Working Paper

THE MULTISTATE LIFE TABLE WITH DURATION-DEPENDENCE

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Foreword

The classical linear multi-state model is represented by an equation due to Kolmogorov, and applied to demography by Andrei Rogers. For many purposes it gives a realistic representation of phenomena, especially in problems in which the population is nearly homogeneous. In that respect it resembles the ordinary life table, of which it is a generalization. But like the life table it acts as though all of the individuals of a given category have the identical probability, so the statistically observed average represents each and every individual in its category.

No demographer has ever regarded this as quite satisfactory; all recognize that individuals within a given cell are different from one another and the average of the cell does not apply to individuals. In a given group every couple may have one chance in 3 of divorcing; or else 1/9 of couples may divorce 3 times each. The overall probability that a couple will divorce is the same in the two cases, but the inference about what will happen to a random couple in the future is very different for the two. Yet to take into account this distinction involves difficulties, both of data and of the model for dealing with the data.

James Vaupel and Anatoli Yashin of this program have made great progress in dealing with this question, and their work will be brought together in a volume now being prepared.

The present paper sets out the theory of a procedure for taking account of a particular kind of heterogeneity--that associated with the length of time in a state. Insofar as people are less likely to divorce the longer they have been married, and if divorce rates by duration are known, separate transition matrices can be set up for different durations. Douglas Wolf ingeniously shows how these separate transition matrices can be combined in a single matrix, and the analysis carried out simply and without further reference to duration.

Thus what follows has a special significance for IIASA's population program, in that it combines lines of thought that go back to the multi-state model introduced by Rogers, and on which many IIASA papers were based in the period 1975-83, and the work on heterogeneity of Vaupel and Yashin, that has been central to IIASA's program in more recent years.

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Acknowledgement

The author has benefited from numerous discussions with Wolfgang Lutz, and from useful comments by Nico Keilman and Andrei Rogers on an earlier draft. The U.S. National Institute of Child Health and Human Development made available the June 1985 CPS data, and Laszlo Zeold and Margaret Kerr helped prepare the data for analysis.

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INTRODUCTION

In recent years techniques for constructing multistate increment-decrement life tables have been extensively developed, and have been fruitfully applied to such demographic phenomena as migration (Rogers and Willekens, 1986), fertility (Suchindran and Koo, 1980) and marital-status dynamics (Espenshade, 1983; 1986). The approach allows one to present a useful tabular summary of a complex demographic process, one in which individual units may make a number of transitions among pairs of discrete states or statuses. One restrictive feature of the approach is the Markov assumption—namely that age-specific transition intensities depend only on the status currently occupied.

Users of the technique have recognized that the Markov assumption is overly restrictive (see, for example, Wijewickrema and Alli, 1984; Espenshade, 1986), yet to date there appears to have been no development of a method of life table construction incorporating such "duration dependence".[1]

This paper describes a new way to formulate a multistate life table such that transition intensities vary by both age and duration of time in status. The solution proposed here is motivated by a desire to utilize insofar as possible the mathematics that have been developed for the usual multistate life table case—that is, the case in which duration-dependence does not appear. It turns out to be rather easy to incorporate the generalization, provided that we are willing to introduce duration in a very specific way: in particular, we use age-specific rates that vary by "duration-category at last birthday". The term "duration-category" will be explained below. There are undoubtedly other possible solutions to the problem of constructing a duration-dependent life table; the virtues of the approach

[1]There has, however, developed a discrete-time semi-Markov approach based upon renewal equations, which like the present approach can handle duration-dependent transitions. See Littman and Mode (1977) and Mode (1980); for some applications see Hennessey (1980) and Rajulton (1985).

described here is that it uses existing mathematical tools, and requires inversion of matrices no larger than those encountered in the standard multistate case.

The method generalizes the linear model whose early development is due primarily to Rogers (1975), and in particular relies on what has been termed the "linear integration hypothesis" (Hoem and Funck Jensen, 1982). The linear model has been criticised, and shown to produce nonsensical results in some cases (see Hoem and Funck Jensen, 1982; Nour and Suchindran, 1983; and Keilman and Gill, 1986). Nonetheless the linear model enjoys widespread use and evidently performs satisfactorily in most applications, and so it seems reasonable to adopt it as the starting point for a more general model. But the shortcomings of the basic model no doubt pertain to the more general one described here, as well.

Before proceeding, it is worthwhile to consider why one might want to incorporate duration-dependence into a life table in the first place. A simple answer to this question is that a life table which incorporates the duration dimension is considerably more informative than is the usual life table. We can, for example, calculate the share of all person-years lived in a given status that are lived prior to the first anniversary, between the first and second anniversary, and so forth. In applications such as tables of working life, this additional information has particular significance: workers typically gain in firm-specific human capital early in their tenure, so the degree of concentration of work experience at low tenures can be used as an index of resources devoted to training costs. Also, since current age plus current duration are sufficient to determine age of most recent transition, a life table which disaggregates survivorship at each age by duration as well as status can be used to study intercohort differences in survivorship.

Another, and a more compelling, reason is that a duration-dependent life table may produce different results than the usual approach, especially with respect to the status distribution at a given age (the $l(x)$ figures). The difference can arise when period data are used, for a population in which the current duration-in-status distribution departs from that for the life table (stable) population. Inferences from a duration-dependent life table may thus prove to be more accurate, and even small differences can prove to be important in practical applications.

The following section describes the formulation of a duration-dependent multistate life table, and provides formulas for the calculation of transition probabilities. Most of the discussion is devoted to the derivation of survivorship figures for a population at a sequence of exact ages 0,1,.... This is followed by a discussion of several summary indices derived from the survivorship figures. We then consider briefly the data requirements of the proposed model, and conclude with an illustration: a simple marital status life table based upon recent US data.

THE MODEL

Preliminaries. We use as a starting point a standard formulation of the multistate life table (MSLT), employing essentially Keyfitz's (1979) notation. Thus, let $\boldsymbol{M}(x)$ be the matrix of transition intensities between pairs of states in the set $1,\ldots,n$, between ages x and $x+1$. The contents of $\boldsymbol{M}(x)$ are depicted in (1); elements $m_{ij}(x)$ correspond to transition rates into state i from state j between exact ages x and $x+1$, while elements $m_{\delta j}(x)$ are death rates in state j between these ages.

$$\boldsymbol{M}(x) = \begin{bmatrix} m_{\delta 1}(x) + \sum_{i \neq 1} m_{i1}(x) & -m_{12}(x) & \cdots & -m_{1n}(x) \\ -m_{21}(x) & m_{\delta 2} + \sum_{i \neq 2} m_{i2}(x) & \cdots & -m_{2n}(x) \\ \vdots & \vdots & & \vdots \\ -m_{n1}(x) & \cdot & & m_{\delta n}(x) + \sum_{i \neq n} m_{in}(x) \end{bmatrix} . \quad (1)$$

The fundamental result used in the sequel is the following:

$$l(x+1) = (I + \frac{\boldsymbol{M}(x)}{2})^{-1}(I - \frac{\boldsymbol{M}(x)}{2})l(x) \quad (2)$$

where $l(x)$ is an array representing survivorship (numbers, or proportions) in states $1,\ldots,n$ at exact age x. The time unit used in this calculation is a single period. Other life-table functions of interest—such as life expectancies, and so on—can be derived from the $l(x)$ arrays. The derivation of (2) is discussed in several sources, including Rogers and Willekens (1986); Keyfitz (1986); and Willekens et al. (1982).

Incorporating duration-dependence. We now consider an extension of the above formulation to the case in which transition rates vary by duration of exposure to risk as well as by age. We denote the model a "duration-dependent multistate life table" (DDMSLT). That is, we suppose that at each age, x, there is a separate matrix of transition rates of the form found in (1), pertaining to persons in duration categories $d = 0,1,\ldots,x$. A person of age x will be in duration category d if the most recent anniversary in the state currently occupied was the

d-th anniversary. Obviously $d \leq x$ at each age x.

Note that "duration category" has a rather special relationship to "duration" in this formulation. At exact age x, someone in duration category d has been in their current status at least d, but less than $d+1$, time units (years). At age $x+\Delta x$ $(0 < \Delta x < 1)$, this person may have passed the $d+1$th anniversary, depending on the exact timing of the previous transition. Yet we classify individuals only with respect to the duration category occupied on a given birthday.

In view of the way in which age- and duration-dependent transition rates are defined here, the essence of the proposed model is as follows. First, the $l(x)$ matrix of survivorship according to status occupied, is modified to accommodate duration-dependent rates. Then, the rates, suitably arranged, are manipulated using essentially the same mathematics as in the usual MSLT case, yielding $l(x+1)$. Someone in state i, and duration category d, at age x, and who survives to age $x+1$ in state i, has necessarily advanced to duration category $d+1$. Thus, the elements of $l(x+1)$ are relabelled at age $x+1$, to reflect the advancement or "promotion" in duration. This process continues until the terminal age of the life table has been reached.

It should be recognized that in the expanded formulation duration has **not** been incorporated into the state space. If duration were to be incorporated into the state space, we would be required to contend with quantities described as the "rate of movement from state i,d"—with d indexing duration categories—"to state i',d'". Instead, we are concerned here with quantities described as the "rate of movement from state i to i', **given** that duration at last anniversary was d at exact age x". The distinction is rather fine—and the verbal description of the rates used here is somewhat cumbersome—but the formulation adapted here greatly facilitates computation, as shall be seen.

We first develop the approach for a simple case in which flows out of all states are governed by duration-dependent transition rates (or, more simply, all states are "duration-dependent states"), with one set of rates at each duration category up to age w (the maximum attainable birthday). The model requires that at age x, we have a sequence of matrices $M_\Delta(x)$, $M_0(x)$, $M_1(x)$,..., $M_x(x)$, each of which is in the form of (1). The subscripts Δ, 0, 1,... refer to duration categories.[2] As noted before, category d refers to those whose last anniversary in the current status

[2] Individual elements of $M_d(x)$ now bear three subscripts: $m_{ijd}(x)$ is the rate of $j \rightarrow i$ movement between exact ages x and $x+1$, given that the duration category in state j at age x is d.

was the dth anniversary.

Duration category Δ plays a special role in the model. This is the category **entered** if a transition occurs between ages x and $x+1$. In words, an off-diagonal element of $M_\Delta(x)$ is the "rate of j-to-i movement between ages x and $x+1$, **given** that a k-to-j move has taken place since exact age x". Someone who has experienced a transition into status i between ages x and $x+1$ will be in duration category 0 at exact age $x+1$. Therefore, what we are calling duration category Δ might as easily (but not as tidily) be called category "-1".

Calculations for the DDMSLT are greatly facilitated if the rates are arranged in the following way. First, let $DM_d(x)$ denote the matrix of $M_d(x)$, with its off-diagonal elements replaced by zeros. Second, let $CM_d(x)$ be the matrix $M_d(x)$ with its main diagonal elements replaced by zeros. Then $M_d(x) = CM_d(x) + DM_d(x)$. All the matrices $DM_d(x)$ and $CM_d(x)$, $d = \Delta, 0, 1, \ldots, x$, are, of course, n-by-n matrices. The full matrix of age- and duration-dependent rates, analogous to (1) but now denoted $M^*(x)$, is defined as follows:

$$M^*(x) = \begin{bmatrix} M_\Delta(x) & CM_0(x) & CM_1(x) & CM_2(x) & \cdots & CM_x(x) \\ 0 & DM_0(x) & 0 & 0 & \cdots & 0 \\ 0 & 0 & DM_1(x) & 0 & \cdots & 0 \\ 0 & 0 & 0 & DM_2(x) & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & DM_x(x) \end{bmatrix} \qquad (3)$$

$M^*(x)$ is thus an $(nx+2n)$-by-$(nx+2n)$ matrix of rates. Moreover, its diagonal is the duration-dependent counterpart to the corresponding diagonal in the usual MSLT case: for a given duration category, d, the diagonal element of $M^*(x)$ equals $m_{\delta jd}(x) + \sum_{i \neq j} m_{ijd}(x)$. The matrices $M_0(x), M_1(x), \ldots$, have thus been pulled apart, with their off-diagonal elements appearing in a band across the top of (3), while their diagonal elements appear as the diagonal of (3).

In order to conform to $M^*(x)$, the $l(x)$ array of survivorship figures must be a column vector of the form

$$[0 \cdots 0 \mid l_{10}(x) \cdots l_{n0}(x) \mid \cdots \mid l_{1x}(x) \cdots l_{nx}(x)]' , \qquad (4)$$

the symbol "|" indicating grouping by duration category; an element $l_{id}(x)$ represents the number (or proportion) of the radix population in state i, duration category d, at exact age x.[3] The first n elements of $l(x)$ are zeros, corresponding to duration category Δ. It is impossible to occupy category Δ at exact age x; rather, as noted above, transitions occurring between ages x and $x+1$ are tantamount to moves **into** category Δ.

Now, let $l^*(x)$ be the result of the following operation, analogous to that given in (2):

$$l^*(x) = (I + \frac{M^*(x)}{2})^{-1}(I - \frac{M^*(x)}{2})l(x) \quad . \qquad (5)$$

Since $M^*(x)$ is a matrix with $(nx+2n)$ rows and columns, the computational requirements necessary to calculate $l^*(x)$ may appear formidable. However, this turns out not to be so. To simplify notation, let $Y = (I + \frac{M^*(x)}{2})$ and $Z = (I - \frac{M^*(x)}{2})$. The patterns of zero and nonzero elements in both Y and Z are the same as in $M^*(x)$. Then Y can be written in partitioned form as

$$\begin{bmatrix} R & S \\ 0 & T \end{bmatrix}$$

where $R = M_\Delta(x)$; $S = [CM_0(x)\ CM_1(x)\ \cdots\ CM_x(x)]$; and

$$T = \begin{bmatrix} DM_0(x) & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & DM_x(x) \end{bmatrix} .$$

S is thus n-by-$(nx+n)$, while T is a diagonal $(nx+n)$-by-$(nx+n)$ matrix. Z can be similarly partitioned, and written as

$$\begin{bmatrix} U & V \\ 0 & W \end{bmatrix}$$

It can easily be verified that

$$Y^{-1} = \begin{bmatrix} R^{-1} & -R^{-1}ST^{-1} \\ 0 & T^{-1} \end{bmatrix} . \qquad (6)$$

[3] Here we consider the simple situation of a single radix state, in which case $l(x)$ is a vector. More generally, $l(x)$ is a matrix with as many columns as there are initial statuses.

Since T is diagonal, its inverse is trivially easy to calculate. Thus, only for R, an n-by-n matrix, is a matrix-inversion algorithm required. This matrix inversion problem is of the same computational order as in the MSLT with the same state-space, but without duration-dependence.

Using (6) we can rewrite (5) as

$$l^*(x) = Y^{-1}Z\, l(x) \tag{7}$$

or

$$l^*(x) = \begin{bmatrix} R^{-1}U & R^{-1}(V - ST^{-1}W) \\ 0 & T^{-1}W \end{bmatrix} l(x) \quad . \tag{8}$$

The data-storage requirements associated with (8) are admittedly much greater than in the usual MSLT case. However, since T and W are both diagonal, they can be stored as vectors [of length $nx+n$]; the diagonal matrix product $T^{-1}W$ can be obtained directly and similarly stored as a vector.

The vector $l^*(x)$ contains, in its first n elements, the array of survivors to age $x+1$ who made transitions between ages x and $x+1$. These individuals necessarily are in duration category 0 at age $x+1$. The next n elements contain the array of survivorship to age $x+1$ of those in duration category 0 at age x; these individuals have **not** made a transition between ages x and $x+1$, and thus have advanced to duration category 1; and so on. Thus, $l^*(x)$ is of the form

$$[l_{10}(x+1) \cdots l_{n0}(x+1) \mid \cdots \mid l_{1,x+1}(x+1) \cdots l_{n,n+1}(x+1)]' \ .$$

The vector $l^*(x)$ must be manipulated into the form given by (4), in order that the computation can proceed to the next age. In other words, the first n elements of $l(x+1)$ must be zeros, the next n must denote survivor in duration-category zero, and so on. To do this, we merely augment $l^*(x)$ above with n zeros, and relabel the augmented vector $l(x+1)$. The manipulation required can be expressed in matrix form as

$$l(x+1) = A_x l^*(x) \quad , \tag{10}$$

where A is an $(nx+3n)$-by-$(nx+2n)$ matrix of the form

$$A_x = \begin{bmatrix} 0_{n,n} & 0_{n,n} & \cdots & 0_{n,n} \\ I_{n,n} & 0_{n,n} & \cdots & 0_{n,n} \\ 0_{n,n} & I_{n,n} & \cdots & 0_{n,n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0_{n,n} & 0_{n,n} & 0_{n,n} & I_{n,n} \end{bmatrix} .$$

Combining (5) and (10), we can represent the basic DDMSLT calculation as

$$l(x+1) = A_x(I + \frac{M^*(x)}{2})^{-1}(I - \frac{M^*(x)}{2})l(x) \quad . \tag{11}$$

Including states whose rates are not duration-dependent. The previous section presented the essential mathematics of the DDMSLT, but only for the case in which all transition rates are duration-dependent. However, such a simple case is often inappropriate in practice. For example, in a marital status life table there can be no independent age and duration-dependence in rates of entry into first union; similarly with first employment in a working-life table. In both examples, the analyst would typically study a synthetic cohort whose life begins in a purely age-dependent state (e.g. "never married" or "never worked").

In contrast, if the states are geographic regions, and the transitions of interest are migration patterns, it is possible to be born into a duration-dependent state. Here, though, another special point must be recognized. If it is possible to be born into a duration-dependent state, then at exact age x one should expect to see individuals in duration category $0,1,\ldots,x$. Those in category d entered their current status between their $x-d-1$th and $x-d$th birthdays. In general we anticipate that the exact durations of those in duration category d are distributed (approximately evenly) on the interval $[d,d+1)$. Yet those in duration category x have in fact never moved, and their exact duration in status is also x, the same as their exact age. This distinction must be born in mind, especially when preparing the input data for a DDMSLT model.

Only a slight modification of the apparatus described above is necessary, if some states are not duration-dependent. Suppose that n_1 states are not duration-dependent, while n_2 are duration-dependent. We simply arrange the $l(x)$ arrays, and the $M^*(x)$ matrices, such that the non-duration-dependent states appear first. Now, the $M_\Delta(x)$ submatrix has the structure

$$\begin{bmatrix} M_\Delta^{(11)}(x) & M_\Delta^{(12)}(x) \\ M_\Delta^{(21)}(x) & M_\Delta^{(22)}(x) \end{bmatrix}$$

where $M_\Delta^{(11)}(x)$—which is n_1-by-n_1—represents flows within the set of non-duration-dependent states, $M_\Delta^{(12)}(x)$—which is n_1-by-n_1—represents flows from duration-dependent to non-duration-dependent states, and so on. However, since the first n_1 states do not depend on duration, there are no corresponding submatrices $M_0^{(11)}(x), \cdots, M_x^{(11)}(x)$ or $M_0^{(21)}(x), \cdots, M_x^{(21)}(x)$. Thus, $M^*(x)$, the full matrix of rates used in the DDMSLT, changes to accommodate the set of non-duration-dependent rates in only the following ways: across the top, we find the sequence $M_{\Delta}^{(11)}(x)$, $M_{\Delta}^{(12)}(x)$, $M_{0}^{(12)}(x)$, $\cdots$, $M_{x}^{(12)}(x)$; down the left side, we find (again) $M_{\Delta}^{(11)}(x)$, then $M_{\Delta}^{(21)}(x)$, and then zeros. With the rates arranged this way, the computational approach described above still applies.

Open-ended duration categories. In some applications duration of exposure to risk will be categorized in the restricted set 0,1,...,u , where u is an open-ended uppermost category. It may be, for example, that theory suggests that the heterogeneity within a cohort sorts itself out in the first few years, after which susceptibility to risk is constant. It may also be the case that available data are insufficient to reveal statistically significant differences in rates of movement by duration for large values of duration.

The effect of using an open-ended upper duration category is that those who are in state i, duration category u, at age x, and who do not exit to some other state between ages x and $x+1$, *remain* in duration category u at age $x+1$; there are no further "promotions". This modification obviously matters only at ages $x > u$. For ages $x > u$, the dimensions of $l(x+1)$ and $l(x)$ are the same, containing $nu+n$ elements. In all other respects, the calculations described above are the same, except that the matrix A_x now is an $(nu+n)$-by-$(nu+n)$ matrix of the form

$$A_x = \begin{bmatrix} 0_{n,n} & 0_{n,n} & \cdots & 0_{n,n} & 0_{n,n} \\ I_{n,n} & 0_{n,n} & \cdots & 0_{n,n} & 0_{n,n} \\ 0_{n,n} & I_{n,n} & \cdots & 0_{n,n} & 0_{n,n} \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0_{n,n} & 0_{n,n} & \cdots & I_{n,n} & I_{n,n} \end{bmatrix},$$

for $x > u$. In this form, A_x causes those newly promoted into duration category u to be added to those already in category u.

Transitions which preserve duration. As a final variant form of the model, we consider the existence of moves between states which do not set the duration clock back to zero; i.e. they do not change the status variable for which the duration clock is running. One example of such a situation is a combined marital status and fertility model, in which marital status transitions depend upon marital status duration (as well as age), while parity transitions either depend upon marital status duration (and age) or on age only. In such a model a birth, unaccompanied by a marital status change, leaves marital duration unchanged. As another (and formally equivalent) example, a table of working life might distinguish the employment statuses "never worked", "employed", "unemployed", and "withdrawn from the labor force", with rates of movement between employment and unemployment duration-dependent, while simultaneously keeping track of the number of job changes experienced; here, increments to the count of job shifts function exactly as parity in the marital status/fertility example given above. Still a third example can be constructed using the same four-state employment model, but distinguishing those with a single concurrent employer from those holding a secondary job. Here, transitions between the states "single concurrent employer" and "multiple jobholder" can take place without setting the duration-of-employment clock back to zero. In each of these examples, the state space is in some sense two-dimensional (marital status plus parity; employment status plus cumulative number of employers, and so on).

Examples such as these can be fit into the basic framework described above, and complicate the mathematics only slightly. In particular, the matrix $M^*(x)$ is now block-diagonal instead of diagonal, the size of each block depending on the number of "duration-preserving" states that are included in the model. Otherwise, the matrix [and the computational approach embodied in equation (8)] is essentially the same. A concrete illustration is provided in the appendix.

Summary Indicators Based on the Model

The sequence $l(0)$, $l(1)$,...,$l(x)$,... obtained as described above can serve as the basis for calculating an extensive set of related quantities, including generalizations of the person-years lived and expectancy calculations associated with the usual MSLT.

The array of survivorship figures at each age produced by equation (10) is grouped by duration rather than by state occupied, and hence is somewhat awkward for purposes of displaying and interpreting the results. A more convenient arrangement is one in which survivorship at each age is grouped by current state, and by duration within state, yielding the following matrix $\tilde{l}$:

$$\tilde{l} = \begin{bmatrix} l_{1\Delta}(0) & l_{1\Delta}(1) & \cdots & l_{1\Delta}(w) \\ 0 & l_{10}(1) & \cdots & l_{10}(w) \\ \\ 0 & 0 & \cdots & l_{1w}(w) \\ l_{2\Delta}(0) & l_{2\Delta}(1) & \cdots & l_{2\Delta}(w) \\ 0 & l_{2}^{1}(1) & \cdots & l_{2}^{1}(w) \\ \\ l_{id}(0) & l_{id}(1) & \cdots & l_{id}(w) \\ \\ 0 & 0 & \cdots & l_{nw}(w) \end{bmatrix} \qquad (12)$$

The $l_{i\Delta}(x)$ entries, which are always zero and therefore convey no real information, are nonetheless included in (12) because they facilitate the later calculations.

In (12), age runs from left to right rather than from top to bottom in the usual case. Within each column of $\tilde{l}$ survivorship is shown by duration within states. More generally, there would be such an $\tilde{l}$ matrix for each radix considered.

Calculation of $L(x)$ and related quantities. In the usual MSLT the $L(x)$ array provides the person-years lived between ages x and $x+1$ in each state. In the DDMSLT, this information is disaggregated by duration category, for duration-dependent states. For simplicity consider a single column $\tilde{l}(x)$ selected from $\tilde{l}$. We also make use of the corresponding column $\tilde{l}^*(x)$, consisting of the end-of-year (prior to relabelling) survivorship figures produced by equation (5), but rearranged as in (12). That is, the first row of $\tilde{l}^*$ contains $l_{1\Delta}^*(0),\ldots,l_{1\Delta}^*(x),\ldots$ —or, equivalently, $l_{10}(1),\ldots,l_{10}(x+1),\ldots$ —the array of survivors to age $x+1$ who are, at age $x+1$, in status 1 and duration category zero. Employing the usual linear survivorship assumption we can calculate $L^*(x) = \frac{1}{2}[\tilde{l}(x) + \tilde{l}^*(x)]$. Here, the asterix indicates that $L^*(x)$ is a provisional quantity. Now, the dth entry in $L^*(x)$ gives the number of person-years lived between ages x and $x+1$, by someone in status 1, duration category d, at exact age x. The $n+d$th entry gives the corresponding figure for someone in status 2, duration category d; and so on.

If someone is in state i, duration category d, at age x, and survives to age $x+1$ and remains in state i then part of that person-year of experience must be allocated to category d, and part to category $d+1$. In keeping with the linear survivorship assumption, we allocate one-half a person-year's experience to each category. However, rows 1, $n+1$, $2n+1, \ldots,$ of $L^*(x)$—which equal, respectively, $\frac{1}{2}[l_{1\Delta}(x) + l_{10}(x+1)]$, $\frac{1}{2}[l_{2\Delta}(x) + l_{20}(x+1)], \ldots$ —consist of experience lived exclusively in duration category zero. Recall that $l_{i\Delta}(x) \equiv 0$ —one cannot occupy duration category Δ on one's birthday—while $l_{i0}(x+1)$ contains the new arrivals in status i as of the $x+1$st birthday. Thus, these rows of $L^*(x)$ must be allocated exclusively to duration category zero.

Let $L(x)$ be the column vector $[L_{10}(x), L_{11}(x), \ldots, L_{id}(x), \ldots, L_{nx}(x)]'$, where $L_{id}(x)$ represents the number of person-years lived in state i, duration category d, between ages x and $x+1$. The reasoning of the preceding paragraphs suggests that we obtain $L(x)$ as follows:

$$L_{i0}(x) = \frac{1}{2}[l_{i\Delta}(x) + l_{i0}(x+1)] + \frac{1}{2}\{\frac{1}{2}[l_{i0}(x) + l_{i1}(x+1)]\} \quad ,$$

and

$$L_{id}(x) = \frac{1}{2}\{\frac{1}{2}[l_{i,d-1}(x) + l_{id}(x+1)]\} + \{\frac{1}{2}\{\frac{1}{2}[l_{id}(x) + l_{i,d+1}(x+1)]\} \quad , \quad (13)$$

for $d > 0$. A compact matrix expression for (13) is

$$L(x) = B[\tilde{l}(x) + \tilde{l}^*(x)] \quad ,$$

where B is a vertical concatenation of n repetitions of the w-by-w matrix

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{4} & & & & & \\ & \frac{1}{4} & \frac{1}{4} & & & & \\ & & \frac{1}{4} & \frac{1}{4} & & & \\ & & & \cdot & \cdot & & \\ & & & & \cdot & \cdot & \\ & & & & & \frac{1}{4} & \frac{1}{4} \\ & & & & & & \frac{1}{4} \end{bmatrix}$$

A matrix L containing, in succession, the column vectors $L(0), L(1), \ldots, L(w-1)$ would display the distribution of the life table population's experience by state and duration category, year by year from birth to the terminal age considered, from the perspective of a specified initial status. As in the MSLT, we can go on to calculate $T_y(x)$, the total number of person years lived in each status/duration category from exact ages y to x, using

$$T_y(x) = \sum_{a=y}^{x-1} L(a) \quad .$$

Of particular interest is $T_0(w)$, the total lifetime person-years of experience, or the expectation of life at birth, also denoted E_0. Again, all those quantities implicitly condition on a single specified initial status.

The lifetime experience of the population in state i is given by

$$E_{0i} = \sum_x \sum_d L_{id}(x) \quad ,$$

while the proportion of this experience that is lived in between the dth and d+1th anniversary is

$$\pi_{0d}(i) = \frac{\sum_x L_{id}(x)}{E_{0i}} \quad . \tag{14}$$

If i is a state which can be reentered, and from which exit occurs fairly rapidly, then a substantial proportion of the total expectation of time spent in i may precede the first anniversary of entry into i. Using (14), it is possible to trace out a frequency distribution, by duration category, of the life-table population's lifetime experience in a given status.

Additional summary indicators. The information contained in the $\tilde{l}$ survivorship matrix can be used to derive further summary indicators, unique to the DDMSLT. First, we can approximate the average duration of current time-in-status (or, in renewal-theoretic language, backwards recurrence times) at each age. At age x, those in status i, duration category d, have on average been in state i for approximately $d+\frac{1}{2}$ years. Using this assumption, we can compute average backwards recurrence times for status i as

$$\bar{d}_i(x) = \frac{1}{2} l_{i0}(x) + \frac{3}{2} l_{i1}(x) + \cdots + \frac{2x-1}{2} l_{i,x-1}(x) + x l_{ix}(x) \quad ,$$

the last term reflecting the fact that those in duration category x at age x must have been in status i continuously since birth.

It is also possible to compare the survivorship of successive age cohorts of entrants into a given status, for example those entering status i at ages a_1 and a_2, by simply reading along the appropriate diagonals of $\tilde{l}$. In the first instance, the survivorship is given by the sequence $l_{i0}(a_1)$, $l_{i1}(a_1+1)$, $l_{i2}(a_1+2)$, $\cdots$; in the second instance, it is the sequence $l_{i0}(a_2)$, $l_{i1}(a_2+1)$, $l_{i2}(a_2+2),\ldots$; and so on. Finally, the median time to exit (by any reason) can be approximated, simply by finding d' such that $l_{id'}(x+d') \approx \frac{1}{2} l_{i0}(x)$.

Data Requirements

The input data required for a DDMSLT might be provided by several potential sources. Retrospective event-history data, collected in surveys with adequate sample sizes, can often be used to estimate the necessary rates directly. A migration survey might, for instance, collect the dates, origins, and destinations of all moves made during the previous 12 months. In combination with date-of-birth data, age- and duration-category-specific rates, as defined previously, can be tabulated. Information on second (and higher-order) moves within the period can also provide direct estimates of the $M_\Delta(x)$ rates.

A second potential source of the necessary data is a population registration system, such as that found in some European and Scandinavian countries. In such a system the population can be classified according to year of birth (yob) and year of last event (yoe) as of the beginning of a calendar year, and events experienced during the year by each yob-yoe combination can be counted. From these counts, in combination with a suitable assumption about exposure (i.e. the distribution of the midyear population) the necessary rates can be computed directly. Data of this sort, from Finland, has in fact been used to construct a marital status/parity DDMSLT, some results from which are reported in Lutz and Wolf (1987). Similar data are also described and utilized by Keilman and Gill (1986), who also provide a more elegant approach to calculating the rates.

A third potential source of data is a vital-event registration system such as that found in the United States. For example, the Standard Certificate of Divorce, Dissolution of Marriage or Annulment used in the U.S. Divorce Registration Area provides for the registration of divorces according to each former spouse's year of birth, and the year of marriage (although the published divorce data are not tabulated according to both time concepts simultaneously).

Although Vital-Event records might serve as a source of occurrence data for rates in the form required by the DDMSLT, the analyst would still face the problem of assembling the requisite exposure data. This is complicated by the presence of a second time dimension—duration, in addition to age—to be allocated to the appropriate units. If, for instance, a midyear survey were taken in which age and duration-category of current status were ascertained, it would be necessary to allocate the population in a given age/duration-category combination to four different yob/yoe combinations. Conversely, it can easily be shown that a given yob/yoe "cohort" passes through four different age (at last birthday)/duration (at last anniversary) combination during a calendar year.

The rates used in the DDMSLT can be viewed as a finite-valued approximation to intensities defined on a continuous set of ages and durations, and denoted as $m_{ij}(a,d)$. The approximation embodies a simplifying assumption, namely that the rates are constant over subregions of a-d space with unit area. The subregions happen to be parallelograms. A more natural simplifying assumption, perhaps, is that the rates are constant over unit subregions defined by age-at-last-birthday, duration-at-last-anniversary integers, that is unit squares. Let us denote rates of the latter form as m_{ijad}, the use of subscripts reflecting the integer nature of the age and duration arguments. Since in some applications rates defined on unit squares may be available, or more easily calculated, it is worth considering how to translate from them to the rates we have denoted $m_{ijd}(a)$—that is, rates in the form required for the DDMSLT.

The translation is rather straightforward. First, since someone who enters state i between ages a and $a+1$ necessarily has a (continuous) duration-in-status less than 1, the following equation holds:

$$m_{ij\Delta}(a) = m_{ija0} \quad .$$

Now, at exact age a, people in duration category d (in the sense used in this paper) can be assumed to be distributed more or less uniformly on the $[d,d+1)$ interval of continuous duration-in-status. In the coming year, about half their exposure will thus be lived beyond the $d+1$th anniversary. Thus, for $d > 0$, we can write

$$m_{ijd}(a) = \frac{1}{2} m_{ijad} + \frac{1}{2} m_{ija,d+1} \quad . \tag{15}$$

If an open-ended duration interval is used in the DDMSLT, (15) holds only for $d > u$; for $d \geq u$ the two alternative approximations are again equal.

An Example

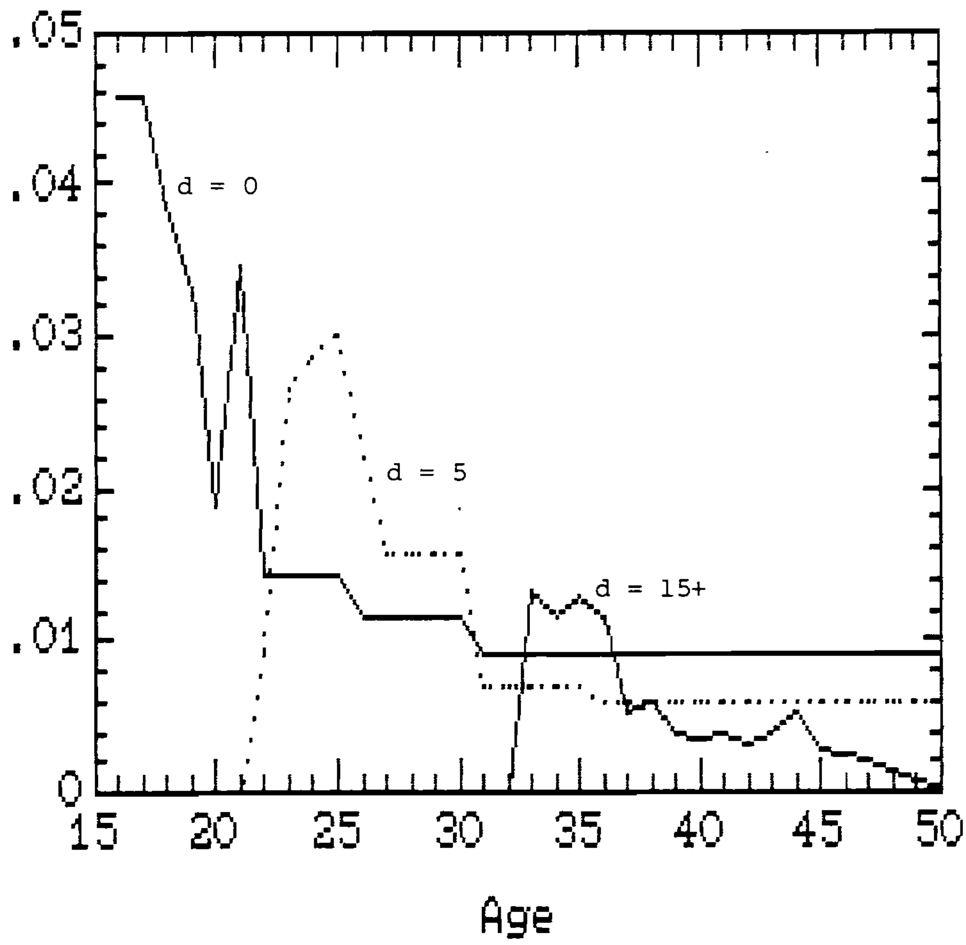
Several of the features of the DDMSLT are illustrated here, using as an example an abbreviated and greatly simplified marital status model, employing data for U.S. females for the 1980-84 period. Only four statuses—single (S), first-married (M), divorced (D), and widowed (W)—are considered. Divorce and widowhood are treated as absorbing states. Of the three possible marital-status transitions, only that from married to divorced is treated as duration-dependent. Only experiences from age 15 to 50 are treated.

For two of the transitions—single to married, and married to divorced—rates were calculated from responses given to the marriage and fertility history questionnaire appended to the June 1985 Current Population Survey (CPS). True occurrence-exposure rates were computed, with exposure, in integer-valued age and duration categories, measured in person-months. For combinations of age and duration intervals with few occurrences and/or little exposure, aggregation over age and duration was used to impose a modest amount of regularity on the data. The DDMSLT calculations were still performed for single-year-of-age/single-duration-category combinations, but with rates treated as constant for some groups of age/duration categories; for example, a single rate was calculated for divorce rates among women aged 30-49, in years 0-4 of marriage (the most extreme case of grouping used). Finally, divorce rates at a given age were treated as constant at durations 15 years and over.

Selected duration-dependent divorce rates used in the example are plotted in Figure 1. The lines show the rates for duration categories 0, 5, and 15+. The three lines plotted make clear the substantial variability found in age-specific divorce rates, according to the duration of marriage. The grouping scheme described above is also reflected in Figure 1: the duration-zero rates, for example, behave as a step-function after age 22. It should be noted that our purpose here is not to interpret or explain such differences--as, for example, period effects, or age-of-marriage effects, or pure duration effects (for efforts in this direction see, for example, Thornton and Rodgers, 1987)—but merely to recognize and take account of the differences, whatever their origin.

For purposes of comparison, divorce rates depending on age only, calculated from the same CPS occurrence-counts and exposure data, are plotted in Figure 2. No grouping or smoothing was imposed on these data. The divorce rates (which pertain to first marriages only, for the period 1980-84) seems somewhat low compared to the U.S. Vital Statistics data for 1982 (which pertain to all marriages);

Figure 1. Selected duration-dependent divorce rates, by age: U.S. females, 1980-84.

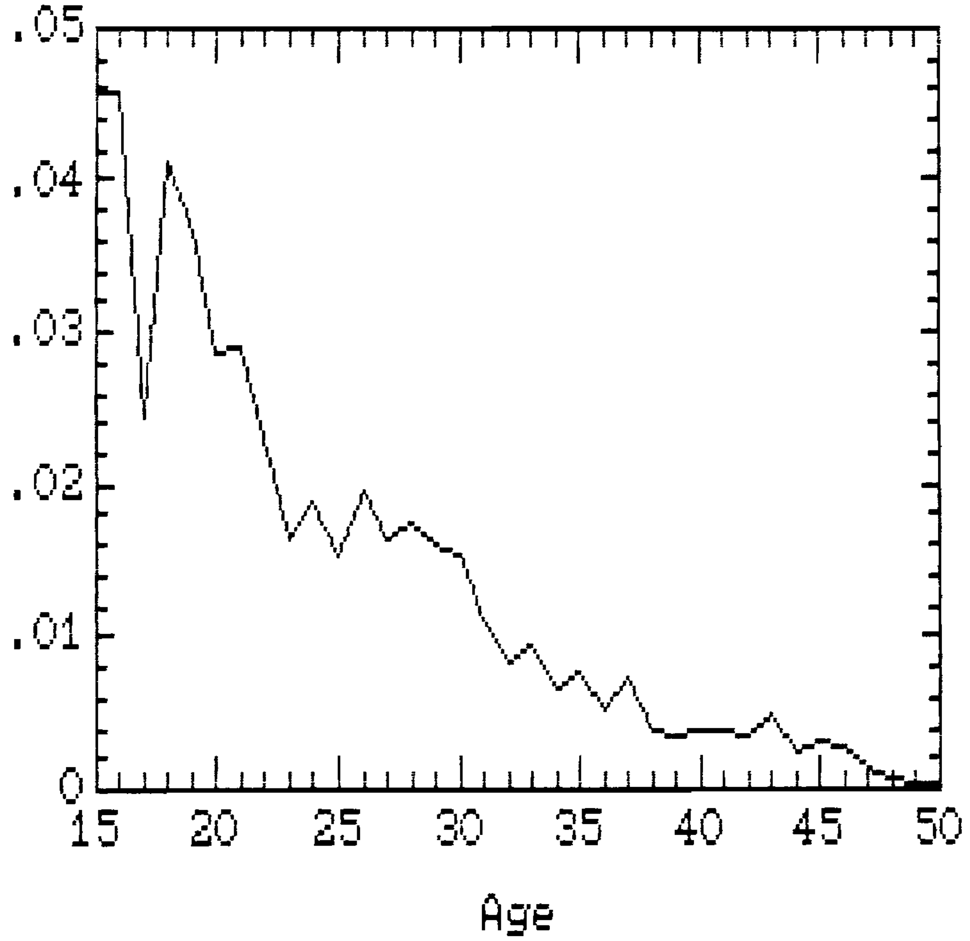


underreporting of divorces has been noticed in previous administrations of the CPS Marital and Fertility History Supplement (see Espenshade and Wolf, 1985), and may be operating here as well.

The illustrative model is further specified by the use of 1982 death rates for all females (treated as constant over marital statuses and durations) and widowhood rates--using the 1982 death rate of men two years older than the married woman's current age--from U.S. Vital Statistics sources.

In Table 1 are shown the columns $\tilde{l}(20)$, $\tilde{l}(25)$,...,$\tilde{l}(50)$ produced by the input rates just described. The radix for this $\tilde{l}$ matrix is 100000 single women aged exactly 15 years old. The status "married,d" means "married, in duration category d". At age 20, married women are found only in duration categories 0,...,4; at each successive 5-year observation period they are arranged over 5 additional duration categories. Irregularities in the duration distribution, at early durations, be-

Figure 2. Age-dependent divorce rates: U.S. females, 1980-84.



gin to appear in the $\tilde{l}(40)$ column, a consequence of the unsmoothed age-at-marriage rates used in the calculation of the table.

The final column of Table 1 gives the status-specific expectation of life (times 100000) from age 15 to 50; thus 12.70 years (per woman) are expected to be lived in the status single, 0.59 years are expected to be lived in the status widowed, and 3.29 years are expected to be lived divorced (recall that widowhood and divorce are treated as absorbing states). Of the total expectation of life in the married state (17.96 years), 0.85 years—4.7 percent—is lived in duration category zero, 0.83 years—4.6 percent—is lived in duration category one, and so on, with a declining percentage lived in each successive duration category. This frequency pattern reflects the combined effects of the pattern of age at first marriage and the differential divorce risks by marital duration. In a true increment-decrement table, with re-entry into the married state a possibility, the frequency pattern of marital-duration experience would also reflect the age pattern of higher-order

Table 1. Selected life table functions, duration-dependent marital status life table for United States.

Status	l(20)	l(25)	l(30)	l(35)	l(40)	l(45)	l(50)	Total Person-years Lived
Single	79286.59	43330.30	25256.08	18879.41	16003.92	14027.55	13362.24	1270257.39
Widowed	51.10	349.05	864.99	1531.07	2443.98	3856.79	6053.25	59137.61
Divorced	1170.84	4992.17	9539.95	12533.12	14326.54	15500.61	15809.60	328732.92
Married,0	7113.67	5439.11	2175.42	678.52	250.10	275.69	83.71	84535.62
Married,1	6398.04	7339.88	2565.36	960.14	558.65	406.28	175.34	83156.65
Married,2	3807.69	7401.68	4077.43	1258.60	576.24	255.10	54.88	81326.33
Married,3	1435.98	7456.43	3993.70	1491.42	582.71	241.74	74.95	79641.77
Married,4	505.86	6653.47	4463.62	1690.11	709.88	588.04	55.06	78171.06
Married,5	0.00	6210.51	5065.56	2052.27	636.99	232.11	250.77	76736.98
Married,6	0.00	5423.39	6805.43	2424.72	904.15	520.08	370.69	75158.45
Married,7	0.00	3191.27	6832.24	3861.20	1188.88	538.12	233.48	73628.40
Married,8	0.00	1230.02	6852.22	3789.07	1413.17	545.84	221.93	72263.56
Married,9	0.00	479.01	6087.16	4242.94	1606.38	667.03	541.53	70837.70
Married,10	0.00	0.00	5564.55	4814.46	1954.12	600.17	214.34	69508.62
Married,11	0.00	0.00	4697.94	6454.07	2309.94	853.89	481.38	68282.00
Married,12	0.00	0.00	2827.53	6465.46	3680.31	1125.42	499.24	66965.24
Married,13	0.00	0.00	1080.92	6470.31	3613.42	1340.88	507.60	65679.92
Married,14	0.00	0.00	436.24	5735.44	4048.33	1527.80	621.75	64368.92
Married,15	0.00	0.00	0.00	5168.18	4599.22	1861.55	561.01	63070.28
Married,16	0.00	0.00	0.00	4327.49	6176.06	2199.85	800.68	61723.82
Married,17	0.00	0.00	0.00	2592.88	6189.99	3501.64	1058.07	60166.53
Married,18	0.00	0.00	0.00	983.42	6193.59	3438.14	1263.00	58411.03
Married,19	0.00	0.00	0.00	405.82	5474.21	3852.41	1441.00	56486.87
Married,20	0.00	0.00	0.00	0.00	4918.03	4376.24	1756.90	54330.62
Married,21	0.00	0.00	0.00	0.00	4118.03	5876.63	2076.17	51870.40
Married,22	0.00	0.00	0.00	0.00	2467.38	5889.89	3304.79	48654.17
Married,23	0.00	0.00	0.00	0.00	935.82	5893.31	3244.85	44878.39
Married,24	0.00	0.00	0.00	0.00	386.18	5208.81	3635.83	40966.53
Married,25	0.00	0.00	0.00	0.00	0.00	4679.59	4130.22	36648.03
Married,26	0.00	0.00	0.00	0.00	0.00	3918.38	5546.25	31422.49
Married,27	0.00	0.00	0.00	0.00	0.00	2347.76	5558.77	25542.22
Married,28	0.00	0.00	0.00	0.00	0.00	890.45	5562.00	19718.02
Married,29	0.00	0.00	0.00	0.00	0.00	367.46	4915.98	14277.96
Married,30	0.00	0.00	0.00	0.00	0.00	0.00	4416.51	9469.60
Married,31	0.00	0.00	0.00	0.00	0.00	0.00	3698.10	5323.31
Married,32	0.00	0.00	0.00	0.00	0.00	0.00	2215.77	2320.37
Married,33	0.00	0.00	0.00	0.00	0.00	0.00	840.39	773.69

marriages.

For purposes of comparison, an ordinary MSLT for the same marital status model, differing only in the use of purely age-dependent divorce rates (illustrated in Figure 2). At each age, the two tables are identical with respect to the status never-married. And, for the first several ages, the two tables are identical, or nearly so, in all other respects as well. Thereafter, the proportions married and divorced differ, as a consequence of controlling for duration of marriage. The proportion married is higher at all ages in the DDMSLT than in the MSLT (with the exceptions of exact ages 19 and 21)—as much as 1.5 percent higher. Greater relative differences are found for the proportion divorced; in the DDMSLT the proportion divorced is as much as 9.2 percent lower (at exact age 32) than the MSLT. Differences this large can, of course, be critical in some projection applications, suggesting the importance of accounting for duration effects when the requisite data can be assembled.

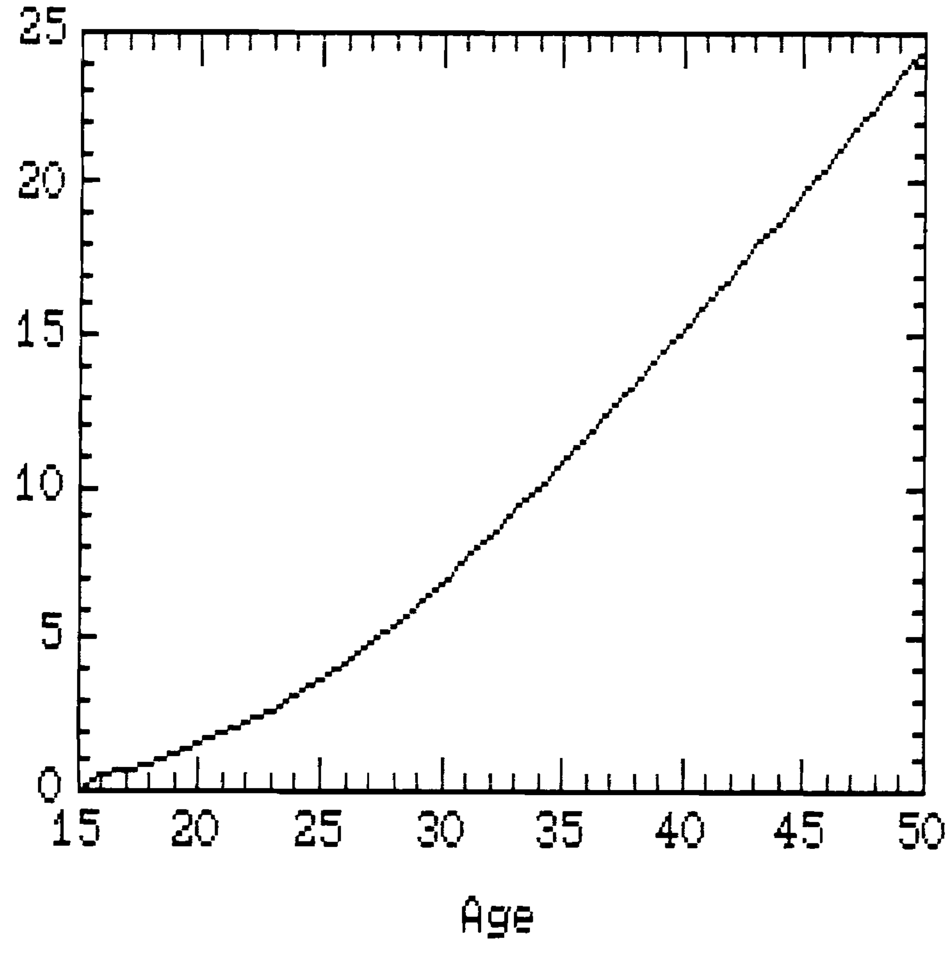
Table 2. Selected transition probabilities (times 100) from marital status DDMSLT; various initial statuses.

		Initial status							
		Age 15	Age 25			Age 35			
		S	S	M,0	M,5	S	M,0	M,5	M,10
Subsequent status									
Age 40:	S	16.0	36.9	—	—	84.8	—	—	—
	D	14.3	4.9	11.2	16.7	0.4	4.2	2.9	2.6
Age 45:	S	14.0	32.4	—	—	74.3	—	—	—
	D	15.5	6.1	12.7	18.0	1.1	6.8	4.5	4.3
Age 50:	S	13.4	30.8	—	—	70.8	—	—	—
	D	15.8	6.7	13.0	18.3	1.8	8.1	5.0	4.8

Selected transition probabilities from the DDMSLT are displayed in Table 2. With the exception of the S → S probabilities, all depend in some way on the presence of duration-dependent rates in the analysis. The duration effects are most pronounced for the M → D transitions from age 25: newly-married women at age 25 are much less likely than 25-year old women in their fifth year of marriage, to be divorced at ages 40, 45, and 50. In contrast, newly-married women at age 35 are **more** likely than 35-year old women in their fifth, or tenth, year of marriage, to be divorced at ages 40, 45, or 50.

Finally, we can compute average backwards recurrence times, which in this example is merely the average duration of marriage, for those currently married, at every age. The averages are plotted in Figure 3, which reveals that the average duration of current marriage rises slowly at first, reaching approximately 7 years at age 30, and rising essentially linearly from ages 30 to 50.

Figure 3. Average duration of marriage, for those currently married.



Summary

A method for generalizing the multistate, increment-decrement life table, to include rates which vary by duration of exposure to risk as well as by age, has been proposed. The method builds upon the linear approximation or linear integration hypothesis developed primarily by Rogers and his colleagues. A computationally-efficient arrangement of the necessary rates has been presented, one which requires inversion of matrices no larger than those one would encounter in the corresponding multistate life table **without** duration-dependence.

The proposed method hinges on the use of rates classified according to age, (at last birthday) and duration-category-at-last-birthday, simultaneously. Duration-category, in turn, is simply a classification of the continuous duration-in-status concept according to duration-at-last-anniversary. The essence of the approach is, first, that given the way in which the rates are defined they are piecewise constant (over unit intervals), and, second, that survivorship in a given status/duration-category from one birthday to the next implies advancement or promotion to the next duration category.

Provided that the necessary data can be assembled, the method outlined here yields a considerably richer array of indices of lifetime experience than does the usual life table. This richer array includes an allocation of status-specific life expectancies according to duration category, median waiting times in each status, mean time-in-status (backwards recurrence times) at every age, and the ability to compare the survivorship of different groups according to their ages of entry into a given status.

The technique was illustrated with a simple 4-state marital-status model, only one transition of which (marriage to divorce) was treated as duration-dependent. Even in this simple example, in which a restricted age range was considered, the new method was found to produce results at considerable variance with the conventional approach. At some ages, the proportion divorced was as much as 9 percent lower with the more general model. Given the widespread use of the ordinary multistate life table in a wide range of substantive applications, the method proposed here would seem to be of considerable practical importance as well.

Appendix

The DDMSLT with Duration-Preserving States

Consider the circular pattern of possible flows in an unlabelled set of five states, depicted with x's in Figure A.1. Suppose that 1 → 2 and 5 → 1 moves are duration-dependent, and that all other moves depend on the duration since moving into state 2. In this illustration, a 1 → 2 move is like a change of marital (employment) status, while 2 → 3 and 3 → 4 moves are like parity progressions (job shifts), as discussed in the text.

Figure A.1 Flows in hypothetical state space.

		Origin state				
		1	2	3	4	5
Destination state	1			x	x	x
	2	x				
	3		x			
	4			x		
	5		x	x	x	

In constructing the $M^*(x)$ matrix, we must arrange the rates such that 2 → 3 and 3 → 4 moves preserve the duration of time since arrival in state 2. This requires arranging the rates as in equation (A.1). To simplify, the dependence on x (age) has been dropped from the notation, and the main diagonal entries are represented in shorthand; that is $f_{jd} = m_{j\delta d}(x) + \sum_{i=1}^{5} m_{ijd}(x)$. The only way in which $M^*(x)$ in (A.1) differs from $M^*(x)$ shown in equation (3) of the text is the way in which the duration-specific submatrices $M_0(x),\ldots,M_x(x)$ are "pulled apart" when forming $M^*(x)$. Rates of duration-preserving moves now appear in the diagonal block of $M^*(x)$ rather than in the band across the top. Reverting to the notation for the partitioned $M^*(x)$ matrix, as used in the text, the submatrix T is no longer a simple diagonal matrix, but rather a block-diagonal matrix. Its inverse must be computed block by block; again, however, no matrix larger than 5-by-5 (the number of states) must be inverted. Given the slightly more complex form of T

(and the corresponding increase in computational requirements) all the rest of the procedures laid out in the main text still apply.

$$
M^{*} = \left[\begin{array}{ccccc|c|ccccc|c|ccccc}
f_{1\Delta} & & & & -m_{15\Delta} & & & & & & -m_{15d} & & & & & & -m_{15x} \\
-m_{21\Delta} & f_{2\Delta} & & & & & -m_{21d} & & & & & & -m_{21x} & & & & \\
 & -m_{32\Delta} & f_{3\Delta} & & & \cdots & & & & & & \cdots & & & & & \\
 & & -m_{43\Delta} & f_{4\Delta} & & & & & & & & & & & & & \\
 & -m_{52\Delta} & -m_{53\Delta} & -m_{54\Delta} & f_{5\Delta} & & & -m_{52d} & -m_{53d} & -m_{54d} & & & & -m_{52x} & -m_{53x} & -m_{54x} & \\
\hline
 & & \vdots & & & & & & \vdots & & & & & & \vdots & & \\
\hline
 & & & & & & f_{1d} & & & & & & & & & & \\
 & & & & & & & f_{2d} & & & & & & & & & \\
 & & & & & & & -m_{32d} & f_{3d} & & & & & & & & \\
 & & & & & & & & -m_{43d} & f_{4d} & & & & & & & \\
 & & & & & & & & & & f_{5d} & & & & & & \\
\hline
 & & \vdots & & & & & & \vdots & & & & & & \vdots & & \\
\hline
 & & & & & & & & & & & & f_{1x} & & & & \\
 & & & & & & & & & & & & & f_{2x} & & & \\
 & & & & & & & & & & & & & -m_{32x} & f_{3x} & & \\
 & & & & & & & & & & & & & & -m_{43x} & f_{4x} & \\
 & & & & & & & & & & & & & & & & f_{5x}
\end{array}\right] \quad (A.1)
$$

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