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THE MULTIPLICITY OF HELICES FOR A REGULARLY INCREASING SEQUENCE OF σ-FIELDS

Dedicated to Professor Tamotsu Tsuchikura on his sixtieth birthday

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1. Introduction. The notion of helices was introduced in the theory of measure-preserving transformations as an application of the martingale theory by J. de Sam Lazaro and P. A. Meyer [2]. The multiplicity of helices has been discussed by the author in the same manner as that of square-integrable martingales [4], [5]. In this paper, we determine the multiplicity of helices under some condition of the regularity on an increasing sequence of sub-σ-fields.

2. Preliminaries. Throughout this paper (Ω, F, P) denotes a complete separable probability space and T an automorphism of Ω, that is, a bimeasurable measure-preserving bijection. Let F_0 be a complete proper sub-σ-field of F such that

$$\text{(a)} \quad F_n \subset F_{n+1} \quad \text{for all} \quad n \in Z\,,$$

$$\text{(b)} \quad \bigvee_{n \in Z} F_n = F$$

where $Z = \{0, \pm 1, \pm 2, \cdots\}$ and F_n denotes the sub-σ-field $T^n F_0$. A pair (T, F_0) is called a system.

Let H denote the class of all squarely integrable real random variables with expectations 0, which is an infinite dimensional Hilbert space under the ordinary inner product, and H_n the subspace of H consisting of all elements measurable with respect to F_n for each $n \in Z$.

DEFINITION 1. A sequence $X = (x_n)_{n \in Z}$ in H is called a helix of (T, F_0) if the following conditions are satisfied:

$$\text{(a)} \quad x_0 = 0\,,$$

$$\text{(b)} \quad x_n - x_{n-1} \in H_n \cap H_{n-1}^{\perp} \quad \text{for all} \quad n \in Z$$

where $\perp$ indicates the orthogonal complementation in H,

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$$\text{(c)}\quad (x_n - x_{n-1})\circ T^{-1} = x_{n+1} - x_n \quad \text{for all} \quad n \in Z\,.$$

By this definition, each helix $X = (x_n)$ can be written as

$$x_0 = 0\,, \quad x_n = \sum_{k=1}^{n} x\circ T^{-(k-1)} \qquad (n > 0)\,,$$
$$x_n = -x_{-n}\circ T^{-n} \qquad (n < 0)$$

for some $x \in H_1 \cap H_0^{\perp}$.

Note that each helix has the property of a martingale, namely $(x_{n+m} - x_m, F_{n+m})_{n\geqq 0}$ is a square-integrable martingale. Thus we can apply the method of martingales to the study of helices.

Given two helices $X = (x_n)$ and $X' = (x'_n)$ of (T, F_0), we define the random variable $\langle X, X'\rangle$ by

$$\langle X, X'\rangle = E[x_1 x'_1 \,|\, F_0]\,.$$

If $X = X'$, then we write simply $\langle X\rangle$ instead of $\langle X, X\rangle$. Consider the process $(\langle X\rangle_n)_{n\geqq 0}$ defined by

$$\langle X\rangle_0 = 0\,, \quad \langle X\rangle_n = \sum_{k=1}^{n} \langle X\rangle\circ T^{-(k-1)} \qquad (n > 0)\,,$$

which is nothing but the predictable increasing process of the Doob-Meyer decomposition for the martingale $(x_n, F_n)_{n\geqq 0}$. We see easily that

$$\langle X, X'\rangle = \langle X', X\rangle$$

and for another helix Y,

$$\langle X + Y, X'\rangle = \langle X, X'\rangle + \langle Y, X'\rangle\,.$$

DEFINITION 2. Two helices X and X' are said to be strictly orthogonal if $\langle X, X'\rangle = 0$.

DEFINITION 3. For two helices X and X', we denote by $\mu_{\langle X,X'\rangle}$ the signed measure on F_0 with density $\langle X, X'\rangle$, that is, for each $B \in F_0$

$$\mu_{\langle X,X'\rangle}(B) = \int_B \langle X, X'\rangle dP = \int_B x_1 x'_1 dP\,.$$

It is called the helix-measure of X and X', and $\mu_{\langle X\rangle}$ is called the helix-measure of X.

DEFINITION 4. For a helix $X = (x_n)$ and a squarely integrable random variable ν on $(\Omega, F_0, \mu_{\langle X\rangle})$, the helix $Y = (y_n)$ given by

$$y_0 = 0\,, \quad y_n = \sum_{k=1}^{n} (\nu\circ T^{-(k-1)})(x_k - x_{k-1}) \qquad (n > 0)\,,$$
$$y_n = -y_{-n}\circ T^{-n} \qquad (n < 0)\,,$$

is called the helix-transform of X by ν and denoted by $\nu * X$.

This notion is analogous to the so-called martingale-transform. We have obviously $\langle \nu * X, X' \rangle = \nu \langle X, X' \rangle$ and $\langle \nu * X \rangle = \nu^2 \langle X \rangle$.

The following result is of fundamental importance in our discussion.

PROPOSITION 1. *Let* (T, F_0) *be a system. Then there exists a sequence of at most countable strictly orthogonal helices* $\mathscr{X} = (X^{(p)})$ *which satisfy the following conditions*:

(a) *Every helix* X *has the representation*

$$X = \sum_p \nu^{(p)} * X^{(p)}, \qquad \nu^{(p)} \in L^2(\Omega, F_0, \mu_{\langle X^{(p)} \rangle}) .$$

(b) $\mu_{\langle X^{(p+1)} \rangle}$ *is absolutely continuous with respect to* $\mu_{\langle Y^{(p)} \rangle}$ *for each* p.

Furthermore, if $\mathscr{Y} = (Y^{(p)})$ *is another such sequence, then* $\mu_{\langle Y^{(p)} \rangle}$ *is equivalent to* $\mu_{\langle X^{(p)} \rangle}$ *for all* p.

Such a sequence of helices is called a strict base of helices of the system. Proposition 1 indicates that the length of a strict base is uniquely determined by the system.

DEFINITION 5. The length of a strict base is called the multiplicity of helices of this system, which is denoted by $M(T, F_0)$.

As for a calculation of the multiplicity, the following two results are known (cf. [4], [5]):

Let (T, F_0) be a system such that

$$F_0 = \bigvee_{n<0} T^n A$$

for some sub-σ-field A of F. Then, it is possible to estimate the multiplicity of helices of this system.

PROPOSITION 2. *Let* (T, F_0) *be the system mentioned above. Then*

$$M(T, F_0) \leqq \dim L_0^2(A)$$

where $L_0^2(A)$ *is the subspace of* H *consisting of all elements measurable with respect to* A.

The equality in the above proposition holds for a special class of systems of the following type (cf. [4], [5]):

DEFINITION 6. Let T be an automorphism of Ω and A a sub-σ-field of F. The pair (T, A) is called a Bernoulli system or simply a B-system if

(a) $(T^n A)_{n \in Z}$ is an independent sequence of sub-σ-fields,

(b) $\bigvee_{n\in Z} T^n A = F$.

If we set $F_0 = \bigvee_{n<0} T^n A$, then (T, F_0) is clearly a system, which is called a Kolmogorov system.

PROPOSITION 3. *Let (T, A) be a B-system and (T, F_0) the Kolmogorov system derived from (T, A). Then all helix-measures of a strict base are equivalent to P on F_0 and*

$$M(T, F_0) = \dim L_0^2(A) .$$

3. Predictable independence. In this section, we define some independence of a sequence of helices and investigate the procedure of Schmidt's orthogonalization for helices.

DEFINITION 7. A sequence $(X^{(p)})$ of helices is said to be predictably independent if $\langle \sum_p \nu^{(p)} * X^{(p)} \rangle$ is not equal to 0 for any $\nu^{(p)} \in L^2(\Omega, F_0, \mu_{\langle X^{(p)} \rangle})$ unless all $\nu^{(p)}$ are equal to 0.

Note that all subsequences of such a sequence of helices are also predictably independent. Further, we remark the following on this independence of helices. If the sequence $(X^{(p)})$ is strictly orthogonal and each $\langle X^{(p)} \rangle$ is positive a.s., then $(X^{(p)})$ is predictably independent. Indeed, if $(X^{(p)})$ is strictly orthogonal, then

$$\begin{aligned}\langle \sum_p \nu^{(p)} * X^{(p)} \rangle &= \sum_p \nu^{(p)2} \langle X^{(p)} \rangle + \sum_{p \neq q} \nu^{(p)} \nu^{(q)} \langle X^{(p)}, X^{(q)} \rangle \\ &= \sum_p \nu^{(p)2} \langle X^{(p)} \rangle .\end{aligned}$$

Hence, if $\langle \sum_p \nu^{(p)} * X^{(p)} \rangle = 0$, then all $\nu^{(p)}$ are equal to 0 since all $\langle X^{(p)} \rangle$ are positive a.s.

Suppose that a sequence $(X^{(p)})_{p=1,2,\cdots,\kappa}$ of helices is predictably independent and each $\langle X^{(p)} \rangle$ is positive a.s. In the case that $\kappa = \infty$, this means simply that the sequence is countably infinite. From such a sequence, we can obtain the strictly orthogonal sequence $(Y^{(p)})_{p=1,2,\cdots,\kappa}$ of helices by the following procedure.

Schmidt's orthogonalization. First put $y^{(1)} = x_1^{(1)} / \langle X^{(1)} \rangle^{1/2}$ and construct a helix $Y^{(1)} = (y_n^{(1)})$ such that $y_1^{(1)} = y^{(1)}$, that is,

$$y_1^{(1)} = 0 , \quad y_n^{(1)} = \sum_{k=1}^{n} y^{(1)} \circ T^{-(k-1)} \qquad (n > 0) ,$$

$$y_n^{(1)} = -y_{-n} \circ T^{-n} \qquad (n < 0) ,$$

so that $\langle Y^{(1)} \rangle = E[y^{(1)2} | F_0] = E[x_1^{(1)2} | F_0] / \langle X^{(1)} \rangle = 1$. Then put

$$z^{(2)} = x_1^{(2)} - \langle X^{(2)}, Y^{(1)} \rangle y^{(1)}$$

and construct a helix $Z^{(2)} = (z_n^{(2)})$ such that $z_1^{(2)} = z^{(2)}$ in the same way as above for $Y^{(1)}$, that is,

$$Z^{(2)} = 1 * X^{(2)} - \langle X^{(2)}, Y^{(1)} \rangle * Y^{(1)} .$$

Then $\langle Z^{(2)}, Y^{(1)} \rangle = \langle X^{(2)}, Y^{(1)} \rangle - \langle X^{(2)}, Y^{(1)} \rangle \cdot \langle Y^{(1)} \rangle = 0$ and $\langle Z^{(2)} \rangle > 0$ a.s., since $X^{(1)}$ and $X^{(2)}$ are predictably independent. Put $y^{(2)} = z^{(2)} / \langle Z^{(2)} \rangle^{1/2}$ and construct a helix $Y^{(2)} = (y_n^{(2)})$ such that $y_1^{(2)} = y^{(2)}$. When $Y^{(1)}, Y^{(2)}, \cdots, Y^{(p-1)}$ are obtained in this way, so that $\langle Y^{(q)}, Y^{(r)} \rangle = \delta_{qr}$ for $1 \leqq q$, $r \leqq p-1$, put

$$z^{(p)} = x_1^{(p)} - \sum_{q=1}^{p-1} \langle X^{(p)}, Y^{(q)} \rangle y^{(q)}$$

and construct a helix $Z^{(p)} = (z_n^{(p)})$ such that $z_1^{(p)} = z^{(p)}$, that is,

$$Z^{(p)} = 1 * X^{(p)} - \sum_{q=1}^{p-1} \langle X^{(p)}, Y^{(q)} \rangle * Y^{(q)} .$$

Then $\langle Z^{(p)}, Y^{(q)} \rangle = 0$ for $1 \leqq q \leqq p-1$ and $\langle Z^{(p)} \rangle > 0$ a.s., since $X^{(1)}$, $X^{(2)}, \cdots, X^{(p)}$ are predictably independent. Put $y^{(p)} = z^{(p)} / \langle Z^{(p)} \rangle^{1/2}$ and construct a helix $Y^{(p)} = (y_n^{(p)})$ such that $y_1^{(p)} = y^{(p)}$ in the same way as above. Hence $Y^{(p)}$ added to $Y^{(1)}, Y^{(2)}, \cdots, Y^{(p-1)}$ retains the property that $\langle Y^{(q)}, Y^{(r)} \rangle = \delta_{qr}$ for $1 \leqq q$, $r \leqq p$, and this procedure can be continued to $p = \kappa$. Thus we obtain a strictly orthogonal sequence $(Y^{(p)})_{p=1,2,\cdots,\kappa}$ of helices such that $\mu_{\langle Y^{(p)} \rangle} = P$ on F_0 for all p.

By this procedure, we can show the following for the multiplicity of helices of a system:

THEOREM 1. *Let (T, F_0) be a system such that*

(a) $F_0 = \bigvee_{n<0} T^n A$ *for some sub-σ-field A and*

(b) $\dim L_0^2(A) = \kappa$.

If $(X^{(p)})_{p=1,2,\cdots,\kappa}$ is a predictably independent sequence of helices of (T, F_0) and each $\langle X^{(p)} \rangle$ is positive a.s., *then all helix-measures of each strict base of helices of (T, F_0) are equivalent to P on F_0 and*

$$M(T, F_0) = \kappa .$$

PROOF. By the procedure of Schmidt's orthogonalization for $(X^{(p)})$, we can obtain a strictly orthogonal sequence$(Y^{(p)})_{p=1,2,\cdots,\kappa}$ of helices such that $\mu_{\langle Y^{(p)} \rangle} = P$ on F_0 for all p. Then we have that $\kappa \leqq M(T, F_0)$. By Proposition 2 and the condition (b) in the statement, we have that $M(T, F_0) \leqq \kappa$ and hence $M(T, F_0) = \kappa$. Thus $(Y^{(p)})$ is a strict base of helices of (T, F_0) such that $\mu_{\langle Y^{(p)} \rangle} = P$ on F_0 for all p. q.e.d.

4. Helices for regularly increasing sub-σ-fields. In this section, we deal with a system (T, F_0) of the following type:

$F_0 = \bigvee_{n<0} T^n A$ where a sub-σ-field A is generated by a partition $\alpha = \{A_0, A_1, \cdots, A_\kappa\}$ of Ω.

In addition, we impose the following condition of regularity on this system:

DEFINITION 8. The system (T, F_0) of the above type is said to be regular if

$$0 < P(A_p | B) < 1 \text{ for all } B \in F_0 \text{ with } P(B) > 0 \text{ and all } A_p \in \alpha$$

where $P(A|B)$ denotes the conditional probability of A under B.

It is obvious that (T, F_0) is regular if (T, A) is a B-system. This definition means that all parts of Ω are *homogeneously* mixed by the transformation T.

THEOREM 2. *If a system (T, F_0) is regular, then all helix-measures of each strict base of helices of (T, F_0) are equivalent to P on F_0 and*

$$M(T, F_0) = \kappa .$$

PROOF. Let $\alpha = \{A_0, A_1, \cdots, A_\kappa\}$ be a partition which generates A. Obviously, $\dim L_0^2(A) = \kappa$. For $1 \leqq p \leqq \kappa$, put

$$x^{(p)} = 1_{A_p} - E[1_{A_p} | F_0]$$

where 1_A denotes an indicator of the event A. Then $x^{(p)} \in H_1 \cap H_0^\perp$. Corresponding to each $x^{(p)}$, construct a helix $X^{(p)} = (x_n^{(p)})$ such that $x_1^{(p)} = x^{(p)}$. To prove the statement under the condition of regularity, it is sufficient to show that the sequence $(X^{(p)})_{p=1,2,\cdots,\kappa}$ is predictably independent and each $\langle X^{(p)} \rangle$ is positive a.s. by Theorem 1 in the preceding section.

First, we shall show that $\langle X^{(p)} \rangle > 0$ a.s. for $1 \leqq p \leqq \kappa$. By the regularity of (T, F_0), it is obvious that

$$0 < E[1_{A_p} | F_0] < 1$$

for $0 \leqq p \leqq \kappa$. Then

$$\langle X^{(p)} \rangle = E[x^{(p)2} | F_0] = E[1_{A_p} | F_0](1 - E[1_{A_p} | F_0])$$

is positive a.s. for $1 \leqq p \leqq \kappa$. Next, to prove that $(X^{(p)})_{p=1,2,\cdots,\kappa}$ is predictably independent, we put

$$B = \left\{ \left\langle \sum_{p=1}^{\kappa} \nu^{(p)} * X^{(p)} \right\rangle = 0 \right\}$$

where $\nu^{(p)} \in L^2(\Omega, F_0, \mu_{\langle X^{(p)} \rangle})$ for $1 \leqq p \leqq \kappa$ and

$$\sum_{p=1}^{\kappa} \int_B \nu^{(p)2} d\mu_{\langle X^{(p)} \rangle} < \infty .$$

Then we have that

$$E\Big[\Big(1_B\cdot\sum_{p=1}^{\kappa}\nu^{(p)}x^{(p)}\Big)^2\Big|F_0\Big] = 1_B\cdot E\Big[\Big(\sum_{p=1}^{\kappa}\nu^{(p)}x^{(p)}\Big)^2\Big|F_0\Big]$$
$$= 1_B\cdot\Big\langle\sum_{p=1}^{\kappa}\nu^{(p)}*X^{(p)}\Big\rangle$$
$$= 0 \quad \text{a.s.}$$

This implies that

$$1_B\cdot\sum_{p=1}^{\kappa}\nu^{(p)}x^{(p)} = 0 \quad \text{a.s.}$$

and hence

$$1_B\cdot\sum_{p=1}^{\kappa}\nu^{(p)}1_{A_p} = 1_B\cdot\sum_{p=1}^{\kappa}\nu^{(p)}E[1_{A_p}|F_0] \quad \text{a.s.}$$

By the measurability of the right hand side of the above formula, the left hand side is also measurable with respect to F_0, which implies that

$$E\Big[\Big(1_B\cdot\sum_{p=1}^{\kappa}\nu^{(p)}1_{A_p}\Big)^2\Big|F_0\Big] = \Big(1_B\cdot\sum_{p=1}^{\kappa}\nu^{(p)}1_{A_p}\Big)^2 .$$

Then we have

$$1_B\cdot\sum_{p=1}^{\kappa}\nu^{(p)2}E[1_{A_p}|F_0] = 1_B\cdot\sum_{p=1}^{\kappa}\nu^{(p)2}1_{A_p}$$

since $A_p\cap A_q=\varnothing$ for $p\neq q$. The right hand side of this formula is equal to 0 on A_0. Then we have

$$1_{A_0}\Big(1_B\cdot\sum_{p=1}^{\kappa}\nu^{(p)2}E[1_{A_p}|F_0]\Big) = 0 \quad \text{a.s.}$$

and hence by conditioning both sides relative to F_0, we obtain

$$E[1_{A_0}|F_0]\Big(1_B\cdot\sum_{p=1}^{\kappa}\nu^{(p)2}E[1_{A_p}|F_0]\Big) = 0 \quad \text{a.s.}$$

Since $E[1_{A_0}|F_0]>0$ by the regularity of (T, F_0), we have

$$1_B\cdot\sum_{p=1}^{\kappa}\nu^{(p)2}E[1_{A_p}|F_0] = 0 \quad \text{a.s.}$$

and since $E[1_{A_p}|F_0]>0$ for $1\leqq p\leqq\kappa$, we have the consequence that $\nu^{(p)}$ is equal to 0 on B for $1\leqq p\leqq\kappa$. q.e.d.

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