

Research Article

The Multiplication Operator from $F(p, q, s)$ Spaces to n th Weighted-Type Spaces on the Unit Disk

Yongmin Liu¹ and Yanyan Yu²

¹ Department of Mathematics, Jiangsu Normal University, Xuzhou 221116, China

² School of Mathematics and Physics Science, Xuzhou Institute of Technology, Xuzhou 221111, China

Correspondence should be addressed to Yongmin Liu, minliu@jsnu.edu.cn

Received 2 February 2012; Revised 11 April 2012; Accepted 30 April 2012

Academic Editor: Amol Sasane

Copyright © 2012 Y. Liu and Y. Yu. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Let $H(\mathbb{D})$ be the space of analytic functions on \mathbb{D} and $u \in H(\mathbb{D})$. The boundedness and compactness of the multiplication operator M_u from $F(p, q, s)$, (or $F_0(p, q, s)$) spaces to n th weighted-type spaces on the unit disk are investigated in this paper.

1. Introduction

Let $H(\mathbb{D})$ denote the space of all analytic functions in the open unit disc \mathbb{D} of the finite-complex plane \mathbb{C} , $\partial\mathbb{D}$ the boundary of \mathbb{D} , \mathbb{N}_0 the set of all nonnegative integers and \mathbb{N} the set of all positive integers. Let $\mu(z)$ be a positive continuous function on \mathbb{D} (weight) such that $\mu(z) = \mu(|z|)$ and $n \in \mathbb{N}_0$. The n th weighted-type spaces on the unit disk \mathbb{D} , denoted by $\mathcal{W}_\mu^{(n)}(\mathbb{D})$ which were introduced in [1], consist of all $f \in H(\mathbb{D})$ such that

$$b_{\mathcal{W}_\mu^{(n)}(\mathbb{D})}(f) = \sup_{z \in \mathbb{D}} \mu(z) |f^{(n)}(z)| < \infty. \quad (1.1)$$

For $n = 0$, the space becomes the weighted-type space $H_\mu^\infty(\mathbb{D})$ (see, e.g., [2–4]), for $n = 1$ the Bloch-type space $\mathcal{B}_\mu(\mathbb{D})$ and for $n = 2$ the Zygmund-type space $\mathcal{Z}_\mu(\mathbb{D})$. For $\mu(z) = 1 - |z|^2$, we obtain correspondingly the classical weighted-type space, the Bloch space $\mathcal{B}(\mathbb{D}) = \mathcal{B}$, and the Zygmund space $\mathcal{Z}(\mathbb{D}) = \mathcal{Z}$. Some information on Zygmund-type spaces on the unit disk and

some operators on them, for example, in [5–9] and on the unit ball, can be found, for example, in [10, 11]. From now on, we will assume that $n \in \mathbb{N}$. Set

$$\|f\|_{\mathcal{W}_\mu^{(n)}(\mathbb{D})} = \sum_{j=0}^{n-1} |f^{(j)}(0)| + b_{\mathcal{W}_\mu^{(n)}(\mathbb{D})}(f). \quad (1.2)$$

With this norm, the n th weighted-type space becomes a Banach space.

The little n th weighted-type space, denoted by $\mathcal{W}_{\mu,0}^{(n)}(\mathbb{D})$, is a closed subspace of $\mathcal{W}_\mu^{(n)}(\mathbb{D})$ consisting of those f for which

$$\lim_{|z| \rightarrow 1} \mu(z) |f^{(n)}(z)| = 0. \quad (1.3)$$

A positive continuous function ϕ on $[0, 1)$ is called a normal if there exist positive numbers a, b , $0 < a < b$, and $t_0 \in [0, 1)$, such that

$$\begin{aligned} \frac{\phi(t)}{(1-t^2)^a} \text{ decreases for } t_0 \leq t < 1 \text{ and } \lim_{t \rightarrow 1^-} \frac{\phi(t)}{(1-t^2)^a} = 0, \\ \frac{\phi(t)}{(1-t^2)^b} \text{ increases for } t_0 \leq t < 1 \text{ and } \lim_{t \rightarrow 1^-} \frac{\phi(t)}{(1-t^2)^b} = \infty \end{aligned} \quad (1.4)$$

(see [12]).

For $0 < p, s < \infty$, $-2 < q < \infty$, a function $f \in H(\mathbb{D})$ is said to belong to the general function space $F(p, q, s) = F(p, q, s)(\mathbb{D})$ if

$$\|f\|_{F(p,q,s)}^p = |f(0)|^p + \sup_{z \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q (1-|\varphi_a(z)|^2)^s dA(z) < \infty, \quad (1.5)$$

where $\varphi_a(z) = (a-z)/(1-\bar{a}z)$, $a \in \mathbb{D}$. An $f \in H(\mathbb{D})$ is said to belong to $F_0(p, q, s) = F_0(p, q, s)(\mathbb{D})$ if

$$\lim_{|a| \rightarrow 1} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q (1-|\varphi_a(z)|^2)^s dA(z) = 0. \quad (1.6)$$

The space $F(p, q, s)$ was introduced by Zhao in [13]. We can get many function spaces if we take some specific parameters of p, q, s ; for example (see [13]), $F(p, q, s) = \mathcal{B}^{(2+q)/p}$ and $F_0(p, q, s) = \mathcal{B}_0^{(2+q)/p}$ for $s > 1$; $F(p, q, s) \subset \mathcal{B}^{(2+q)/p}$ and $F_0(p, q, s) \subset \mathcal{B}_0^{(2+q)/p}$ for $0 < s \leq 1$; $F(2, 0, s) = Q_s$ and $F_0(2, 0, s) = Q_{s,0}$; $F(2, 0, 1) = \text{BMOA}$ and $F_0(2, 0, 1) = \text{VMOA}$. Since for $q + s \leq -1$, $F(p, q, s)$ is the space of constant functions, we assume that $q + s > -1$.

The multiplication operator M_u is defined by $M_u f = uf$. It is interesting to provide a function theoretic characterization of when u induces a bounded or compact composition operator on various spaces (see, e.g., [14–20]). Yu and Liu in [21] studied the boundedness and compactness of the operator DM_u from mixed-norm spaces to the Bloch-type space. Stević in [22] studied the boundedness and compactness of the product of the differentiation

and composition operator from the space of bounded analytic functions, the Bloch space and the little Bloch space to n th weighted-type spaces on the unit disk. Zhang and Xiao in [23] studied the bounded and compact-weighted composition operator uC_φ from the $F(p, q, s)$ space to the Bloch-type space in the unit disc. Zhang and Zeng in [24] studied the boundedness and compactness of weighted differentiation composition operators from weighted bergman space to n th weighted-type spaces on the unit disk. Ye in [25] studied the boundedness and compactness of the weighted composition operator uC_φ from $F(p, q, s)$ into the logarithmic Bloch space \mathcal{B}_{\log} on the unit disk. Yang in [26] studied the boundedness and compactness of weighted differentiation composition operators from the $F(p, q, s)$ space to the Bloch-type space. Yang in [27] studied the boundedness and compactness of the composition operator from the $F(p, q, s)$ space to n th weighted-type spaces on the unit disk. Zhou and Chen in [28] studied the weighted composition operator from the $F(p, q, s)$ space to the Bloch-type space in the unit ball. Zhu in [29] studied the weighted composition operator from the $F(p, q, s)$ space to F_μ^∞ space in the unit ball. Stević in [30, 31] studied the boundedness and compactness of the integral operators between $F(p, q, s)$ spaces and Bloch-type spaces in the unit ball. This paper focuses on the boundedness and compactness of the operators M_u from $F(p, q, s)$ (or $F_0(p, q, s)$) to n th weighted-type spaces on the unit disk.

From now on, we will always assume that $0 < p, s < \infty$, $-2 < q < \infty$, $q + s > -1$, $\mu(z) = \mu(|z|)$ is normal and $n \in \mathbb{N}$. Further, for the sake of simplicity, C will always denote an independent constant, which can be different from one display to another.

2. Auxiliary Results

In this section we formulate some auxiliary results which will be used in the proofs of the main results.

Lemma 2.1 (see [13, 27]). *Assume that $f \in F(p, q, s)$. Then, for each $n \in \mathbb{N}$, there is a positive constant C , independent of f such that $\|f\|_{\mathcal{B}^{(2+q)/p}} \leq C\|f\|_{F(p,q,s)}$ and*

$$|f^{(n)}(z)| \leq \frac{C\|f\|_{F(p,q,s)}}{(1-|z|^2)^{((2+q-p)/p)+n}}, \quad z \in \mathbb{D}. \quad (2.1)$$

Moreover, if $f \in F_0(p, q, s)$, then $f \in \mathcal{B}_0^{(2+q)/p}$.

Lemma 2.2 (see [32, 33]). *Let $\alpha > 0$ and $f \in \mathcal{B}^\alpha$. Then,*

$$|f(z)| \leq \begin{cases} C\|f\|_{\mathcal{B}^\alpha}, & \text{if } 0 < \alpha < 1, \\ C \log \frac{2}{1-|z|^2} \|f\|_{\mathcal{B}^\alpha}, & \text{if } \alpha = 1, \\ \frac{C}{(\alpha-1)(1-|z|^2)^{\alpha-1}} \|f\|_{\mathcal{B}^\alpha}, & \text{if } \alpha > 1. \end{cases} \quad (2.2)$$

Lemma 2.3 (see [1]). Assume $a > 0$ and

$$D_n(a) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a & (a+1) & \cdots & (a+n-1) \\ a(a+1) & (a+1)(a+2) & \cdots & (a+n-1)(a+n) \\ \vdots & \vdots & \ddots & \vdots \\ \prod_{j=0}^{n-2} (a+j) & \prod_{j=0}^{n-2} (a+j+1) & \cdots & \prod_{j=0}^{n-2} (a+j+n-1) \end{vmatrix}. \quad (2.3)$$

Then, $D_n(a) = \prod_{j=0}^{n-1} j!$.

By standard arguments (see, e.g., [34] or Lemma 3 in [35]) the following lemma follows.

Lemma 2.4. Assume that $u \in H(\mathbb{D})$. Then, $M_u : F(p, q, s)$ (or $F_0(p, q, s)$) $\rightarrow \mathcal{W}_\mu^{(n)}(\mathbb{D})$ is compact if and only if $M_u : F(p, q, s)$ (or $F_0(p, q, s)$) $\rightarrow \mathcal{W}_\mu^{(n)}(\mathbb{D})$ is bounded and for any bounded sequence $\{f_k\}$ in $F(p, q, s)$ (or $F_0(p, q, s)$) which converges to zero uniformly on compact subsets of \mathbb{D} as $k \rightarrow \infty$, one has $\|M_u f_k\|_{\mathcal{W}_\mu^{(n)}(\mathbb{D})} \rightarrow 0$ as $k \rightarrow \infty$.

3. The Boundedness and Compactness of M_u from $F(p, q, s)$ to $\mathcal{W}_\mu^{(n)}(\mathbb{D})$

In this section, we characterize the boundedness and compactness of $M_u : F(p, q, s) \rightarrow \mathcal{W}_\mu^{(n)}(\mathbb{D})$.

Theorem 3.1. Assume that $u \in H(\mathbb{D})$ and μ is normal.

(a) If $2 + q > p$, then $M_u : F(p, q, s) \rightarrow \mathcal{W}_\mu^{(n)}(\mathbb{D})$ is bounded if and only if

$$\sup_{z \in \mathbb{D}} \frac{\mu(|z|)|u(z)|}{(1 - |z|^2)^{(2+q-p)/p+n}} < \infty. \quad (3.1)$$

(b) If $2 + q < p$, then $M_u : F(p, q, s) \rightarrow \mathcal{W}_\mu^{(n)}(\mathbb{D})$ is bounded if and only if (3.1) holds and $u \in \mathcal{W}_\mu^{(n)}(\mathbb{D})$.

(c) If $2 + q = p$, $s > 1$, then $M_u : F(p, q, s) \rightarrow \mathcal{W}_\mu^{(n)}(\mathbb{D})$ is bounded if and only if (3.1) holds and

$$\sup_{z \in \mathbb{D}} \mu(|z|) \left| u^{(n)}(z) \right| \log \frac{2}{1 - |z|^2} < \infty. \quad (3.2)$$

Proof. Let $0 < p, s < \infty, -2 < q < \infty, q + s > -1$. Assume that conditions (3.1) holds. Then for all $z \in \mathbb{D}$

$$|u(z)| \leq C \frac{(1 - |z|^2)^{(2+q-p)/p+n}}{\mu(|z|)}. \quad (3.3)$$

Since

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|e^{i\theta} - z|^2} d\theta = \frac{1}{1 - |z|^2}, \quad \forall z \in \mathbb{D}, \quad (3.4)$$

let $\delta_z = (1 + |z|)/2$, then we have $|z/\delta_z| = 2|z|/(1 + |z|) < 1$, so

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|\delta_z e^{i\theta} - z|^2} d\theta = \frac{1}{\delta_z^2 - |z|^2}. \quad (3.5)$$

By the Cauchy integral formula and (3.3), we obtain

$$\begin{aligned} |u'(z)| &= \frac{1}{2\pi} \left| \int_0^{2\pi} \frac{u(\delta_z e^{i\theta})}{(\delta_z e^{i\theta} - z)^2} \delta_z e^{i\theta} d\theta \right| \\ &\leq C \frac{(1 - \delta_z^2)^{(2+q-p)/p+n}}{\mu(\delta_z)} \frac{1}{2\pi} \int_0^{2\pi} \frac{\delta_z}{|\delta_z e^{i\theta} - z|^2} d\theta \\ &= C \frac{(1 - \delta_z^2)^{(2+q-p)/p+n}}{\mu(\delta_z)} \frac{\delta_z}{\delta_z^2 - |z|^2} \\ &\leq C \frac{(1 - \delta_z^2)^{(2+q-p)/p+n}}{\mu(\delta_z)} \frac{1}{1 - \delta_z}. \end{aligned} \quad (3.6)$$

Note that

$$\frac{1}{2}(1 - |z|) \leq 1 - \delta_z^2 = (1 + \delta_z)(1 - \delta_z) \leq (1 - |z|), \quad (3.7)$$

and μ are normal, we have

$$|u'(z)| \leq C \frac{(1 - |z|^2)^{(2+q-p)/p+n-1}}{\mu(|z|)}, \quad (3.8)$$

hence

$$\sup_{z \in \mathbb{D}} \frac{\mu(|z|)|u'(z)|}{(1-|z|^2)^{((2+q-p)/p)+n-1}} < \infty. \quad (3.9)$$

Similarly, for $j \in \{2, 3, \dots, n\}$, we have that

$$\sup_{z \in \mathbb{D}} \frac{\mu(|z|)|u^{(j)}(z)|}{(1-|z|^2)^{((2+q-p)/p)+n-j}} < \infty. \quad (3.10)$$

Hence, if $2 + q > p$, by Lemmas 2.1 and 2.2, the Leibnitz formula, (3.1), (3.9), and (3.10), we have that

$$\begin{aligned} & \left| \mu(|z|)(M_u f)^{(n)}(z) \right| \\ &= \mu(|z|) \left| (u(z)f(z))^{(n)} \right| \\ &= \mu(|z|) \left| \sum_{j=0}^n C_n^j u^{(j)}(z) f^{(n-j)}(z) \right| \\ &\leq \mu(|z|) \left| u^{(n)}(z) f(z) \right| + C \sum_{j=0}^{n-1} \frac{|C_n^j \mu(|z|) u^{(j)}(z)|}{(1-|z|^2)^{((2+q-p)/p)+n-j}} \|f\|_{F(p,q,s)} \\ &\leq C \frac{\mu(|z|) |u^{(n)}(z)|}{(1-|z|^2)^{(2+q-p)/p}} \|f\|_{\mathcal{B}^{(2+q)/p}} + C \sum_{j=0}^{n-1} \frac{|C_n^j \mu(|z|) u^{(j)}(z)|}{(1-|z|^2)^{((2+q-p)/p)+n-j}} \|f\|_{F(p,q,s)} \\ &\leq C \|f\|_{F(p,q,s)}, \end{aligned} \quad (3.11)$$

for every $z \in \mathbb{D}$ and $f \in F(p, q, s)$, if $2 + q < p$, using $u \in \mathcal{W}_\mu^{(n)}(\mathbb{D})$, we have that

$$\begin{aligned} & \left| \mu(|z|)(M_u f)^{(n)}(z) \right| \\ &\leq C \mu(|z|) \left| u^{(n)}(z) \right| \|f\|_{\mathcal{B}^{(2+q)/p}} + C \sum_{j=0}^{n-1} \frac{|C_n^j \mu(|z|) u^{(j)}(z)|}{(1-|z|^2)^{((2+q-p)/p)+n-j}} \|f\|_{F(p,q,s)} \\ &\leq C \|u\|_{\mathcal{W}_\mu^{(n)}(\mathbb{D})} \|f\|_{\mathcal{B}^{(2+q)/p}} + C \sum_{j=0}^{n-1} \frac{|C_n^j \mu(|z|) u^{(j)}(z)|}{(1-|z|^2)^{((2+q-p)/p)+n-j}} \|f\|_{F(p,q,s)} \\ &\leq C \|f\|_{F(p,q,s)}, \end{aligned} \quad (3.12)$$

for every $z \in \mathbb{D}$ and $f \in F(p, q, s)$, if $2 + q = p$, by (3.2), we have that

$$\begin{aligned}
& \left| \mu(|z|)(M_u f)^{(n)}(z) \right| \\
& \leq C \mu(|z|) \left| u^{(n)}(z) \right| \log \frac{2}{1 - |z|^2} \|f\|_{\mathcal{B}} + C \sum_{j=0}^{n-1} \frac{|C_n^j \mu(|z|) u^{(j)}(z)|}{(1 - |z|^2)^{(2+q-p)/p+n-j}} \|f\|_{F(p,q,s)} \\
& \leq C \|f\|_{\mathcal{B}} + C \sum_{j=0}^{n-1} \frac{|C_n^j \mu(|z|) u^{(j)}(z)|}{(1 - |z|^2)^{n-j}} \|f\|_{F(p,q,s)} \\
& \leq C \|f\|_{F(p,q,s)},
\end{aligned} \tag{3.13}$$

for every $z \in \mathbb{D}$ and $f \in F(p, q, s)$. We also have that

$$|(M_u f)(0)| = |u(0)f(0)| \leq C \|f\|_{F(p,q,s)}, \tag{3.14}$$

and for each $s \in \{1, 2, \dots, n-1\}$,

$$\left| (M_u f)^{(s)}(0) \right| = \left| \sum_{j=0}^s C_s^j u^{(j)}(0) f^{(s-j)}(0) \right| \leq C \sum_{j=0}^s |C_s^j u^{(j)}(0)| \|f\|_{F(p,q,s)}. \tag{3.15}$$

Using (3.11), (3.12), (3.13), (3.14), and (3.15) it follows that the operator $M_u : F(p, q, s) \rightarrow \mathcal{W}_\mu^{(n)}(\mathbb{D})$ is bounded.

On the other hand, suppose that $M_u : F(p, q, s) \rightarrow \mathcal{W}_\mu^{(n)}(\mathbb{D})$ is bounded, that is, there exists a constant C such that

$$\|M_u f\|_{\mathcal{W}_\mu^{(n)}(\mathbb{D})} \leq C \|f\|_{F(p,q,s)} \tag{3.16}$$

for all $f \in F(p, q, s)$. Then, we can easily obtain $u \in \mathcal{W}_\mu^{(n)}(\mathbb{D})$ by taking $f(z) = 1$. For a fixed $w \in \mathbb{D}$, and constants c_1, c_2, \dots, c_n , set

$$f_w(z) = \sum_{j=1}^n c_j \frac{(1 - |w|^2)^{j+1}}{(1 - \bar{w}z)^{\alpha+j}}, \tag{3.17}$$

where $\alpha = (2 + q)/p$. A straightforward calculation shows that, for $l \in \{1, \dots, n\}$,

$$f_w^{(l)}(z) = \sum_{j=1}^n c_j \prod_{k=0}^{l-1} (\alpha + j + k) (\bar{w})^l \frac{(1 - |w|^2)^{j+1}}{(1 - \bar{w}z)^{\alpha+j+l}}, \tag{3.18}$$

so

$$f_w^{(l)}(w) = \frac{(\bar{w})^l}{(1 - |w|^2)^{\alpha-1+l}} \sum_{j=1}^n c_j \prod_{k=0}^{l-1} (\alpha + j + k). \quad (3.19)$$

It is easy to see that $f_w \in F(p, q, s)$ for each $w \in \mathbb{D}$ and $\|f_w\|_{F(p, q, s)} \leq C$ by using the same methods in [23]. By Lemma 2.3, using the same method in [22, 36], we can choose c_1, c_2, \dots, c_n , the corresponding function is denoted by f_w , such that

$$f_w^{(n)}(w) = \sum_{j=1}^n c_j \prod_{k=0}^{l-1} (\alpha + j + k) \frac{(\bar{w})^n}{(1 - |w|^2)^{\alpha-1+n}}, \quad f_w^{(l)}(w) = 0, \quad l \in \{0, 1, \dots, n-1\}. \quad (3.20)$$

Therefore,

$$\begin{aligned} C &\geq \|M_u f_w\|_{\mathcal{W}_\mu^{(n)}(\mathbb{D})} \geq \mu(|w|) |u(w) f_w^{(n)}(w)| \\ &= \sum_{j=1}^n c_j \prod_{k=0}^{l-1} (\alpha + j + k) \frac{\mu(|w|) |w|^n |u(w)|}{(1 - |w|^2)^{(2+q-p)/p+n}}. \end{aligned} \quad (3.21)$$

From this, we obtain

$$\sup_{|w|>1/2} \frac{\mu(|w|) |u(w)|}{(1 - |w|^2)^{(2+q-p)/p+n}} \leq C. \quad (3.22)$$

Since μ is normal and $u \in H(\mathbb{D})$, we get

$$\sup_{|w|\leq 1/2} \frac{\mu(|w|) |u(w)|}{(1 - |w|^2)^{(2+q-p)/p+n}} \leq C \sup_{|w|\leq 1/2} \mu(|w|) |u(w)| \leq C, \quad (3.23)$$

which along with (3.22) implies that (3.1) is necessary for all case. Let $2 + q = p, s > 1$, and $M_u : F(p, q, s) \rightarrow \mathcal{W}_\mu^{(n)}(\mathbb{D})$ be bounded. To prove (3.2), we set

$$g_w(z) = A \log \frac{2}{1 - \bar{w}z} + B \frac{(\log(2/(1 - \bar{w}z)))^2}{\log(2/(1 - |w|^2))}, \quad z, w \in \mathbb{D}. \quad (3.24)$$

By a direct calculation, we get

$$|g_w'(z)| \leq \frac{|A|}{|1 - \bar{w}z|} + \frac{2|B \log(2/(1 - \bar{w}z))|}{\left| \log(2/(1 - |w|^2)) \right| |1 - \bar{w}z|} \leq \frac{C}{|1 - \bar{w}z|}, \quad (3.25)$$

thus we have $g_w \in F(p, q, s)$ and $\sup_{w \in \mathbb{D}} \|g_w\|_{F(p, q, s)} \leq C < \infty$ (see [19, 30]). On the other hand, we have that

$$\begin{aligned}
g'_w(z) &= A \frac{\bar{w}}{1 - \bar{w}z} + 2B \frac{\bar{w} \log(2/(1 - \bar{w}z))}{\left(\log\left(2/(1 - |w|^2)\right)\right)(1 - \bar{w}z)}, \\
g''_w(z) &= A \frac{(\bar{w})^2}{(1 - \bar{w}z)^2} + 2B \frac{(\bar{w})^2}{\left(\log\left(2/(1 - |w|^2)\right)\right)(1 - \bar{w}z)^2} \\
&\quad + 2B \frac{(\bar{w})^2 \log(2/(1 - \bar{w}z))}{\left(\log\left(2/(1 - |w|^2)\right)\right)(1 - \bar{w}z)^2}, \\
g'''_w(z) &= 2A \frac{(\bar{w})^3}{(1 - \bar{w}z)^3} + 6B \frac{(\bar{w})^3}{\left(\log\left(2/(1 - |w|^2)\right)\right)(1 - \bar{w}z)^3} \\
&\quad + 2 \cdot 2B \frac{(\bar{w})^3 \log(2/(1 - \bar{w}z))}{\left(\log\left(2/(1 - |w|^2)\right)\right)(1 - \bar{w}z)^3}, \\
g^{(4)}_w(z) &= 3 \cdot 2A \frac{(\bar{w})^4}{(1 - \bar{w}z)^4} + 22B \frac{(\bar{w})^4}{\left(\log\left(2/(1 - |w|^2)\right)\right)(1 - \bar{w}z)^4} \\
&\quad + 3!2B \frac{(\bar{w})^4 \log(2/(1 - \bar{w}z))}{\left(\log\left(2/(1 - |w|^2)\right)\right)(1 - \bar{w}z)^4}, \\
g^{(5)}_w(z) &= 4!A \frac{(\bar{w})^5}{(1 - \bar{w}z)^5} + 100B \frac{(\bar{w})^5}{\left(\log\left(2/(1 - |w|^2)\right)\right)(1 - \bar{w}z)^5} \\
&\quad + 4!2B \frac{(\bar{w})^5 \log(2/(1 - \bar{w}z))}{\left(\log\left(2/(1 - |w|^2)\right)\right)(1 - \bar{w}z)^5}, \\
&\quad \vdots \\
g^{(m)}_w(z) &= (m-1)!A \frac{(\bar{w})^m}{(1 - \bar{w}z)^m} + A_m B \frac{(\bar{w})^m}{\left(\log\left(2/(1 - |w|^2)\right)\right)(1 - \bar{w}z)^m} \\
&\quad + 2(m-1)!B \frac{(\bar{w})^m \log(2/(1 - \bar{w}z))}{\left(\log\left(2/(1 - |w|^2)\right)\right)(1 - \bar{w}z)^m}.
\end{aligned} \tag{3.26}$$

Moreover, we have that

$$g_w(w) = (A + B) \log \frac{2}{1 - |w|^2},$$

$$\begin{aligned}
g'_w(w) &= (A + 2B) \frac{\bar{w}}{1 - |w|^2}, \\
g''_w(w) &= (A + 2B) \frac{(\bar{w})^2}{(1 - |w|^2)^2} + 2B \frac{(\bar{w})^2}{(\log(2/(1 - |w|^2))) (1 - |w|^2)^2}, \\
g'''_w(w) &= 2(A + 2B) \frac{(\bar{w})^3}{(1 - |w|^2)^3} + 6B \frac{(\bar{w})^3}{(\log(2/(1 - |w|^2))) (1 - |w|^2)^3}, \\
g_w^{(4)}(w) &= 6(A + 2B) \frac{(\bar{w})^4}{(1 - |w|^2)^4} + 22B \frac{(\bar{w})^4}{(\log(2/(1 - |w|^2))) (1 - |w|^2)^4}, \\
g_w^{(5)}(w) &= 24(A + 2B) \frac{(\bar{w})^5}{(1 - |w|^2)^5} + 100B \frac{(\bar{w})^5}{(\log(2/(1 - |w|^2))) (1 - |w|^2)^5}, \\
&\vdots \\
g_w^{(m)}(w) &= (m - 1)!(A + 2B) \frac{(\bar{w})^m}{(1 - |w|^2)^m} + A_m B \frac{(\bar{w})^m}{(\log(2/(1 - |w|^2))) (1 - |w|^2)^m}.
\end{aligned} \tag{3.27}$$

Taking $A = 2, B = -1$, if $n = 1$, we have

$$\mu(|w|) \left| u'(w) \log \frac{2}{1 - |w|^2} \right| \leq \|M_u g_w\|_{\mathcal{H}_\mu^{(n)}(\mathbb{D})} \leq C \|M_u\| < \infty, \tag{3.28}$$

if $n \neq 1$, we have

$$\begin{aligned}
\|M_u g_w\|_{\mathcal{H}_\mu^{(n)}(\mathbb{D})} &\geq \mu(|w|) \left| \sum_{j=0}^n C_n^j u^{(j)}(w) g_w^{(n-j)}(w) \right| \\
&\geq \mu(|w|) \left| u^{(n)}(w) g_w(w) \right| - \mu(|w|) \left| \sum_{j=0}^{n-1} C_n^j u^{(j)}(w) g_w^{(n-j)}(w) \right| \\
&\geq \mu(|w|) \left| u^{(n)}(w) g_w(w) \right| - \mu(|w|) \\
&\quad \times \left| \sum_{j=0}^{n-1} C_n^j u^{(j)}(w) A_{n-j} B \frac{(\bar{w})^{n-j}}{(\log(2/(1 - |w|^2))) (1 - |w|^2)^{n-j}} \right|.
\end{aligned} \tag{3.29}$$

From (3.29), (3.1), (3.9), and (3.10) we have

$$\begin{aligned} & \mu(|w|) \left| u^{(n)}(w) \log \frac{2}{1-|w|^2} \right| \\ & \leq \|M_u \mathcal{G}_w\|_{\mathcal{W}_\mu^{(n)}(\mathbb{D})} + C \left| \sum_{j=0}^{n-1} \frac{\mu(|w|) u^{(j)}(w)}{(1-|w|^2)^{n-j}} \right| \\ & \leq C \|M_u\| + C < \infty. \end{aligned} \quad (3.30)$$

Using (3.28) and (3.30), it is easy to get that (3.2) holds, finishing the proof of the theorem. \square

Theorem 3.2. *Assume that $u \in H(\mathbb{D})$ and μ is normal.*

(a) *If $2 + q > p$, then $M_u : F(p, q, s) \rightarrow \mathcal{W}_\mu^{(n)}(\mathbb{D})$ is compact if and only if*

$$\lim_{|z| \rightarrow 1} \frac{\mu(|z|) |u(z)|}{(1-|z|^2)^{(2+q-p)/p+n}} = 0. \quad (3.31)$$

(b) *If $2 + q < p$, then $M_u : F(p, q, s) \rightarrow \mathcal{W}_\mu^{(n)}(\mathbb{D})$ is compact if and only if (3.31) holds and $u \in \mathcal{W}_\mu^{(n)}(\mathbb{D})$.*

(c) *If $2 + q = p$, $s > 1$, then $M_u : F(p, q, s) \rightarrow \mathcal{W}_\mu^{(n)}(\mathbb{D})$ is compact if and only if (3.31) holds and*

$$\lim_{|z| \rightarrow 1} \mu(|z|) \left| u^{(n)}(z) \right| \log \frac{2}{1-|z|^2} = 0. \quad (3.32)$$

Proof. Assume that conditions (3.31) hold. By Theorem 3.1, $M_u : F(p, q, s) \rightarrow \mathcal{W}_\mu^{(n)}(\mathbb{D})$ is bounded. For any bounded sequence $\{f_k\}$ in $F(p, q, s)$ with $f_k \rightarrow 0$ uniformly on compact subsets of \mathbb{D} establish the assertion, it suffices, in view of Lemma 2.4, to show that

$$\|M_u f_k\|_{\mathcal{W}_\mu^{(n)}(\mathbb{D})} \rightarrow 0 \quad \text{as } k \rightarrow \infty. \quad (3.33)$$

We assume that $\|f_k\|_{F(p, q, s)} \leq 1$. From (3.31) and (3.32), given $\epsilon > 0$, there exists a $\delta \in (0, 1)$, when $\delta < |z| < 1$, we have

$$\frac{\mu(|z|) |u(z)|}{(1-|z|^2)^{(2+q-p)/p+n}} < \epsilon, \quad (3.34)$$

$$\mu(|z|) \left| u^{(n)}(z) \right| \log \frac{2}{1-|z|^2} < \epsilon. \quad (3.35)$$

By (3.34) and the Cauchy integral formula, we obtain

$$\begin{aligned}
|u'(z)| &= \frac{1}{2\pi} \left| \int_0^{2\pi} \frac{u(\delta_z e^{i\theta})}{(\delta_z e^{i\theta} - z)^2} \delta_z e^{i\theta} d\theta \right| \\
&< e^{\frac{(1 - \delta_z^2)^{((2+q-p)/p)+n}}{\mu(\delta_z)}} \frac{1}{2\pi} \int_0^{2\pi} \frac{\delta_z}{|\delta_z e^{i\theta} - z|^2} d\theta \\
&= e^{\frac{(1 - \delta_z^2)^{((2+q-p)/p)+n}}{\mu(\delta_z)}} \frac{\delta_z}{\delta_z^2 - |z|^2} \\
&\leq \epsilon C \frac{(1 - |z|^2)^{((2+q-p)/p)+n-1}}{\mu(|z|)},
\end{aligned} \tag{3.36}$$

when $\delta < |z| < 1$. Hence,

$$\frac{\mu(|z|)|u'(z)|}{(1 - |z|^2)^{((2+q-p)/p)+n-1}} < C\epsilon, \tag{3.37}$$

when $\tau < |z| < 1$. Similarly, for $j \in \{2, \dots, n\}$, we have that

$$\frac{\mu(|z|)|u^{(j)}(z)|}{(1 - |z|^2)^{((2+q-p)/p)+n-j}} < C\epsilon, \tag{3.38}$$

when $\delta < |z| < 1$. Since $f_k \rightarrow 0$ uniformly on compact subsets of \mathbb{D} , Cauchy's estimate gives that $f_k^{(j)}$ converges to 0 uniformly on compact subsets of \mathbb{D} for each $j \in \{1, \dots, n\}$, there exists a $K_0 \in \mathbb{N}$ such that $k > K_0$ implies that

$$\sum_{j=0}^{n-1} \sum_{m=0}^{j-1} C_j^m \left| u^{(m)}(0) f_k^{(j-m)}(0) \right| + \sup_{|z| \leq \delta} \mu(|z|) \left| \sum_{j=0}^n C_n^j u^{(j)}(z) f_k^{(n-j)}(z) \right| < C\epsilon. \tag{3.39}$$

If $2 + q > p$, from (3.34), (3.38), (3.39), and Lemmas 2.1 and 2.2, we have

$$\begin{aligned}
&\|M_u f_k\|_{\mathcal{W}_\mu^{(n)}(\mathbb{D})} \\
&= \sum_{j=0}^{n-1} \left| (u f_k)^{(j)}(0) \right| + \sup_{z \in \mathbb{D}} \mu(|z|) \left| (u f_k)^{(n)}(z) \right|
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{j=0}^{n-1} \sum_{m=0}^{j-1} C_n^m \left| u^{(m)}(0) f_k^{(j-m)}(0) \right| + \sup_{|z| \leq \delta} \mu(|z|) \left| \sum_{j=0}^n C_n^j u^{(j)}(z) f_k^{(n-j)}(z) \right| \\
&\quad + \sup_{|z| > \delta} \mu(|z|) \left| \sum_{j=0}^n C_n^j u^{(j)}(z) f_k^{(n-j)}(z) \right| \\
&< C\epsilon + C \sup_{|z| > \delta} \left| \mu(|z|) u^{(n)}(z) f_k(z) \right| + C \sup_{|z| > \delta} \sum_{j=0}^{n-1} \left| \frac{C_n^j \mu(|z|) u^{(j)}(z)}{(1-|z|^2)^{(2+q-p)/p+n-j}} \right| \|f_k\|_{F(p,q,s)} \\
&\leq C\epsilon + C \sup_{|z| > \delta} \left| \frac{\mu(|z|) u^{(n)}(z)}{(1-|z|^2)^{(2+q-p)/p}} \right| \|f_k\|_{\mathcal{B}^{(2+q)/p}} + C \sup_{|z| > \delta} \sum_{j=0}^{n-1} \left| \frac{C_n^j \mu(|z|) u^{(j)}(z)}{(1-|z|^2)^{(2+q-p)/p+n-j}} \right| \\
&\leq C\epsilon + C \sup_{|z| > \delta} \left| \frac{\mu(|z|) u^{(n)}(z)}{(1-|z|^2)^{(2+q-p)/p}} \right| + C \sup_{|z| > \delta} \sum_{j=0}^{n-1} \left| \frac{C_n^j \mu(|z|) u^{(j)}(z)}{(1-|z|^2)^{(2+q-p)/p+n-j}} \right| \\
&< \left(2 + C \sum_{j=0}^n C_n^j \right) C\epsilon,
\end{aligned} \tag{3.40}$$

when $k > K_0$. If $2+q < p$, then $f_k \in F(p, q, s) \subset \mathcal{B}^{(2+q)/p}$, $k \in \mathbb{N}$. By [37, Lemma 3.2], we have

$$\lim_{k \rightarrow \infty} \sup_{z \in \mathbb{D}} |f_k(z)| = 0. \tag{3.41}$$

By (3.38), (3.39), Lemma 2.1, and $u \in \mathcal{W}_\mu^{(n)}(\mathbb{D})$, we get

$$\begin{aligned}
&\|M_u f_k\|_{\mathcal{W}_\mu^{(n)}(\mathbb{D})} \\
&= \sum_{j=0}^{n-1} \left| (u f_k)^{(j)}(0) \right| + \sup_{z \in \mathbb{D}} \mu(|z|) \left| (u f_k)^{(n)}(z) \right| \\
&< C\epsilon + C \sup_{|z| > \delta} \left| \mu(|z|) u^{(n)}(z) f_k(z) \right| + C \sup_{|z| > \delta} \sum_{j=0}^{n-1} \left| \frac{C_n^j \mu(|z|) u^{(j)}(z)}{(1-|z|^2)^{(2+q-p)/p+n-j}} \right| \|f_k\|_{F(p,q,s)} \\
&\leq C\epsilon + C \sup_{z \in \mathbb{D}} |f_k(z)| + C \sup_{|z| > \delta} \sum_{j=0}^{n-1} \left| \frac{C_n^j \mu(|z|) u^{(j)}(z)}{(1-|z|^2)^{(2+q-p)/p+n-j}} \right|
\end{aligned}$$

$$\begin{aligned}
&\leq C\epsilon + C\epsilon + C\sup_{|z|>\delta} \sum_{j=0}^{n-1} \left| \frac{C_n^j \mu(|z|) u^{(j)}(z)}{(1-|z|^2)^{(2+q-p)/p+n-j}} \right| \\
&< \left(2 + C \sum_{j=0}^n C_n^j \right) C\epsilon,
\end{aligned} \tag{3.42}$$

when $k > K_0$. If $2 + q = p$, from (3.35), (3.38), (3.39), and Lemmas 2.1 and 2.2, we get

$$\begin{aligned}
&\|M_u f_k\|_{\mathcal{W}_\mu^{(n)}(\mathbb{D})} \\
&= \sum_{j=0}^{n-1} \left| (u f_k)^{(j)}(0) \right| + \sup_{z \in \mathbb{D}} \mu(|z|) \left| (u f_k)^{(n)}(z) \right| \\
&< C\epsilon + C\sup_{|z|>\delta} \left| \mu(|z|) u^{(n)}(z) f_k(z) \right| + C\sup_{|z|>\delta} \sum_{j=0}^{n-1} \left| \frac{C_n^j \mu(|z|) u^{(j)}(z)}{(1-|z|^2)^{(2+q-p)/p+n-j}} \right| \|f_k\|_{F(p,q,s)} \\
&< C\epsilon + C\sup_{|z|>\delta} \left| \mu(|z|) u^{(n)}(z) \right| \log \frac{2}{1-|z|^2} \|f_k\|_{\mathcal{B}} + C\sup_{|z|>\delta} \sum_{j=0}^{n-1} \left| \frac{C_n^j \mu(|z|) u^{(j)}(z)}{(1-|z|^2)^{(2+q-p)/p+n-j}} \right| \\
&\leq C\epsilon + C\sup_{|z|>\delta} \left| \mu(|z|) u^{(n)}(z) \right| \log \frac{2}{1-|z|^2} \|f_k\|_{F(p,q,s)} + C\sup_{|z|>\delta} \sum_{j=0}^{n-1} \left| \frac{C_n^j \mu(|z|) u^{(j)}(z)}{(1-|z|^2)^{n-j}} \right| \\
&< \left(2 + C \sum_{j=0}^n C_n^j \right) C\epsilon,
\end{aligned} \tag{3.43}$$

when $k > K_0$. It follows that the operator $M_u : F(p, q, s) \rightarrow \mathcal{W}_\mu^{(n)}(\mathbb{D})$ is compact.

Conversely, assume that $M_u : F(p, q, s) \rightarrow \mathcal{W}_\mu^{(n)}(\mathbb{D})$ is compact. Taking $f = 1$, we get $u \in \mathcal{W}_\mu^{(n)}(\mathbb{D})$. Let $\{z_k\}$ be a sequence in \mathbb{D} such that $|z_k| \rightarrow 1$ as $k \rightarrow \infty$. Taking the test functions h_{z_k} , where h_{z_k} is defined by (3.17), we will write

$$h_k(z) = f_{z_k}(z). \tag{3.44}$$

We obtain that $h_k \in F(p, q, s)$,

$$\sup_{k \in \mathbb{N}} \|h_k\|_{F(p,q,s)} \leq C, \tag{3.45}$$

and for $t \in \{1, \dots, n-1\}$,

$$h_k^{(n)}(z_k) = \sum_{j=1}^n c_j \prod_{k=0}^{l-1} (\alpha + j + k) \frac{(\bar{z}_k)^n}{(1 - |z_k|^2)^{((2+q-p)/p)+n}}, \quad h_k^{(t)}(z_k) = 0. \quad (3.46)$$

For $|z| = r < 1$, we have

$$|h_k(z)| = \left| \sum_{j=1}^n c_j \frac{(1 - |z_k|^2)^{j+1}}{(1 - \bar{z}_k z)^{\alpha+j}} \right| \leq \sum_{j=1}^n c_j \frac{(1 - |z_k|^2)^{j+1}}{(1 - r)^{\alpha+j}} \rightarrow 0 \quad \text{as } k \rightarrow \infty, \quad (3.47)$$

that is, h_k converges to 0 uniformly on compact subsets of \mathbb{D} , using the compactness of $M_u : F(p, q, s) \rightarrow \mathcal{W}_\mu^{(n)}(\mathbb{D})$, we obtain

$$\frac{\mu(z_k) |u(z_k)| |z_k|^n}{(1 - |z_k|^2)^{((2+q-p)/p)+n}} \leq C \|M_u h_k\|_{\mathcal{W}_\mu^{(n)}(\mathbb{D})} \rightarrow 0 \quad \text{as } k \rightarrow \infty, \quad (3.48)$$

and consequently (3.31) holds.

Assume that $2 + q = p, s > 1$ and $M_u : F(p, q, s) \rightarrow \mathcal{W}_\mu^{(n)}(\mathbb{D})$ is compact, we only need to prove (3.32) holds. To do this, let $\{z_k\}$ be a sequence in \mathbb{D} such that $|z_k| \rightarrow 1$ as $k \rightarrow \infty$. Set

$$a_k = \log \frac{2}{1 - |z_k|^2}, \quad (3.49)$$

$$J_k(z) = \frac{E}{a_k} \left(\log \frac{2}{1 - \bar{z}_k z} \right)^2 + \frac{F}{a_k^2} \left(\log \frac{2}{1 - \bar{z}_k z} \right)^3.$$

Since

$$|J'_k(z)| \leq \left| \frac{2E}{a_k} \left(\log \frac{2}{1 - \bar{z}_k z} \right) \frac{\bar{z}_k}{1 - \bar{z}_k z} \right| + \left| \frac{3F}{a_k^2} \left(\log \frac{2}{1 - \bar{z}_k z} \right)^2 \frac{\bar{z}_k}{1 - \bar{z}_k z} \right| \leq \frac{C}{|1 - \bar{z}_k z|}, \quad (3.50)$$

we have $J_k \in F(p, q, s)$ and $\sup_{k \in \mathbb{N}} \|J_k\|_{F(p, q, s)} \leq C$ (see [19, 30]), and $J_k \rightarrow 0$ uniformly on compact subsets of \mathbb{D} . Since $M_u : F(p, q, s) \rightarrow \mathcal{W}_\mu^{(n)}(\mathbb{D})$ is compact, we have

$$\|M_u J_k\|_{\mathcal{W}_\mu^{(n)}(\mathbb{D})} \rightarrow 0 \quad \text{as } k \rightarrow \infty. \quad (3.51)$$

We also have that

$$\begin{aligned}
J'_k(z) &= \frac{2E}{a_k} \left(\log \frac{2}{1-\bar{z}_k z} \right) \frac{\bar{z}_k}{1-\bar{z}_k z} + \frac{3F}{a_k^2} \left(\log \frac{2}{1-\bar{z}_k z} \right)^2 \frac{\bar{z}_k}{1-\bar{z}_k z}, \\
J''_k(z) &= \frac{2E}{a_k} \frac{(\bar{z}_k)^2}{(1-\bar{z}_k z)^2} + \frac{2E}{a_k} \left(\log \frac{2}{1-\bar{z}_k z} \right) \frac{(\bar{z}_k)^2}{(1-\bar{z}_k z)^2} \\
&\quad + \frac{3 \cdot 2F}{a_k^2} \left(\log \frac{2}{1-\bar{z}_k z} \right) \frac{(\bar{z}_k)^2}{(1-\bar{z}_k z)^2} + \frac{3F}{a_k^2} \left(\log \frac{2}{1-\bar{z}_k z} \right)^2 \frac{(\bar{z}_k)^2}{(1-\bar{z}_k z)^2}, \\
J'''_k(z) &= \frac{2 \cdot 2E}{a_k} \frac{(\bar{z}_k)^3}{(1-\bar{z}_k z)^3} + \frac{2E}{a_k} \frac{(\bar{z}_k)^3}{(1-\bar{z}_k z)^3} + \frac{4E}{a_k} \left(\log \frac{2}{1-\bar{z}_k z} \right) \frac{(\bar{z}_k)^3}{(1-\bar{z}_k z)^3} \\
&\quad + \frac{3 \cdot 2F}{a_k^2} \frac{(\bar{z}_k)^3}{(1-\bar{z}_k z)^3} + \frac{3 \cdot 2 \cdot 2F}{a_k^2} \left(\log \frac{2}{1-\bar{z}_k z} \right) \frac{(\bar{z}_k)^3}{(1-\bar{z}_k z)^3} \\
&\quad + \frac{3 \cdot 2F}{a_k^2} \left(\log \frac{2}{1-\bar{z}_k z} \right) \frac{(\bar{z}_k)^3}{(1-\bar{z}_k z)^3} + \frac{6F}{a_k^2} \left(\log \frac{2}{1-\bar{z}_k z} \right)^2 \frac{(\bar{z}_k)^3}{(1-\bar{z}_k z)^3}, \\
&= \frac{6E}{a_k} \frac{(\bar{z}_k)^3}{(1-\bar{z}_k z)^3} + \frac{4E}{a_k} \left(\log \frac{2}{1-\bar{z}_k z} \right) \frac{(\bar{z}_k)^3}{(1-\bar{z}_k z)^3} + \frac{6F}{a_k^2} \frac{(\bar{z}_k)^3}{(1-\bar{z}_k z)^3} \\
&\quad + \frac{18F}{a_k^2} \left(\log \frac{2}{1-\bar{z}_k z} \right) \frac{(\bar{z}_k)^3}{(1-\bar{z}_k z)^3} + \frac{6F}{a_k^2} \left(\log \frac{2}{1-\bar{z}_k z} \right)^2 \frac{(\bar{z}_k)^3}{(1-\bar{z}_k z)^3}, \\
J_k^{(4)}(z) &= \frac{22E}{a_k} \frac{(\bar{z}_k)^4}{(1-\bar{z}_k z)^4} + \frac{3!2E}{a_k} \left(\log \frac{2}{1-\bar{z}_k z} \right) \frac{(\bar{z}_k)^4}{(1-\bar{z}_k z)^4} + \frac{36F}{a_k^2} \frac{(\bar{z}_k)^4}{(1-\bar{z}_k z)^4} \\
&\quad + \frac{66F}{a_k^2} \left(\log \frac{2}{1-\bar{z}_k z} \right) \frac{(\bar{z}_k)^4}{(1-\bar{z}_k z)^4} + \frac{3!3F}{a_k^2} \left(\log \frac{2}{1-\bar{z}_k z} \right)^2 \frac{(\bar{z}_k)^4}{(1-\bar{z}_k z)^4}, \\
&\quad \vdots \\
J_k^{(m)}(z) &= \frac{b_m E}{a_k} \frac{(\bar{z}_k)^m}{(1-\bar{z}_k z)^m} + \frac{(m-1)!2E}{a_k} \left(\log \frac{2}{1-\bar{z}_k z} \right) \frac{(\bar{z}_k)^m}{(1-\bar{z}_k z)^m} + \frac{d_m F}{a_k^2} \frac{(\bar{z}_k)^m}{(1-\bar{z}_k z)^m} \\
&\quad + \frac{c_m F}{a_k^2} \left(\log \frac{2}{1-\bar{z}_k z} \right) \frac{(\bar{z}_k)^m}{(1-\bar{z}_k z)^m} + \frac{(m-1)!3F}{a_k^2} \left(\log \frac{2}{1-\bar{z}_k z} \right)^2 \frac{(\bar{z}_k)^m}{(1-\bar{z}_k z)^m}.
\end{aligned} \tag{3.52}$$

Therefore,

$$\begin{aligned}
J_k(z_k) &= (E + F)a_k, \\
J'_k(z_k) &= (2E + 3F) \frac{\bar{z}_k}{1-|z_k|^2},
\end{aligned}$$

$$\begin{aligned}
J_k''(z_k) &= (2E + 3F) \frac{(\overline{z_k})^2}{(1 - |z_k|^2)^2} + \frac{2(E + 3F)}{a_k} \frac{(\overline{z_k})^2}{(1 - |z_k|^2)^2}, \\
J_k'''(z_k) &= 2(2E + 3F) \frac{(\overline{z_k})^3}{(1 - |z_k|^2)^3} + \frac{6(E + 3F)}{a_k} \frac{(\overline{z_k})^3}{(1 - |z_k|^2)^3} + \frac{6F}{a_k^2} \frac{(\overline{z_k})^3}{(1 - |z_k|^2)^3}, \\
J_k^{(4)}(z_k) &= 3!(2E + 3F) \frac{(\overline{z_k})^4}{(1 - |z_k|^2)^4} + \frac{22(E + 3F)}{a_k} \frac{(\overline{z_k})^4}{(1 - |z_k|^2)^4} + \frac{36F}{a_k^2} \frac{(\overline{z_k})^4}{(1 - |z_k|^2)^4}, \\
&\vdots \\
J_k^{(m)}(z_k) &= (m - 1)!(2E + 3F) \frac{(\overline{z_k})^m}{(1 - |z_k|^2)^m} + \frac{b_m E + c_m F}{a_k} \frac{(\overline{z_k})^m}{(1 - |z_k|^2)^m} + \frac{d_m F}{a_k^2} \frac{(\overline{z_k})^m}{(1 - |z_k|^2)^m}.
\end{aligned} \tag{3.53}$$

Taking $E = 3$ and $F = -2$, if $n = 1$, we have

$$\mu(|z_k|) \left| u'(z_k) \log \frac{2}{1 - |z_k|^2} \right| \leq \|M_u J_k\|_{\mathcal{H}_\mu^{(n)}(\mathbb{D})} \rightarrow 0 \quad \text{as } k \rightarrow \infty, \tag{3.54}$$

if $n \neq 1$, we have

$$\begin{aligned}
\|M_u J_k\|_{\mathcal{H}_\mu^{(n)}(\mathbb{D})} &\geq \mu(|z_k|) \left| \sum_{j=0}^n C_n^j u^{(j)}(z_k) J_k^{(n-j)}(z_k) \right| \\
&\geq \mu(|z_k|) \left| u^{(n)}(z_k) J_k(z_k) \right| - \mu(|z_k|) \left| \sum_{j=0}^{n-1} C_n^j u^{(j)}(z_k) J_k^{(n-j)}(z_k) \right| \\
&\geq \mu(|z_k|) \left| u^{(n)}(z_k) \log \frac{2}{1 - |z_k|^2} \right| - \mu(|z_k|) \left| \sum_{j=0}^{n-1} C_n^j u^{(j)}(z_k) J_k^{(n-j)}(z_k) \right|.
\end{aligned} \tag{3.55}$$

From (3.31), (3.35), (3.37), and (3.38), we have that for sufficiently large k :

$$\begin{aligned}
&\mu(|z_k|) \left| u^{(n)}(z_k) \log \frac{2}{1 - |z_k|^2} \right| \\
&\leq \|M_u J_k\|_{\mathcal{H}_\mu^{(n)}(\mathbb{D})} + \mu(|z_k|) \left| \sum_{j=0}^{n-1} C_n^j u^{(j)}(z_k) J_k^{(n-j)}(z_k) \right|
\end{aligned}$$

$$\begin{aligned}
&\leq \|M_u J_k\|_{\mathcal{W}_\mu^{(n)}(\mathbb{D})} + C \sum_{j=0}^{n-1} \left(\frac{\mu(|z_k|) |u^{(j)}(z_k)|}{a_k (1 - |z_k|^2)^{n-j}} + \frac{\mu(|z_k|) |u^{(j)}(z_k)|}{a_k^2 (1 - |z_k|^2)^{n-j}} \right) \\
&\leq \|M_u J_k\|_{\mathcal{W}_\mu^{(n)}(\mathbb{D})} + C \sum_{j=0}^{n-1} \frac{\mu(|z_k|) |u^{(j)}(z_k)|}{(1 - |z_k|^2)^{n-j}} \\
&< C\epsilon.
\end{aligned} \tag{3.56}$$

Using (3.54) and (3.56), it is easy to get that (3.32) holds. From which we obtain the desired result. \square

4. The Boundedness of M_u from $F(p, q, s)$ (or $F_0(p, q, s)$) to $\mathcal{W}_{\mu,0}^{(n)}(\mathbb{D})$

In this section, we characterize the boundedness of $M_u : F(p, q, s)$ (or $F_0(p, q, s)$) $\rightarrow \mathcal{W}_{\mu,0}^{(n)}(\mathbb{D})$.

Theorem 4.1. *Assume that $u \in H(\mathbb{D})$ and μ is normal. Then,*

- (1) $M_u : F(p, q, s) \rightarrow \mathcal{W}_{\mu,0}^{(n)}(\mathbb{D})$ is bounded if and only if $M_u : F(p, q, s) \rightarrow \mathcal{W}_\mu^{(n)}(\mathbb{D})$ is bounded and

$$\lim_{|z| \rightarrow 1} \mu(z) |u^{(n)}(z)| = 0. \tag{4.1}$$

- (2) $M_u : F_0(p, q, s) \rightarrow \mathcal{W}_{\mu,0}^{(n)}(\mathbb{D})$ is bounded if and only if $M_u : F_0(p, q, s) \rightarrow \mathcal{W}_\mu^{(n)}(\mathbb{D})$ is bounded and (4.1) holds.

Proof. (1) Assume that $M_u : F(p, q, s) \rightarrow \mathcal{W}_{\mu,0}^{(n)}(\mathbb{D})$ is bounded and condition (4.1) holds. Since

$$u(z) - u(0) = \int_0^1 x u'(zx) dx, \tag{4.2}$$

it follows that

$$\begin{aligned}
\mu(z) |u(z)| &\leq \mu(z) |u(0)| + \mu(z) \left| \int_0^1 x u'(zx) dx \right| \\
&\leq \mu(z) |u(0)| + \mu(z) \left| \int_0^{1/2} x u'(zx) dx \right| + \mu(z) \left| \int_{1/2}^1 x u'(zx) dx \right| \\
&\leq \mu(z) |u(0)| + \mu(z) \max_{|z| \leq 1/2} |u'(z)| + \mu(z) \left| \int_{1/2}^1 x u'(zx) dx \right|.
\end{aligned} \tag{4.3}$$

Since μ is normal, by the monotonicity of $\mu(t)/(1-t^2)^a$, for $t_0 \leq t_1 < t < 1$, we have

$$\mu(t) = (1-t^2)^a \frac{\mu(t)}{(1-t^2)^a} \leq (1-t^2)^a \frac{\mu(t_1)}{(1-t_1^2)^a} < \mu(t_1), \quad (4.4)$$

that is, μ is decreasing on $[t_0, 1)$, and for any $\epsilon > 0$, there is a $\tau > 0$ such that

$$0 < \mu(|z|) < \epsilon(1-|z|^2)^a, \quad (\tau < |z| < 1), \quad (4.5)$$

which implies $\lim_{|z| \rightarrow 1} \mu(|z|) = 0$. Since for $1/2 < x < 1$ and $2t_0 < |z| < 1$, we have $\mu(|z|) \leq \mu(x|z|)$. From (4.3), it follows that

$$\mu(|z|)|u(|z|)| \leq \mu(|z|)|u(0)| + \mu(|z|) \max_{|z| \leq 1/2} |u'(z)| + \int_{1/2}^1 \mu(x|z|) |u'(zx)| dx. \quad (4.6)$$

For $j \in \{2, 3, \dots, n\}$, by applying formula (4.6) to the function $u^{(j-1)}$, we get

$$\mu(|z|) |u^{(j-1)}(z)| \leq \mu(|z|) |u^{(j-1)}(0)| + \mu(|z|) \max_{|z| \leq 1/2} |u^{(j)}(z)| + \int_{1/2}^1 \mu(|z|x) |u^{(j)}(zx)| dx, \quad (4.7)$$

when $2t_0 < |z| < 1$. It follows from (4.1), (4.6), and (4.7) that $\lim_{|z| \rightarrow 1} \mu(|z|) |u^{(j-1)}(z)| = 0$ for $j \in \{1, 3, \dots, n\}$. Since, for each polynomial p , we have

$$\begin{aligned} \left| \mu(|z|) (M_u p)^{(n)}(z) \right| &= \mu(|z|) \left| (u(z)p(z))^{(n)} \right| \\ &= \mu(|z|) \left| \sum_{j=0}^n C_n^j u^{(j)}(z) p^{(n-j)}(z) \right| \\ &\leq C \sum_{j=0}^n \left| C_n^j \mu(|z|) u^{(j)}(z) \right| \|p^{(n-j)}\|_{\infty}, \end{aligned} \quad (4.8)$$

hence, $M_u p \in \mathcal{W}_{\mu,0}^{(n)}(\mathbb{D})$. Since the set of all polynomials is dense in $F(p, q, s)$, we have that for every $f \in F(p, q, s)$ there is a sequence of polynomials $\{p_k\}$ such that

$$\lim_{k \rightarrow \infty} \|p_k - f\|_{F(p,q,s)} = 0. \quad (4.9)$$

From this and since the operator $M_u : F(p, q, s) \rightarrow \mathcal{W}_{\mu}^{(n)}(\mathbb{D})$ is bounded, we have that

$$\|M_u p_k - M_u f\|_{\mathcal{W}_{\mu}^{(n)}(\mathbb{D})} \leq \|M_u\| \|p_k - f\|_{F(p,q,s)} \rightarrow 0 \quad \text{as } k \rightarrow \infty. \quad (4.10)$$

Since $\mathcal{W}_{\mu,0}^{(n)}(\mathbb{D})$ is a closed subspace of $\mathcal{W}_{\mu}^{(n)}(\mathbb{D})$, therefore, we have $M_u(F(p, q, s)) \subset \mathcal{W}_{\mu,0}^{(n)}(\mathbb{D})$, from which the boundedness of $M_u : F(p, q, s) \rightarrow \mathcal{W}_{\mu,0}^{(n)}(\mathbb{D})$ follows.

On the other hand, assume that $M_u : F(p, q, s) \rightarrow \mathcal{W}_{\mu,0}^{(n)}(\mathbb{D})$ is bounded, then $M_u : F(p, q, s) \rightarrow \mathcal{W}_{\mu}^{(n)}(\mathbb{D})$ is bounded. By taking the function given by $f(z) = 1$, we obtain

$$\mu(|z|) \left| u^{(n)}(z) \right| = \left| \mu(|z|) (M_u f)^{(n)}(z) \right| \rightarrow 0 \quad (\text{as } |z| \rightarrow 1), \quad (4.11)$$

as desired.

(2) The proof is similar to that of the case (1). We leave the details to the interested reader. \square

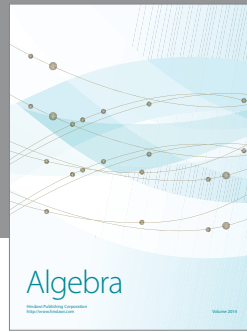
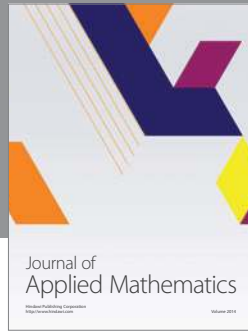
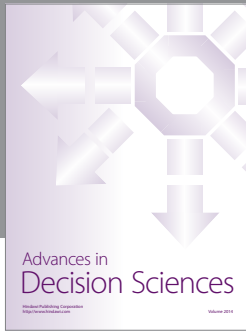
Acknowledgments

The authors acknowledge gratefully the support in part by the National Natural Science Foundation of China (no. 11171285) and the Grant of Natural Science Basic Research of Jiangsu Province of China for Colleges and Universities (nos. 06KJD110175; 07KJB110115). The authors also thank the referees for their thoughtful comments and helpful suggestions which greatly improved the final version of this paper.

References

- [1] S. Stević, "Composition operators from the weighted Bergman space to the n th weighted spaces on the unit disc," *Discrete Dynamics in Nature and Society*, vol. 2009, Article ID 742019, 11 pages, 2009.
- [2] K. L. Avetisyan, "Hardy-Bloch type spaces and lacunary series on the polydisk," *Glasgow Mathematical Journal*, vol. 49, no. 2, pp. 345–356, 2007.
- [3] K. D. Bierstedt and W. H. Summers, "Biduals of weighted Banach spaces of analytic functions," *Journal of the Australian Mathematical Society A*, vol. 54, no. 1, pp. 70–79, 1993.
- [4] J. S. Manhas, "Compact differences of weighted composition operators on weighted Banach spaces of analytic functions," *Integral Equations and Operator Theory*, vol. 62, no. 3, pp. 419–428, 2008.
- [5] S. Li and S. Stević, "Generalized composition operators on Zygmund spaces and Bloch type spaces," *Journal of Mathematical Analysis and Applications*, vol. 338, no. 2, pp. 1282–1295, 2008.
- [6] X. Liu and Y. Yu, "The product of differentiation operator and multiplication operator from H^∞ to Zygmund spaces," *Journal of Xuzhou Normal University (Natural Science Edition)*, vol. 29, no. 1, pp. 37–39, 2011.
- [7] Y. Liu and Y. Yu, "Composition followed by differentiation between H^∞ and Zygmund spaces," *Complex Analysis and Operator Theory*, vol. 6, no. 1, pp. 121–137, 2012.
- [8] Y. Zhang, "Weighted composition operator from B^α space to Zygmund space," *Journal of Xuzhou Normal University (Natural Science Edition)*, vol. 28, no. 4, pp. 38–42, 2010.
- [9] X. Zhu, "Volterra type operators from logarithmic Bloch spaces to Zygmund type spaces," *International Journal of Modern Mathematics*, vol. 3, no. 3, pp. 327–336, 2008.
- [10] S. Li and S. Stević, "Integral-type operators from Bloch-type spaces to Zygmund-type spaces," *Applied Mathematics and Computation*, vol. 215, no. 2, pp. 464–473, 2009.
- [11] S. Stević, "On an integral operator from the Zygmund space to the Bloch-type space on the unit ball," *Glasgow Mathematical Journal*, vol. 51, no. 2, pp. 275–287, 2009.
- [12] A. L. Shields and D. L. Williams, "Bonded projections, duality, and multipliers in spaces of analytic functions," *Transactions of the American Mathematical Society*, vol. 162, pp. 287–302, 1971.
- [13] R. Zhao, "On a general family of function spaces," *Annales Academiæ Scientiarum Fennicæ. Mathematica Dissertationes*, no. 105, p. 56, 1996.
- [14] H. Chen and M. Zhang, "Boundedness from below of multiplication operators between α -Bloch spaces," *Canadian Mathematical Bulletin*, vol. 53, no. 1, pp. 23–36, 2010.
- [15] F. Colonna and G. R. Easley, "Multiplication operators on the Lipschitz space of a tree," *Integral Equations and Operator Theory*, vol. 68, no. 3, pp. 391–411, 2010.

- [16] R. G. Douglas, S. Sun, and D. Zheng, "Multiplication operators on the Bergman space via analytic continuation," *Advances in Mathematics*, vol. 226, no. 1, pp. 541–583, 2011.
- [17] B. Yousefi and A. Khaksari, "Multiplication operators on analytic functional spaces," *Taiwanese Journal of Mathematics*, vol. 13, no. 4, pp. 1159–1165, 2009.
- [18] M. Zhang, "Boundedness from below of multiplication operators between α -Bloch spaces on the unit ball," *Acta Mathematica Scientia B*, vol. 30, no. 5, pp. 1619–1630, 2010.
- [19] X. Zhang, "Multipliers on some holomorphic function spaces," *Chinese Annals of Mathematics A*, vol. 26, no. 4, pp. 477–486, 2005.
- [20] X. Zhang, J. Xiao, and Z. Hu, "The multipliers between the mixed norm spaces in \mathbb{C}^n ," *Journal of Mathematical Analysis and Applications*, vol. 311, no. 2, pp. 664–674, 2005.
- [21] Y. Yu and Y. Liu, "The product of differentiation and multiplication operator from the mixed-norm to the Bloch-type space (in Chinese)," *Acta Mathematica Scientia*, vol. 32A, no. 1, pp. 68–79, 2012.
- [22] S. Stević, "Composition followed by differentiation from H^∞ and the Bloch space to n th weighted-type spaces on the unit disk," *Applied Mathematics and Computation*, vol. 216, no. 12, pp. 3450–3458, 2010.
- [23] X. Zhang and J. Xiao, "Weighted composition operator between two analytic function spaces," *Advances in Mathematics (China)*, vol. 35, no. 4, pp. 477–486, 2006.
- [24] L. Zhang and H. Zeng, "Weighted differentiation composition operators from weighted bergman space to n th weighted space on the unit disk," *Journal of Inequalities and Applications*, vol. 2011, no. 1, article 65, 2011.
- [25] S. Ye, "Weighted composition operators from $F(p,q,s)$ into logarithmic Bloch space," *Journal of the Korean Mathematical Society*, vol. 45, no. 4, pp. 977–991, 2008.
- [26] W. Yang, "Generalized weighted composition operators from the $F(p,q,s)$ space to the Bloch-type space," *Applied Mathematics and Computation*, vol. 218, no. 9, pp. 4967–4972, 2012.
- [27] W. Yang, "Composition operators from $F(p,q,s)$ spaces to the n th weighted-type spaces on the unit disc," *Applied Mathematics and Computation*, vol. 218, no. 4, pp. 1443–1448, 2011.
- [28] Z. Zhou and R. Chen, "Weighted composition operators from $F(p,q,s)$ to Bloch type spaces on the unit ball," *International Journal of Mathematics*, vol. 19, no. 8, pp. 899–926, 2008.
- [29] X. Zhu, "Weighted composition operators from $F(p,q,s)$ spaces to H_μ^∞ spaces," *Abstract and Applied Analysis*, vol. 2009, Article ID 290978, 14 pages, 2009.
- [30] S. Stević, "On some integral-type operators between a general space and Bloch-type spaces," *Applied Mathematics and Computation*, vol. 218, no. 6, pp. 2600–2618, 2011.
- [31] S. Stević, "Boundedness and compactness of an integral-type operator from Bloch-type spaces with normal weights to $F(p,q,s)$ space," *Applied Mathematics and Computation*, vol. 218, no. 9, pp. 5414–5421, 2012.
- [32] S. Ye, "Multiplication operators between the little Bloch type and the Bloch type spaces," *Journal of Fujian Normal University(Natural Science Edition)*, vol. 22, no. 2, 2006.
- [33] X. Zhang, "Composition operators and weighted composition operators on p -Bloch spaces," *Chinese Annals of Mathematics A*, vol. 24, no. 6, pp. 711–720, 2003.
- [34] B. R. Choe, H. Koo, and W. Smith, "Composition operators on small spaces," *Integral Equations and Operator Theory*, vol. 56, no. 3, pp. 357–380, 2006.
- [35] S. Stević, "Composition operators between H^∞ and α -Bloch spaces on the polydisc," *Zeitschrift für Analysis und ihre Anwendungen*, vol. 25, no. 4, pp. 457–466, 2006.
- [36] S. Stević, "Weighted differentiation composition operators from H^∞ and Bloch spaces to n th weighted-type spaces on the unit disk," *Applied Mathematics and Computation*, vol. 216, no. 12, pp. 3634–3641, 2010.
- [37] X. Zhang, "Weighted Cesàro operators on Dirichlet type spaces and Bloch type spaces of \mathbb{C}^n ," *Chinese Annals of Mathematics A*, vol. 26, no. 1, pp. 139–150, 2005.



Hindawi

Submit your manuscripts at
<http://www.hindawi.com>

