The multiplier problem for the ball

By CHARLES FEFFERMAN*

1. Introduction

Define an operator T on $L^p(\mathbf{R}^n)$ by the equation $\widehat{Tf}(x) = \chi_B(x)\hat{f}(x)$, where χ_B is the characteristic function of the unit ball. This operator and its variants play the role of the Hilbert transform for a number of problems on multiple Fourier series and boundary behavior of analytic functions of several complex variables. (See [6], [7], for instance.) Therefore, one would like to know whether T is a bounded operator on $L^p(\mathbf{R}^n)$. For the analogue of T in which the ball is replaced by a cube, an affirmative answer has been known for many years: T_{cube} is bounded on all L^p $(1 < p < \infty)$. See [6]. However, T is essentially different from T_{cube}. One sees this almost immediately, by applying T, say, to the function

$$f(x) = \begin{cases} 1 & \text{if } |x| < 1/10 \\ 0 & \text{otherwise .} \end{cases}$$

Tf dies too slowly at infinity to be a good function, and it follows that T cannot be bounded on $L^p(\mathbf{R}^n)$ for any p outside the range $2n/(n+1) < p < 2n/(n-1)$. The well-known "disc conjecture" asserts that T is bounded on all $L^p(\mathbf{R}^n)$ with $2n/(n+1) < p < 2n/(n-1)$. Only the trivial case $p = 2$ has ever been decided.

In [5] and [3], heartening progress was made on the disc conjecture. Instead of the difficult operator T, Stein and I dealt with the slightly less singular Bochner-Riesz spherical summation operators, defined by $\widehat{T_\delta f}(x) = [\max(1 - |x|^2, 0)]^\delta \hat{f}(x)$. For each $\delta > 0$, one expects T_δ to be bounded on a certain range of L^p's, and [3] proves the analogue of the disc conjecture for all T_δ with δ greater than a critical value δ_c.

It therefore comes as a surprise, at least to me, that the disc conjecture is false.

THEOREM 1. *T is bounded only on L^2 $(n > 1)$.*

We shall prove this unfortunate fact in section 2 below. Note that it is enough to disprove L^p-boundedness of T on $\mathbf{R}^2$ for $p > 2$. For, L^p-boundedness of T on $\mathbf{R}^n$ implies boundedness on $\mathbf{R}^{n-1}$ by a theorem of de Leeuw; and

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the case $p > 2$ yields the case $p < 2$ by duality. (See Jodeit [4] for an enlightening proof of de Leeuw's theorem.)

How badly does T fail to be bounded on $L^p(\mathbf{R}^n)$, $2n/(n+1) < p < 2n/(n-1)$? Just barely, at least in $\mathbf{R}^2$, according to the following result.

THEOREM 2. *T_δ is bounded on $L^p(\mathbf{R}^2)$, for $\delta > 0$, $4/3 \leqq p \leqq 4$.*

L. Carleson and P. Sjölin proved Theorem 2 by means of Carleson's recent theory of "saddle-point" integrals. This news was a great psychological stimulus to the author as he was working his way through the disc problem. He is profoundly grateful for this.

With greatest pleasure, we acknowledge E. M. Stein's decisive contributions to this research. His remarkable "Restriction Theorem" on Fourier transforms was the key insight that opened up the disc problem to serious attack. He provided me with the moral and mathematical support without which that problem would today be as open as ever. Lastly, he simplified my proof of Theorem 1 to its present digestible form.

2. Disproof of the disc conjecture

Assume that $\| Tf \|_p \leqq C \| f \|_p$, where $p > 2$. From this would follow:

LEMMA 1. (Y. Meyer.) *Let $v_1, v_2, \cdots$ be a sequence of unit vectors in $\mathbf{R}^2$, and let H_j be the half-plane $\{x \in \mathbf{R}^2 \mid x \cdot v_j \geqq 0\}$. Define a sequence of operators $T_1, T_2, \cdots$ on $L^p(\mathbf{R}^2)$ by setting $\widehat{T_j f}(x) = \chi_{H_j}(x)\hat{f}(x)$. Then for any sequence of functions $f_1, f_2, \cdots$, the following inequality holds:*

$$\text{(A)} \qquad \| (\textstyle\sum_j | T_j f_j |^2)^{1/2} \|_p \leqq C \| (\sum_j | f_j |^2)^{1/2} \|_p \,.$$

Proof. The idea is to replace T_j by an operator more closely related to the disc. Define T_j^r by the equation $\widehat{T_j^r f}(x) = \chi_{D_j^r}(x)\hat{f}(x)$, where D_j^r is the disc of radius r and center rv_j. D_j^r looks much like the half-plane H_j for enormous r, so that we might expect $T_j f(x) = \lim_{r\to\infty} T_j^r f(x)$ for $x \in \mathbf{R}^2$. Indeed, when $f \in C_0^\infty(\mathbf{R}^2)$, this is immediate from routine estimates on $\widehat{T_j f} - \widehat{T_j^r f}$. Fatou's Lemma now shows that

$$\| (\textstyle\sum_j | T_j f_j |^2)^{1/2} \|_p \leqq \liminf_{r\to\infty} \| (\sum_j | T_j^r f_j |^2)^{1/2} \|_p \,.$$

So in order to prove (A), it is enough to prove

$$\text{(A}'\text{)} \qquad \| (\textstyle\sum_j | T_j^r f_j |^2)^{1/2} \|_p \leqq C \| (\sum_j | f_j |^2)^{1/2} \|_p$$

with C independent of r. By dilating $\mathbf{R}^2$, we may assume that $r = 1$. However, $T_j^1 f(x) = e^{iv_j \cdot x} T(e^{-iv_j \cdot y} f(y))$, so that the left-hand side of (A$'$) is nothing but

$$\text{(B)} \qquad \| (\textstyle\sum_j | T(e^{iv_j \cdot y} f_j(y)) |^2)^{1/2} \|_p \,,$$

and our problem is to dominate (B) by $C\,\|(\sum_j |f_j|^2)^{1/2}\|_p$.

We are assuming that $\|Tf\|_p \leqq C\|f\|_p$ for all $f \in L^p(\mathbf{R}^2)$. An application of the Rademacher functions shows that the vector analogue

$$\|(\sum_j |Tf_j|^2)^{1/2}\|_p \leqq C\,\|(\sum_j |f_j|^2)^{1/2}\|_p$$

holds also (see [7, vol. 2, p. 224]). Therefore, the expression (B) is dominated by

$$C\,\|(\sum_j |e^{v_j \cdot y} f_j(y)|^2)^{1/2}\|_p = C\,\|(\sum_j |f_j|^2)^{1/2}\|_p ,$$

which is just what we had to prove. Q.E.D.

To prove Theorem 1, we shall exhibit a counterexample to (A) for any $p > 2$. Our example is based on a slight variant of (Schönberg's improvement of) Besicovitch's construction for the Kakeya needle problem.

LEMMA 2. *Fix a small number* $\eta > 0$. *There is a set* $E \subseteqq \mathbf{R}^2$ *and a collection* $\mathcal{R} = \{R_j\}$ *of pairwise disjoint rectangles, with the properties:*

(1) *At least one-tenth the area of each* $\tilde{R}_j$ *lies in* E.

(2) $|E| \leqq \eta \sum_j |R_j|$.

Here, $\tilde{R}_j$ *is the shaded region in Figure* 1.

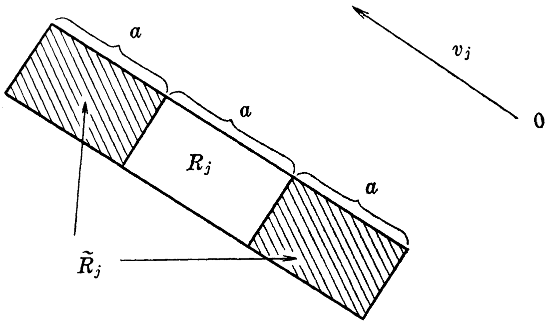

FIGURE 1

As soon as we know Lemma 2, we can immediately disprove (A) and thus establish Theorem 1. For, we simply set $f_j = \chi_{R_j}$, and let v_j be parallel to the longer sides of R_j, as in Figure 1. Direct computation shows that $|T_j f_j| \geqq 1/2$ on $\tilde{R}_j$, so that

$$\text{(C)}\quad \begin{aligned} \int_E (\sum_j |T_j f_j(x)|^2) dx &= \sum_j \int_E |T_j f_j(x)|^2 dx \geqq \frac{1}{4} \sum_j |E \cap \tilde{R}_j| \\ &\geqq \frac{1}{40} \sum_j |\tilde{R}_j| = \frac{1}{20} \sum_j |R_j| , \end{aligned}$$

(by (1)).

On the other hand, if (A) were true, Hölder's inequality would show that

$$\text{(D)}\qquad \begin{aligned} \int_E \left(\sum_j \mid T_j f_j(x) \mid^2\right) dx &\leqq \mid E \mid^{(p-2)/p} \left\| \left(\sum_j \mid T_j f_j \mid^2\right)^{1/2} \right\|_p^2 \\ &\leqq C \mid E \mid^{(p-2)/p} \left\| \left(\sum_j \mid f_j \mid^2\right)^{1/2} \right\|_p^2 \\ &= C \mid E \mid^{(p-2)/p} \left(\sum_j \mid R_j \mid\right)^{2/p} \leqq C\eta^{(p-2)/p} \sum \mid R_j \mid \end{aligned}$$

(by (2)).

For small η, (C) contradicts (D), which disproves (A). Therefore, Theorem 1 follows from Lemma 2.

Proof of Lemma 2. We shall follow closely the excellent exposition in [2]. See also the classic paper of Busemann and Feller [1].

Consider the following process: We are given a triangle T as in Figure 2a, with horizontal base ab and height h. Extend the lines ac and bc to points a' and b' of height $h' > h$. Let d be the midpoint of ab. (See Figure 2b).

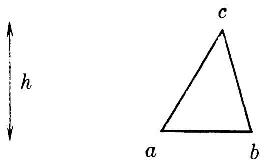

FIGURE 2a

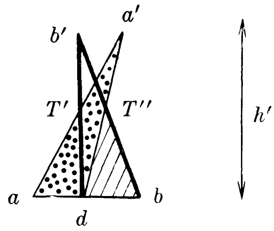

FIGURE 2b

We say that the two triangles $T' = ada'$ and $T'' = bdb'$ arise as T *sprouts* from height h to height h'.

Now we can construct the Besicovitch set E. Begin with an equilateral triangle T° whose base is the interval [0, 1] on the x-axis, and pick an increasing sequence of numbers $h_0, h_1, h_2, \cdots, h_k$, where $h_0 = \sqrt{3}/2 = \text{height}\,(T^\circ)$. Sprout T° from height h_0 to height h_1, to obtain two triangles, T' and T''. Sprout both T' and T'' from height h_1 to height h_2 to obtain four triangles T^1, T^2, T^3, T^4, all of height h_2. Continue sprouting, obtaining at stage n, 2^n triangles of height h_n and base 2^{-n}. Finally, set E equal to the union of all 2^k triangles $T_1, T_2, \cdots, T_{2^k}$ arising at stage k.

For the case $h_0 = \frac{\sqrt{3}}{2}$, $h_1 = \frac{\sqrt{3}}{2}\left(1 + \frac{1}{2}\right)$, $h_2 = \frac{\sqrt{3}}{2}\left(1 + \frac{1}{2} + \frac{1}{3}\right), \cdots$, Busemann and Feller compute the area of E, and find it to be at most 17. (Actually, Busemann and Feller use a sprouting procedure slightly different from ours. However, since their sprouted triangles are strictly larger than ours, their estimates apply here.)

Now we have built the set E and found its measure. It remains to construct a collection $\mathcal{R}$ of pairwise disjoint rectangles satisfying (1) and (2).

To do so, note that each dyadic interval $I \subset [0, 1]$, of length 2^{-k}, is the base of exactly one T_j. Denote that T_j by $T(I)$, and call its upper vertex $P(I)$. Using $T(I)$, we construct the rectangle $R(I)$ as in Figure 3.

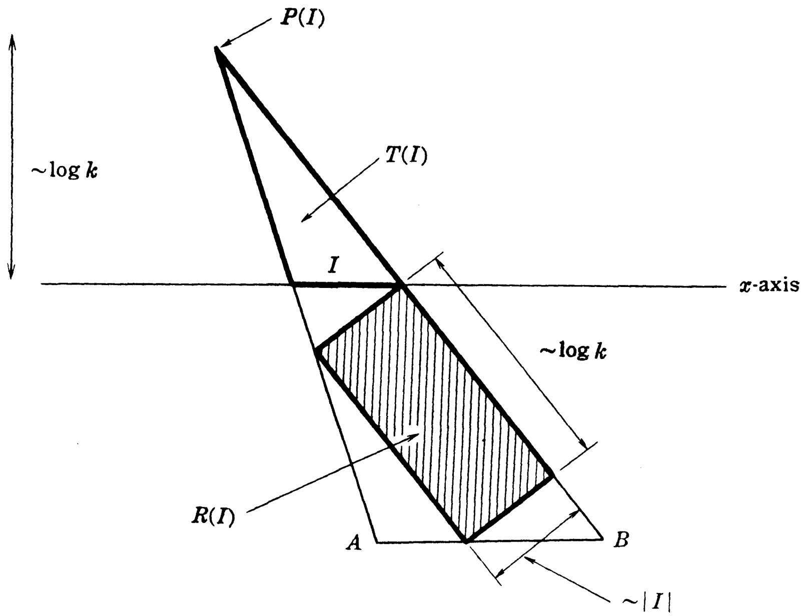

FIGURE 3

It doesn't matter how $R(I)$ is placed, as long as it stays inside triangle $P(I)BA$. Now define $\mathcal{R} = \{R(I) \mid I$ is a dyadic subinterval of $[0, 1]$, of length $2^{-k}\}$.

Let us check (1) and (2). (1) is obvious from Figure 3, since $T(I) \subseteqq E$. To check (2), we note that each $R(I)$ has area $\sim (\log k) \cdot 2^{-k}$, and that there are 2^k $R(I)$'s. So

$$\sum_I R(I) \sim (\log k) \cdot 2^{-k} \cdot 2^k = \log k ,$$

whereas we saw that the area of E is at most 17. Thus, (2) holds if we take k so large that $\log k > 17/\eta$.

It remains only to show that the different $R(I)$ are pairwise disjoint. This is geometrically obvious from the following.

LEMMA 3. *Let I and I' be two dyadic subintervals of* $[0, 1]$ *of length* 2^{-k}. *If I lies to the left of I', then $P(I')$ lies to the left of $P(I)$.*

Proof of Lemma 3. For any triangle T with horizontal base, define the region $\tilde{T}$ as in Figure 4.

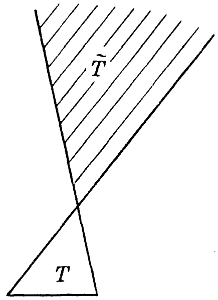

FIGURE 4

We have:

(α) If $I_1 \subseteqq I_2$ then $\widetilde{T(I_1)} \subseteqq \widetilde{T(I_2)}$. This is clear by induction.

(β) Let I_1 and I_2 be two halves of the dyadic interval I. Say that I_1 is to the left of I_2. Then $\widetilde{T(I_1)}$ and $\widetilde{T(I_2)}$ are disjoint, and $\widetilde{T(I_2)}$ is to the left of $\widetilde{T(I_1)}$. (See Figure 2b).

Therefore,

(γ) If I_1 and I_2 are disjoint dyadic subintervals of $[0, 1]$, and I_1 lies to the left of I_2, then $\widetilde{T(I_1)}$ and $\widetilde{T(I_2)}$ are pairwise disjoint and $\widetilde{T(I_2)}$ lies to the left of $\widetilde{T(I_1)}$.

Lemma 3 follows at once from (γ), since $P(I) \in \widetilde{T(I)}$. Q.E.D.

Lemma 3 is trivial, and its proof should not be taken too seriously. In any event, we now know that the rectangles of $\mathcal{R}$ are pairwise disjoint. The proofs of Lemma 2 and Theorem 1 are complete. Q.E.D.

We explain briefly how to construct explicit counterexamples to the disc conjecture in $\mathbf{R}^2$. By carrying out the construction of Lemma 2 and then dilating the plane, we obtain a set E' and a collection of pairwise disjoint rectangles $\{R'_j\}$ satisfying (1) and (2) above, with the further property that all the R'_j have dimension roughly $N^{1/2} \times N$. Here N is as large as we please. Also, we may replace E' by a slightly smaller set E'' which is a union of squares of length $N^{1/2}$, without destroying (1) and (2). As above, set v_j equal to a unit vector parallel to the longer sides of R'_j. Then the function

$$f(x) = \sum_j e^{iv_j \cdot k} \chi_{R'_j}(x) \equiv \sum_j f_j(x)$$

satisfies

$$\| Tf \|_p > C \| f \|_p \qquad (p > 2) ,$$

provided we take $N = N(C, p)$ large enough. To prove this, one first computes Tf_j and verifies that for $j \neq j'$, Tf_j and $Tf_{j'}$ are essentially orthogonal on every square of length $\geqq N^{1/2}$. (On each square of side $N^{1/2}$, Tf_j and $Tf_{j'}$

turn out, more or less, to be plane waves travelling in different directions. It is helpful to look at $\widehat{Tf_j}$ and $\widehat{Tf_{j'}}$.) Therefore,

$$\int_E |Tf(x)|^2 dx = \int_E \Big|\sum\nolimits_j Tf_j(x)\Big|^2 dx \approx \sum\nolimits_j \int_E |Tf_j(x)|^2 dx \,.$$

As in the deduction of Theorem 1 from Lemma 2, the left-hand integral turns out to be so large that $\| Tf \|_p \leqq C \| f \|_p$ is impossible.

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