be obtained by using the leading asymptotic term in Eq. (2.3a), with $M = 3$, $\Omega = \Gamma$ and integrand $f = \dfrac{f_1 f_2}{\varphi\,\mu\,\psi}\;\dfrac{(\nabla g\cdot\vec{n})}{|\nabla g|^2}\;\dfrac{\rho_1\rho_2}{\sqrt{a^2-\rho_1^2}\sqrt{a^2-\rho_2^2}}$. For this, one needs to calculate the components of the 3×3 Hessian matrix $\tilde{g}''_{1*}$:

$$\left(\frac{\partial^2\tilde{g}_1}{\partial\theta_1^2}\right)_* = \frac{\eta_0 d}{\varphi_*} + \frac{\rho_2^* + \sqrt{2}a}{\mu_*}\,d, \quad \left(\frac{\partial^2\tilde{g}_1}{\partial\theta_1\partial\rho_2}\right)_* = \left(\frac{\partial^2\tilde{g}_1}{\partial\rho_2\partial\theta_2}\right)_* = 0,$$

$$\left(\frac{\partial^2\tilde{g}_1}{\partial\theta_1\partial\theta_2}\right)_* = \frac{\rho_2^*\,d}{\mu_*}, \quad \left(\frac{\partial^2\tilde{g}_1}{\partial\theta_2^2}\right)_* = -\frac{\eta\rho_2^*}{\psi_*} + \frac{d-\sqrt{2}a}{\mu_*}\,\rho_2^*,$$

$$\left(\frac{\partial^2\tilde{g}_1}{\partial\rho_2^2}\right)_* = \frac{a^2\zeta}{\psi_*(a^2-\rho_2^{*2})^{3/2}} - \frac{1}{\psi_*^3}\left[\frac{\zeta\rho_2^*}{\sqrt{a^2-\rho_2^{*2}}}+\eta\right]^2 + \frac{a^2}{(a^2-\rho_2^{*2})^{3/2}}$$

$$\times\frac{(2+\sqrt{2})a-\sqrt{a^2-d^2}}{\mu_*} - \frac{1}{\mu_*^3}\left\{[(2+\sqrt{2})a-\sqrt{a^2-d^2}]\frac{\rho_2^*}{\sqrt{a^2-\rho_2^{*2}}}+\sqrt{2}a-d\right\}^2, \tag{3.14}$$

from which we can deduce

$$\det\tilde{g}''_{1*} = \left(\frac{\partial^2\tilde{g}_1}{\partial\rho_2^2}\right)_*\left[\left(\frac{\partial^2\tilde{g}_1}{\partial\theta_1^2}\right)_*\left(\frac{\partial^2\tilde{g}_1}{\partial\theta_2^2}\right)_* - \left(\frac{\partial^2\tilde{g}_1}{\partial\theta_1\partial\theta_2}\right)_*^2\right]\ (>0) \tag{3.15}$$

along with $\operatorname{sign}(\tilde{g}''_{1*}) = 3$. As a consequence, the corresponding contribution is given by applying Eq. (2.3a) to Eq. (3.10), as follows:

$$J(\mathrm{i}_1) = \frac{ika^2}{4\pi^2}\left(\frac{2\pi}{k}\right)^{3/2}\left[\frac{f_1\,f_2}{\varphi\,\mu\,\psi\,(\det\tilde{g}''_1)^{1/2}}\,\frac{\nabla g\cdot\vec{n}}{|\nabla g|^2}\,\frac{\rho_1\rho_2\,e^{i(k\tilde{g}_1+3\pi/4)}}{\sqrt{a^2-d^2}\,\sqrt{a^2-\rho_2^2}}\right]_*\left[1+O\left(\frac{1}{k}\right)\right]$$

$$= \frac{1}{\sqrt{k}}\,\frac{ia^2 f_1^* f_2^*}{\sqrt{6\pi}\ \varphi_*\,\mu_*\,\psi_*\,(\det\tilde{g}''_{1*})^{1/2}}\left(\frac{\partial g/\partial\rho_1}{|\nabla g|^2}\right)_*\frac{\rho_2^*\,e^{i(k\tilde{g}_{1*}+3\pi/4)}}{\sqrt{a^2-\rho_2^{*2}}}\left[1+O\left(\frac{1}{k}\right)\right] \equiv J_1(k)\,, \tag{3.16}$$

where the asterisks mean that all quantities involved are to be calculated for $\theta_1 = \theta_1^* = \pi/2,\ \theta_2 = \theta_2^* = -\pi/2,\ \rho_2 = \rho_2^* = 0.07792a$.

(i_2) The values $\theta_1^* = -\pi/2$, $\theta_2^* = \pi/2$ also automatically provide vanishing expressions (3.12b) and (3.12d). By analogy to case (i_1), it can be proved that again only one value $\rho_2^* = 0.03581a$ exists for which expression (3.12c) is trivial.

The elements of the Hessian now are

$$\left(\frac{\partial^2\tilde{g}_1}{\partial\theta_1^2}\right)_* = -\frac{\eta_0 d}{\varphi_*} + \frac{\rho_2^* - \sqrt{2}a}{\mu_*}\,d, \quad \left(\frac{\partial^2\tilde{g}_1}{\partial\theta_1\partial\rho_2}\right)_* = \left(\frac{\partial^2\tilde{g}_1}{\partial\rho_2\partial\theta_2}\right)_* = 0,$$

$$\left(\frac{\partial^2\tilde{g}_1}{\partial\theta_1\partial\theta_2}\right)_* = \frac{\rho_2^*\,d}{\mu_*}, \quad \left(\frac{\partial^2\tilde{g}_1}{\partial\theta_2^2}\right)_* = \frac{\eta\rho_2^*}{\psi_*} + \frac{d+\sqrt{2}a}{\mu_*}\,\rho_2^*,$$

$$\left(\frac{\partial^2\tilde{g}_1}{\partial\rho_2^2}\right)_* = \frac{a^2\zeta}{\psi_*(a^2-\rho_2^{*2})^{3/2}} - \frac{1}{\psi_*^3}\left[\frac{\zeta\rho_2^*}{\sqrt{a^2-\rho_2^{*2}}}-\eta\right]^2 + \frac{a^2}{(a^2-\rho_2^{*2})^{3/2}}$$

$$\times\frac{(2+\sqrt{2})a-\sqrt{a^2-d^2}}{\mu_*} - \frac{1}{\mu_*^3}\left\{[(2+\sqrt{2})a-\sqrt{a^2-d^2}]\frac{\rho_2^*}{\sqrt{a^2-\rho_2^{*2}}}-\sqrt{2}a-d\right\}^2, \tag{3.17}$$

whence determinant (>0) and sign ($=3$) can be calculated. The corresponding contribution is given by

$$J(\mathrm{i}_2)=\frac{1}{\sqrt{k}}\,\frac{ia^2 f_1^* f_2^*}{\sqrt{6\pi}\;\varphi_*\,\mu_*\,\psi_*\,(\det \tilde{g}''_{1*})^{1/2}}\left(\frac{\partial g/\partial\rho_1}{|\nabla g|^2}\right)_*\frac{\rho_2^*\,e^{i(k\tilde{g}_{1*}+3\pi/4)}}{\sqrt{a^2-\rho_2^{*2}}}\left[1+O\left(\frac{1}{k}\right)\right]\equiv J_2(k), \tag{3.18}$$

where the asterisks mean that the quantities involved are to be calculated for $\theta_1=\theta_1^*=-\pi/2,\ \theta_2=\theta_2^*=\pi/2,\ \rho_2=\rho_2^*=0.03581a$.

A detailed analysis of Eqs. (3.12) shows that in case (i) two other (less evident) stationary points exist, additionally to those of sub-cases (i_1) and (i_2). They are:

(i_3) $\theta_1^*=-19.596^0$, $\rho_2^*=0.07370a$, $\theta_2^*=-0.363^0$.

(i_4) $\theta_1^*=-160.404^0$, $\rho_2^*=0.07370a$, $\theta_2^*=-179.637^0$.

These two points possess the same value of ρ_2^* but angular arguments θ_1^* and θ_2^* shifted by 180° with respect to each other. The contributions from them, namely $J_3\equiv J(\mathrm{i}_3)$ and $J_4\equiv J(\mathrm{i}_4)$, are rather similar to those given by expressions (3.16), (3.18). The only difference is that the elements of the Hesssian (and, as a result, some related quantities) have a much more complex form, which is omitted for brevity.

(ii) $\rho_2=d$. Again, for the sake of brevity, we give only a brief sketch of the four analogous boundary stationary points.

First of all, the phase function here is three-dimensional too: $\tilde{g}_2(\rho_1,\theta_1,\theta_2)=g(\rho_1,\theta_1,d,\theta_2)$, and in Eq. (3.10) it holds that $d\sigma_{\rho\theta}=d\rho_1\,d\theta_1\,d\theta_2$, $(\nabla g\cdot\vec{n})=\partial g/\partial\rho_2$. The stationary points of function $\tilde{g}_2(\rho_1,\theta_1,\theta_2)$ are defined by Eqs. (3.10a,b,d) set equal to zero. The analysis, like the previous one performed above, shows evident properties of symmetry with respect to the following change: $\rho_1\leftrightarrows\rho_2$, $\theta_1\leftrightarrows\theta_2$. In fact, all functions involved in integral (3.10) have that symmetry, except functions f_1 and f_2. If these functions also would have the same symmetry, then case (ii) would be absolutely the same as case (i), and the final result could be simply obtained by taking into account the above quantities $J_1,\dots,J_4$ two times. Nonetheless, it can be easily understood how to correctly calculate the arising quantities $J_5,\dots,J_8$, analogous to $J_1,\dots,J_4$, in the case when they are slightly different, due to the presence of non-symmetric functions f_1 and f_2 (we omit such calculations).

As a consequence, the final asymptotic estimate of the diffraction integral (3.2), through Eqs. (3.9) and (3.10), consists of the following expression for the total scattered wave field:

$$p^{sc}(x)\sim\frac{e^{4ika}}{4a}\left[\frac{2}{(3+2\sqrt{2})(6+5\sqrt{2})}\right]^{1/2}+\sum_{h=1}^{8}J_h(k)+O\left(\frac{1}{k}\right),\qquad k\to\infty, \tag{3.19}$$

where all quantities J have the order $O(1/\sqrt{k})$. One thus can see that the contribution given by the edge effects has an asymptotic order which is more significant than the neglected residue $O(1/k)$.

A comparison of this explicit expression (versus wave number) with respect to other results including the direct numerical evaluation of integral (3.2) is reflected in Figure 3.

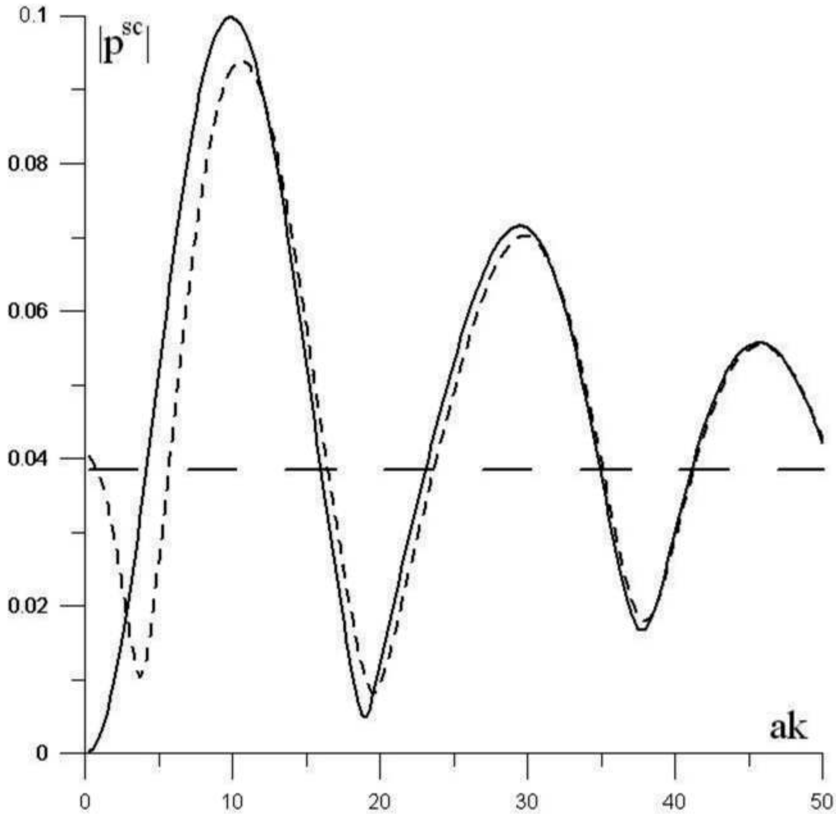

FIG. 3. Comparison of explicit asymptotic result (3.19) (short dashed line: - - -) with exact numerical treatment of integral (3.2) (solid line: ——) and leading asymptotic term (3.9b) (long dashed line: — — —).

4. Concluding remarks and numerical investigation. 1. In the present paper a new asymptotic method is proposed to study high-frequency multiple diffraction by curved surfaces, in frames of the Kirchhoff diffraction theory. For this aim, we use an iterated multi-fold version of the standard Kirchhoff integral (early known for single reflection) suitable for arbitrary number of re-reflections, as established in some authors' previous works. We show that, in the case of curved reflecting surfaces, two types of stationary points may arise, and accept the assumption that only the second-type ones are important for the present study. The main goal of the paper is to give the basics for an analytical construction of the two leading terms arising from Kirchhoff theory when applied to multiple diffraction by non-plane surfaces. It is proved that the full value of the diffracted wave's amplitude, at high frequencies, is given by the sum of the (leading) specular reflection term plus the boundary edges contribution. An example of double reflection from two spherical domes shows the application of such a general approach.

2. It is shown that the main contribution from the boundary edges comes out from the stationary points of the corresponding phase functions, which are functions of three variables. When such special points exist, the respective contributions become more significant than the (neglected) second asymptotic term of the classical ray-theory prediction. More precisely, the neglected terms have the order $O\,(1/k)$, while the stationary points of the boundary phase functions give a (greater) contribution of the order $O\left(1/\sqrt{k}\right)$, as $k \to \infty$. In all cases, the asymptotic estimate of the edge effects is made through the application of the multidimensional stationary phase method [16]–[18].

3. There are three lines in Figure 3. The solid line reflects a simple direct numerical calculation of the pertinent four-dimensional Kirchhoff's diffraction integral; this has been performed by applying a standard quadrature formula over all four variables, with a dense mesh of nodes (the number of nodes should be very large when one operates with a short-wavelength analysis, to keep at least 10 nodes per wave length; concrete examples show that the desired precision in drawing the graph, with a relative error of 10^{-3}, can be attained with a few days' calculation implemented on a personal computer). The short dashed line shows indeed the key result of the present paper, since it is obtained as a sum of the specular reflection term plus the explicit asymptotic estimate of the boundary integral giving the edge effects, here analytically calculated by the stationary phase method. The long dashed line simply represents the specular reflection leading term.

4. From the figure, it is clearly seen that the "specular reflection" term (typical of Ray theory procedures) provides only a poor precision, even for sufficiently high frequencies. In the approach proposed here, where the improvement in precision is achieved by an explicit analytical estimate of the edge effects, the results obtained are quite precise, as shown by the comparison between lines 1 and 2. In the considered example, the relative error does not exceed a few percents for $ak > 12$. Finally, we would emphasize that, when double reflection from curved surfaces is involved, the authors do not know of any other published theory which can predict correctly in explicit form even the first leading ray-theory asymptotic term.

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References

[1] H. Hönl, A. W. Maue, and R. Westpfahl, *Theory of Diffraction*, Naval Intelligence Support Center, New York, 1978.

[2] L. B. Felsen and N. Marcuvitz, *Radiation and Scattering of Waves*, John Wiley, Hoboken, New Jersey, 2003.

[3] V. A. Borovikov and B. Ye. Kinber, *Geometrical theory of diffraction*, IEE Electromagnetic Waves Series, vol. 37, Institution of Electrical Engineers (IEE), London, 1994. Translated and revised from the Russian original. MR1292857 (95j:78003)

[4] M. Born and E. Wolf, *Principles of Optics (7th ed.)*, Cambridge University Press, 1999.

[5] R. Guenther, Diffraction and Gaussian Beams. In: *Modern Optics*, John Wiley, New York, 1990, 323-360.

[6] V. M. Babich, V. S. Buldyrev, *Asymptotic Methods in Short-Wavelength Diffraction Theory*, Springer-Verlag, Berlin / Heidelberg, 1989.

[7] H. Kuttruff, *Room Acoustics*, Applied Science, London, 1973.
[8] Joseph B. Keller, *Geometrical theory of diffraction*, J. Opt. Soc. Amer. **52** (1962), 116–130. MR0135064 (24 #B1115)
[9] D. A. McNamara, C. W. I. Pistorius, and J. A. G. Malherbe, *Introduction to the uniform geometrical theory of diffraction*, The Artech House Antennas and Propagation Library, Artech House Inc., Boston, MA, 1990. MR1118379 (92m:78017)
[10] Eugen Skudrzyk, *The foundations of acoustics*, Springer-Verlag, New York, 1971. Basic mathematics and basic acoustics. MR0502736 (58 #19668)
[11] I. Tolstoy, *Wave Propagation*, McGraw-Hill, New York, 1973.
[12] Michael E. Taylor, *Pseudodifferential operators*, Princeton Mathematical Series, vol. 34, Princeton University Press, Princeton, N.J., 1981. MR618463 (82i:35172)
[13] Mezhlum A. Sumbatyan and Antonio Scalia, *Equations of mathematical diffraction theory*, Differential and Integral Equations and Their Applications, vol. 5, Chapman & Hall/CRC, Boca Raton, FL, 2005. MR2099746 (2005i:74044)
[14] Gary M. Jebsen and Herman Medwin, *On the failure of the Kirchhoff assumption in backscatter*, J. Acoust. Soc. Amer. **72** (1982), no. 5, 1607–1611, DOI 10.1121/1.388496. MR677408 (83m:73030)
[15] Joachim Focke, *Asymptotische Entwicklungen mittels der Methode der stationären Phase* (German), Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Nat. Kl. **101** (1954), no. 3, 48. MR0068650 (16,919a)
[16] Nicholas Chako, *Asymptotic expansions of double and multiple integrals occurring in diffraction theory*, J. Inst. Math. Appl. **1** (1965), 372–422. MR0204944 (34 #4779)
[17] M. V. Fedoryuk, *The stationary phase method and pseudodifferential operators*, Russian Math. Surveys **26** (1971), 65-115.
[18] J. C. Cooke, *Stationary phase in two dimensions*, IMA J. Appl. Math. **29** (1982), no. 1, 25–37, DOI 10.1093/imamat/29.1.25. MR667780 (84d:41056)
[19] E. Scarpetta and M. A. Sumbatyan, *Explicit analytical representations in the multiple high-frequency reflection of acoustic waves from curved surfaces: the leading asymptotic term*, Acta Acustica united with Acustica **97** (2011), 115-127.
[20] M. A. Sumbatyan and N. V. Boyev, *High-frequency diffraction by non-convex obstacles*, J. Acoust. Soc. Amer. **95** (1994), 2346-2353.
[21] Edoardo Scarpetta and Mezhlum A. Sumbatyan, *An asymptotic estimate of the edge effects in the high frequency Kirchhoff diffraction theory for 3-d problems*, Wave Motion **48** (2011), no. 5, 408–422, DOI 10.1016/j.wavemoti.2011.02.003. MR2793773 (2012j:35298)
[22] E. Scarpetta and M. A. Sumbatyan, *High-frequency multiple Kirchhoff diffraction by flat reflectors with a change of ray-path planes*, Acta Acustica united with Acustica **98** (2012), 700-712.
[23] E. Scarpetta and M. A. Sumbatyan, *The contribution of the edge effects in the multiple high-frequency Kirchhoff diffraction by plane surfaces*, Wave Motion **50** (2013), 210–225.
[24] M. V. Fedorjuk, *The method of stationary phase in the multidimensional case. Contribution from the boundary of the domain* (Russian), Ž. Vyčisl. Mat. i Mat. Fiz. **10** (1970), 286–299. MR0273263 (42 #8144)