THE $N = 1$ TRIPLET VERTEX OPERATOR SUPERALGEBRAS

DRAŽEN ADAMOVIĆ AND ANTUN MILAS

ABSTRACT. We introduce a new family of C_2-cofinite $N = 1$ vertex operator superalgebras $\mathcal{SW}(m)$, $m \geq 1$, which are natural super analogs of the triplet vertex algebra family $\mathcal{W}(p)$, $p \geq 2$, important in logarithmic conformal field theory. We classify irreducible $\mathcal{SW}(m)$-modules and discuss logarithmic modules. We also compute bosonic and fermionic formulas of irreducible $\mathcal{SW}(m)$ characters. Finally, we contemplate possible connections between the category of $\mathcal{SW}(m)$-modules and the category of modules for the quantum group $U_q^{small}(sl_2)$, $q = e^{\frac{2\pi i}{2m+1}}$, by focusing primarily on properties of characters and the Zhu's algebra $A(\mathcal{SW}(m))$. This paper is a continuation of our paper Adv. Math. 217 (2008), no.6, 2664-2699.

CONTENTS

1. INTRODUCTION

Compared to rational vertex algebras, significantly less is known about the structure of modules for general vertex algebras. Recently, geared up with clues from the physics literature, some breakthrough has been achieved in understanding at least quasi-rational vertex algebras (i.e., C_2-cofinite irrational vertex algebras), and in particular the triplet vertex algebras $\mathcal{W}(p)$, $p \geq 2$ [AM2], [FHST], [CF] (cf. also [Ab] for $p = 2$). But apart from the symplectic fermions $\mathcal{W}(2)$, the description of categories of weak (logarithmic) modules for other triplets $\mathcal{W}(p), p \geq 3$ remains open, even though there is strong evidence for Kazhdan-Lusztig correspondence between the category of logarithmic $\mathcal{W}(p)$-modules and certain categories of modules for quantum groups (for these and related developments we refer the reader to [FHST], and especially [FGST1], [FGST2], [Se], and references therein).

In [AM2] we obtained several useful results about the structure of the category of $\mathcal{W}(p)$-modules by using primarily Zhu's algebra and Miyamoto's pseudocharacters [Miy]. Eventually, we will require more-or-less explicit knowledge of "higher" Zhu's algebras for the triplet. But several obstacles (e.g., explicit realization of certain logarithmic modules) prevents us for taking this theory to the next level. We hope that this approach, in particular, will give an additional evidence for Kazhdan-Lusztig correspondence, because believe that proper understanding of the relationship between quantum groups and triplets.

As with other familiar rational vertex operator algebras (e.g. Virasoro minimal models), one may also wonder if the triplet has (interesting!) $N = k$ super extensions, and whether those exhibit similar properties (e.g., C_2-cofiniteness). In this paper we solve this problem for $k = 1$,

by constructing a family of $N = 1$ vertex operator superalgebras $\mathcal{SW}(m)$, $m \geq 1$, which share many similarities with the triplet family.

In what follows we briefly recall the construction of $\mathcal{SW}(m)$ and present our main results.

Let us recall that the triplet vertex algebra $\mathcal{W}(p)$ [FHST], [AM2] is defined as the kernel of a screening operator acting from V_L to $V_{L-\frac{\alpha}{p}}$ where V_L is the vertex algebra associated to rank one even lattice $\mathbb{Z}\alpha$, $\langle \alpha, \alpha \rangle = 2p$, and $V_{L-\frac{\alpha}{p}}$ is a certain V_L-module. To construct an $N = 1$ super triplet we replace the even lattice with an odd lattice such that $\langle \alpha, \alpha \rangle = 2m + 1$, so that V_L has a natural vertex operator superalgebra structure. Then we tensor V_L with F, the free fermion vertex operator superalgebra (cf. [KWn]), and again, there is a screening operator

$$\tilde{Q} : \ V_L \otimes F \longrightarrow V_{L-\frac{\alpha}{2m+1}} \otimes F.$$

The kernel of this operator, denoted by $\mathcal{SW}(m)$, is what we call $N = 1$ triplet vertex operator superalgebra (or simply, $N = 1$ super triplet). If we restrict the kernel of $\tilde{Q}$ on the charge zero subspace we obtain another vertex operator superalgebra

$$\overline{SM(1)} \subset \mathcal{SW}(m),$$

called $N = 1$ singlet vertex operator superalgebra. Both vertex operator superalgebras contain Neveu-Schwarz vector τ, giving a representation of $\mathfrak{ns}$ Lie superalgebra of central charge $\frac{3}{2} - \frac{12m^2}{2m+1}$. This is precisely the central charge of $(1, 2m + 1)$ Neveu-Schwarz (degenerate) minimal modules. A different $N = 1$ extension of the symplectic fermion $\mathcal{W}$-algebra $\mathcal{W}(2)$ was considered in [MS]

By using the notation used by physicists, our super triplet would be an example of a super $\mathcal{W}$-algebra of type $\mathcal{W}(\frac{3}{2}, 2m + \frac{1}{2}, 2m + \frac{1}{2}, 2m + \frac{1}{2})$. Similarly, the $N = 1$ singlet algebra $\overline{SM(1)}$ is an example of a super $\mathcal{W}$-algebra of type $\mathcal{W}(\frac{3}{2}, 2m + \frac{1}{2})$. We should say that for low m (e.g., $m = 1$) some general properties of $\mathcal{W}$ superalgebras with two generators were also discussed in the physics literature, but mostly by using Jacobi identity and methods of Lie algebras (cf. [BS] and references therein). We should also mention that several general results about $\mathcal{W}$-superalgebras associated to affine superalgebras were recently obtained in [Ar], [KWak] (see also [HK]). However, super singlet and super triplet vertex superalgebras do not appear in these works.

Because of the similarity between $\mathcal{W}(p)$ and $\mathcal{SW}(m)$ many results we obtain here are intimately related to those for the triplet [AM2] (cf. [FGST1], [CF]), but there are some subtle

differences which we address at various stages. However, to keep the paper self-contained at many places we gave proofs that are almost identical to those in [AM2].

Let us first consider the super singlet $\overline{SM(1)}$. This vertex superalgebra is too small to be C_2-cofinite (let alone rational!), which is evident from the following result.

Theorem 1.1. *Assume that $m \geq 1$.*
  (i) *The singlet vertex superalgebra $\overline{SM(1)}$ is a simple $N = 1$ vertex operator superalgebra generated by τ and a primary vector H of conformal weight $2m + \frac{1}{2}$.*
 (ii) *The associative Zhu's algebra $A(\overline{SM(1)})$ is isomorphic to the commutative algebra $\mathbb{C}[x,y]/\langle P(x,y)\rangle$ where $\langle P(x,y)\rangle$ is the ideal in $\mathbb{C}[x,y]$ generated by the polynomial*

$$P(x,y) = y^2 - C_m \prod_{i=0}^{2m} (x - h^{2i+1,1}),,$$

  where C_m is a non-trivial constant and $h^{2i+1,1} = \frac{i(i-2m)}{2(2m+1)}$.

So the structure and representation theory of $\overline{SM(1)}$ is quite similar to that of $\overline{M(1)}$ investigated in [A3] and [AM1]. In particular we can construct interesting logarithmic $\overline{SM(1)}$–modules and *logarithmic intertwining operators* as defined in [HLZ].

Next we study the vertex operator superalgebra $\mathcal{SW}(m)$. The main result on the structure on this vertex superalgebra is

Theorem 1.2. *Assume that $m \geq 1$.*
  (i) *$\mathcal{SW}(m)$ is a simple $N = 1$ vertex operator superalgebra generated by τ and three primary vectors E, F, H of conformal weight $2m + \frac{1}{2}$.*
 (ii) *The vertex operator superalgebra $\mathcal{SW}(m)$ is irrational and C_2-cofinite.*
(iii) *$\mathcal{SW}(m)$ has precisely $2m + 1$ inequivalent irreducible modules.*

Our proof of (iii) imitates the proof of C_2-cofiniteness for the triplet $\mathcal{W}(p)$ [AM2] (for a different proof see [CF]). The rest is done by combining methods of Zhu's associative algebra and our knowledge of irreducible $V_L \otimes F$-modules. In parallel with the triplet vertex algebra, we do not have an explicit description of $A(\mathcal{SW}(m))$, but we believe that the following conjecture should hold true.

Conjecture 1.1. *The Zhu's algebra decomposes as a sum of ideals*

$$A(\mathcal{SW}(m)) = \bigoplus_{i=2m+1}^{3m} \mathbb{M}_{h^{2i+1,1}} \oplus \bigoplus_{i=0}^{m-1} \mathbb{I}_{h^{2i+1,1}} \oplus \mathbb{C}_{h^{2m+1,1}},$$

where $\mathbb{M}_{h^{2i+1,1}} \cong M_2(\mathbb{C})$, $\dim(\mathbb{I}_{h^{2i+1,1}}) = 2$ *and* $\mathbb{C}_{h^{2m+1,1}}$ *is one-dimensional. In particular* $A(\mathcal{SW}(m))$ *is* $6m + 1$-*dimensional.*

In view of the classification result (cf. Theorem 1.1), it is important to compute irreducible characters and study their modular transformation properties. As with the triplet vertex algebra [F1], irreducible $\mathcal{SW}(m)$-characters are often expressible as sums of modular forms of unequal weight. Also, because we are working within vertex operator superalgebras the $SL(2,\mathbb{Z})$ group should be replaced with the θ-group Γ_θ. Then we have

Theorem 1.3. *The* Γ_θ-*closure of the space spanned by irreducible* $\mathcal{SW}(m)$-*characters is* $3m + 1$-*dimensional.*

For a more precise statement see Theorem 12.1. Our result should be compared with [F1], where it was observed that the $SL(2,\mathbb{Z})$ closure of the vector space of $\mathcal{W}(p)$ characters is $3p - 1$ dimensional. Finally, in parallel with [FGK] and [FFT], we also obtain (see Section 13) fermionic formulas for characters of irreducible $\mathcal{SW}(m)$-modules. Our main results indicate that there is an interesting relationship between characters of irreducible $\mathcal{SW}(m)$-modules and irreducible characters of $\mathcal{W}(2m + 1)$-modules. It is not clear if there is a deeper connection between these two $\mathcal{W}$-algebras.

Notice that if Conjecture 1.1 is true, then the center of $A(\mathcal{SW}(m))$ is $3m + 1$-dimensional, which is precisely the dimension of the center of the small quantum group $U_q^{small}(sl_2)$, $q = e^{\frac{2\pi i}{2m+1}}$ [Ker]. It is no accident that this dimension matches the dimension in Theorem 1.3 (similar phenomena occurs for the triplet vertex algebra [FGST1]). Furthermore, both $U_q^{small}(sl_2)$ and $\mathcal{SW}(m)$ have the same number of irreducible modules [Ker] (see also [La]). Thus, motivated by conjectures in [FGST1], we expect the following (rather bold) conjecture to be true.

Conjecture 1.2. *The category of weak* $\mathcal{SW}(m)$-*modules is equivalent to the category of modules for the quantum group* $U_q^{small}(sl_2)$, *where* $q = e^{\frac{2\pi i}{2m+1}}$.

2. PRELIMINARIES

In this section we briefly discuss the definition of vertex operator superalgebras, their modules and intertwining operators as developed in [FFR], [K], [KWn], [Li], [HLZ], [HM], etc. We assume the reader is familiar with basics of vertex algebra theory (cf. [FHL], [FLM], [FB], [K], [LL], etc.).

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be any $\mathbb{Z}_2$–graded vector space. Then any element $u \in V_{\bar{0}}$ (resp. $u \in V_{\bar{1}}$) is said to be even (resp. odd). We define $|u| = \bar{0}$ if u is even and $|u| = \bar{1}$ if u is odd. Elements in $V_{\bar{0}}$ or $V_{\bar{1}}$ are called homogeneous. Whenever $|u|$ is written, it is understood that u is homogeneous.

The notion of vertex operator superalgebra is a natural (and straightforward) generalization of the notion of vertex algebra where the vector space V in the definition is assumed to be $\mathbb{Z}_2$-graded, where the vertex operator map

$$Y(\cdot, z) : V \longrightarrow \operatorname{Hom}(V, V((z))), \;\; Y(u, z) = \sum_{n \in \mathbb{Z}} u_n z^{-n-1}$$

is compatible with the $\mathbb{Z}_2$-grading, and where Jacobi identity for a pair of homogeneous elements is adjusted with an appropriate sign.

A vertex superalgebra V is called a vertex operator superalgebra if there is a special element $\omega \in V_{\bar{0}}$ (called *conformal vector*) whose vertex operator we write in the form

$$Y(\omega, z) = \sum_{n \in \mathbb{Z}} \omega_n z^{-n-1} = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2},$$

such that $L(n)$ close the Virasoro algebra representation on V, and where V is $\frac{1}{2}\mathbb{Z}$-graded (by *weight*), truncated from below, with finite-dimensional vector spaces. Also, the grading is determined with the action of the Virasoro operator $L(0)$. In this paper, we shall assume that

$$V_{\bar{0}} = \coprod_{n \in \mathbb{Z}_{\geq 0}} V(n), \;\; V_{\bar{1}} = \coprod_{n \in \frac{1}{2} + \mathbb{Z}_{\geq 0}} V(n), \;\; \text{where} \;\; V(n) = \{a \in V \mid L(0)a = na\}.$$

For $a \in V(n)$, we shall write $\operatorname{wt}(a) = n$ or $\deg(a) = n$. We shall sometimes refer to vertex operator superalgebra V as a quadruple $(V, Y, \mathbf{1}, \omega)$, where $\mathbf{1}$ is the vacuum vector (as for vertex operator algebras).

We say that the vertex operator superalgebra V is generated by the set $S \subset V$ if

$$V = \operatorname{span}_{\mathbb{C}}\{u^1_{n_1} \cdots u^r_{n_r} \mathbf{1} \mid u^1, \ldots, u^r \in S, \; n_1, \ldots, n_r \in \mathbb{Z}, r \in \mathbb{Z}_{>0}\}.$$

The vertex operator algebra V is said to be *strongly generated* (cf. [K]) by the set R if

$$V = \operatorname{span}_{\mathbb{C}}\{u^1_{n_1} \cdots u^r_{n_r} \mathbf{1} \mid u^1, \ldots, u^r \in R, \; n_i < 0, r \in \mathbb{Z}_{>0}\}.$$

In parallel with vertex algebras we can define the notion of *weak module* for vertex operator superalgebras. Again, the only new requirement is that the vector space M in the definition is $\mathbb{Z}_2$-graded, with grading compatible with respect to the action of V, and where the Jacobi identity is adjusted as in the case of vertex superalgebras. The vertex operator acting on M is usually denoted by Y_M.

A weak V–module (M, Y_M) is called an (ordinary) V-module if M carries an action of the Virasoro algebra via the expansion of $Y_M(\omega, x)$, and in addition M is equipped with a $\mathbb{R}$-grading (or even $\mathbb{C}$-grading) determined by the Virasoro operator $L(0)$. In addition, the grading is truncated from below, with finite dimensional graded subspaces.

As usual, we say that a V-module M is irreducible (or simple) if M has no proper submodules. We say that a vertex operator superalgebra is *rational* if every V-module M is semisimple (i.e., M decomposes as a direct sum of irreducible modules) and if V has only finitely many (inequivalent) irreducible modules.

Definition 2.1. Let V be a vertex operator superalgebra. We say that a weak V-module M is *logarithmic*, if it carries an action of the Virasoro algebra and if it admits decomposition

$$M = \coprod_{r \in \mathbb{C}} M_r,$$

where

$$M_r = \{v \ : \ (L(0) - r)^k v = 0, \text{ for some } k \in \mathbb{N}\}.$$

2.1. **Zhu's algebra $A(V)$.** We define two bilinear maps $* : V \times V \to V$, $\circ : V \times V \to V$ as follows. For homogeneous $a, b \in V$ let

(2.1)
$$a * b \quad = \begin{cases} \operatorname{Res}_x Y(a, x) \dfrac{(1+x)^{\deg(a)}}{x} b & \text{if } a, b \in V_{\bar{0}} \\ 0 & \text{if } a \text{ or } b \in V_{\bar{1}} \end{cases}$$

(2.2)
$$a \circ b \quad = \begin{cases} \operatorname{Res}_x Y(a, x) \dfrac{(1+x)^{\deg(a)}}{x^2} b & \text{if } a \in V_{\bar{0}} \\ \operatorname{Res}_x Y(a, x) \dfrac{(1+x)^{\deg(a) - \frac{1}{2}}}{x} b & \text{if } a \in V_{\bar{1}} \end{cases}$$

Next, we extend $*$ and $\circ$ on $V \otimes V$ linearly, and denote by $O(V) \subset V$ the linear span of elements of the form $a \circ b$, and by $A(V)$ the quotient space $V/O(V)$. The image of $v \in V$, under the natural map $V \mapsto A(V)$ will be denoted by $[v]$. The space $A(V)$ has a unital associative algebra structure, with the product $*$ and $[\mathbf{1}]$ as the unit element. The associative algebra $A(V)$ is called the Zhu's algebra of V.

Assume that $M = \oplus_{n \in \frac{1}{2}\mathbb{Z}_{\geq 0}} M(n)$ is a $\frac{1}{2}\mathbb{Z}_{\geq 0}$–graded V–module. Then the top component $M(0)$ of M is a $A(V)$–module under the action $[a] \mapsto o(a) = a_{\operatorname{wt}(a)-1}$ for homogeneous a in $V_{\bar{0}}$. We shall sometimes write $a(0)$ for $o(a)$. (Note that if $a \in V_{\bar{1}}$, then $[a] = 0$ in $A(V)$. We formally set $o(a) = a(0) = 0$ in this case.)

Moreover, there is one-to-one correspondence between irreducible $A(V)$–modules and irreducible $\frac{1}{2}\mathbb{Z}_{\geq 0}$–graded V–modules (cf. [KWn]).

As usual, for a vertex operator superalgebra V we let

$$C_2(V) = \{a_{-2}b \ : \ a, b \in V\}.$$

Then it is not hard to see that

$$\mathcal{P}(V) = V/C_2(V)$$

has a super Poisson algebra structure with the multiplication

$$\bar{a} \cdot \bar{b} = \overline{a_{-1}b},$$

and the Lie bracket

$$[\bar{a}, \bar{b}] = \overline{a_0 b},$$

where $-$ denotes the natural projection from V to $\mathcal{P}(V)$ (see for instance [Z]). Therefore we have a decomposition $\mathcal{P}(V) = \mathcal{P}(V)_0 \oplus \mathcal{P}(V)_1$ into even and odd subspace, respectively. If $V/C_2(V)$ is finite-dimensional we say that V is C_2-cofinite. Let $a, b \in V$, be $\mathbb{Z}_2$ homogeneous. Then by using super-commutator formulae in vertex operator superalgebras one can easily see that

(2.3) $$\bar{a} \cdot \bar{b} - (-1)^{|a||b|}\bar{b} \cdot \bar{a} = 0 \quad \text{in } V/C_2(V).$$

The following result was proved in [DK], and it is a generalization of Proposition 2.2 in [Ab].

Proposition 2.1. *Let V be strongly generated by the set S. Then we have:*

(1) *$\mathcal{P}(V)$ is generated by the set $\{\bar{a}, a \in S\}$.*

(2) *$A(V)$ is generated by the set $\{[a], a \in S\}$.*

(3) *If V is C_2-cofinite*

$$\dim(\mathcal{P}(V)_0) \geq \dim(A(V)).$$

2.2. **Intertwining operators among vertex operator superalgebra modules.** Intertwining operators for superconformal vertex operator algebras were introduced in [KWn]. Their theory is further developed in [HM] by using both even and odd formal variables. We briefly outline the definition here.

Definition 2.2. *Let V be a vertex operator superalgebra and M_1, M_2 and M_3 a triple of V–module. An intertwining operator $\mathcal{Y}(\cdot, z)$ of type $\binom{M_3}{M_1 \ M_2}$ is a linear map*

$$\mathcal{Y} : M_1 \rightarrow End(M_2, M_3)\{z\},$$

$$w_1 \mapsto \mathcal{Y}(w_1, z) = \sum_{n \in \mathbb{C}} (w_1)_n z^{-n-1},$$

satisfying the following conditions for $w_i \in M_i$, $i = 1, 2$ and $a \in V$:

(I1) $\mathcal{Y}(L(-1)w_1, z) = \frac{d}{dz}\mathcal{Y}(w_1, z)$.

(I2) $(w_1)_n(w_2) = 0$ for $\mathrm{Re}(n)$ *sufficiently large.*

(I3) *The following Jacobi identity holds*

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)Y_{M_3}(a, z_1)\mathcal{Y}(w_1, z_2)w_2 - (-1)^{|a||w_1|}z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right)\mathcal{Y}(w_1, z_2)Y_{M_2}(a, z_1)w_2$$

$$= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)\mathcal{Y}(Y_{M_1}(a, z_0)w_1, z_2)w_2,$$

for $\mathbb{Z}_2$-homogeneous a and w_1.

We shall denote by

$$I\begin{pmatrix} M_3 \\ M_1\ M_2 \end{pmatrix}$$

the vector space of intertwining operators of type $\binom{M_3}{M_1\ M_2}$. Their dimensions are known as the "fusion rules".

3. $N = 1$ NEVEU-SCHWARZ VERTEX OPERATOR SUPERALGEBRAS

The $N = 1$ Neveu-Schwarz (or simply NS) algebra is the Lie superalgebra

$$\mathfrak{ns} = \bigoplus_{n \in \mathbb{Z}}\mathbb{C}L(n)\bigoplus \bigoplus_{m \in \frac{1}{2} + \mathbb{Z}}\mathbb{C}G(m)\bigoplus\mathbb{C}C$$

with commutation relations ($m, n \in \mathbb{Z}$):

$$[L(m), L(n)] = (m - n)L(m + n) + \delta_{m+n,0}\frac{m^3 - m}{12}C,$$

(3.1)
$$[G(m + \frac{1}{2}), L(n)] = (m + \frac{1}{2} - \frac{n}{2})G(m + n + \frac{1}{2}),$$

(3.2)
$$\{G(m + \frac{1}{2}), G(n - \frac{1}{2})\} = 2L(m + n) + \frac{1}{3}m(m + 1)\delta_{m+n,0}C,$$

$$[L(m), C] = 0, \quad [G(m + \frac{1}{2}), C] = 0.$$

It is important to consider vertex algebras which admit an action of the $N = 1$ Neveu-Schwarz algebras (cf. [HM]). These vertex operator superalgebras are called $N = 1$ Neveu-Schwarz vertex operator superalgebras and are subject to an additional axiom:

There exists $\tau \in V_{3/2}$ (superconformal vector) such that

$$Y(\tau, z) = \sum_{n \in \mathbb{Z} + 1/2}G(n)z^{-n-3/2}, \quad G(n) \in \mathrm{End}(V)$$

where $G(n)$ satisfy bracket relations as in (3.1) and (3.2).

The simplest examples of $N = 1$ vertex operator superalgebras are $\mathfrak{ns}$-modules $L^{\mathfrak{ns}}(c, 0)$, $c \neq 0$ where we use the standard notation and for any $(c, h) \in \mathbb{C}^2$ we denote by $L^{\mathfrak{ns}}(c, h)$ the corresponding irreducible highest weight $\mathfrak{ns}$–module with central charge c and highest weight h (cf. [KWn], [Li], [A1], [HM]). It is well-known that the vertex operator superalgebra $L^{\mathfrak{ns}}(c, 0)$, $c \neq 0$ is simple.

Set
$$c_{p,q} = \frac{3}{2}(1 - \frac{2(p-q)^2}{pq}),$$
$$h_{p,q}^{r,s} = \frac{(sp - rq)^2 - (p-q)^2}{8pq}.$$
In the rest of the paper we shall focus on certain $\mathfrak{ns}$ modules of central charge $c_{2m+1,1}$, $m \geq 1$.

4. FUSION RULES FOR $N = 1$ SUPERCONFORMAL $(2m+1, 1)$-MODELS

From now on we will mostly focus on (non-minimal) $(2m+1, 1)$-models, so that $p = 2m+1$, $q = 1$. Relevant lowest weights are $h^{r,s} := h_{2m+1,1}^{r,s}$, $r, s \in \mathbb{Z}$.

It will be of great use to determine the fusion rules

(4.1)
$$I \begin{pmatrix} L(c_{2m+1,1}, h^{r'',s''}) \\ L(c_{2m+1,1}, h^{r,s}) \; L(c_{2m+1,1}, h^{r',s'}) \end{pmatrix}$$

for certain triples (r, s), (r', s') and $(r'', s'') \in \mathbb{Z}^2$. For $m = 0$ (i.e., the $c = 3/2$ case) these numbers were computed in (see [M1]). In particular, for every $s > 0$ we have:

(4.2)
$$L(\frac{3}{2}, h^{1,3}) \times L(\frac{3}{2}, h^{1,2s+1}) = L(\frac{3}{2}, h^{1,2s-1}) \oplus L(\frac{3}{2}, h^{1,2s+1}) \oplus L(\frac{3}{2}, h^{1,2s+3}),$$

where $\times$ is just a formal product indicating which triples of irreducible modules admit non-trivial fusion rules (all with multiplicity one). As shown in [M1], the fusion rules for $m = 0$ can be computed by using certain projection formulas for singular vectors combined with Frenkel-Zhu's formula. It is not hard to see that the same approach extends to $m \geq 1$ as well. We only have to apply appropriate projection formulas as in Lemma 3.1 of [IK1]. Actually, for purposes of this paper we do not need any of results from [IK1], because we are interested only in special properties of "fusion rules" (4.1) (nevertheless, see Remark 4.1).

Proposition 4.1. *For every $i = 0, ..., m - 1$ and $n \geq 1$ we have: the space*

$$I \begin{pmatrix} L(c_{2m+1,1}, h) \\ L(c_{2m+1,1}, h^{1,3}) \; L(c_{2m+1,1}, h^{2i+1,2n+1}) \end{pmatrix}$$

is nontrivial only if $h \in \{h^{2i+1,2n-1}, h^{2i+1,2n+1}, h^{2i+1,2n+3}\}$, and

$$I \begin{pmatrix} L(c_{2m+1,1}, h) \\ L(c_{2m+1,1}, h^{1,3}) \; L(c_{2m+1,1}, h^{2i+1,1}) \end{pmatrix}$$

is nontrivial only if $h = h^{2i+1,3}$.

Similarly, for every $i = 0, ..., m - 1$ and $n \geq 2$ we have: the space

$$I \begin{pmatrix} L(c_{2m+1,1}, h) \\ L(c_{2m+1,1}, h^{1,3}) \; L(c_{2m+1,1}, h^{2i+1,-2n+1}) \end{pmatrix}$$

is nontrivial only if $h \in \{h^{2i+1,-2n-1}, h^{2i+1,-2n+1}, h^{2i+1,-2n+3}\}$, and

$$I \begin{pmatrix} L(c_{2m+1,1}, h) \\ L(c_{2m+1,1}, h^{1,3}) \; L(c_{2m+1,1}, h^{2i+1,-1}) \end{pmatrix}$$

is nontrivial only if $h \in \{h^{2i+1,-3}, h^{2i+1,-1}\}$.

For a stronger statement see Remark 4.1.

Proof. We assume that $n \geq 1$ (for other cases essentially the same argument works). Let $A(L(c_{2m+1,0}, 0))$ be the Zhu's algebra of $L(c_{2m+1,0}, 0)$ (polynomial algebra in one variable) and $A(L(c_{2m+1}, h))$ the $A(L(c_{2m+1,0}, 0))$-bimodule of $L(c_{2m+1}, h)$ [FZ].

As in [M1], it is sufficient to analyze the structure of the $A(L(c_{2m+1,0}, 0))$-module

(4.3) $$A(L(c_{2m+1}, h^{1,3})) \otimes_{A(L(c_{2m+1,0},0))} L(c_{2m+1,1}, h^{2i+1,2n+1})(0),$$

where $L(c_{2m+1,1}, h)(0)$ denotes the top weight component of $L(c_{2m+1,1}, h)$ (cf. [FZ]). From [IK1] (or elsewhere) it follows that the Verma module $M(c_{2m+1,0}, h^{1,3})$ combines in the following short exact sequence

$$0 \longrightarrow M(c_{2m+1,0}, h^{1,3} + \frac{3}{2}) \longrightarrow M(c_{2m+1,0}, h^{1,3}) \longrightarrow L(c_{2m+1,0}, h^{1,3}) \longrightarrow 0.$$

Thus the maximal submodule of $M(c_{2m+1,0}, h^{1,3})$ is generated by a singular vector of weight $h^{1,3} + \frac{3}{2}$ (explicitly, $(-L(-1)G(-1/2) + (2m + 1)G(-3/2))v_{1,3}$ where $v_{1,3}$ is the highest weight vector in $M(c_{2m+1,0}, h^{1,3})$). Now, as in [M1], it is not hard to see that the space (4.3) is three-dimensional and that all fusion rules covered by the statements are at most 1 (actually, they are all one; see Remark 4.1).

$\square$

Remark 4.1. We can actually prove "if and only if" statement in Proposition 4.1 by using at least two different methods. On one hand we would have to combine methods from [M1] and projection formula in Lemma 3.1 [IK1] (we do not have explicit singular vectors to work

with!). Alternatively, with Proposition 4.1, it is sufficient to construct non-trivial intertwining operators for all types covered in Proposition 4.1. This was actually done in later sections.

We should say that our fusion rules formulas coincide with Iohara-Koga's fusion rule formula in the generic case, which are computed by using coinvariants and projection formulas rather than Frenkel-Zhu's formula [IK1]. But as we know the coinvariant approach and Frenkel-Zhu's formulas yield the same answer in practically all known examples (for further examples see [W], [M1], [M4]).

5. LATTICE AND FERMIONIC VERTEX SUPERALGEBRAS

We shall first recall some basic facts about lattice and fermionic vertex superalgebras.

Let $m \in \mathbb{Z}_{\geq 0}$. Let $\widetilde{L} = \mathbb{Z}\beta$ be a rational lattice of rank one with nondegenerate bilinear form $\langle \cdot, \cdot \rangle$ given by

$$\langle \beta, \beta \rangle = \frac{1}{2m+1}.$$

Let $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} \widetilde{L}$. Extend the form $\langle \cdot, \cdot \rangle$ on $\widetilde{L}$ to $\mathfrak{h}$. Let $\hat{\mathfrak{h}} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{h} \oplus \mathbb{C}c$ be the affinization of $\mathfrak{h}$. Set $\hat{\mathfrak{h}}^+ = t\mathbb{C}[t] \otimes \mathfrak{h}$; $\hat{\mathfrak{h}}^- = t^{-1}\mathbb{C}[t^{-1}] \otimes \mathfrak{h}$. Then $\hat{\mathfrak{h}}^+$ and $\hat{\mathfrak{h}}^-$ are abelian subalgebras of $\hat{\mathfrak{h}}$. Let $U(\hat{\mathfrak{h}}^-) = S(\hat{\mathfrak{h}}^-)$ be the universal enveloping algebra of $\hat{\mathfrak{h}}^-$. Let $\lambda \in \mathfrak{h}$. Consider the induced $\hat{\mathfrak{h}}$-module

$$M(1, \lambda) = U(\hat{\mathfrak{h}}) \otimes_{U(\mathbb{C}[t] \otimes \mathfrak{h} \oplus \mathbb{C}c)} \mathbb{C}_\lambda \simeq S(\hat{\mathfrak{h}}^-) \quad \text{(linearly)},$$

where $t\mathbb{C}[t] \otimes \mathfrak{h}$ acts trivially on $\mathbb{C}_\lambda \cong \mathbb{C}$, $\mathfrak{h}$ acting as $\langle h, \lambda \rangle$ for $h \in \mathfrak{h}$ and c acts on $\mathbb{C}_\lambda$ as multiplication by 1. We shall write $M(1)$ for $M(1, 0)$. For $h \in \mathfrak{h}$ and $n \in \mathbb{Z}$ write $h(n) = t^n \otimes h$. Set $h(z) = \sum_{n \in \mathbb{Z}} h(n)z^{-n-1}$. Then $M(1)$ is a vertex algebra which is generated by the fields $h(z)$, $h \in \mathfrak{h}$, and $M(1, \lambda)$, for $\lambda \in \mathfrak{h}$, are irreducible modules for $M(1)$.

As in [DL] (see also [D], [FLM], [GL], [K]), we have the generalized vertex algebra

$$V_{\widetilde{L}} = M(1) \otimes \mathbb{C}[\widetilde{L}],$$

where $\mathbb{C}[\widetilde{L}]$ is a group algebra of $\widetilde{L}$ with a generator e^β. For $v \in V_{\widetilde{L}}$, let $Y(v, z) = \sum_{s \in \frac{1}{2m+1}\mathbb{Z}} v_s z^{-s-1}$ be the corresponding vertex operator (for precise formulae see [DL]).

Define $\alpha = (2m+1)\beta$. Then $\langle \alpha, \alpha \rangle = 2m+1$, implying $L = \mathbb{Z}\alpha \subset \widetilde{L}$ is an integer lattice. Therefore the subalgebra $V_L \subset V_{\widetilde{L}}$ has the structure of a vertex superalgebra.

Define the Schur polynomials $S_r(x_1, x_2, \cdots)$ in variables $x_1, x_2, \cdots$ by the following equation:

(5.1)
$$\exp\left(\sum_{n=1}^{\infty} \frac{x_n}{n} y^n\right) = \sum_{r=0}^{\infty} S_r(x_1, x_2, \cdots) y^r.$$

For any monomial $x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}$ we have an element

$$h(-1)^{n_1} h(-2)^{n_2} \cdots h(-r)^{n_r} \mathbf{1}$$

in $M(1)$ for $h \in \mathfrak{h}$. Then for any polynomial $f(x_1, x_2, \cdots)$, $f(h(-1), h(-2), \cdots)\mathbf{1}$ is a well-defined element in $M(1)$. In particular, $S_r(h(-1), h(-2), \cdots)\mathbf{1} \in M(1)$ for $r \in \mathbb{Z}_{\geq 0}$. Set $S_r(h)$ for $S_r(h(-1), h(-2), \cdots)\mathbf{1}$.

The following relations in the generalized vertex operator algebra $V_{\widetilde{L}}$ are of great importance:

(5.2) $$e_i^\gamma e^\delta = 0 \quad \text{for } i \geq -\langle \gamma, \delta \rangle.$$

Especially, if $\langle \gamma, \delta \rangle \geq 0$, we have $e_i^\gamma e^\delta = 0$ for $i \in \mathbb{Z}_{\geq 0}$, and if $\langle \gamma, \delta \rangle = -n < 0$, we get

(5.3) $$e_{i-1}^\gamma e^\delta = S_{n-i}(\gamma) e^{\gamma+\delta} \quad \text{for } i \in \{0, \ldots, n\}.$$

5.1. **Fermionic vertex operator superalgebra F.**

In what follows we consider the Clifford algebra CL, generated by $\{\phi(n), n \in \frac{1}{2} + \mathbb{Z}\} \cup \{1\}$ and relations

$$\{\phi(n), \phi(m)\} = \delta_{n,-m}, \quad n, m \in \frac{1}{2} + \mathbb{Z}.$$

Let F be the CL–module generated by the vector $\mathbf{1}$ such that

$$\phi(n)\mathbf{1} = 0, \; n > 0.$$

Then the field

$$Y(\phi(-\tfrac{1}{2})\mathbf{1}, z) = \phi(z) = \sum_{n \in \frac{1}{2}+\mathbb{Z}} \phi(n) z^{-n-\frac{1}{2}},$$

generate a unique vertex operator superalgebra structure on F. We choose

$$\omega^{(s)} = \frac{1}{2}\phi(-\tfrac{3}{2})\phi(-\tfrac{1}{2})\mathbf{1}$$

for the Virasoro element giving central charge $\frac{1}{2}$. Moreover, F is a rational vertex operator superalgebra, and F is up to equivalence the unique irreducible F–module (see [FRW], [KWn], [Li]).

5.2. **Vertex superalgebra** $SM(1)$. In this subsection we study the vertex superalgebra $SM(1) :=$ $M(1) \otimes F$. We shall first define a family of $N = 1$ superconformal vectors in $SM(1)$. For every $m \in \mathbb{Z}_{\geq 0}$, we define (see also [MR], [K],[IK2])

$$\tau = \frac{1}{\sqrt{2m+1}} \left(\alpha(-1)\mathbf{1} \otimes \phi(-\tfrac{1}{2})\mathbf{1} + 2m\mathbf{1} \otimes \phi(-\tfrac{3}{2})\mathbf{1}. \right)$$

$$\omega = \frac{1}{2(2m+1)}(\alpha(-1)^2 + 2m\alpha(-2))\mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes \omega^{(s)}.$$

Set

$$Y(\tau, z) = G(z) = \sum_{n \in \mathbb{Z}} G(n + \tfrac{1}{2})z^{-n-2}, \;\; Y(\omega, z) = L(z) = \sum_{n \in \mathbb{Z}} L(n)z^{-n-2}.$$

Then τ is an $N = 1$ superconformal vector, and the vertex subalgebra of $SM(1)$ strongly generated by the fields $G(z)$ and $L(z)$ is isomorphic to the Neveu-Schwarz vertex operator superalgebra $L^{\mathfrak{ns}}(c_{2m+1,1}, 0)$, where $c_{2m+1,1} = \frac{3}{2}(1 - \frac{8m^2}{2m+1})$. In other words, $SM(1)$ becomes a Fock module for the Neveu-Schwarz algebra with central charge $c_{2m+1,1}$. Moreover, for every $\lambda \in \mathfrak{h}$, the $SM(1)$–modules $SM(1, \lambda) := M(1, \lambda) \otimes F$ is also a Fock module with central charge $c_{2m+1,1}$ and conformal weight

(5.4) $$\frac{1}{2(2m+1)}(\langle \lambda, \alpha \rangle^2 - 2m\langle \lambda, \alpha \rangle).$$

Now we want to describe the structure of these Fock modules viewed as $\mathfrak{ns}$-modules. For this purpose we need the concept of screening operators. As in [A3], we shall construct these operators using generalized vertex algebras.

The $N = 1$ superconformal vector $\tau \in M(1) \otimes F$ also defines an $N = 1$ superconformal structure on $V_{\widetilde{L}} \otimes F$ and $V_L \otimes F$. In particular, $V_L \otimes F$ is an $N = 1$ vertex operator superalgebra. The operator $L(0)$ defines a $\frac{1}{2}\mathbb{Z}_{\geq 0}$–gradation on $V_L \otimes F$. Recall that $\mathrm{wt}(v) = n$ if $L(0)v = nv$.

Define

$$s^{(1)} = e^{\alpha} \otimes \phi(-\tfrac{1}{2})\mathbf{1} \in V_L \otimes F,$$

$$s^{(2)} = e^{-\beta} \otimes \phi(-\tfrac{1}{2})\mathbf{1} \in V_{\widetilde{L}} \otimes F.$$

By using the Jacobi identity in the (generalized) vertex algebras $V_L \otimes F$ and $V_{\widetilde{L}} \otimes F$ we get the following formulas

(5.5) $$[G(n + \tfrac{1}{2}), s_i^{(1)}] = -\frac{i}{\sqrt{2m+1}}e_{i+n}^{\alpha}, \quad [L(n), s_i^{(1)}] = -i \, s_{i+n}^{(1)} \quad (i \in \mathbb{Z})$$

(5.6) $$[G(n + \tfrac{1}{2}), s_r^{(2)}] = r\sqrt{2m+1}e_{i+n}^{-\beta}, \quad [L(n), s_r^{(2)}] = -r \, s_{r+n}^{(2)} \quad (r \in \frac{1}{2m+1}\mathbb{Z}).$$

Let

$$Q = s_0^{(1)} = \text{Res}_z\, Y(s^{(1)}, z),$$
$$\widetilde{Q} = s_0^{(2)} = \text{Res}_z Y(s^{(2)}, z).$$

From relations (5.5) and (5.6) we see that the operators Q and $\widetilde{Q}$ commute with the action of the Neveu-Schwarz algebra (see also [IK2]).

We are interested in the action of these operators on $SM(1)$. In fact, Q and $\widetilde{Q}$ are the screening operators, and therefore $\text{Ker}_{SM(1)}Q$ and $\text{Ker}_{SM(1)}\widetilde{Q}$ are vertex subalgebras of $SM(1)$ (for details see Section 14 in [FB] and reference therein).

The following lemma gives the basic properties of the operators Q and $\widetilde{Q}$. The proof is similar to that of Lemma 2.1 in [A3].

Lemma 5.1.
(i) *If $m \neq 0$, $[Q, \widetilde{Q}] = 0$.*
(ii) *$\widetilde{Q}e^{n\alpha} \neq 0$, $n \in \mathbb{Z}_{>0}$.*
(iii) *$\widetilde{Q}e^{-n\alpha} = 0$, $n \in \mathbb{Z}_{\geq 0}$.*

We now define the following three (non-zero) elements in the vertex operator superalgebras $V_L \otimes F$:

$$F = e^{-\alpha}, \quad H = QF, \quad E = Q^2 F.$$

By using expression for conformal weights (5.4) and Lemma 5.1, we conclude that these vectors are singular vectors for the action of the Neveu-Schwarz algebra, and

$$\text{wt}(F) = \text{wt}(H) = \text{wt}(E) = h^{1,3} = 2m + \tfrac{1}{2}.$$

It is also important to notice that $H \in SM(1)$.

The proof of the following result is similar to that of Lemma 3.1 in [A3].

Lemma 5.2. *In the vertex operator superalgebra $V_L \otimes F$ the following relations hold:*
(i) *$Q^3 F = 0$.*
(ii) *$E_i E = F_i F = 0$, for every $i \geq -2m - 1$.*
(iii) *$Q(H_i H) = 0$, for every $i \geq -2m - 1$.*

We define

$$(5.7) \qquad \widehat{F} = e^{-\alpha} \otimes \phi(-\tfrac{1}{2}), \quad \widehat{H} = Q\widehat{F}, \quad \widehat{E} = Q^2 \widehat{F}.$$

These vectors are even and have conformal weight $2m+1$. We will need the following result.

Lemma 5.3. *We have*

$$\widehat{F}_i \widehat{F} = 0, \quad \widehat{E}_i \widehat{E} = 0, \quad i \geq -2m.$$

Also,

$$Q(\widehat{H}_i \widehat{H}) = 0, \quad i \geq -2m.$$

Proof. Since Q acts as a derivation if $\widehat{F}_i \widehat{F} = 0$, for $i \geq -2m$ then $Q^4(\widehat{F}_i \widehat{F}) = 6\widehat{E}_i \widehat{E} = 0$, for $i \geq -2m$. We only have to notice relations

$$\widehat{F}_k \widehat{F} = \operatorname{Res}_x x^k Y(e^{-\alpha}, x)e^{-\alpha} \otimes Y(\phi(-1/2), x)\phi(-1/2)\mathbf{1},$$

$$\operatorname{Res}_x x^i Y(e^{-\alpha}, x)e^{-\alpha} = 0, \quad i \geq -2m - 1,$$

proven in Lemma 5.2, and

$$\operatorname{Res}_x x^j Y(\phi(-1/2), x)\phi(-1/2)\mathbf{1} = 0, \quad j \geq 1.$$

The last formula follows from $Q^3(\widehat{F}_i \widehat{F}) = 0$ for $i \geq -2m$. $\qquad\square$

6. THE $N = 1$ NEVEU-SCHWARZ MODULE STRUCTURE OF $V_L \otimes F$-MODULES

For $i \in \mathbb{Z}$, we set

$$(6.1) \qquad\qquad\qquad \gamma_i = \frac{i}{2m+1}\alpha.$$

We shall first present results on the structure of $V_L \otimes F$–modules as modules for the $N = 1$ Neveu-Schwarz algebra. It is a known fact that irreducible $V_L \otimes F$-modules are given by

$$V_{L+\gamma_i} \otimes F, \quad i = 0, \ldots, 2m.$$

Each $V_{L+\gamma_i}$ is a direct sum of super Feigin-Fuchs modules via

$$V_{L+\gamma_i} \otimes F = \bigoplus_{n \in \mathbb{Z}} (M(1) \otimes e^{\gamma_i + n\alpha}) \otimes F.$$

We shall now investigate the action of the operator Q. Since operators Q^j, $j \in \mathbb{Z}_{>0}$, commute with the action of the Neveu-Schwarz algebra, they are actually intertwiners between super Feigin-Fuchs modules inside $V_{L+\gamma_i} \otimes F$. Assume that $0 \leq i \leq m$. If $Q^j e^{\gamma_i - n\alpha}$ is nontrivial, it is a singular vector in the Fock module $SM(1, \gamma_i + (j-n)\alpha)$ of weight

$$\operatorname{wt}(Q^j e^{\gamma_i - n\alpha}) = \operatorname{wt}(e^{\gamma_i - n\alpha}) = h^{2i+1,2n+1},$$

where $h^{2i+1,2n+1} := h_{1,2m+1}^{2i+1,2n+1}$. Since $\operatorname{wt}(e^{\gamma_i + (j-n)\alpha}) > \operatorname{wt}(e^{\gamma_i - n\alpha})$ if $j > 2n$, we conclude that

$$(6.2) \qquad\qquad\qquad Q^j e^{\gamma_i - n\alpha} = 0 \quad \text{for } j > 2n.$$

One can similarly see that for $m + 1 \le i \le 2m$:

(6.3) $$Q^j e^{\gamma_i - n\alpha} = 0 \quad \text{for } j > 2n + 1.$$

The following lemma is useful for constructing singular vectors in $V_{L+\gamma_i} \otimes F$:

Lemma 6.1.
(1) $Q^{2n} e^{\gamma_i - n\alpha} \ne 0$ *for* $0 \le i \le m$.
(2) $Q^{2n+1} e^{\gamma_i - n\alpha} \ne 0$ *for* $m + 1 \le i \le 2m$.

Proof. We shall prove the assertion (1) by induction on $n \in \mathbb{Z}_{>0}$.

For $n = 1$ we can see directly that $Q^2 e^{\gamma_i - \alpha} \ne 0$ (or see below).

Assume now that (1) holds for certain $n \in \mathbb{Z}_{>0}$. Since $V_{L+\gamma_i} \otimes F$ is a simple module for the simple vertex operator superalgebra $V_L \otimes F$ we have that

$$Y(E, z) Q^{2n} e^{\gamma_i - n\alpha} \ne 0,$$

(for the proof see [DL]). So there is $j_0 \in \mathbb{Z}$ such that

$$E_{j_0} Q^{2n} e^{\gamma_i - n\alpha} \ne 0 \quad \text{and} \quad E_j Q^{2n} e^{\gamma_i - n\alpha} = 0 \text{ for } j > j_0.$$

Since

$$E_{j_0} Q^{2n} e^{\gamma_i - n\alpha} = \frac{1}{(n+1)(2n+1)} Q^{2n+2} (e_{j_0}^{-\alpha} e^{\gamma_i - n\alpha}),$$

we have that $j_0 \le i - 1 - (2m+1)n$. By using the fusion rules from Proposition 4.1, we conclude that

$$e_{j_0}^{-\alpha} e^{\gamma_i - n\alpha} \in U(\mathfrak{ns}).e^{\gamma_i - (n+1)\alpha}$$

and therefore $Q^{2n+2} e^{\gamma_i - (n+1)\alpha} \ne 0$, which proves (1). Notice that the idea used in the induction step, and fusion rules from Proposition 4.1 can be alternatively used to show that $Q^2 e^{\gamma_i - \alpha} \ne 0$.

The proof of (2) is similar so we omit it here.

□

Remark 6.1. It would be desirable - in parallel with the Virasoro algebra case - to have a direct proof of Lemma 6.1 with no reference to fusion rules. However, the Virasoro algebra approach based on matrix coefficients does not apply verbatim to superconformal $(1, 2m + 1)$-models, so we decided to give a proof which uses the theory of vertex algebras and fusion rules. We found this approach to be quite elegant. We also remark that Iohara and Koga proved certain properties of screening operators among super Feigin-Fuchs modules in Theorem 3.1, [IK2] (see also [MR]), but it is not clear whether these results can be used to prove Lemma 6.1.

As in the Virasoro algebra case the $N = 1$ Feigin-Fuchs modules are classified according to their embedding structure. For our purposes we shall focus only on modules of certain type (Type 4 and 5 in [IK2]). These modules are either semisimple (Type 5) or they become semisimple after quotienting with the maximal semisimple submodule (Type 4). As usual the singular vectors will be denoted by • and cosingular vectors with ∘.

The following result follows directly from Lemma 6.1 and the structure theory of super Feigin-Fuchs modules [IK2] after some minor adjustments of parameters (cf. Type 4 embedding structure).

Theorem 6.1. *Assume that $i \in \{0, \ldots, m-1\}$.*

(i) *As a module for the Neveu-Schwarz algebra, $V_{L+\gamma_i} \otimes F$ is generated by the family of singular and cosingular vectors $\widetilde{Sing}_i \bigcup \widetilde{CSing}_i$, where*

$$\widetilde{Sing}_i = \{u_i^{(j,n)} \mid j, n \in \mathbb{Z}_{\geq 0},\ 0 \leq j \leq 2n\}; \quad \widetilde{CSing}_i = \{w_i^{(j,n)} \mid n \in \mathbb{Z}_{>0}, 0 \leq j \leq 2n-1\}.$$

These vectors satisfy the following relations:

$$u_i^{(j,n)} = Q^j e^{\gamma_i - n\alpha}, \quad Q^j w_i^{(j,n)} = e^{\gamma_i + n\alpha}.$$

The submodule generated by singular vectors $\widetilde{Sing}_i$, denoted by $S\Lambda(i+1)$, is isomorphic to [1]

$$\bigoplus_{n=0}^{\infty} (2n+1) L^{\mathfrak{ns}}(c_{2m+1,1}, h^{2i+1,2n+1}).$$

(ii) *For the quotient module we have*

$$S\Pi(m-i) := (V_{L+\gamma_i} \otimes F)/S\Lambda(i+1) \cong \bigoplus_{n=1}^{\infty} (2n) L^{\mathfrak{ns}}(c_{2m+1,1}, h^{2i+1,-2n+1}).$$

The situation described in Theorem 6.1 can be depicted by the following diagram:

---

[1] In this section notation $k\, L^{\mathfrak{ns}}(c, h)$ means $L^{\mathfrak{ns}}(c, h)^{\oplus k}$, $k \in \mathbb{Z}_{\geq 0}$.

$$(6.4) \qquad j = -2 \qquad j = -1 \qquad j = 0 \qquad j = 1 \qquad j = 2$$

Let M' be the contragradient V-module, where V is a vertex operator superalgebra. Then we have an isomorphism of $M(1) \otimes F$-modules.

$$(M(1) \otimes e^{\frac{j}{2m+1}\alpha+i\alpha} \otimes F)' \cong (M(1) \otimes e^{\frac{2m-j}{2m+1}\alpha-i\alpha} \otimes F).$$

By taking direct sums we obtain the following isomorphism of ns-modules

$$(6.5) \qquad (V_{L+\gamma_i} \otimes F)' \cong V_{L+\gamma_{2m-i}} \otimes F.$$

Since the dual functor interchanges cosingular and singular vectors, Theorem 6.1 implies the next result (alternatively, use Type 4 embedding structure in [IK2]):

Theorem 6.2. *Assume that $i \in \{0, \ldots, m-1\}$.*
(i) *As a module for the Neveu-Schwarz algebra, $V_{L+\gamma_{2m-i}} \otimes F$ is generated by the family of singular and cosingular vectors $\widetilde{Sing}_i' \bigcup \widetilde{CSing}_i'$, where*

$$\widetilde{Sing}_i' = \{u_i'^{(j,n)} \mid n \in \mathbb{Z}_{>0}, 0 \le j \le 2n-1\}; \quad \widetilde{CSing}_i' = \{w_i'^{(j,n)} \mid j, n \in \mathbb{Z}_{\ge 0}, \ 0 \le j \le 2n\}.$$

These vectors satisfy the following relations:

$$u_i'^{(j,n)} = Q^j e^{\gamma_{2m-i}-n\alpha}, \quad Q^j w_i'^{(j,n)} = e^{\gamma_{2m-i}+n\alpha}.$$

The submodule generated by singular vectors $\widetilde{Sing}_i$ is is isomorphic to
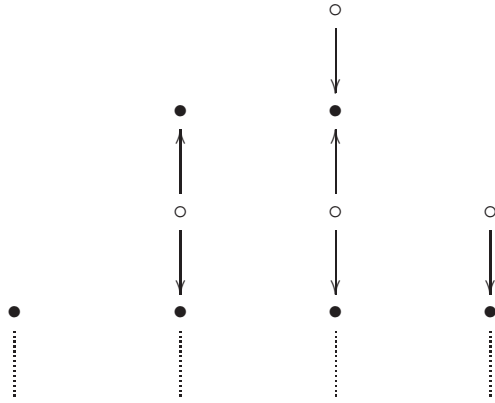
$$S\Pi(m-i) \cong \bigoplus_{n=1}^{\infty}(2n)L^{\mathfrak{ns}}(c_{2m+1,1}, h^{2i+1,-2n+1}).$$

(ii) *For the quotient module we have*

$$S\Lambda(i+1) \cong (V_{L+\gamma_i} \otimes F)/S\Pi(m-i) \cong \bigoplus_{n=0}^{\infty}(2n+1)L^{\mathfrak{ns}}(c_{2m+1,1}, h^{2i+1,2n+1}).$$

The embedding diagram for $V_{L+\gamma_{2m-i}} \otimes F$, $i = 0, ..., m-1$ is now

(6.6)
$$j = -1 \qquad j = 0 \qquad j = 1 \qquad j = 2$$

Finally, (6.5) imply that $V_{L+\gamma_m} \otimes F$ is a self-dual $V_L \otimes F$-module. In view of that, it is not surprising that $V_{L+\gamma_m} \otimes F$ is a semisimple $\mathfrak{ns}$-module. More precisely, we have the following result (for the proof see embedding structure in Type 5 case in [IK2])

Theorem 6.3. *As a module for the Neveu-Schwarz algebra $V_{L+\gamma_m} \otimes F$ is completely reducible and generated by the family of singular vectors*

$$\widetilde{Sing}_m = \{u_m^{(j,n)} := Q^j e^{\gamma_m - n\alpha} \mid j, n \in \mathbb{Z}_{\geq 0},\ 0 \leq j \leq 2n\};$$

and it is isomorphic to

$$S\Lambda(m+1) := V_{L+\gamma_m} \otimes F \cong \bigoplus_{n=0}^{\infty}(2n+1)L^{\mathfrak{ns}}(c_{2m+1,1}, h^{2m+1,2n+1}).$$

The embedding structure in the last case is a totally disconnected diagram

$$(6.7) \qquad j = -2 \qquad j = -1 \qquad j = 0 \qquad j = 1 \qquad j = 2$$

7. The vertex operator superalgebra $\overline{SM(1)}$

Let us fix a positive integer m. We shall first present the structure of the vertex operator superalgebra $SM(1)$ as a module for the Neveu-Schwarz algebra. The next result follows directly from Theorem 6.1.

Theorem 7.1. *For every* $n \in \mathbb{Z}_{\geq 0}$, *set*

$$u_n := u_0^{(n,n)} = Q^n e^{-n\alpha}, \quad w_{n+1} := w_0^{(n+1,n+1)}.$$

(i) *The vertex operator superalgebra* $SM(1)$, *as a module for the vertex operator superalgebra* $L^{\mathfrak{ns}}(c_{2m+1,1}, 0)$, *is generated by the family of singular and cosingular vectors* $\widetilde{Sing} \bigcup \widetilde{CSing}$, *where*

$$\widetilde{Sing} = \{u_n \mid n \in \mathbb{Z}_{\geq 0}\}; \quad \widetilde{CSing} = \{w_n \mid n \in \mathbb{Z}_{>0}\}.$$

Moreover, $U(\mathfrak{ns})u_n \cong L^{\mathfrak{ns}}(c_{2m+1,1}, h^{1,2n+1})$.

(ii) *The submodule generated by vectors* $u_n, n \in \mathbb{Z}_{\geq 0}$ *is isomorphic to*

$$[Sing] \cong \bigoplus_{n=0}^{\infty} L^{\mathfrak{ns}}(c_{2m+1,1}, h^{1,2n+1}).$$

(iii) *The quotient module is isomorphic to*

$$M(1)/[Sing] \cong \bigoplus_{n=1}^{\infty} L^{\mathfrak{ns}}(c_{2m+1,1}, h^{1,-2n+1}).$$

(iv) $Qu_0 = Q\mathbf{1} = 0$, *and* $Qu_n \neq 0$, $Qw_n \neq 0$ *for every* $n \geq 1$.

Our Theorem 7.1 immediately gives the following result.

Proposition 7.1. *We have*

$$L^{\text{ns}}(c_{2m+1,1},0) \cong W_0 = Ker_{SM(1)}Q .$$

Define the following vertex algebra

$$\overline{SM(1)} = \text{Ker}_{SM(1)}\widetilde{Q}.$$

Since $\widetilde{Q}$ commutes with the action of the Neveu-Schwarz algebra, we have

$$L^{\text{ns}}(c_{2m+1,1},0) \cong W_0 \subset \overline{SM(1)}.$$

This implies that $\overline{SM(1)}$ is a vertex operator subalgebra of $SM(1)$ in the sense of [FHL] (i.e., $\overline{SM(1)}$ has the same Virasoro element as $SM(1)$).

The following theorem will describe the structure of the vertex operator superalgebra $\overline{SM(1)}$ as a $L^{\text{ns}}(c_{2m+1,1},0)$–module.

Theorem 7.2. *The vertex operator superalgebra* $\overline{SM(1)}$ *is isomorphic to* [Sing] *as a* $L^{\text{ns}}(c_{2m+1,1},0)$–*module, i.e.,*

$$\overline{SM(1)} \cong \bigoplus_{n=0}^{\infty} L^{\text{ns}}(c_{2m+1,1}, h^{1,2n+1}).$$

Proof. By Theorem 7.1 we know that the $L^{\text{ns}}(c_{2m+1,1},0)$–submodule generated by the set $\widetilde{Sing}$ is completely reducible. So to prove the assertion, it suffices to show that the operator $\widetilde{Q}$ annihilates vector $v \in \widetilde{Sing} \cup \widetilde{CSing}$ if and only if $v \in \widetilde{Sing}$. Let $v \in \widetilde{Sing}$, then $v = Q^n e^{-n\alpha}$ for certain $n \in \mathbb{Z}_{\geq 0}$. Since by Lemma 5.1 $\widetilde{Q}e^{-n\alpha} = 0$, we have

$$\widetilde{Q}v = \widetilde{Q}Q^n e^{-n\alpha} = Q^n \widetilde{Q}e^{-n\alpha} = 0.$$

Let now $v \in \widetilde{CSing}$. Then there is $n \in \mathbb{Z}_{>0}$ such that $Q^n v = e^{n\alpha}$. Assume that $\widetilde{Q}v = 0$. Then we have that

$$0 = Q^n \widetilde{Q}v = \widetilde{Q}Q^n v = \widetilde{Q}e^{n\alpha},$$

contradicting Lemma 5.1 (iii). This proves the theorem.   □

Next we shall prove that the vertex operator algebra $\overline{SM(1)}$ is generated by only two generators.

Theorem 7.3.
(1) *The vertex operator superalgebra* $\overline{SM(1)}$ *is generated by* τ *and* H.
(2) *The vertex operator superalgebra* $\overline{SM(1)}$ *is strongly generated by the set*

$$\{\tau, \omega, H, G(-\tfrac{1}{2})H\}.$$

Proof. Let U be the vertex subalgebra of $\overline{SM(1)}$ generated by τ and H. We need to prove that $U = \overline{SM(1)}$. Let W_n by the (irreducible) ns–submodule of $\overline{SM(1)}$ generated by vector u_n. Then $W_n \cong L^{\mathsf{ns}}(c_{2m+1,1}, h^{1,2n+1})$. Using Lemma 6.1 we see that

$$\text{Ker}_{\overline{SM(1)}} \, Q^n \cong \bigoplus_{i=0}^{n-1} W_i.$$

To prove (1) suffices to show that $u_n \in U$ for every $n \in \mathbb{Z}_{\geq 0}$. We shall prove this claim by induction. By definition we have that $u_0, u_1 (= H) \in U$. Assume that we have $k \in \mathbb{Z}_{\geq 0}$ such that $u_n \in U$ for $n \leq k$. In other words, the inductive assumption is $\oplus_{i=0}^k W_i \subset U$.

We shall now prove that $u_{k+1} \in U$. Set $j = -(2m+1)k - 1$. By Lemma 6.1 we have

$$Q^{2k+2} e^{-(k+1)\alpha} = Q^{2k+2} \left(e_j^{-\alpha} e^{-k\alpha} \right) \neq 0.$$

Next we notice that

$$Q^{k+1}(H_j u_k) = Q^{k+1} \left(Q e^{-\alpha} \right)_j \left(Q^k e^{-k\alpha} \right) = \frac{1}{2k+1} Q^{2k+2} \left(e_j^{-\alpha} e^{-k\alpha} \right),$$

which implies that

$$Q^{k+1}(H_j u_k) \neq 0.$$

So we have found vector $H_j u_k \in U$ such that

$$\text{wt}(H_j u_k) = \text{wt}(u_{k+1}).$$

This implies

$$H_j u_k \in \bigoplus_{i=0}^{k+1} W_i \text{ and } H_j u_k \notin \bigoplus_{i=0}^{k} W_i \, .$$

Since $Q^{k+1} \left(\oplus_{i=0}^k W_i \right) = 0$ and $\text{wt}(H_j u_k) = \text{wt}(u_{k+1})$ we conclude that there is a constant C, $C \neq 0$, such that

$$H_j u_k = C u_{k+1} + u', \; u' \in \bigoplus_{i=0}^{k} W_i \subset U.$$

Since $H_j u_k \in U$, we conclude that $u_{k+1} \in U$.

Therefore, the claim is verified, and the proof of (1) is complete.

The proof of (1) shows that $\overline{SM(1)}$ is spanned by the vectors

(7.1) $$u_{n_1}^1 \cdots u_{n_r}^r \mathbf{1}, \quad u^i \in \{\tau, H\},$$

such that for $1 \leq i \leq r$:

(7.2) $$n_i \leq -1 \quad \text{if } u^i = H \quad \text{and} \quad n_i \leq 0 \quad \text{if } u^i = \tau.$$

This implies that $\overline{SM(1)}$ is strongly generated by the set $\{\tau, \omega, H, G(-\frac{1}{2})H\}$, and (2) holds. $\square$

The following lemma imply that for $i \geq -(2m+1)$ vectors $H_i H$ and $\widehat{H}_{i+1}\widehat{H}$ can be constructed using only the action of the Neveu-Schwarz operators $L(n)$ and $G(n+\frac{1}{2})$ on the vacuum vector $\mathbf{1}$.

Lemma 7.1. *We have:*

$$H_i H \in W_0 \cong L^{\mathrm{ns}}(c_{2m+1,1}, 0) \quad \text{for every } i \geq -(2m+1),$$

$$\widehat{H}_i \widehat{H} \in W_0 \cong L^{\mathrm{ns}}(c_{2m+1,1}, 0) \quad \text{for every } i \geq -2m.$$

Remark 7.1. If we adopt notation used by physicists, then Theorem 7.3 implies that $\overline{SM(1)}$ is a $\mathcal{W}(\frac{3}{2}, 2m+\frac{1}{2})$ superalgebra, meaning that it is generated by primary fields of weight $\frac{3}{2}$ and $2m+\frac{1}{2}$. In some physics papers $\mathcal{W}(\frac{3}{2}, 2m+\frac{1}{2})$ super algebras are studied by using general principles (e.g., Jacobi identities) but only for low m. Because $\overline{SM(1)}$ shares many similarities with the singlet algebra $\overline{M(1)}$ [AM1] we call $\overline{SM(1)}$ *super singlet* vertex algebra.

8. ZHU'S ALGEBRA $A(\overline{SM(1)})$ AND CLASSIFICATION OF IRREDUCIBLE $\overline{SM(1)}$–MODULES

In this section we completely determine Zhu's algebra $A(\overline{SM(1)})$ and classify all irreducible $\overline{SM(1)}$–modules. It turns out that the structure of Zhu's algebra $A(\overline{SM(1)})$ is similar to the structure of Zhu's algebra for $A(\overline{M(1)})$ studied in [A3] and the proofs of the main results are completely analogous.

Recall that $\widehat{H} = Q(e^{-\alpha} \otimes \phi(-\frac{1}{2}))$. Clearly, $\widehat{H}$ is proportional to $G(-\frac{1}{2})H$ and therefore $\widehat{H} \in \overline{SM(1)}$.

Next result shows that Zhu's algebra $A(\overline{SM(1)})$ is commutative.

Theorem 8.1. *Zhu's algebra $A(\overline{SM(1)})$ is spanned by the set*

$$\{[\omega]^{*s}[\widehat{H}]^{*t} \mid s, t \geq 0\}.$$

In particular, Zhu's algebra $A(\overline{SM(1)})$ is isomorphic to a certain quotient of the polynomial algebra $\mathbb{C}[x, y]$, where x and y correspond to $[\omega]$ and $[\widehat{H}]$.

Proof. The proof follows from Proposition 2.1, Theorem 7.3 and because τ and H are odd vectors. $\square$

Let $h^{r,s} = h^{r,s}_{2m+1,1}$, so that $h^{2i+1,1} = \frac{i(i-2m)}{2(2m+1)}$.

As in [A3], for $X = F$, E or H we let $\widehat{X}(n) := \widehat{X}_{2m+n}$ (here as usual $Y(\widehat{X}, z) = \sum_{n \in \mathbb{Z}} \widehat{X}_n z^{-n-1}$). In particular, $\widehat{H}(0)$ is a degree zero operator acting on $\overline{SM(1)}$. Since $\overline{SM(1)} \subset M(1)$ every $M(1, \lambda) \otimes F$ is naturally an $\overline{SM(1)}$-module.

Let $\mathcal{T}$ be the subspace of $M(1) \otimes F$ linearly spanned by the vectors

$$a \otimes b, \quad \text{where } a \in M(1), \ b \in F, \ \deg(b) > 0.$$

(So we only assume that b is homogeneous in F and that it is not proportional to $\mathbf{1}$).

The proof of the following lemma is a consequence of the definition of vertex superalegbra structure on $M(1) \otimes F$.

Lemma 8.1. *Let $\lambda \in \mathfrak{h}^*$ and v_λ be the highest weight vector in $M(1, \lambda) \otimes F$. Assume that $w \in \mathcal{T}$. Then $o(w)v_\lambda = 0$.*

We have the following proposition about the action of "Cartan subalgebra" of $\overline{SM(1)}$ on the top component.

Proposition 8.1. *Let $\lambda \in \mathfrak{h}^*$, $t = \langle \alpha, \lambda \rangle$ and v_λ the highest weight vector in $M(1, \lambda) \otimes F$. Then we have*

$$L(0) \cdot v_\lambda = \frac{t(t - 2m)}{2(2m + 1)} v_\lambda,$$

$$\widehat{H}(0) \cdot v_\lambda = \binom{t}{2m + 1} v_\lambda.$$

Proof. From the very definition of Q and H we see that

$$\widehat{H} = \phi(1/2) S_{2m+1}(\alpha) \phi(-1/2) + w = S_{2m+1}(\alpha) + w,$$

where

$$w = S_{2m-1}(\alpha) \otimes \phi(-\tfrac{3}{2}) \phi(-\tfrac{1}{2}) + \cdots + \mathbf{1} \otimes \phi(-2m - \tfrac{1}{2}) \phi(-\tfrac{1}{2}) \in \mathcal{T}.$$

On the other hand it is known (cf. Proposition 3.1 in [A2]) that

$$S_r(\alpha)(0)v_\lambda = \binom{t}{r} v_\lambda, \ \ r \geq 1.$$

The proof follows. $\square$

It is not hard to see that $x(t) = \frac{t(t-2m)}{2(2m+1)}$ and $y(t) = \binom{t}{2m+1}$ parametrize the genus zero curve $P(x, y) = 0$ where

(8.1) $$P(x, y) = y^2 - C_m \left(x + \frac{m^2}{2(2m + 1)} \right) \prod_{i=0}^{m-1} \left(x - \frac{i(i - 2m)}{2(2m + 1)} \right)^2,$$

where $C_m = \frac{2^{2m+1}(2m+1)^{2m+1}}{(2m+1)!}$. Alternatively, notice that we can write

$$(8.2) \qquad P(x,y) = y^2 - C_m \prod_{i=0}^{2m} \left(x - h^{2i+1,1} \right),$$

By using arguments analogous to those in the proof of Lemma 6.1 from [A3], we obtain the following result:

Lemma 8.2. *In Zhu's algebra $A(\overline{SM(1)})$ we have the following relation*

$$[\widehat{H}] * [\widehat{H}] = C_m \prod_{i=0}^{2m} ([\omega] - h^{2i+1,1}),$$

where C_m is as above.

By using Theorem 8.1, Lemma 8.2 and the same proof as that of Theorem 6.1. from [A3] we get:

Theorem 8.2. *Zhu's algebra $A(\overline{SM(1)})$ is isomorphic to the commutative, associative algebra $\mathbb{C}[x,y]/\langle P(x,y)\rangle$ where $\langle P(x,y)\rangle$ is the ideal in $\mathbb{C}[x,y]$ generated by the polynomial*

$$P(x,y) = y^2 - C_m \prod_{i=0}^{2m}(x - h^{2i+1,1}).$$

The fact that Zhu's algebra $A(\overline{SM(1)})$ is commutative, enable us to study irreducible lowest weight representations of the vertex operator superalgebra $\overline{SM(1)}$. For given $(r,s) \in \mathbb{C}^2$ such that $P(r,s) = 0$ let $\mathcal{L}(r,s)$ be the irreducible lowest weight $\overline{SM(1)}$–module generated by the vector $v_{r,s}$ such that

$$L(m)v = r\delta_{m,0}v_{r,s}, \quad \widehat{H}(m)v = s\delta_{m,0}rv_{r,s} \quad (m \geq 0).$$

Our Theorem 8.2 and standard Zhu's theory imply the following classification result.

Theorem 8.3. *The set*

$$\{\mathcal{L}(r,s) \mid P(r,s) = 0\}$$

provides all non-isomorphic irreducible $\frac{1}{2}\mathbb{Z}_{\geq 0}$-gradable $\overline{SM(1)}$-modules.

By using classification of irreducible $\overline{SM(1)}$–modules and the same proof to that of Theorem 4.3 of [AM1] we get:

Corrolary 8.1. *The vertex operator superalgebra $\overline{SM(1)}$ is simple.*

8.1. **Logarithmic $\overline{SM(1)}$-modules.** In [AM1] we studied logarithmic modules for the singlet vertex algebra $\overline{M(1)}_p$. Here we have a similar result.

As in [AM1], let $M(1, \lambda) \otimes \Omega$ be an $\hat{\mathfrak{h}}$-module, where Ω is a two-dimensional vector space and where $\alpha(0)|_\Omega$ is given by formula

(8.3)
$$\begin{bmatrix} \langle \alpha, \lambda \rangle & 1 \\ 0 & \langle \alpha, \lambda \rangle \end{bmatrix}$$

in some basis $\{w_1, w_2\}$ of Ω (see also [M3]). Then $M(1, \lambda) \otimes F \otimes \Omega$ carries an $\mathfrak{ns}$-module structure.

Proposition 8.2. *The vector space $M(1, \lambda) \otimes F \otimes \Omega$, $\lambda \neq \frac{m}{2m+1}\alpha$ is a genuine logarithmic $\overline{SM(1)}$-module [2], while for $\lambda = \frac{m}{2m+1}\alpha$, $M(1, \lambda) \otimes F \otimes \Omega$ is an ordinary $\overline{SM(1)}$-module.*

Notice that the previous result is in agreement with Theorem 8.2. More precisely, because of the linear term $(x + \frac{m^2}{2(2m+1)})$ in $P(x, y)$, as in the proof of Proposition 7.1 [AM1], Theorem 8.2 can be now used to show that there are no logarithmic self-extension of $M(1, \frac{m}{2m+1}\alpha) \otimes F$.

8.2. **Further properties of $A(\overline{SM(1)})$.** In the next sections we shall make use of the following important technical results.

Proposition 8.3. *In Zhu's algebra $A(\overline{SM(1)})$ we have*

$$[Q^2 e^{-2\alpha}] = B_m f_m([\omega])$$

where

$$f_m([\omega]) = \prod_{i=0}^{3m} ([\omega] - h^{2i+1,1})$$

and

$$B_m = (-1)^m \frac{\binom{2m}{m}(2(2m+1))^{3m+1}}{\binom{4m+1}{m}(3m+1)!^2}.$$

Proof. First we notice that

$$Q^2 e^{-2\alpha} = \nu H_{-2m-2} H + v$$

where $\nu \neq 0$ and $v \in U(\mathfrak{ns}).\mathbf{1}$ (see also [AM2], Lemma 3.3). The above results on the structure of $A(\overline{SM(1)})$ implies that

$$[Q^2 e^{-2\alpha}] = \Phi_m([\omega])$$

---

[2]In other words, the module involves nontrivial Jordan blocks with respect to the action of $L(0)$.

for certain $\Phi_m \in \mathbb{C}[x]$, $\deg \Phi_m \leq 3m + 1$. We shall evaluate the action of $Q^2 e^{-2\alpha}$ on top levels of $\overline{SM(1)}$–modules $M(1, \lambda) \otimes F$. Let v_λ be the highest weight vector in $M(1, \lambda) \otimes F$. First we notice that

$$Q^2 e^{-2\alpha} = \sum_{i=0}^{4m+1} e^\alpha_{-i-1} e^\alpha_i e^{-2\alpha} + w, \quad \text{where } w \in \mathcal{T}.$$

By using a direct calculation similar to that of [AM2] we see that

$$o(Q^2 e^{-2\alpha}) v_\lambda = \sum_{i=0}^{\infty} o(e^\alpha_{-i-1} e^\alpha_i e^{-2\alpha}) v_\lambda$$

$$= \operatorname{Res}_{z_1} \operatorname{Res}_{z_2} \sum_{i=0}^{\infty} z_1^{-i-1} z_2^i (z_1 - z_2)^{2m+1} (z_1 z_2)^{-4m-2} (1 + z_1)^t (1 + z_2)^t v_\lambda$$

$$= \operatorname{Res}_{z_1} \operatorname{Res}_{z_2} (z_1 - z_2)^{2m} (z_1 z_2)^{-4m-2} (1 + z_1)^t (1 + z_2)^t v_\lambda$$

$$= \Phi_m \left(\tfrac{1}{2(2m+1)} (t^2 - 2tm) \right) v_\lambda = \widetilde{\Phi_m}(t) v_\lambda,$$

$$\text{where} \quad \widetilde{\Phi_m}(t) = \sum_{k=0}^{2m} (-1)^k \binom{2m}{k} \binom{t}{4m + 1 - k} \binom{t}{2m + 1 + k},$$

$t = \langle \lambda, \alpha \rangle$.

As in the proof of Lemma 3.4 in [AM2] one can prove the following identity

(8.4) $$\widetilde{\Phi_m}(t) = \bar{A}_m \binom{t}{3m + 1} \binom{t + m}{3m + 1}, \quad \text{where } \bar{A}_m = \frac{(-1)^m \binom{2m}{m}}{\binom{4m+1}{m}}.$$

This implies

$$\Phi_m \left(\tfrac{1}{2(2m+1)} (t^2 - 2mt) \right) = \widetilde{\Phi_m}(t) = B_m f_m \left(\tfrac{1}{2(2m+1)} (t^2 - 2mt) \right).$$

Consequently, Φ_m is a non-trivial polynomial of degree $3m + 1$ and in $A(\overline{SM(1)})$ we have

(8.5) $$[Q^2 e^{-2\alpha}] = \Phi_m([\omega]) = B_m f_m([\omega]), \quad B_m \neq 0 \quad \square$$

Define the following non-trivial vector

$$U^{F,E} := \operatorname{Res}_z Y(F, z) E \frac{(z + 1)^{2m}}{z} \in \overline{SM(1)}.$$

Set $U^{F,E}(0) := o(U^{F,E}) = \sum_{i \geq 0} \binom{2m}{i} o(F_{i-1} E)$.

Proposition 8.4. *In Zhu's algebra $A(\overline{SM(1)})$ we have:*

$$[U^{F,E}] = g([\omega])[\widehat{H}]$$

where $g(x) \in \mathbb{C}[x]$ is of degree at most m.

Proof. First we notice that

$$U^{F,E} = aH - H \circ H$$

for certain $a \in U(\mathfrak{ns})$. This implies that in Zhu's algebra $A(\overline{SM(1)})$, we have

(8.6) $$[U^{F,E}] = g([\omega])[\widehat{H}]$$

where $g \in \mathbb{C}[x]$ is a polynomial of degree at most m. (Here we used the relation $[H \circ H] = 0$, which holds in $A(\overline{SM(1)})$.)

$\square$

It is not at all clear that $g(x)$ is a nonzero polynomial.

9. THE $N = 1$ TRIPLET VERTEX ALGEBRA $\mathcal{SW}(m)$

Define the following vertex superalgebra

$$\mathcal{SW}(m) = \mathrm{Ker}_{V_L \otimes F}\widetilde{Q}.$$

Recall definition (5.7). For any $X \in \{E, F, H\}$, $\widehat{X}$ is proportional to $G(-\frac{1}{2})X$, and therefore $\widehat{X} \in \mathcal{SW}(m)$.

Theorem 9.1.

(1) *For every $m \geq 1$, $\mathcal{SW}(m)$ is an $N = 1$ vertex operator superalgebra and $\mathcal{SW}(m) \cong S\Lambda(1)$.*

(2) *The vertex operator superalgebra $\mathcal{SW}(m)$ is generated by E, F, H and τ.*

(3) *The vertex operator superalgebra $\mathcal{SW}(m)$ is strongly generated by the set*

$$\{\tau, \omega, E, F, H, \widehat{E}, \widehat{F}, \widehat{H}\}.$$

Proof. Recall the structure of $V_L \otimes F$ as a module for the Neveu-Schwarz algebra from Theorem 6.1. By using Lemma 5.1 , similarly to the proof of Theorem 7.2, we conclude that $\mathcal{SW}(m)$ is a completely reducible module for the Neveu-Schwarz algebra, generated by the family of singular vectors:

(9.1) $$Q^j e^{-n\alpha}, \quad n \in \mathbb{Z}_{\geq 0}, \ j \in \{0, \cdots, 2n\}.$$

This proves (1).

Let Z_n be the Neveu-Schwarz module generated by singular vectors

$$Q^j e^{-\ell\alpha}, \quad \ell \leq n, \ j \in \mathbb{Z}_{\geq 0}.$$

Therefore $\mathcal{SW}(m) = \bigcup_{n \in \mathbb{Z}_{\geq 0}} Z_n$. Let now $\mathcal{U}$ be the vertex subalgebra of $\mathcal{SW}(m)$ generated by τ, E, F, H. Clearly, $\mathcal{U} \subseteq \mathcal{SW}(m)$. We shall prove that in fact $\mathcal{U} = \mathcal{SW}(m)$. In order to do so it is

sufficient to show that $Z_n \subseteq \mathcal{U}$ for every $n \in \mathbb{Z}_{>0}$. We shall prove this claim by induction on n. By the definition, the claim holds for $n = 1$. Assume now that $Z_n \subseteq \mathcal{U}$. Set $j_0 = (2m+1)n+1$. As in the proof of Theorem 7.3 we have

$$
\begin{aligned}
F_{-j_0} e^{-n\alpha} &= e^{-(n+1)\alpha}, \\
E_{-j_0} Q^{2n} e^{-n\alpha} &= B_{2n+1} Q^{2n+2} e^{-(n+1)\alpha},
\end{aligned}
$$

where $B_{2n+1} \neq 0$ and

$$
H_{-j_0} Q^j e^{-n\alpha} = B_j Q^{j+1} e^{-(n+1)\alpha} + v'_j,
$$

where $v'_j \in Z_n$, $B_j \neq 0$, $0 \leq j \leq 2n$. These relations imply that $Z_{n+1} \subseteq \mathcal{U}$. By induction we conclude that $Z_n \subseteq \mathcal{U}$ for every $n \in \mathbb{Z}_{>0}$ and therefore $\mathcal{U} = \mathcal{SW}(m)$. This proves (2). The proof of (2) actually gives that $\mathcal{SW}(m)$ is spanned by the vectors

$$
\tag{9.2} u^1_{n_1} \cdots u^r_{n_r} \mathbf{1}, \quad u^i \in \{\tau, E, F, H\}
$$

such that for $1 \leq i \leq r$:

$$
\tag{9.3} n_i \leq -1 \quad \text{if } u^i \in \{E, F, H\} \qquad \text{and} \qquad n_i \leq 0 \quad \text{if } u^i = \tau.
$$

The assertion (3) follows. $\square$

Theorem 9.2. *Assume that $m \geq 1$. Then we have*

(1) *The vertex operator superalgebra $\mathcal{SW}(m)$ is C_2–cofinite.*

(2) *The vertex operator superalgebra $\mathcal{SW}(m)$ is irrational.*

Proof. By using Proposition 2.1, relation (2.3) and Theorem 7.2 we conclude that $\mathcal{SW}(m)/C_2(\mathcal{SW}(m))$ is generated by

$$
\tag{9.4} \overline{\tau}, \overline{\varpi}, \overline{E}, \overline{\widehat{E}}, \overline{F}, \overline{\widehat{F}}, \overline{H}, \overline{\widehat{H}},
$$

and that every two generators either commute or anti-commute. In order to prove C_2–cofiniteness it suffices to prove that every generator (9.4) is nilpotent in $\mathcal{SW}(m)/C_2(\mathcal{SW}(m))$. Let X be either E or F. From Lemma 5.2 we see that $X_{-1} X = 0$, and thus $\overline{X}^2 = 0$. By using

$$
G(-i - 1/2)^2 = L(-2i - 1) \in U(\mathfrak{ns})
$$

we get $\overline{\tau}^2 = 0$. Similarly, from

$$
H_{-1} H \in U(\mathfrak{ns}) \cdot \mathbf{1},
$$

$$
H_{-1} H = \sum_{k/2 + i_1 + \cdots + i_k + j_1 + \cdots + j_s = 4m+1} a_{i_1, \ldots, i_k} G(-i_1 - 1/2) \cdots G(-i_k - 1/2) L(-j_1) \cdots L(-j_s) \mathbf{1},
$$

where

$$i_1 > i_2 > \cdots > i_k \geq 1, \ j_1, \ldots, j_s \geq 2, \ a_{i_1,\ldots,i_k} \in \mathbb{C},$$

it follows that $H_{-1}H \in C_2(\mathcal{SW}(m))$, and thus

$$\overline{H}^2 = \overline{H_{-1}H} = 0.$$

We also have $X_{-1}X = 0$, $X \in \{\widehat{F}, \widehat{E}\}$ in $\mathcal{SW}(m)$ (cf. Lemma 5.3), so that

$$\overline{X}^2 = 0 \ \text{ in } \ \mathcal{SW}(m)/C_2(\mathcal{SW}(m)).$$

Thus, it remains to prove that $\overline{\varpi}$ and $\overline{\widehat{H}}$ are nilpotent. We prove this as in [AM2]. Since

$$\widehat{E}_{-1}\widehat{F} + \widehat{F}_{-1}\widehat{E} + 2\widehat{H}_{-1}\widehat{H} = 0$$

we get

$$\overline{\widehat{H}}^4 = 0.$$

Moreover, the description of Zhu's algebra from Theorem 8.2 implies that

$$\overline{\widehat{H}}^2 = C_m \overline{\varpi}^{2m+1}, \quad (C_m \neq 0),$$

which implies that $\overline{\varpi}^{4m+2} = 0$. Therefore, every generator of $\mathcal{SW}(m)/C_2(\mathcal{SW}(m))$ is nilpotent and $\mathcal{SW}(m)$ is C_2–cofinite. This proves (1).

Assertion (2) follows from the fact that $V_L \otimes F$ is not completely reducible, viewed as $\mathcal{SW}(m)$–module. $\qquad\square$

10. CLASSIFICATION OF IRREDUCIBLE $\mathcal{SW}(m)$–MODULES

From the definition of Zhu's algebra and the structure of the vertex operator superalgebra $\mathcal{SW}(m)$ follows:

Proposition 10.1. *The associative algebra $A(\mathcal{SW}(m))$ is generated by $[\widehat{E}]$, $[\widehat{H}]$, $[\widehat{F}]$ and $[\omega]$.*

Proof. The proof follows from Proposition 2.1, Theorem 9.1 and the fact that τ, E, F and H are all odd. $\qquad\square$

Theorem 10.1. *In Zhu's algebra $A(\mathcal{SW}(m))$ we have the following relation*

$$f_m([\omega]) = 0$$

where

$$f_m(x) = \prod_{i=0}^{3m} (x - h^{2i+1,1}).$$

Proof. Since $O(\overline{SM(1)}) \subset O(\mathcal{SW}(m))$, the embedding $\overline{SM(1)} \subset \mathcal{SW}(m)$ induces an algebra homomorphism $A(\overline{SM(1)}) \rightarrow A(\mathcal{SW}(m))$. Applying this homomorphism to Proposition 8.3 and using the fact that $Q^2 e^{-2\alpha} \in O(\mathcal{SW}(m))$ we get that $f_m([\omega]) = 0$ in $A(\mathcal{SW}(m))$.   □

Alternatively, we can write the polynomial $f_m(x)$ as

$$(10.1) \qquad f_m(x) = (x - h^{2m+1,1}) \prod_{i=0}^{m-1} (x - h^{2i+1,1})^2 \prod_{i=2m+1}^{3m} (x - h^{2i+1,1}),$$

indicating possibility of existence of logarithmic modules of generalized lowest conformal weight $h^{2i+1,1}$, $i = 0, ..., m - 1$.

Theorem 10.2.

(1) *For every $0 \leq i \leq m$, $S\Lambda(i+1)$ is an irreducible $\frac{1}{2}\mathbb{Z}_{\geq 0}$–gradable $\mathcal{SW}(m)$–module, with the top component $S\Lambda(i+1)(0)$ of lowest weight $h^{2i+1,1}$. Moreover, $S\Lambda(i+1)(0)$ is an 1–dimensional irreducible $A(\mathcal{SW}(m))$–module.*

(2) *For every $0 \leq j \leq m-1$, $S\Pi(m-j)$ is an irreducible $\frac{1}{2}\mathbb{Z}_{\geq 0}$–gradable $\mathcal{SW}(m)$–module, with the top component $S\Pi(m-j)(0)$ of lowest weight $h^{2i+1,1}$ where $i = 2m+1+j$. Moreover, $S\Pi(m-j)(0)$ is an 2–dimensional irreducible $A(\mathcal{SW}(m))$–module.*

Proof. Proof is similar to that of Theorem 3.7 in [AM2] so we omit it here.   □

Applying the previous theorem in the case of $\mathcal{SW}(m) = S\Lambda(1)$ we get:

Corrolary 10.1. *The vertex operator superalgebra $\mathcal{SW}(m)$ is simple.*

As in [AM2] we have the following result

Proposition 10.2. *In Zhu's associative algebra we have*

$$(10.2) \qquad\qquad [\widehat{H}] * [\widehat{F}] - [\widehat{F}] * [\widehat{H}] = -2q([\omega])[\widehat{F}],$$

$$(10.3) \qquad\qquad [\widehat{H}] * [\widehat{E}] - [\widehat{E}] * [\widehat{H}] = 2q([\omega])[\widehat{E}]$$

$$(10.4) \qquad\qquad [\widehat{E}] * [\widehat{F}] - [\widehat{F}] * [\widehat{E}] = -2q([\omega])[\widehat{H}].$$

where q is a certain polynomial.

Theorem 10.3. *The set*

$$\{S\Pi(i)(0) : 1 \leq i \leq m\} \cup \ S\Lambda(i)(0) : 1 \leq i \leq m+1\}$$

provides, up to isomorphism, all irreducible modules for Zhu's algebra $A(\mathcal{SW}(m))$.

Proof. The proof is similar to that of Theorem 3.11 in [AM2]. Assume that U is an irreducible $A(\mathcal{SW}(m))$–module. Relation $f_m([\omega]) = 0$ in $A(\mathcal{SW}(m))$ implies that

$$L(0)|U = h^{2i+1,1} \, \text{Id}, \quad \text{for} \quad i \in \{0, \ldots, m\} \cup \{2m+1, \ldots, 3m\}.$$

Assume first that $i = 2m + 1 + j$ for $0 \leq j \leq m - 1$. By combining Propositions 10.2 and Theorem 10.2 we have that $q(h^{2i+1,1}) \neq 0$. Define

$$e = \frac{1}{\sqrt{2}q(h^{2i+1,1})}[\widehat{E}], \quad f = -\frac{1}{\sqrt{2}q(h^{2i+1,1})}[\widehat{F}], \quad h = \frac{1}{q(h^{2i+1,1})}[\widehat{H}].$$

Therefore U carries the structure of an irreducible, $\mathfrak{sl}_2$–module with the property that $e^2 = f^2 = 0$ and $h \neq 0$ on U. This easily implies that U is a 2–dimensional irreducible $\mathfrak{sl}_2$–module. Moreover, as an $A(\mathcal{SW}(m))$–module U is isomorphic to $S\Pi(m - j)(0)$.

Assume next that $0 \leq i \leq m$. If $q(h^{2i+1,1}) \neq 0$, as above we conclude that U is an irreducible 1–dimensional sl_2–module. Therefore $U \cong S\Lambda(i + 1)(0)$.

If $q(h^{2i+1,1}) = 0$, from Proposition 10.2 we have that the action of generators of $A(\mathcal{SW}(m))$ commute on U. Irreducibility of U implies that U is 1-dimensional. Since $[\widehat{H}], [\widehat{E}]^2, [\widehat{F}]^2$ must act trivially on U, we conclude that $[\widehat{H}], [\widehat{E}], [\widehat{F}]$ also act trivially on U. Therefore $U \cong S\Lambda(i + 1)(0)$. $\square$

As a consequence of the previous theorem we have.

Theorem 10.4. *The set*

$$\{S\Pi(i) : 1 \leq i \leq m\} \cup \{S\Lambda(i) : 1 \leq i \leq m + 1\}$$

provides, up to isomorphism, all irreducible modules for the vertex operator superalgebra $\mathcal{SW}(m)$.

11. ON THE STRUCTURE OF ZHU'S ALGEBRA $A(\mathcal{SW}(m))$

As in [AM2], the main difficulty in description of Zhu's algebra $A(\mathcal{SW}(m))$ is that of not having a good understanding of logarithmic $\mathcal{SW}(m)$-modules. For the triplet $\mathcal{W}(p)$ this problem can be resolved, at least if p is prime, by using modular invariance. We believe the same approach can be applied for $\mathcal{SW}(m)$, which would require a super version of Miyamoto's result [Miy]. This is the main reason why in this part we focus mostly on the case $2m+1$ is prime, but we expect all results to be true in general.

In many ways this section is analogous to Section 5 (and Appendix) in [AM2], but as we shall see there are some important differences.

First a few generalities regarding the Lagrange interpolation polynomial.

Proposition 11.1. *Let $S = \{(x_1, y_1), ..., (x_n, y_n)\}$, $x_i \neq x_j$ be a set of points in $\mathbb{C}^2$ such that their Lagrange interpolation polynomial $L_n(x)$ is of degree exactly $n-1$. Then every interpolation polynomial of degree exactly n is given by*

$$Q_\lambda(x) = L_n(x) + \lambda \prod_{i=1}^{n}(x - x_i), \ \ \lambda \neq 0.$$

Proof. Let $P(x)$ be an arbitrary interpolation polynomial of degree n. Then for some λ, the polynomial $P(x) - \lambda \prod_{i=1}^{n}(x - x_i)$ is of degree less or equal $n - 1$, but not zero. But then $P(x) - \lambda \prod_{i=1}^{n}(x - x_i) = L_n(x)$. $\hspace{2cm}$ $\square$

Lemma 11.1. *Let $L_m(x)$ be the Lagrange interpolation polynomial for $(h^{2i+1,1}, \binom{i}{2m+1})$, where $2m + 1 \leq i \leq 3m$. If we let $r(t) = L_n(\frac{t(t-2m)}{2(2m+1)})$, then*

$$r(t) = \frac{\prod_{i=2m+1}^{3m}(t - i)(t - 2m + i)}{(2m + 1)!}$$

$$\times \sum_{i=2m+1}^{3m} \left(\frac{(i!)^2(-1)^{i+m}}{(i - 2m - 1)!^2(3m - i)!(i + m)!} \left(\frac{1}{t - i} - \frac{1}{t - 2m + i}\right) \right) \in \mathbb{C}[t].$$

Now, we have an important technical result (in a slightly different setup a similar result has been proven in Appendix of [AM2]).

Proposition 11.2. *For every $m \geq 1$ we have*

$$L_m(h^{2i+1,1}) \neq 0, \ \ 0 \leq i \leq m.$$

Proof. As in [AM2] it suffices to let

$$s(t) = \frac{r(t)}{\prod_{i=2m+1}^{3m}(t - i)(t - 2m + i)}$$

and check first

$$s(0) < 0, \ \ s(1) < 0,$$

which follows by using hypergeometric summations. That $r(h^{2i+1,1}) \neq 0$ for $0 \leq i \leq m$ follows now from the recursion

$$s(t)(m+t)(2m+1-t)^2 = 2(m+1-t)(2m^2+2tm-2-t^2+2t)s(t-1)+(t-1)^2(3m+2-t)s(t-2),$$

because all coefficients in the recursion are positive for $1 \leq t \leq m$. $\hspace{1cm}$ $\square$

As in Appendix of [AM2] we now observe that

$$\widehat{H} * \widehat{F} = a.F,$$

where

$$a \in U(\mathfrak{ns}).$$

From

$$\deg(\widehat{H}_{-1}\widehat{F}) = 4m + 2,$$

and

(11.1) $$[\widehat{H}] * [\widehat{F}] = -q([\omega])[\widehat{F}],$$

for some $q \in \mathbb{C}[x]$. It follows that $q([\omega])$ is a polynomial of degree at most m. In [AM2] this observation was sufficient to argue that q has to be the interpolation polynomial. However, in view of Proposition 11.1 and Lemma 11.1, we are unable to argue that $q = L_m$, because L_m is of degree $m - 1$. Thus, it is not clear what the q polynomial should be.

Proposition 11.3. *Let $g(x)$ be as in Proposition 8.4 and*

$$u(x) = \prod_{i=2m+1}^{3m} (x - h^{2i+1,1}).$$

Then

$$g(x) = D_m u(x),$$

for some constant D_m. Moreover,

(11.2) $$D_m u([\omega]) * [\widehat{X}] = 0, \quad X \in \{F, H, E\}.$$

Proof. First we notice that $U^{F,E} = F \circ E \in O(\mathcal{SW}(m))$. Then Proposition 8.4 implies that

$$g([\omega]) * [\widehat{H}] = 0 \quad \text{in} \quad A(\mathcal{SW}(m))$$

for some polynomial of degree at most m. Because we already know all irreducible $\mathcal{SW}(m)$-modules we also know that $g([\omega])$ must act as zero on all $\mathcal{SW}(m)$-modules with two-dimensional highest weight subspaces (here $[\widehat{H}]$ acts nontrivially). Thus we know that

$$g([\omega]) = D_m u([\omega])$$

for some constant D_m. Since Q preserves $O(\mathcal{SW}(m))$ we get (11.2). $\square$

It is crucial for our considerations to show that $D_m \neq 0$ (i.e., $g(x) \neq 0$). This will requires an explicit computation of $U^{F,E}(0)$ on the top degree subspaces of certain $\overline{SM(1)}$-modules. We have the following result.

Theorem 11.1. *If $m \in \mathbb{N}$ such that $2m + 1$ is a prime integer, then $g(x) \neq 0$.*

For the proof of this important technical result we refer the reader to Appendix.

If $D_m \neq 0$, Proposition 11.3 and (11.1) we get

$$[\widehat{H}] * [\widehat{F}] = -q([\omega])[\widehat{F}] = -q'([\omega])[\widehat{F}],$$

where $q'([\omega])$ is a polynomial of degree $m - 1$, which forces $q' = L_m$. We should say here that in [AM2] the formula (11.2) was a consequence of a formula analogous to (11.1).

Theorem 11.2. *Assume that $2m + 1$ is prime or $D_m \neq 0$. Then we have*

(i) $[\widehat{E}]^2 = [\widehat{F}]^2 = 0$

(ii) $[\widehat{H}]^2 = C_m P([\omega])$, *where*

$$P(x) = \prod_{i=0}^{2m}(x - h^{2i+1,1}) \in \mathbb{C}[x]$$

and C_m is a nonzero constant.

(iii)

$$[\widehat{H}] * [\widehat{F}] = -[\widehat{F}] * [\widehat{H}] = -q([\omega]) * [\widehat{F}],$$

$$[\widehat{H}] * [\widehat{E}] = -[\widehat{E}] * [\widehat{H}] = q([\omega]) * [\widehat{E}],$$

where $q(x)$ is a nonzero polynomial of degree $m - 1$ and

$$q(h^{2i+1,1}) \neq 0, \ \ 0 \leq i \leq m.$$

(iv)

$$[\widehat{H}] * [\widehat{F}] - [\widehat{F}] * [\widehat{H}] = -2q([\omega])[\widehat{F}],$$

$$[\widehat{H}] * [\widehat{E}] - [\widehat{E}] * [\widehat{H}] = 2q([\omega])[\widehat{E}],$$

$$[\widehat{E}] * [\widehat{F}] - [\widehat{F}] * [\widehat{E}] = -2q([\omega])[\widehat{H}],$$

where $q(x)$ is as in (iii).

(v)

$$\prod_{i=2m+1}^{3m}([\omega] - h^{2i+1,1}) * [X] = 0, \ \ X \in \{\widehat{E}, \widehat{F}, \widehat{H}\}.$$

(vi) *The center of $A(\mathcal{SW}(m))$ is a subalgebra generated by $[\omega]$.*

Proof. We recall that $\mathcal{SW}(m)$ is generated by $[\omega]$ and $[\widehat{X}]$, $X = F$, H and E (see Proposition 10.1).

For (i) we recall [AM2] that Q lifts to a derivation of $A(\mathcal{SW}(m))$, denoted by the same symbol. Now, because of Lemma 5.3 we have

$$[\widehat{F}] * [\widehat{F}] = [\widehat{E}] * [\widehat{E}] = 0.$$

Part (ii) has been proven in Lemma 8.2.

It is left to show relations (iii), (iv) and (v). As in [AM2] we compute

$$0 = Q([\widehat{F}] * [\widehat{F}]) = [\widehat{H}] * [\widehat{F}] + [\widehat{F}] * [\widehat{H}],$$

which yields

$$[\widehat{H}] * [\widehat{F}] = -[\widehat{F}] * [\widehat{H}].$$

After an application of Q^2 on the previous equation we get

$$[\widehat{H}] * [\widehat{E}] = -[\widehat{E}] * [\widehat{H}].$$

Two remaining formulas in (iii)

(11.3) $$[\widehat{H}] * [\widehat{F}] = -q([\omega]) * [\widehat{F}],$$

$$[\widehat{H}] * [\widehat{E}] = q([\omega]) * [\widehat{E}],$$

have already been proven in the discussion preceding the theorem.

The relation (iv) follow from (iii) (cf. [AM2]). Part (v) follows directly from Proposition 11.3. Part (vi) follows from the fact that $q([\omega])$ is a unit in $A(\mathcal{SW}(m))$. $\qquad\square$

Corrolary 11.1. *Under the assumptions of Theorem 11.2, the associative algebra $A(\mathcal{SW}(m))$ is spanned by*

$$\{[\omega]^i,\ 0 \le i \le 3m\} \cup \{[\omega]^i * [X], 0 \le i \le m-1,\ X = \widehat{E}, \widehat{F} \text{ or } \widehat{H}\}.$$

Thus, $A(\mathcal{SW}(m))$ is at most $6m+1$-dimensional.

By using the same ideas as in [AM2] it is not hard to show that

Theorem 11.3. *Assume $2m+1$ be prime or $D_m \ne 0$. Then the Zhu's algebra decomposes as a sum of ideals*

$$A(\mathcal{SW}(m)) = \bigoplus_{i=2m+1}^{3m} \mathbb{M}_{h^{2i+1,1}} \oplus \bigoplus_{i=0}^{m-1} \mathbb{I}_{h^{2i+1,1}} \oplus \mathbb{C}_{h^{2m+1,1}},$$

where $\mathbb{M}_{h^{2i+1,1}} \cong M_2(\mathbb{C})$, $1 \le \dim(\mathbb{I}_{h^{2i+1,1}}) \le 2$ and $\mathbb{C}_{h^{2m+1,1}}$ is one-dimensional.

It is also not hard to find explicit generators for every ideal, in parallel with [AM2].

As with the triplet we expect that all $\mathbb{I}_{h^{2i+1,1}}$ are two-dimensional (which is related to exis-
tence of logarithmic modules). This is equivalent to

Conjecture 11.1. *The associative algebra* $A(\mathcal{SW}(m))$ *is* $6m + 1$*-dimensional.   Then the center of*
$A(\mathcal{SW}(m))$ *is* $3m + 1$*-dimensional.*

Remark 11.1. Dong and Jiang have recently proven [DJ] that if $A(V)$ is semisimple and every
irreducible admissible module is an ordinary module, then V is rational. It is feasible to assume
that their result applies for vertex operator superalgebras. This would imply $\dim(\mathbb{I}_{h^{2i+1,1}}) = 2$
for at least one i, and in particular

$$\dim A(\mathcal{SW}(1)) = 7.$$

(Note that in the case $m = 1$, $D_1 \neq 0$ certainly holds.)

12. MODULAR PROPERTIES OF CHARACTERS OF IRREDUCIBLE $\mathcal{SW}(m)$-MODULES

We first introduce several basic facts regarding classical modular forms needed for descrip-
tion of irreducible $\mathcal{SW}(m)$ characters. The Dedekind η-function is usually defined as the infi-
nite product

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty}(1 - q^n),$$

an automorphic form of weight $\frac{1}{2}$. As usual in all these formulas $q = e^{2\pi i \tau}$, $\tau \in \mathbb{H}$ [3]. We also
introduce

$$(12.1) \qquad \mathfrak{f}(\tau) \;=\; q^{-1/48} \prod_{n=0}^{\infty}(1 + q^{n+1/2}),$$

$$(12.2) \qquad \mathfrak{f}_1(\tau) \;=\; q^{-1/48} \prod_{n=1}^{\infty}(1 - q^{n-1/2}),$$

$$(12.3) \qquad \mathfrak{f}_2(\tau) \;=\; q^{1/24} \prod_{n=1}^{\infty}(1 + q^n).$$

These (slightly normalized) Weber functions form a vector-valued modular form of weight
zero. More precisely,

$$\mathfrak{f}(-1/\tau) = \mathfrak{f}(\tau), \; \mathfrak{f}_2(-1/\tau) = \frac{1}{\sqrt{2}}\mathfrak{f}_1(\tau), \; \mathfrak{f}_1(-1/\tau) = \sqrt{2}\mathfrak{f}_2(\tau),$$

---

[3]Here τ - the coordinate of $\mathbb{H}$ - should not be confused with the superconformal vector used in previous sections.

$$\mathfrak{f}(\tau+1) = e^{-2\pi i/48}\mathfrak{f}_1(\tau), \ \mathfrak{f}_2(\tau+1) = e^{2\pi i/24}\mathfrak{f}_2(\tau), \ \mathfrak{f}_1(\tau+1) = e^{-2\pi i/48}\mathfrak{f}(\tau).$$

In what follows, we denote by

$$\Theta_{j,k}(\tau) = \sum_{n\in\mathbb{Z}} q^{(2kn+j)^2/4k}$$

Jacobi-Riemann Θ-series where $j \in \mathbb{Z}$ and $k \in \mathbb{N}/2$. We also let

$$(\partial\Theta)_{j,k}(\tau) = \sum_{n\in\mathbb{Z}} (2kn+j)q^{(2kn+j)^2/4k}.$$

Then we have transformation formulas (notice that here $k \in \mathbb{N}/2$ so $\Theta_{j,k}(\tau)$ is not invariant under $\tau \longrightarrow \tau+1$ in general):

$$(12.4) \qquad \eta(-1/\tau) = \sqrt{-i\tau}\eta(\tau), \ \ \eta(\tau+1) = e^{\pi i/12}\eta(\tau)$$

$$(12.5) \qquad \Theta_{j,k}(-1/\tau) = \sqrt{\frac{-i\tau}{2k}} \sum_{j'=0}^{2k-1} e^{i\pi jj'/k}\Theta_{j',k}(\tau)$$

$$(12.6) \qquad \Theta_{j,k}(\tau+2) = e^{i\pi j^2/k}\Theta_{j,k}(\tau)$$

$$(12.7) \qquad (\partial\Theta)_{j,k}(\tau+2) = e^{i\pi j^2/k}(\partial\Theta)_{j,k}(\tau),$$

$$(12.8) \qquad (\partial\Theta)_{j,k}(-1/\tau) = (-\tau)\sqrt{-i\tau/2k} \sum_{j'=1}^{2k-1} e^{i\pi jj'/k}(\partial\Theta)_{j',k}(\tau).$$

For a vertex operator algebra module M we define its graded-dimension or simply character

$$\chi_M(\tau) = \mathrm{tr}|_M q^{L(0)-c/24}.$$

If $V = L^{\mathrm{ns}}(c_{2m+1,0}, 0)$ and $M = L(c_{2m+1,0}, h^{2i+1,2n+1})$, then (see [IK2], for instance)

$$(12.9) \qquad \chi_{L^{\mathrm{ns}}(c_{2m+1,1}, h^{2i+1,2n+1})}(\tau) = q^{\frac{m^2}{2(2m+1)}} \frac{\mathfrak{f}(\tau)}{\eta(\tau)} \left(q^{h^{2i+1,2n+1}} - q^{h^{2i+1,-2n-1}} \right).$$

By combining Theorem 6.1, 6.2 and 6.3, and formula (12.9) we obtain

Proposition 12.1. *For $i = 0, ..., m-1$*

$$(12.10) \qquad \chi_{S\Lambda(i+1)}(\tau) = \frac{\mathfrak{f}(\tau)}{\eta(\tau)} \left(\frac{2i+1}{2m+1}\Theta_{m-i,\frac{2m+1}{2}}(\tau) + \frac{2}{2m+1}(\partial\Theta)_{m-i,\frac{2m+1}{2}}(\tau) \right),$$

$$(12.11) \qquad \chi_{S\Pi(m-i)}(\tau) = \frac{\mathfrak{f}(\tau)}{\eta(\tau)} \left(\frac{2m-2i}{2m+1}\Theta_{m-i,\frac{2m+1}{2}}(\tau) - \frac{2}{2m+1}(\partial\Theta)_{m-i,\frac{2m+1}{2}}(\tau) \right).$$

Also,

$$(12.12) \qquad \chi_{S\Lambda(m+1)}(\tau) = \frac{\mathfrak{f}(\tau)}{\eta(\tau)}\Theta_{0,\frac{2m+1}{2}}(\tau).$$

For purposes of modular invariance, it is also important to compute *supercharacters* of irreducible modules. Let us recall that a supercharacter of a V-module M is defined

$$\chi_M^F(\tau) = \mathrm{tr}|_M \sigma q^{L(0)-c/24},$$

where σ is the sign operator taking values 1 (resp. -1) on even (resp. odd) vectors.

In parallel with Proposition 12.1, it is not hard to compute irreducible supercharacters of $\mathcal{SW}(m)$-modules. Here is an explicit description in terms of Θ-constants and their derivatives.

Proposition 12.2. *For $i = 0, ..., m-1$*

$$\chi_{S\Lambda(i+1)}^F(\tau) =$$

(12.13)
$$\frac{\mathfrak{f}_2(\tau)}{\eta(\tau)} \left(\frac{2i+1}{2m+1} \left(\Theta_{2(m-i),2(2m+1)}(\tau) - \Theta_{2(m+i+1),2(2m+1)}(\tau) \right) + \right.$$

(12.14)
$$\left. + \frac{1}{2m+1} \left((\partial\Theta)_{2(m-i),2(2m+1)}(\tau) - (\partial\Theta)_{2(m+i+1),2(2m+1)}(\tau) \right) \right).$$

$$\chi_{S\Pi(m-i)}^F(\tau) =$$

(12.15)
$$\frac{\mathfrak{f}_2(\tau)}{\eta(\tau)} \left(\frac{2m-2i}{2m+1} \left(\Theta_{2(m-i),2(2m+1)}(\tau) - \Theta_{2(m+i+1),2(2m+1)}(\tau) \right) + \right.$$

(12.16)
$$\left. - \frac{1}{2m+1} \left((\partial\Theta)_{2(m-i),2(2m+1)}(\tau) - (\partial\Theta)_{2(m+i+1),2(2m+1)}(\tau) \right) \right).$$

Also,

(12.17)
$$\chi_{S\Lambda(m+1)}^F(\tau) = \frac{\mathfrak{f}_2(\tau)}{\eta(\tau)} \left(\Theta_{0,2(2m+1)}(\tau) - \Theta_{2(2m+1),2(2m+1)}(\tau) \right).$$

As in [F2] we now study modular invariance properties of irreducible $\mathcal{SW}(m)$ characters and supercharacters. We only consider some special modular transformations. For example,

$$\chi_{S\Lambda(i+1)}(-1/\tau) = \frac{\mathfrak{f}(\tau)}{\eta(\tau)} \sum_{k=0}^{2m} \lambda_k \Theta_{k,\frac{2m+1}{2}}(\tau) +$$

$$+ \frac{\mathfrak{f}(\tau)}{\eta(\tau)}(-\tau) \sum_{j=1}^{2m} \nu_j (\partial\Theta)_{j,\frac{2m+1}{2}}(\tau),$$

for some constants λ_k and ν_j. Because of

$$\Theta_{j,k} = \Theta_{-j,k} = \Theta_{2k-j,k} = \Theta_{2k+j,k},$$

$$(\partial\Theta)_{j,k} = -(\partial\Theta)_{-j,k}$$

the previous formula indicates that

$$\text{(12.18)} \qquad \tau \frac{\mathfrak{f}(\tau)}{\eta(\tau)} (\partial\Theta)_{j, \frac{2m+1}{2}}(\tau), \ \ j = 1, ..., m$$

have to be added to the vector space spanned by irreducible $\mathcal{SW}(m)$ characters in order to preserve modular invariance. In the case of the triplet vertex algebra expressions similar to (12.18) could be interpreted as Miyamoto's pseudocharacters (cf. [AM2]). On the other hand, the T transformation $\tau \mapsto \tau + 1$, maps characters to supercharacters (multiplied with appropriate scalars). In order to find an $SL(2, \mathbb{Z})$-closure, we would have to apply the S transformation on the space of supercharacters, but this requires a knowledge of irreducible σ-twisted characters. Since we do not study σ-twisted $\mathcal{SW}(m)$-modules in this paper, at this point we record modular invariance property for the untwisted sector only.

Theorem 12.1. *The vector space $\mathcal{NS}$ spanned by:*

$$\chi_{S\Lambda(m+1)}(\tau), \ \chi_{S\Lambda(i+1)}(\tau), \ \chi_{S\Pi(m-i)}(\tau), \ \ i = 0, ..., m-1,$$

$$\text{(12.19)} \qquad \tau \frac{\mathfrak{f}(\tau)}{\eta(\tau)} (\partial\Theta)_{m-i, \frac{2m+1}{2}}(\tau), \ \ i = 0, ..., m-1$$

is $(3m + 1)$-dimensional and invariant under the subgroup $\Gamma_\theta \subset SL(2, \mathbb{Z})$, where $\Gamma_\theta = \langle S, T^2 \rangle$.

Remark 12.1. We expect that S-transforms of (generalized) supercharacters are expressible in terms of characters and generalized characters of σ-twisted $\mathcal{SW}(m)$-modules. More precisely, appropriately defined vector space spanned by characters and generalized supercharacters , denoted by $\widetilde{\mathcal{NS}}$, and the vector space spanned by characters and generalized characters of σ-twisted modules, denoted by $\mathcal{R}$, should be inter-related as on the diagram
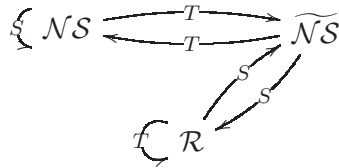
It is known that (super)characters of $N = 1$ minimal models in NS and R sector transform according to this picture (see [IK1]).

13. $\mathcal{SW}(m)$-CHARACTERS AND q-SERIES IDENTITIES

In this section we discuss fermionic expressions for irreducible characters of $\mathcal{SW}(m)$-modules. As we shall see irreducible $\mathcal{SW}(m)$-modules admit q-series formulas similar to those for the triplet, conjectured by Flohr-Grabov-Koehn [FGK], and proven by Warnaar [Wa] (Feigin et al.

independently obtained similar identities by using different methods [FFT]). More precisely, the characters of irreducible modules for the super triplet $\mathcal{SW}(m)$ are intimately related to characters of irreducible $\mathcal{W}(2m+1)$-modules. It is not clear whether a deeper connection persists beyond characters.

13.1. **The $m=1$ case: first computation.** Motivated by computations in [FGK] for $\mathcal{W}(2)$, here we probe double-sum fermionic expressions of irreducible characters of $\mathcal{SW}(1)$-modules.

As usual, we will be using

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}),$$

$$(a;q)_\infty = \prod_{i=1}^\infty (1-aq^{i-1}),$$

and sometimes we shall write

$$(q)_n = (q;q)_n,$$

for simplicity.

We start a basic relation

(13.1)
$$\frac{\prod_{n=1}^\infty (1+q^{n-1/2})}{\prod_{n=1}^\infty (1-q^n)} = \frac{1}{\prod_{n=1}^\infty (1-q^{n/2})(1+q^n)}.$$

We shall also use a Durfee rectangle identities which hold for every $k \in \mathbb{Z}_{\geq 0}$,

$$\frac{1}{\prod_{n\geq 1}(1-q^{n/2})} = \sum_{n=0}^\infty \frac{q^{(n^2+kn)/2}}{(q^{1/2})_n (q^{1/2})_{n+k}}$$

(13.2)
$$= \sum_{n=0}^\infty \frac{(-q^{1/2})_n (-q^{1/2})_{n+k} q^{(n^2+kn)/2}}{(q)_n (q)_{n+k}}.$$

Another useful elementary formula due to Euler is

(13.3)
$$\eta(q) = q^{1/24} \sum_{n=0}^\infty \frac{(-1)^n q^{(n+1)n/2}}{(q)_n}.$$

For $m=1$ there are three irreducible characters. We will focus here on

(13.4)
$$\chi_{S\Lambda(1)}(\tau) = \frac{\mathfrak{f}(\tau)}{\eta(\tau)}\left(\frac{1}{3}\Theta_{1,\frac{3}{2}}(\tau) + \frac{2}{3}(\partial\Theta)_{1,\frac{3}{2}}(\tau)\right).$$

We first notice a theta-function identity

$$(\partial\Theta)_{1,3/2}(\tau) = \frac{\eta(\tau)^3}{\mathfrak{f}(\tau)^2},$$

(essentially, a consequence of the Jacobi triple product identity) or equivalently

$$\frac{\mathfrak{f}(\tau)}{\eta(\tau)}(\partial\Theta)_{1,3/2}(\tau) = \frac{\eta(\tau)^2}{\mathfrak{f}(\tau)}.$$

Now, we apply the relation

$$\mathfrak{f}(\tau) = \frac{\eta(\tau)^2}{\eta(\tau/2)\eta(2\tau)}$$

and (13.3), so we obtain

(13.5)
$$\frac{\mathfrak{f}(\tau)}{\eta(\tau)}(\partial\Theta)_{1,3/2}(\tau) = \eta(2\tau)\eta(\tau/2) = q^{5/48}\prod_{n=1}^{\infty}(1-q^{2n})(1-q^{n/2})$$

$$= q^{5/48}\sum_{(m_1,m_2)\in\mathbb{Z}_{\geq 0}^2}\frac{(-1)^{m_1+m_2}(-q^{1/2};q^{1/2})_{m_2}q^{m_1(m_1+1)+m_2(m_2+1)/4}}{(q^2)_{m_1}(q)_{m_2}}.$$

On the other hand Durfee square identity (13.2) yields (after some computation)

$$\frac{\mathfrak{f}(\tau)}{\eta(\tau)}\Theta_{1,3/2}(\tau)$$

(13.6)
$$= \frac{q^{5/48}}{(-q;q)_{\infty}}\sum_{\substack{(m_1,m_2)\in\mathbb{Z}_{\geq 0}^2 \\ m_1\equiv m_2\,(2)}}\frac{(-q^{1/2};q^{1/2})_{m_1}(-q^{1/2};q^{1/2})_{m_2}q^{\frac{3(m_1-m_2)^2}{8}+\frac{(m_1-m_2)}{2}+\frac{m_1m_2}{2}}}{(q)_{m_1}(q)_{m_2}}.$$

Evidently, double fermionic expressions for $(\partial\Theta)_{1,3/2}(\tau)$ and $\Theta_{1,3/2}(\tau)$ (cf. formulas (13.5) and (13.6), respectively) appear to have little in common, so it is unclear to us that (13.4) admits representation as a closed *double* fermionic sum. Thus, it appears that the $m = 1$ case is rather different compared to the triplet $\mathcal{W}(2)$. This is perhaps reflected by the fact that $p = 2$ triplet admits a fermionic construction, while such a realization seems to be absent for $\mathcal{SW}(1)$ and its modules.

13.2. **Irreducible $\mathcal{SW}(m)$ characters from $\mathcal{W}(2m + 1)$ characters.** In this part we will be using character formulas of irreducible $\mathcal{W}(p)$-modules (see for instance (6.34) and (6.35) in [AM2], or [FHST]). Recall

$$\mathfrak{f}_2(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1 + q^n).$$

The first result in this part is

Proposition 13.1.   (i) *For $0 \leq i \leq m$, we have*

$$\chi_{S\Lambda(i+1)}(\tau) = \frac{\chi_{\Lambda(2i+1)}\left(\frac{\tau}{2}\right)}{\mathfrak{f}_2(\tau)}.$$

(ii) *For $0 \leq i \leq m - 1$, we also have*

$$\chi_{S\Pi(m-i)}(\tau) = \frac{\chi_{\Pi(2m-2i)}\left(\frac{\tau}{2}\right)}{\mathfrak{f}_2(\tau)}.$$

Here $\Lambda(i)$ and $\Pi(2m + 2 - i)$, $i = 1, \ldots, 2m + 1$, are irreducible $\mathcal{W}(2m + 1)$-modules [AM2].

Proof. The proof follows from character formulas for irreducible $\mathcal{W}(p)$-modules, Theorem 12.1, and the following transformation formulas

$$\Theta_{2j,2m+1}(\tau/2) = \Theta_{j,\frac{2m+1}{2}}(\tau),$$

$$(\partial\Theta)_{2j,2m+1}(\tau/2) = 2(\partial\Theta)_{j,\frac{2m+1}{2}}(\tau),$$

$$\frac{1}{\eta(\tau/2)} = \frac{\mathfrak{f}(\tau)}{\eta(\tau)}\mathfrak{f}_2(\tau).$$

$\square$

We recall two multi-sum identities obtain recently by Warnaar [Wa] (these identities are essentially conjectures from [FGK]):

Theorem 13.1. *For $\lambda = 0, \ldots, p$ and $\sigma \in \{0, 1\}$ we have*

(13.7)
$$\sum_{\substack{n_1, \ldots, n_p = 0 \\ n_{p-1} + n_p \equiv 0 \, (2)}} \frac{q^{\sum_{i,j=1}^{p} B_{i,j} n_i n_j + \lambda/2(n_{p-1} - n_p + \sigma) - \sigma p/4}}{(q; q)_{n_1} \cdots (q; q)_{n_k}}$$

$$= \frac{1}{(q; q)_{\infty}} \sum_{n \in \mathbb{Z}} q^{pn^2 + (\lambda - \sigma p)n}$$

and

(13.8)
$$\sum_{\substack{n_1, \ldots, n_p = 0, \\ n_{p-1} + n_p \equiv 0 \, (2)}} \frac{q^{\sum_{i,j=1}^{p} B_{i,j} n_i n_j + \lambda/2(n_{p-1} + n_p + \sigma) + \sum_{p-\lambda}^{p-2} (i-p+\lambda+1)n_i - \sigma p/4}}{(q; q)_{n_1} \cdots (q; q)_{n_k}}$$

$$= \frac{1}{(q; q)_{\infty}} \sum_{n \in \mathbb{Z}} (2n - \sigma + 1) q^{pn^2 + (\lambda - \sigma p)n},$$

where $B_{i,j}$ are entries of the inverse Cartan matrix of the Lie algebra D_p.

Equipped with Warnaar's formulas and Proposition 13.1 it is now not hard to prove the next result

Theorem 13.2. *We have the following formulas for irreducible* $\mathcal{SW}(m)$*-characters:*

(13.9)
$$q^{-1/16}\chi_{S\Lambda(m+1)}(\tau)$$

$$= \sum_{\substack{n_1,\ldots,n_{2m+1}=0,\\ n_{2m}+n_{2m+1}\equiv 0\,(2)}} \frac{(-q^{1/2};q^{1/2})_{n_1}\cdots(-q^{1/2};q^{1/2})_{n_{2m+1}}q^{\sum_{k,l=1}^{2m+1}B_{k,l}n_k n_l/2}}{(-q;q)_\infty(q;q)_{n_1}\cdots(q;q)_{n_{2m+1}}}.$$

For $i=0,\ldots,m-1$*, we have*

$$q^{-a_{i,m}}\chi_{S\Lambda(i+1)}(\tau)$$

$$= \sum_{\substack{n_1,\ldots,n_{2m+1}=0\\ n_{2m}+n_{2m+1}\equiv 0\,(2)}} \frac{(-q^{1/2};q^{1/2})_{n_1}\cdots(-q^{1/2};q^{1/2})_{n_{2m+1}}q^{\sum_{k,l=1}^{2m+1}B_{k,l}n_k n_l/2+(m-i)(n_{2m}+n_{2m+1})/2+\sum_{k=2i+1}^{2m-1}(k-2i)n_k/2}}{(-q;q)_\infty(q;q)_{n_1}\cdots(q;q)_{n_{2m+1}}},$$

and

$$q^{-b_{i,m}}\chi_{S\Pi(m-i)}(\tau)$$

$$= \sum_{\substack{n_1,\ldots,n_{2m+1}=0\\ n_{2m}+n_{2m+1}\equiv 1\,(2)}} \frac{(-q^{1/2};q^{1/2})_{n_1}\cdots(-q^{1/2};q^{1/2})_{n_{2m+1}}q^{\sum_{k,l=1}^{2m+1}B_{k,l}n_k n_l/2+(m-i)(n_{2m}+n_{2m+1})/2+\sum_{k=2i+1}^{2m-1}(k-2i)n_k/2}}{(-q;q)_\infty(q;q)_{n_1}\cdots(q;q)_{n_{2m+1}}},$$

where $a_{i,m}$ *and* $b_{i,m}$ *are certain rational numbers.*

Proof. We prove the middle formula only. The other two formulas follow along the same lines.

Recall that

(13.10)
$$\chi_{S\Lambda(i+1)}(\tau) = \frac{f(\tau)}{\eta(\tau)}\left(\frac{2i+1}{2m+1}\Theta_{m-i,\frac{2m+1}{2}}(\tau) + \frac{2}{2m+1}(\partial\Theta)_{m-i,\frac{2m+1}{2}}(\tau)\right).$$

Now,

$$\frac{2i+1}{2m+1}\Theta_{m-i,\frac{2m+1}{2}}(\tau) + \frac{2}{2m+1}(\partial\Theta)_{m-i,\frac{2m+1}{2}}(\tau)$$

$$= q^{(m-i)^2/(2(2m+1))}\sum_{n\in\mathbb{Z}}(2n+1)q^{\frac{(2m+1)n^2+2(m-i)n}{2}}.$$

Finally, if we substitute $q^{1/2}$ for q in (13.8), and let $p=2m+1$, $\sigma=0$, $\lambda=2m-2i$, and apply formula (13.1) and simple identity

$$\frac{1}{(q^{1/2};q^{1/2})_n} = \frac{(-q^{1/2};q^{1/2})_n}{(q)_n}.$$

the proof automatically follows. □

14. A CONJECTURAL RELATION OF $\mathcal{SW}(m)$ WITH QUANTUM GROUPS

Let $\hat{\mathfrak{g}}$ be an untwisted affine Kac-Moody Lie algebra. Then there is a well-known (Kazhdan-Lusztig) equivalence between the tensor category of $L_{\mathfrak{g}}(k, 0)$-modules $k \in \mathbb{N}$, and the semisimple part of the tensor category of $U_q(\mathfrak{g})$-modules where q is a certain root of unity (not to be confused with $q = e^{2\pi i \tau}$ used in the previous section) depending on the level k and $\mathfrak{g}$ [Fi]. Notice that on the quantum group side we have a semisimplified category, and not the full category of $U_q(\mathfrak{g})$-modules.

In [FGST1] and [FGST2] (see also [Se]) the authors proposed a remarkable equivalence between the (enhanced) tensor category of $\mathcal{W}(p)$-modules and the category of $\overline{U_q(sl_2)}$-modules, $q = e^{i\pi/p}$, where $\overline{U_q(sl_2)}$ is the *restricted* finite-dimensional quantum group. While this is still a conjecture for $p > 2$, the same authors established an important weaker equivalence among the $SL(2, \mathbb{Z})$-module $\mathfrak{Z}_{cft}$ formed by generalized $\mathcal{W}(p)$ characters and the $SL(2, \mathbb{Z})$-module $\mathfrak{Z}$, the center of $\overline{U_q(sl_2)}$. Thus, it is natural question to find Kazhdan-Lusztig dual of the category or ordinary and logarithmic $\mathcal{SW}(m)$-modules. In our case the relevant space of generalized characters is the Γ_θ invariant subspace described in Theorem 12.1, which is $3m + 1$-dimensional.

As indicated in the introduction, we believe that the quantum group $U_q^{small}(sl_2)$, $q = e^{\frac{2i\pi}{2m+1}}$ is relevant for the supertriplet $\mathcal{SW}(m)$. Here are some evidences. Firstly, both $\mathcal{SW}(m)$ and $U_q^{small}(sl_2)$ have the same number of inequivalent irreducible representations. Also, in [Ker] (see also [La]) it was proven that the center of $U_q^{small}(sl_2)$ is $3m + 1$-dimensional, and that it carries a projective action of the modular group. Notice that $3m + 1$ is also (conjecturally) the dimension of the center of $A(\mathcal{SW}(m))$. Thus, in parallel with [FGST1], we expect the following conjecture to be true.

Conjecture 14.1. *The category of weak $\mathcal{SW}(m)$-modules is equivalent to the category of $U_q^{small}(sl_2)$-modules, where $q = e^{\frac{2\pi i}{2m+1}}$.*

Finally, Proposition 13.1 is a strong indication for a possible that the category of $\mathcal{SW}(m)$-modules should be related to a subcategory of $\mathcal{W}(2m + 1)$ and $\overline{U_q(sl_2)}$-modules, $q = e^{\pi i/(2m+1)}$.

15. OUTLOOK AND FINAL REMARKS

There are several research directions we plan to pursue in the future. Let us mention only a few we found the most interesting.

(i) The most important problem that we left open is the existence and description of logarithmic $\mathcal{SW}(m)$-modules. We strongly believe the ideas based on modular invariance as in [AM2] could be successfully applied for the super triplet.

(ii) As with any $N = 1$ vertex operator superalgebra, the most obvious next step would be to examine the category of σ-twisted $\mathcal{SW}(m)$-modules, where σ is the parity automorphisms. As we already indicated (cf. Theorem 12.1) the space of $SL(2, \mathbb{Z})$-transforms of irreducible $\mathcal{SW}(m)$-modules should close a finite-dimensional vector space. Supposedly characters of irreducible σ-twisted modules are included in the same vector space (cf. Remark 12.1).

(iii) Singular vectors in Feigin-Fuchs modules for the $N = 1$ Neveu-Schwarz algebra certainly deserve more attention. We expect these vectors to have description in terms of modified Jack polynomials and as kernels of super Calogero-Sutherland operators. Similar results for the Virasoro algebra have been obtained in [MY].

(iv) Our fermionic expressions for the $\mathcal{SW}(m)$-characters indicate a possibility of parafermionic (or quasiparticle) bases for $\mathcal{SW}(m)$-modules. For the triplet $\mathcal{W}(p)$ this problem has been resolved in [FFT].

16. Appendix

Here we prove Theorem 11.1 and give a strong evidence that in Proposition 11.3 the polynomial $g(x)$ is nonzero for every m. In the process of proving these results we discovered certain constant term identities which are of independent interest.

We recall

$$U^{F,E} := \operatorname{Res}_z \frac{(1+z)^{2m}}{z} Y(F, z) E \in \overline{SM(1)}.$$

Then we have

$$U^{F,E}(0) := o(U^{F,E}) = \sum_{i \geq 0} \binom{2m}{i} o(F_{i-1} E).$$

In Proposition 8.4 we proved that inside $A(\overline{SM(1)})$ we have the relation

$$[U^{F,E}] = g([\omega])[\widehat{H}].$$

Because of the homomorphism from $A(\overline{SM(1)})$ to $A(\mathcal{SW}(m))$ and Proposition 11.3 it is sufficient to show that $U^{F,E}(0)$ acts nontrivially on the top components of at least one $SM(1)$-modules $M(1, \lambda) \otimes F$.

Proposition 16.1. *Let v_λ be the highest weight vector in $M(1, \lambda) \otimes F$. Then we have*

$$U^{F,E}(0) \cdot v_\lambda = -G_m(t)v_\lambda,$$

where $t = \langle \lambda, \alpha \rangle$ and

$$G_m(t) = \sum_{l=1}^{2m+1} \sum_{i=0}^{l-1} \sum_{j=0}^{2m+1+i-l} \sum_{k=0}^{l-1-i} (-1)^{j+k+l} \binom{2m+1}{l} \binom{-2m-1}{j} \cdot$$

$$\binom{-2m-1}{k} \binom{2m-t}{j+k+2m+1} \binom{t}{i-j-l+2m+1} \binom{t}{l-k-1-i}.$$

Proof. It is not hard to see that

$$U^{F,E} = \left(\sum_{i=0}^{2m} \text{Res}_{z_1} \text{Res}_{z_2} \text{Res}_{z_3} \frac{(1+z_1)^{2m}}{z_1} z_2^{-i-1} z_3^i Y(e^{-\alpha}, z_1) Y(e^\alpha, z_2) Y(e^\alpha, z_3) e^{-\alpha} \right) + w,$$

where $w \in \mathcal{T}$. By repeatedly using the well-known formula (cf. [LL])

$$E^+(\delta, x)E^-(\gamma, y) = (1 - y/x)^{\langle \delta, \gamma \rangle} E^-(\gamma, y)E^+(\delta, x),$$

which holds for every $\delta, \gamma \in \mathbb{Z}\beta$, we get

$$U^{F,E} = \sum_{i=0}^{2m} \text{Res}_{z_1} \text{Res}_{z_2} \text{Res}_{z_3} \frac{(1+z_1)^{2m}}{z_1} z_2^{-i-1} z_3^i (z_1 z_2 z_3)^{-2m-1}.$$

$$(1 - z_2/z_1)^{-2m-1}(1 - z_3/z_1)^{-2m-1}(z_2 - z_3)^{2m+1} E^-(\alpha, z_1) E^-(-\alpha, z_2) E^-(-\alpha, z_3) + w.$$

Previous formula together with

$$o(E^-(\beta, x)) \cdot v_\lambda = (1+x)^{-\langle \beta, \lambda \rangle} v_\lambda$$

and

$$o(w)v_\lambda = 0$$

implies

$$U^{F,E}(0) \cdot v_\lambda = \sum_{i=0}^{2m} \text{Res}_{z_1} \text{Res}_{z_2} \text{Res}_{z_3} \frac{(1+z_1)^{2m}}{z_1} z_2^{-i-1} z_3^i (z_1 z_2 z_3)^{-2m-1}.$$

$$(1 - z_2/z_1)^{-2m-1}(1 - z_3/z_1)^{-2m-1}(z_2 - z_3)^{2m+1}(1+z_1)^{-t}(1+z_2)^t(1+z_3)^t v_\lambda.$$

The rest follows by expanding generalized rational functions with respect to standard conventions in vertex algebra theory and extracting the residues in all three variables. $\square$

If we view parameter t as a variable, the expression $G_m(t)$ is a polynomial in t of degree at most $4m + 1$. However, it is a priori not clear that the polynomial $G_m(t)$ is nonzero. We made some computations for small m and we came up with the following hypothesis.

Conjecture 16.1.
$$G_m(t) = \binom{2m}{m}^2 \binom{t+m}{4m+1}.$$

We checked this conjecture by using Mathematica package for every $m \le 20$.

As in Section 11, by using representation theory of $\mathcal{SW}(m)$ it is not hard to see that $\binom{t+m}{4m+1}$ must divide $G_m(t)$ for every m. Since $\deg(G_m(t)) \le 4m+1$, then we have

(16.1)
$$G_m(t) = A_m \binom{t+m}{4m+1},$$

for some constant A_m. But even proving $A_m \ne 0$ seems to be a nontrivial problem.

Proposition 16.2. *Let $2m+1$ be prime. Then $G_m(t) \ne 0$.*

Proof. We will prove this result by virtue of reduction mod $2m+1$. Let

$$p = 2m+1$$

be a prime. It is not hard to see that in fact $G_m(a) \in \mathbb{Z}_{(p)}$, for every $a \in \mathbb{Z}$ (in other words, $G_m(a)$ is p-integral). Thus it is sufficient to prove that for some $t = t_0$ we have $G_m(t_0) \ne 0 \bmod p$. We take $t_0 = 3m+1$ and examine

$$G_m(t) = \sum_{l=1}^{2m+1} \sum_{i=0}^{l-1} \sum_{j=0}^{2m+1+i-l} \sum_{k=0}^{l-1-i} (-1)^{j+k+l} \binom{2m+1}{l} \binom{-2m-1}{j}.$$

$$\binom{-2m-1}{k} \binom{-m-1}{j+k+2m+1} \binom{3m+1}{i-j-l+2m+1} \binom{3m+1}{l-k-1-i}.$$

The finite sum $G_m(3m+1)$ has many terms divisible by p. For instance, in the summation, all terms $\binom{2m+1}{l} \equiv 0 \bmod p$ unless $l = 2m+1$. After some analysis it is not hard to see that possible nontrivial $(\bmod\ p)$ contribution comes only if $k = j = 0$ and $l = 2m+1$ (in other cases at least one binomial coefficient is divisible by p). Thus we get:

$$G_m(3m+1) \equiv \sum_{i=0}^{2m} (-1) \binom{-m-1}{2m+1} \binom{3m+1}{i} \binom{3m+1}{2m-i} \bmod p.$$

Observe the basic relation

$$\binom{-m-1}{2m+1} = -\binom{3m+1}{2m+1} = -\binom{3m+1}{m}.$$

Also, for i as in the summation we have

$$\binom{3m+1}{i} \binom{3m+1}{2m-i} \equiv 0 \bmod p, \quad i \ne m.$$

However for $i = m$ we have

$$\binom{3m+1}{m} \equiv 11^{-1}22^{-1}\cdots mm^{-1} \equiv 1 \bmod p.$$

Consequently, the summation reduces to a single term

$$G_m(3m+1) \equiv \binom{3m+1}{m}^3 \equiv 1 \bmod p.$$

$\square$

Notice that the previous computations support our Conjecture 16.1 because

$$\binom{2m}{m} \equiv \pm 1 \bmod p,$$

so that for $t = 3m + 1$

$$\binom{2m}{m}^2 \binom{t+m}{4m+1} = \binom{2m}{m}^2 \binom{4m+1}{4m+1} \equiv 1 \bmod p.$$

Remark 16.1. Because of interesting arithmetics involved in Propositions 16.1 and 16.2, we plan to return to Conjecture 16.1 in our future work.

REFERENCES

[Ab]   T. Abe, A $\mathbb{Z}_2$-orbifold model of the symplectic fermionic vertex operator superalgebra. *Math. Z.* **255** (2007), 755–792.

[A1]   D. Adamović, Rationality of Neveu-Schwarz vertex operator superalgebras, Internat. Math. Res. Notices No. 17 (1997), 865-874.

[A2]   D. Adamović, Representations of the vertex algebra $\mathcal{W}_{1+\infty}$ with a negative integer central charge, Comm. Algebra 29 (2001), no. 7, 3153-3166.

[A3]   D. Adamović, Classification of irreducible modules of certain subalgebras of free boson vertex algebra, J. Algebra 270 (2003) 115-132.

[AM1]   D. Adamović and A. Milas, Logarithmic intertwining operators and $\mathcal{W}(2, 2p-1)$-algebras, *Journal of Math. Physics* **48**, 073503 (2007).

[AM2]   D. Adamović and A. Milas, On the triplet vertex algebra $\mathcal{W}(p)$, Adv. Math. 217 (2008) 2664-2699; arxiv:0707.1857.

[AM3]   D. Adamović and A. Milas, work in progress.

[Ar]   T. Arakawa, Representation theory of superconformal algebras and the Kac-Roan-Wakimoto conjecture, *Duke Math. J.* **130** (2005), 435-478.

[BS]   P. Bouwknegt and K. Schoutens, $\mathcal{W}$–symmetry in Conformal Field Theory, Phys.Rept. 223 (1993) 183-276

[CF]   N. Carqueville and M.Flohr, Nonmeromorphic operator product expansion and C_2-cofiniteness for a family of $\mathcal{W}$-algebras, *Phys. A: Math. Gen.* **39**, (2006), 951-966.

[D]   C. Dong, Vertex algebras associated with even lattices, J. Algebra **160** (1993), 245-265.

[DL]   C. Dong and J. Lepowsky, Generalized vertex algebras and relative vertex operators, Birkhäuser, Boston, 1993.

[DJ]   C. Dong and C. Jiang, Rationality of vertex operator algebras, arxiv.QA/0607679.

[DK]   A. De Sole and V. Kac, Finite vs. affine W-algebras, *Japanese Journal of Math.* **1** (2006) 137-261; arxiv:math/05110055.

[EFHHNV] W. Eholzer, M. Flohr, A. Honecker, R. Hubel, W. Nahm and R. Vernhagen, Representations of $\mathcal{W}$–algebras with two generators and new rational models, Nucl. Phys. B 383 (1992), 249-288.

[FFR]   A. J. Feingold, I. B. Frenkel and J. Ries, Spinor Construction of Vertex Operator Algebras, Triality, and $E_8^{(1)}$, Contemporary Math. 121, Amer. Math. Soc., Providence, RI, 1991.

[FRW]   A. J. Feingold, J. Ries and M. Weiner, Spinor construction of the $c = \frac{1}{2}$ minimal model, Moonshine, The Monster and related topics, Contemporary Math. 193, (Chongying Dong and Geoffrey Mason, eds.), Amer. Math. Soc., Providence, RI, 1995, pp. 45-92.

[FF]   B. Feigin and D. B. Fuchs, Representations of the Virasoro algebra, in *Representations of infinite-dimensional Lie groups and Lie algebras*, Gordon andd Breach, New York (1989).

[FFT]   B. Feigin, E. Feigin and I. Tipunin, Fermionic formulas for $(1, p)$ logarithmic model characters in $\Phi_{2,1}$ quasiparticle realisation, arxiv/0704.2464.

[FGST1]   B.L. Feigin, A.M. Gaĭnutdinov, A. M. Semikhatov, and I. Yu Tipunin, Modular group representations and fusion in logarithmic conformal field theories and in the quantum group center. *Commun. Math. Phys.* **265** (2006), 47–93.

[FGST2]   B.L. Feigin, A.M. Gaĭnutdinov, A. M. Semikhatov, and I. Yu Tipunin, The Kazhdan-Lusztig correspondence for the representation category of the triplet W-algebra in logorithmic conformal field theories, *Theoret. and Math. Phys.* **148** (2006), 1210–1235.

[FHL]   I. B. Frenkel, Y.-Z. Huang and J. Lepowsky, On axiomatic approaches to vertex operator algebras and modules, Mem. Amer. Math. Soc. **104**, 1993.

[Fi]   M. Finkelberg, Am equivalence of fusion categories, *Geometric And Functional Analysis*, (1996), 249-267.

[F1]   M. Flohr, On modular invariant partition functions of conformal field theories with logarithmic operators, Internat. J. Modern Phys. A **11** (1996), 4147–4172.

[F2]   M. Flohr, Bits and pieces in logarithmic conformal field theory. Proceedings of the School and Workshop on Logarithmic Conformal Field Theory and its Applications (Tehran, 2001), Internat. J. Modern Phys. A **18** (2003), 4497–4591.

[FGK]   M. Flohr, C. Grabow and M. Koehn, Fermionic formulas for the characters of $c_{p,1}$ logarithmic field theory, `arxiv.hep-th/0611241`.

[FB]   E. Frenkel and D. Ben-Zvi, Vertex algebras and algebraic curves, Mathematical Surveys and Monographs; no. 88, AMS, 2001.

[FLM]   I. B. Frenkel, J. Lepowsky and A. Meurman, Vertex Operator Algebras and the Monster, Pure and Applied Math., Vol. **134**, Academic Press, New York, 1988.

[FZ]   I. B. Frenkel and Y. Zhu, Vertex operator algebras associated to representations of affine and Virasoro algebras, Duke Math. J. 66 (1992), 123–168.

[FHST]   J. Fuchs, S. Hwang, A.M. Semikhatov and I. Yu. Tipunin, Nonsemisimple Fusion Algebras and the Verlinde Formula, Commun. Math. Phys. **247** (2004), no. 3, 713–742.

[GK1]   M. Gaberdiel and H. G. Kausch, A rational logarithmic conformal field theory, Phys. Lett B 386 (1996), 131-137, hep-th/9606050

[GK2]   M. Gaberdiel and H. G. Kausch, A local logarithmic conformal field theory, Nucl. Phys. B 538 (1999), 631-658, hep-th/9807091

[GL]   Y. Gao and H. Li, Generalized vertex algebras generated by parafermion-like vertex operators, J. Algebra 240 (2001), 771-807.

[HK]   R. Heluani and V. Kac, SUSY lattice vertex algebras, arxiv:0710.1587.

[H]   A. Honecker, Automorphisms of $\mathcal{W}$ algebras and extended rational conformal field theories, Nucl. Phys. B 400 (1993), 574-596.

[HLZ]   Y-Z. Huang, J. Lepowsky and L. Zhang, Logarithmic tensor product theory for generalized modules for a conformal vertex algebra, `arXiv:0710.2687`.

[HM]   Y.-Z. Huang and A. Milas, Intertwining operator superalgebras and vertex tensor categories for superconformal algebras, I, *Commun. Contemp. Math.* **4** (2002), 327–355.

[IK1]   K. Iohara and Y. Koga, Fusion algebras for $N = 1$ superconformal field theories through coinvariants, II, $N = 1$ super-Virasoro-symmetry, *J. Lie Theory* **11** (2001), 305–337.

[IK2]   K. Iohara and Y. Koga, Representation theory of the Neveu-Schwarz and Ramond Algebras II: Fock modules, Ann. Inst. Fourier, Grenoble, 53, 6 (2003), 1755-1818.

[K]   V. G. Kac, Vertex Algebras for Beginners, University Lecture Series, Second Edition, AMS, vol. 10 (1998).

[KWn]   V. G. Kac and W. Wang, Vertex operator superalgebras and their representations, Contemp. Math. **175** (1994), 161-191.

[KWak]   V. Kac and M. Wakimoto, Quantum Reduction and Representation Theory of Superconformal Algebras, *Adv. Math.* **185**, (2004), 400-458.

[Ker]   T. Kerler, Mapping class group actions on quantum doubles. *Commun. Math. Phys.* **168** (1995), no. 2, 353–388.

[La]   A. Lachowska, On the center of the small quantum group. *J. Algebra* **262** (2003), 313–331.

[LL]   J. Lepowsky and H. Li, *Introduction to Vertex Operator Algebras and Their Representations*, Progress in Mathematics, Vol. 227, Birkhäuser, Boston, 2003.

[Li]   H. Li, Local systems of vertex operators, vertex superalgebras and modules, J. Pure Appl. Algebra 109 (1996), 143–195.

[MS]   N. Mavromatos, R. Szabo, The Neveu-Schwarz and Ramond Algebras of Logarithmic Superconformal Field Theory, *JHEP* **0301** (2003) 041.

[MR]   A. Meurman and A. Rocha-Caridi, Highest weight representations of the Neveu-Schwarz and Ramond algebras, *Commun. Math. Physics*, **107** (1986), 263-294.

[M1]   A. Milas, Fusion rings for degenerate minimal models, J. Algebra **254** (2002), no. 2, 300–335.

[M2]   A. Milas, Weak modules and logarithmic intertwining operators for vertex operator algebras. Recent developments in infinite-dimensional Lie algebras and conformal field theory (Charlottesville, VA, 2000), 201–225, *Contemp. Math.* **297**, Amer. Math. Soc., Providence, RI, 2002.

[M3]   A. Milas, Logarithmic intertwining operators and vertex operators, *Commun. Math. Physics*, **277** (2008), 497-529; `math.QA/0609306,`

[M4]   A. Milas, Characters, Supercharacters and Weber modular functions, *Crelle's Journal*, **608**,(2007), 35-64.

[MY]   K. Mimachi and Y. Yamada, Singular vectors of the Virasoro algebra in terms of Jack symmetric polynomials, Commun. Math. Phys. **174** (1995), 447-455.

[Miy]   M. Miyamoto, Modular invariance of vertex operator algebras satisfying C_2-cofiniteness. *Duke Math. J.* **122** (2004), 51–91.

[Se]   A. Semikhatov, Factorizable ribbon quantum groups in logarithmic conformal field theories, `arXiv:0705.4267.`

[Wa]   S. Ole Warnaar, Proof of the Flohr-Grabow-Koehn conjecture for characters of logarithmic field theory, J. Phys. A: Math. Theor. 40 (2007), 12243-12254; `arxiv:0704311.`

[W]   W.Wang, Rationality of Virasoro Vertex operator algebras, Internat. Math. Res. Notices , Vol **71**, No.1 (1993), 197-211.

[Z]   Y.-C. Zhu, Modular invariance of characters of vertex operator algebras, J. Amer. Math. Soc. 9 (1996), 237-302.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ZAGREB, CROATIA

E-mail address: `adamovic@math.hr`

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY AT ALBANY (SUNY), ALBANY, NY 12222

E-mail address: `amilas@math.albany.edu`