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The n-Body Problem in Spaces of Constant Curvature. Part I: Relative Equilibria

Florin Diacu · Ernesto Pérez-Chavela · Manuele Santoprete

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Abstract We extend the Newtonian n-body problem of celestial mechanics to spaces of curvature $\kappa =$ constant and provide a unified framework for studying the motion. In the 2-dimensional case, we prove the existence of several classes of relative equilibria, including the Lagrangian and Eulerian solutions for any $\kappa \neq 0$ and the hyperbolic rotations for $\kappa < 0$. These results lead to a new way of understanding the geometry of the physical space. In the end we prove Saari's conjecture when the bodies are on a geodesic that rotates elliptically or hyperbolically.

Keywords n-body problem · Spaces of constant curvature · Relative equilibria · Lagrangian and Eulerian orbits

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F. Diacu (✉)
Pacific Institute for the Mathematical Sciences and Department of Mathematics and Statistics, University of Victoria, P.O. Box 3060 STN CSC, Victoria V8W 3R4, BC, Canada
e-mail: diacu@math.uvic.ca

E. Pérez-Chavela
Departamento de Matemáticas, Universidad Autónoma Metropolitana-Iztapalapa, Apdo 55534, México, D.F., México
e-mail: epc@xanum.uam.mx

M. Santoprete
Department of Mathematics, Wilfrid Laurier University, 75 University Avenue West, Waterloo N2L 3C5, ON, Canada
e-mail: msantopr@wlu.ca

1 Introduction

This paper extends the Newtonian n-body problem to spaces of curvature $\kappa =$ constant $\neq 0$ and studies its relative equilibria. This problem has its roots in the 2-body case, independently proposed by Bolyai (1913), and Lobachevsky (1949). Their geometric ideas were expressed analytically by Schering (1870, 1873), who showed that the natural extension of the Newtonian potential is given by the cotangent of the distance, as we will later explain. Like in the Euclidean Kepler problem, the potential is a harmonic function in 3-space and generates a central field for which bounded orbits are closed. Liebmann generalized Kepler's laws and proved that the orbits of the corresponding Kepler problem are conics (Liebmann 1902, 1903, 1905). A more recent paper is Dombrowski and Zitterbarth (1991), but we were mostly inspired by Cariñena et al. (2005). The research presented here is also the basis of an investigation on the homographic solutions of the curved 3-body problem (Diacu and Pérez-Chavela 2011).

After obtaining the equations of motion in Hamiltonian form, we study relative equilibria. We show that we can restrict our investigations to the sphere $\mathbf{S}^2$ ($\kappa = 1$) and the hyperbolic plane $\mathbf{H}^2$ ($\kappa = -1$), which we identify with the upper sheet of the hyperboloid of two sheets corresponding to Weierstrass's model of hyperbolic geometry (see the Appendix). But the generalization of our results to any $\kappa \neq 0$ is straightforward.

We show that the Lagrangian equilateral triangle can be a solution for $\kappa = \pm 1$ only if the masses are equal. We also prove the existence of hyperbolic rotation orbits for $\kappa = -1$, as well as the existence of fixed points for $\kappa = 1$. These results have some interesting consequences towards understanding the shape of the physical space. Gauss allegedly tried to determine if space is Euclidean, spherical, or hyperbolic by measuring the angles of triangles having the vertices some miles apart (Miller 1972). His results were inconclusive. Unfortunately his idea does not apply to cosmic triangles because we cannot reach distant stars. Our results, however, show that cosmic travel is not necessary for understanding the nature of space. Since there exist specific celestial orbits for each of the cases $\kappa < 0$, $\kappa = 0$, and $\kappa > 0$, we could, in principle, determine the geometry of our universe through astronomical observations of trajectories. The existence of Lagrangian orbits of non-equal masses in our solar system, such as the triangle formed by the Sun, Jupiter, and the Trojan asteroids, proves that, at distances of a few astronomical units, the physical space is Euclidean. This conclusion can be drawn under the reasonable assumption that the universe has constant curvature.

Our extension of the Newtonian n-body problem to curved space also reveals new aspects of Saari's conjecture (Saari 2005). Proposed in 1970 by Don Saari in the Euclidean case, this conjecture claims that solutions with constant moment of inertia are relative equilibria. Rick Moeckel solved the case $n = 3$ in 2005 (Moeckel 2005); the collinear case, for any number of bodies and more general potentials, was settled the same year by the authors of this paper, (Diacu et al. 2008). Saari's conjecture is also connected to the Chazy–Wintner–Smale conjecture (Smale 1998; Wintner 1947), which asks us to determine whether the number of central configurations is finite for n given bodies in Euclidean space.

Since relative equilibria have elliptic and hyperbolic versions in $\mathbf{H}^2$, Saari's conjecture raises new questions for $\kappa < 0$. We answer them here for the case when the bodies are restrained to a geodesic that rotates elliptically or hyperbolically.

2 Equations of Motion

We derive in this section a Newtonian n-body problem on surfaces of constant curvature. The equations of motion are simple enough to allow an analytic approach. At the end, we provide a straightforward generalization of these equations to spaces of constant curvature of any finite dimension.

2.1 Unified Trigonometry

Let us first consider what, following Cariñena et al. (2005), we call trigonometric κ-functions, which unify circular and hyperbolic trigonometry. We define the κ-sine, sn_κ, as

$$\mathrm{sn}_\kappa(x) := \begin{cases} \kappa^{-1/2}\sin\kappa^{1/2}x & \text{if } \kappa > 0, \\ x & \text{if } \kappa = 0, \\ (-\kappa)^{-1/2}\sinh(-\kappa)^{1/2}x & \text{if } \kappa < 0, \end{cases}$$

the κ-cosine, csn_κ, as

$$\mathrm{csn}_\kappa(x) := \begin{cases} \cos\kappa^{1/2}x & \text{if } \kappa > 0, \\ 1 & \text{if } \kappa = 0, \\ \cosh(-\kappa)^{1/2}x & \text{if } \kappa < 0, \end{cases}$$

as well as the κ-tangent, tn_κ, and κ-cotangent, ctn_κ, as

$$\mathrm{tn}_\kappa(x) := \frac{\mathrm{sn}_\kappa(x)}{\mathrm{csn}_\kappa(x)} \quad \text{and} \quad \mathrm{ctn}_\kappa(x) := \frac{\mathrm{csn}_\kappa(x)}{\mathrm{sn}_\kappa(x)},$$

respectively. We can also define inverse functions; for example $\mathrm{csn}_\kappa^{-1}(y)$ is equal to $(\kappa)^{-1/2}\cos^{-1}y$ if $\kappa > 0$, $(-\kappa)^{-1/2}\cosh^{-1}y$ if $\kappa < 0$ and 1 if $\kappa = 0$. The entire trigonometry can be rewritten in this unified context, but the only identity we will further need (to obtain the expression of (1), below) is the fundamental formula

$$\kappa\,\mathrm{sn}_\kappa^2(x) + \mathrm{csn}_\kappa^2(x) = 1.$$

2.2 The Potential

To obtain the differential equations of the n-body problem on surfaces of constant curvature, we start with some notations. Consider n bodies of masses $m_1, \ldots, m_n$ moving on a surface of constant curvature κ. When $\kappa > 0$, the surfaces are spheres of radii $\kappa^{-1/2}$ given by the equation $x^2 + y^2 + z^2 = \kappa^{-1}$; for $\kappa = 0$, we recover the Euclidean plane; and if $\kappa < 0$, we consider the Weierstrass model of hyperbolic geometry (see the Appendix), which is devised on the sheets with $z > 0$ of the hyperboloids of two sheets $x^2 + y^2 - z^2 = \kappa^{-1}$. The coordinates of the body of mass m_i are given by $\mathbf{q}_i = (x_i, y_i, z_i)$ and a constraint, depending on κ, that restricts the motion of this body to one of the above described surfaces.

We denote by $\widetilde{\nabla}_{\mathbf{q}_i} = (\partial_{x_i}, \partial_{y_i}, \sigma\partial_{z_i})$ the gradient with respect to $\mathbf{q}_i$, where $\sigma = 1$ for $\kappa \geq 0$ and $\sigma = -1$ for $\kappa < 0$. We will use $\nabla_{\mathbf{q}_i}$ to denote the gradient for $\kappa \geq 0$ and $\overline{\nabla}_{\mathbf{q}_i}$ for $\kappa < 0$. $\widetilde{\nabla}$ stands for $(\widetilde{\nabla}_{\mathbf{q}_1}, \dots, \widetilde{\nabla}_{\mathbf{q}_n})$. For $\mathbf{a} = (a_x, a_y, a_z)$ and $\mathbf{b} = (b_x, b_y, b_z)$ in $\mathbb{R}^3$, we define the inner product and the cross product as follows:

$$\mathbf{a} \odot \mathbf{b} := (a_x b_x + a_y b_y + \sigma a_z b_z),$$

$$\mathbf{a} \otimes \mathbf{b} := \big(a_y b_z - a_z b_y, a_z b_x - a_x b_z, \sigma(a_x b_y - a_y b_x)\big).$$

For $\kappa \geq 0$ ($\kappa < 0$) these operations reduce to the Euclidean (Lorentz) inner product and cross product, which we denote by $\cdot$ and $\times$ ($\boxdot$ and $\boxtimes$), respectively.

We will define the potential in $\mathbb{R}^3$ if $\kappa > 0$, and in the Minkowski space $\mathbb{M}^3$ (see the Appendix) if $\kappa < 0$, so that we can use a variational method to derive the equations of motion. For this purpose we define the distance as

$$d_\kappa(\mathbf{a}, \mathbf{b}) := \begin{cases} \operatorname{csn}_\kappa^{-1}(\Gamma_k(\mathbf{a}, \mathbf{b})), & \text{for } \kappa \neq 0, \\ |\mathbf{a} - \mathbf{b}|, & \text{for } \kappa = 0, \end{cases}$$

where $\Gamma_k(\mathbf{a}, \mathbf{b}) = \frac{\kappa \mathbf{a} \odot \mathbf{b}}{\sqrt{\kappa \mathbf{a} \odot \mathbf{a}}\sqrt{\kappa \mathbf{b} \odot \mathbf{b}}}$ and $|\cdot|$ is the Euclidean norm. This definition matches the standard distance on the sphere $x^2 + y^2 + z^2 = \kappa^{-1}$ or on the hyperboloid $x^2 + y^2 - z^2 = \kappa^{-1}$, when we restrict the vectors $\mathbf{a}$ and $\mathbf{b}$ to those surfaces. We also rescale the units such that the gravitational constant G is 1, and define the potential as $-U_\kappa$, where

$$U_\kappa(\mathbf{q}) := \frac{1}{2}\sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} m_i m_j \operatorname{ctn}_\kappa\big(d_\kappa(\mathbf{q}_i, \mathbf{q}_j)\big)$$

stands for the force function, and $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_n)$ is the configuration of the system. Notice that $\operatorname{ctn}_0(d_0(\mathbf{q}_i, \mathbf{q}_j)) = |\mathbf{q}_i - \mathbf{q}_j|^{-1}$, which means that we recover the Newtonian potential in the Euclidean case. Therefore the potential U_κ varies continuously with the curvature κ.

Remark 1 The Newtonian potential has two fundamental properties: it is a harmonic function in 3-space and it is one of the two potentials (the other corresponds to the elastic spring) that generates a central field in which all bounded orbits are closed. The cotangent potential generalizes these properties to spaces of constant curvature, and thus it is widely accepted as an extension of the Newtonian potential.

Now that we have defined a potential that satisfies the basic continuity condition required of any extension of the n-body problem beyond the Euclidean space, we focus on the case $\kappa \neq 0$. A straightforward computation shows that

$$U_\kappa(\mathbf{q}) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} \frac{m_i m_j (\sigma\kappa)^{1/2}\, \Gamma_\kappa(\mathbf{q}_i, \mathbf{q}_j)}{\sqrt{\sigma - \sigma\, (\Gamma_\kappa(\mathbf{q}_i, \mathbf{q}_j))^2}}, \quad \kappa \neq 0. \tag{1}$$

2.3 Derivation of the Equations of Motion

Using Euler's formula for homogeneous functions (in our case U_κ is homogeneous of degree zero), we obtain

$$\mathbf{q} \odot \widetilde{\nabla} U_\kappa(\mathbf{q}) = 0, \tag{2}$$

or, in terms of coordinates,

$$\mathbf{q}_i \odot \widetilde{\nabla}_{\mathbf{q}_i} U_\kappa(\mathbf{q}) = 0, \quad i = \overline{1, n}. \tag{3}$$

Notice that potential is a homogeneous function of degree zero in intrinsic coordinates too, so all the properties connected to this feature are preserved (Diacu et al. 2011; Pérez Chavela and Reyes Victoria 2012).

To derive the equations of motion for $\kappa \neq 0$, we apply a variational method to the force function (1). The Lagrangian of the n-body system has the form

$$L_\kappa(\mathbf{q}, \dot{\mathbf{q}}) = T_\kappa(\mathbf{q}, \dot{\mathbf{q}}) + U_\kappa(\mathbf{q}),$$

where $T_\kappa(\mathbf{q}, \dot{\mathbf{q}}) := \frac{1}{2}\sum_{i=1}^{n} m_i(\dot{\mathbf{q}}_i \odot \dot{\mathbf{q}}_i)(\kappa \mathbf{q}_i \odot \mathbf{q}_i)$ is the kinetic energy of the system (we introduced the factors $\kappa \mathbf{q}_i \odot \mathbf{q}_i = 1$ to endow the equations of motion with a Hamiltonian structure). Then, according to the theory of constrained Lagrangian dynamics (see, e.g. Gelfand and Fomin 1963), the equations of motion are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_\kappa}{\partial \dot{\mathbf{q}}_i}\right) - \frac{\partial L_\kappa}{\partial \mathbf{q}_i} - \lambda_\kappa^i(t)\frac{\partial f_\kappa^i}{\partial \mathbf{q}_i} = \mathbf{0}, \quad i = \overline{1, n}, \tag{4}$$

where $f_\kappa^i = \mathbf{q}_i \odot \mathbf{q}_i - \kappa^{-1}$ gives the constraint $f_\kappa^i = 0$, which keeps the body of mass m_i on the surface of constant curvature κ, and λ_κ^i is the corresponding Lagrange multiplier. Since $\mathbf{q}_i \odot \mathbf{q}_i = \kappa^{-1}$ implies that $\dot{\mathbf{q}}_i \odot \mathbf{q}_i = 0$, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_\kappa}{\partial \dot{\mathbf{q}}_i}\right) = m_i \ddot{\mathbf{q}}_i(\kappa \mathbf{q}_i \odot \mathbf{q}_i) + 2m_i(\kappa \dot{\mathbf{q}}_i \odot \mathbf{q}_i) = m_i \ddot{\mathbf{q}}_i.$$

This relation, together with

$$\frac{\partial L_\kappa}{\partial \mathbf{q}_i} = m_i \kappa(\dot{\mathbf{q}}_i \odot \dot{\mathbf{q}}_i)\mathbf{q}_i + \widetilde{\nabla}_{\mathbf{q}_i} U_\kappa(\mathbf{q}),$$

implies that (4) are equivalent to

$$m_i \ddot{\mathbf{q}}_i - m_i \kappa(\dot{\mathbf{q}}_i \odot \dot{\mathbf{q}}_i)\mathbf{q}_i - \widetilde{\nabla}_{\mathbf{q}_i} U_\kappa(\mathbf{q}) - 2\lambda_\kappa^i(t)\mathbf{q}_i = \mathbf{0}, \quad i = \overline{1, n}. \tag{5}$$

By straightforward computation we get $\lambda_\kappa^i = -\kappa m_i(\dot{\mathbf{q}}_i \odot \dot{\mathbf{q}}_i)$. Substituting $\lambda_\kappa^i, i = \overline{1,n}$, into (5), the equations of motion, which can be put in Hamiltonian form, become

$$m_i\ddot{\mathbf{q}}_i = \widetilde{\nabla}_{\mathbf{q}_i}U_\kappa(\mathbf{q}) - m_i\kappa(\dot{\mathbf{q}}_i \odot \dot{\mathbf{q}}_i)\mathbf{q}_i, \qquad \mathbf{q}_i \odot \mathbf{q}_i = \kappa^{-1}, \quad \kappa \neq 0,\ i = \overline{1,n}. \tag{6}$$

Consider the coordinate and time-rescaling transformations

$$\mathbf{q}_i = |\kappa|^{-1/2}\mathbf{r}_i, \quad i = \overline{1,n}, \quad \text{and} \quad \mathrm{d}\tau = |\kappa|^{3/4}\,\mathrm{d}t. \tag{7}$$

Let $\mathbf{r}_i'$ and $\mathbf{r}_i''$ denote the first and second derivative of $\mathbf{r}_i$ with respect to the rescaled time variable τ. Then the equations of motion (6) take the form

$$\mathbf{r}_i'' = \sum_{j=1, j\neq i}^{N} \frac{m_j[\mathbf{r}_j - \sigma(\mathbf{r}_i \odot \mathbf{r}_j)\mathbf{r}_i]}{[\sigma - \sigma(\mathbf{r}_i \odot \mathbf{r}_j)^2]^{3/2}} - \sigma\left(\mathbf{r}_i' \odot \mathbf{r}_i'\right)\mathbf{r}_i, \quad i = \overline{1,n}. \tag{8}$$

Notice that the explicit dependence on κ vanishes; it shows up only as $\sigma = 1$ for $\kappa > 0$ and $\sigma = -1$ for $\kappa < 0$. Moreover, the change of coordinates (7) shows that

$$\mathbf{r}_i \odot \mathbf{r}_i = |\kappa|\mathbf{q}_i \odot \mathbf{q}_i = |\kappa|\kappa^{-1} = \sigma.$$

It follows that $\mathbf{r}_i \in \mathbf{S}^2$ for $\kappa > 0$ and $\mathbf{r}_i \in \mathbf{H}^2$ for $\kappa < 0$, $i = \overline{1,n}$. Thus, the qualitative behavior of the orbits is independent of the value of the curvature, and we can restrict our study to $\mathbf{S}^3$ and $\mathbf{H}^2$. So the equations to study are

$$\ddot{\mathbf{q}}_i = \sum_{j=1, j\neq i}^{N} \frac{m_j[\mathbf{q}_j - \sigma(\mathbf{q}_i \odot \mathbf{q}_j)\mathbf{q}_i]}{[\sigma - \sigma(\mathbf{q}_i \odot \mathbf{q}_j)^2]^{3/2}} - \sigma(\dot{\mathbf{q}}_i \odot \dot{\mathbf{q}}_i)\mathbf{q}_i, \qquad \mathbf{q}_i \odot \mathbf{q}_i = \sigma, \quad i = \overline{1,n}. \tag{9}$$

The force function, its gradient, and the kinetic energy then have the form

$$U(\mathbf{q}) = \sum_{1\le i<j\le N} \frac{\sigma m_i m_j \mathbf{q}_i \odot \mathbf{q}_j}{[\sigma(\mathbf{q}_i \odot \mathbf{q}_i)(\mathbf{q}_j \odot \mathbf{q}_j) - \sigma(\mathbf{q}_i \odot \mathbf{q}_j)^2]^{1/2}}, \tag{10}$$

$$\widetilde{\nabla}_{\mathbf{q}_i}U(\mathbf{q}) = \sum_{j=1, j\neq i}^{N} \frac{m_i m_j[\mathbf{q}_j - \sigma(\mathbf{q}_i \odot \mathbf{q}_j)\mathbf{q}_i]}{[\sigma - \sigma(\mathbf{q}_i \odot \mathbf{q}_j)^2]^{3/2}}, \tag{11}$$

$$T(\mathbf{q}, \dot{\mathbf{q}}) := \frac{1}{2}\sum_{i=1}^{N} m_i(\dot{\mathbf{q}}_i \odot \dot{\mathbf{q}}_i)(\sigma\mathbf{q}_i \odot \mathbf{q}_i). \tag{12}$$

From Noether's theorem, the system (8) has the energy and total angular-momentum integrals (with integration constants $h \in \mathbb{R}$ and $\mathbf{c} \in \mathbb{R}^3$) given by

$$T(\mathbf{q}, \dot{\mathbf{q}}) - U(\mathbf{q}) = h \quad \text{and} \quad \sum_{i=1}^{n} \mathbf{q}_i \otimes \mathbf{p}_i = \mathbf{c}.$$

Remark 2 The equations of motion encounter singularities when, for $i \neq j$, we have $\mathbf{q}_i \cdot \mathbf{q}_j = 1$ (collisions) or, in $\mathbf{S}^2$ only, we have $\mathbf{q}_i \cdot \mathbf{q}_j = -1$ (antipodal configurations). These singularities are studied in detail in Diacu et al. (2012), Diacu (2011).

Remark 3 If $U =$ constant, the equations of motion reduce to the geodesic equations determined by T, therefore a free body on a surface of constant curvature is either at rest or moves uniformly along a geodesic. Moreover, for $\kappa > 0$, every orbit is closed.

3 Relative Equilibria in $\mathbf{S}^2$

We prove in this section some results about fixed points and relative equilibria, orbits which are invariant to rotations about a fixed axis in $\mathbf{S}^2$. In general, a relative equilibrium is a trajectory contained in a single group orbit, a fact which implies that the mutual distances between bodies remain constant. In our case, however, we would like a clear representation of the elliptic relative equilibria in order to prove their existence. Since, by Euler's theorem (see the Appendix), every element of the group $SO(3)$ can be written, in an orthonormal basis, as a rotation about the z axis, we can define elliptic relative equilibria as follows.

Definition 1 An elliptic relative equilibrium in $\mathbf{S}^2$ is a solution of system (9) with $\sigma = 1$ of the form $\mathbf{q}_i = (x_i, y_i, z_i)$, $i = \overline{1, n}$, where $x_i = r_i \cos(\omega t + \alpha_i)$, $y_i = r_i \sin(\omega t + \alpha_i)$, $z_i = constant$, where ω, α_i, r_i, with $0 \leq r_i = (1 - z_i^2)^{1/2} \leq 1$, $i = \overline{1, n}$, are constants.

3.1 Fixed Points

The simplest solutions of the equations of motion are fixed points. They can be seen as trivial relative equilibria that correspond to $\omega = 0$.

Definition 2 A solution of system (9) with $\sigma = 1$ is called a fixed point if

$$\nabla_{\mathbf{q}_i} U(\mathbf{q})(t) = \mathbf{p}_i(t) = \mathbf{0} \quad \text{for all } t \in \mathbb{R} \text{ and } i = \overline{1, n}.$$

Let us find the simplest fixed points, those that occur when all the masses are equal.

Theorem 1 *Consider the n-body problem in $\mathbf{S}^2$ with n odd. If $m_1 = \cdots = m_n$, the regular n-gon lying on any geodesic is a fixed point of the equations of motion. For $n = 4$, the regular tetrahedron is a fixed point too.*

Proof Consider an n-gon with an odd number of sides inscribed in a geodesic of $\mathbf{S}^2$ with a body, initially at rest, at each vertex. (The assumption that n is odd is imposed to avoid antipodal configurations.) In general, two forces act on the body of mass m_i: $\nabla_{\mathbf{q}_i} U(\mathbf{q})$, which is due to the interaction with the other bodies, and $-m_i(\dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i)\mathbf{q}_i$, which is due to the constraints. The latter force is zero at $t = 0$ because the bodies are initially at rest. Since $\mathbf{q}_i \cdot \nabla_{\mathbf{q}_i} U(\mathbf{q}) = 0$, it follows that $\nabla_{\mathbf{q}_i} U(\mathbf{q})$ is orthogonal to $\mathbf{q}_i$, and thus tangent to $\mathbf{S}^2$. Then the symmetry of the n-gon implies that, at the initial moment $t=0$, $\nabla_{\mathbf{q}_i}U(\mathbf{q})$ is the sum of forces that cancel each other, so $\nabla_{\mathbf{q}_i}U(\mathbf{q}(0))=\mathbf{0}$. From the equations of motion and the fact that the bodies are initially at rest, it follows that

$$\ddot{\mathbf{q}}_i(0) = -\big(\dot{\mathbf{q}}_i(0)\cdot\dot{\mathbf{q}}_i(0)\big)\mathbf{q}_i(0) = \mathbf{0}, \quad i=\overline{1,n}.$$

But then no force acts on m_i at time $t=0$; consequently, the body does not move. So $\dot{\mathbf{q}}_i(t)=\mathbf{0}$ for all $t\in\mathbb{R}$. Then $\ddot{\mathbf{q}}_i(t)=\mathbf{0}$ for all $t\in\mathbb{R}$, therefore $\nabla_{\mathbf{q}_i}U(\mathbf{q}(t))=\mathbf{0}$ for all $t\in\mathbb{R}$, so the n-gon is a fixed point of (9) with $\sigma=1$.

The regular tetrahedron is a fixed point because four bodies of equal masses with initial coordinates $\mathbf{q}_1=(0,0,1)$, $\mathbf{q}_2=(0,2\sqrt{2}/3,-1/3)$, $\mathbf{q}_3=(-\sqrt{6}/3, -\sqrt{2}/3,-1/3)$, $\mathbf{q}_4=(\sqrt{6}/3,-\sqrt{2}/3,-1/3)$ satisfy system (9) with $\sigma=1$. □

Remark 4 Equal masses placed at the vertices of the other convex regular polyhedra: octahedron (six bodies), cube (eight bodies), dodecahedron (12 bodies), and icosahedron (20 bodies), have antipodal singularities, so they do not form fixed points.

3.2 Polygonal Solutions

We further show that fixed points lying on geodesics of spheres can generate relative equilibria. The first result is an immediate consequence of the fact that the equator S^1 is invariant to the flow generated by (6).

Proposition 1 *Consider a fixed point given by the masses $m_1,\dots,m_n$ that lie on a great circle of $\mathbf{S}^2$. Then for every nonzero angular velocity, this configuration generates a relative equilibrium that rotates along the great circle.*

We can now state and prove the following result.

Theorem 2 *Place an odd number of equal bodies at the vertices of a regular n-gon inscribed in a great circle of $\mathbf{S}^2$. Then the only elliptic relative equilibria that can be generated from this configuration are those that rotate along the original great circle.*

Proof Without loss of generality, we prove this result for the equator, $z=0$. Consider an elliptic relative equilibrium of the form

$$\begin{aligned} &x_i = r_i\cos(\omega t+\alpha_i), \qquad y_i = r_i\sin(\omega t+\alpha_i),\\ &z_i = \pm\big(1-r_i^2\big)^{1/2}, \quad i=\overline{1,n}, \end{aligned} \tag{13}$$

with $+$ taken for $z_i>0$ and $-$ for $z_i<0$. The only condition we impose on this solution is that r_i and α_i, $i=\overline{1,n}$, are chosen such that the configuration is a regular n-gon inscribed in a moving great circle of $\mathbf{S}^2$ at all times. Therefore the plane of the n-gon can have any angle with, say, the z-axis. This solution has the derivatives

$$\dot{x}_i = -r_i\omega\sin(\omega t+\alpha_i), \qquad \dot{y}_i = r_i\omega\cos(\omega t+\alpha_i), \qquad \dot{z}_i = 0, \quad i=\overline{1,n},$$
$$\ddot{x}_i = -r_i\omega^2\cos(\omega t+\alpha_i), \qquad \ddot{y}_i = -r_i\omega^2\sin(\omega t+\alpha_i), \qquad \ddot{z}_i = 0, \quad i=\overline{1,n}.$$

Then

$$\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2 = r_i^2\omega^2, \quad i = \overline{1,n}.$$

Since, by symmetry, any n-gon solution with n odd satisfies the conditions $\nabla_{\mathbf{q}_i} U(\mathbf{q}) = \mathbf{0}$, $i = \overline{1,n}$, system (9) with $\sigma = 1$ reduces to

$$\ddot{\mathbf{q}}_i = -\left(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2\right)\mathbf{q}_i, \quad i = \overline{1,n}.$$

Then the substitution of (13) into the above equations leads to

$$\begin{cases} r_i\left(1 - r_i^2\right)\omega^2\cos(\omega t + \alpha_i) = 0, & i = \overline{1,n}, \\ r_i\left(1 - r_i^2\right)\omega^2\sin(\omega t + \alpha_i) = 0, & i = \overline{1,n}. \end{cases}$$

But assuming $\omega \neq 0$, this system is nontrivially satisfied if and only if $r_i = 1$, conditions which are equivalent to $z_i = 0,\ i = \overline{1,n}$. Therefore the bodies must rotate along the equator $z = 0$. □

Theorem 2 raises the question whether elliptic relative equilibria given by regular polygons can rotate on other curves than geodesics. The answer is given here.

Theorem 3 *Consider* $m_1 = \cdots = m_n =: m$ *in* $\mathbf{S}^2$. *Then, for any* n *odd,* $m > 0$ *and* $z \in (-1, 1)$, *there are a positive and a negative* ω *that produce elliptic relative equilibria in which the bodies are at the vertices of an* n*-gon rotating in the plane* $z =$ constant. *If* n *is even, this property is still true if we exclude the case* $z = 0$.

Proof Substitute into system (9) with $\sigma = 1$ a solution of the form (13) with $\alpha_i = \frac{2\pi i}{n}$, $i = \overline{1,n}$. The equation for z_1, similar to any z_i, $i = \overline{2,n}$, is

$$\sum_{j=2}^{n} \frac{m(z - k_{1j}z)}{(1 - k_{1j}^2)^{3/2}} - r^2\omega^2 z = 0,$$

where $k_{1j} = x_1x_j + y_1y_j + z_1z_j = (1 - z^2)\cos\alpha_j + z^2$. For $z \neq 0$, this equation becomes

$$\sum_{j=2}^{\nu} \frac{2(1 - \cos\alpha_j)}{(1 - k_{1j}^2)^{3/2}} + \frac{2\eta}{(1 - k_{1(\nu+1)}^2)^{3/2}} = \frac{\omega^2}{m},$$

where ν is the integer part of $n/2$, $\eta = 0$ for n even, and $\eta = 1$ for n odd. The coordinates x_1 and y_1 lead to the same equation. Writing the denominators explicitly, we obtain

$$\sum_{j=2}^{\nu} \frac{2}{(1 - \cos\alpha_j)^{1/2}\{(1 - z^2)[2 - (1 - \cos\alpha_j)(1 - z^2)]\}^{3/2}} + \frac{\eta}{4z^2|z|(1 - z^2)^{3/2}} = \frac{\omega^2}{m}.$$

The left hand side is positive, so, if n is even, for any $m > 0$ and $z \in (-1, 1)$ fixed, there are a positive and a negative ω that satisfy the equation. If n is odd the same result holds except for the case $z = 0$, which introduces antipodal singularities. □

3.3 Lagrangian Solutions

The 3-body problem presents particular interest in the Euclidean case because the equilateral triangle is an elliptic relative equilibrium for any values of the masses. But before we check whether this fact holds in $\mathbf{S}^2$, let us consider the equal-mass case in more detail.

Corollary 1 *Consider the* 3*-body problem with equal masses*, $m := m_1 = m_2 = m_3$, *in* $\mathbf{S}^2$. *Then for any* $m > 0$ *and* $z \in (-1, 1)$, *there are a positive and a negative* ω *that produce elliptic relative equilibria in which the bodies are at the vertices of an equilateral triangle that rotates in the plane* $z =$ *constant. Moreover, for every* ω^2/m *there are two values of* z *that lead to relative equilibria if* $\omega^2/m \in (8/\sqrt{3}, \infty) \cup \{3\}$, *three values if* $\omega^2/m = 8/\sqrt{3}$, *and four values if* $\omega^2/m \in (3, 8/\sqrt{3})$.

Proof The first part of the statement is a consequence of Theorem 3 for $n = 3$. Alternatively, we can substitute into system (9) with $\sigma = 1$ a solution of the form (13) with $i = 1, 2, 3$, $r := r_1 = r_2 = r_3$, $z = \pm(1 - r^2)^{1/2}$, $\alpha_1 = 0$, $\alpha_2 = 2\pi/3$, $\alpha_3 = 4\pi/3$, and we obtain the equation

$$\frac{8}{\sqrt{3}(1 + 2z^2 - 3z^4)^{3/2}} = \frac{\omega^2}{m}. \tag{14}$$

The left hand side is positive for $z \in (-1, 1)$ and tends to infinity when $z \to \pm 1$. So for any z in this interval and $m > 0$, there are a positive and a negative ω for which the above equation is satisfied. □

Remark 5 A similar result to Corollary 1 can be proved for two equal masses that rotate on a non-geodesic circle, when the bodies are situated at opposite ends of a rotating diameter. Then, for $z \in (-1, 0) \cup (0, 1)$, the analogue of (14) is the equation

$$\frac{1}{4z^2|z|(1 - z^2)^{3/2}} = \frac{\omega^2}{m}.$$

The case $z = 0$ yields no solution because it involves an antipodal singularity.

We can now decide if the equilateral triangle of non-equal masses is an elliptic relative equilibrium in $\mathbf{S}^2$. The following result shows that, unlike in the Euclidean case, the answer is negative when the bodies move on the sphere in the same Euclidean plane.

Proposition 2 *In the* 3*-body problem in* $\mathbf{S}^2$, *if the bodies* m_1, m_2, m_3 *are initially at the vertices of an equilateral triangle in the plane* $z =$ *constant for some* $z \in (-1, 1)$, *then there are initial velocities that lead to an elliptic relative equilibrium in which the triangle rotates in its own plane if and only if* $m_1 = m_2 = m_3$.

Proof The implication showing that for equal masses the rotating equilateral triangle is a relative equilibrium follows from Theorem 1. To prove the converse, substitute into system (9) with $\sigma = 1$ a solution of the form (13) with $i = 1, 2, 3,\ r := r_1, r_2, r_3,\ z := z_1 = z_2 = z_3 = \pm(1 - r^2)^{1/2}$, and $\alpha_1 = 0, \alpha_2 = 2\pi/3, \alpha_3 = 4\pi/3$. Then

$$m_1 + m_2 = \gamma\omega^2, \qquad m_2 + m_3 = \gamma\omega^2, \qquad m_3 + m_1 = \gamma\omega^2, \tag{15}$$

where $\gamma = \sqrt{3}(1 + 2z^2 - 3z^4)^{3/2}/4$. But for any $z =$ constant in the interval $(-1, 1)$, the above system has a solution only for $m_1 = m_2 = m_3 = \gamma\omega^2/2$. □

Our next result leads to the conclusion that Lagrangian solutions in $\mathbf{S}^2$ can occur only in Euclidean planes of $\mathbb{R}^3$. This property has its analogue in the Euclidean case (Wintner 1947), but Wintner's proof does not work in our case because it uses the integrals of the centre of mass, which do not exist here. Most importantly, our result also implies that Lagrangian orbits with non-equal masses cannot exist in $\mathbf{S}^2$.

Theorem 4 *For all Lagrangian solutions in $\mathbf{S}^2$, the masses m_1, m_2, and m_3 have to rotate on the same circle, whose plane must be orthogonal to the rotation axis, and therefore $m_1 = m_2 = m_3$.*

Proof Consider a Lagrangian solution in $\mathbf{S}^2$ with bodies of masses m_1, m_2, and m_3. Then the solution, which is an elliptic relative equilibrium, must have the form

$$x_i = r_i \cos(\omega t + \phi_i), \qquad y_i = r_i \sin(\omega t + \phi_i), \qquad z_i = \left(1 - r_i^2\right)^{1/2},$$

where $i = 1, 2, 3$, $\phi_1 = 0$, $\phi_2 = a$, and $\phi_3 = b$, with $b > a > 0$. In other words, we assume that this equilateral triangle forms a constant angle with the rotation axis, z, such that each body describes its own circle on $\mathbf{S}^2$. But for such a solution to exist it is necessary that the total angular momentum is either zero or is given by a vector parallel with the z axis. Otherwise this vector rotates around the z axis, in violation of the angular-momentum integrals. This means that at least the first two components of the vector $\sum_{i=1}^{3} m_i \mathbf{q}_i \times \dot{\mathbf{q}}_i$ are zero. Then

$$m_1 r_1 z_1 \sin\omega t + m_2 r_2 z_2 \sin(\omega t + a) + m_3 r_3 z_3 \sin(\omega t + b) = 0,$$

assuming that $\omega \neq 0$. For $t = 0$, this equation becomes

$$m_2 r_2 z_2 \sin a = -m_3 r_3 z_3 \sin b. \tag{16}$$

Using now the fact that

$$\alpha := x_1 x_2 + y_1 y_2 + z_1 z_2 = x_1 x_3 + y_1 y_3 + z_1 z_3 = x_3 x_2 + y_3 y_2 + z_3 z_2$$

is constant because the triangle is equilateral, the equation of motion corresponding to $\ddot{y}_1$ takes the form

$$K r_1\left(r_1^2 - 1\right)\omega^2 \sin\omega t = m_2 r_2 \sin(\omega t + a) + m_3 r_3 \sin(\omega t + b),$$

where K is a nonzero constant. For $t = 0$, this equation becomes

$$m_2 r_2 \sin a = -m_3 r_3 \sin b. \tag{17}$$

Dividing (16) by (17), we find that $z_2 = z_3$. Similarly, we can show that $z_1 = z_2 = z_3$, therefore the motion must take place in the same Euclidean plane on a circle orthogonal to the rotation axis. Proposition 2 then implies that $m_1 = m_2 = m_3$. □

3.4 Eulerian Solutions

In agreement with the Euclidean case, the bodies of an Eulerian solution lie on the same rotating geodesic, so it is now natural to ask whether such elliptic relative equilibria exist. The answer in the case $n = 3$ of equal masses is given by the following result.

Theorem 5 *Consider the* 3*-body problem in* $\mathbf{S}^2$ *with masses* $m_1 = m_2 = m_3 =: m$. *Fix* m_1 *at* $(0, 0, 1)$ *and* m_2 *and* m_3 *at the opposite ends of a diameter on the circle* $z = $ *constant. Then, for any* $m > 0$ *and* $z \in (-0.5, 0) \cup (0, 1)$, *there are a positive and a negative* ω *that produce elliptic relative equilibria.*

Proof Substituting into the equations of motion (9) with $\sigma = 1$ a solution of the form

$$\begin{aligned} &x_1 = 0, \quad y_1 = 0, \quad z_1 = 1, \\ &x_2 = r\cos\omega t, \quad y_2 = r\sin\omega t, \quad z_2 = z, \\ &x_3 = r\cos(\omega t + \pi), \quad y_3 = r\sin(\omega t + \pi), \quad z_3 = z, \end{aligned}$$

with the constants r and z satisfying $r^2 + z^2 = 1$, we are led either to identities or to the algebraic equation

$$\frac{4z + |z|^{-1}}{4z^2(1 - z^2)^{3/2}} = \frac{\omega^2}{m}. \tag{18}$$

The function on the left hand side is negative for $z \in (-1, -0.5)$, 0 at $z = -0.5$, positive for $z \in (-0.5, 0) \cup (0, 1)$, and undefined at $z = 0$. Therefore, for every $m > 0$ and $z \in (-0.5, 0) \cup (0, 1)$, there are a positive and a negative ω that lead to a geodesic relative equilibrium. For $z = -0.5$, we recover the equilateral fixed point. □

Remark 6 If in Theorem 5 we take the masses $m_1 =: m$ and $m_2 = m_3 =: M$, the analogue of (18) is

$$\frac{4mz + M|z|^{-1}}{4z^2(1 - z^2)^{3/2}} = \omega^2.$$

Then solutions exist for any $z \in (-\sqrt{M/m}/2, 0) \cup (0, 1)$. From the above equation we see that there are no fixed points for $M \geq 4m$.

4 Relative Equilibria in $\mathbf{H}^2$

In this section we prove some results about elliptic and hyperbolic relative equilibria in $\mathbf{H}^2$. Since, by the Principal Axis theorem for the Lorentz group, every Lorentzian rotation (see the Appendix) can be written, in some basis, as an elliptic rotation about the z axis, a hyperbolic rotation about the x axis, or a parabolic rotation about the line $x = 0$, $y = z$, we can define three kinds of relative equilibrium: elliptic, hyperbolic, and parabolic, in agreement with the terminology of hyperbolic geometry (Henle 2001).

Definition 3 An elliptic relative equilibrium in $\mathbf{H}^2$ is a solution $\mathbf{q}_i = (x_i, y_i, z_i),\ i = \overline{1, n}$, of system (9) with $\sigma = -1$, where $x_i = \rho_i \cos(\omega t + \alpha_i)$, $y_i = \rho_i \sin(\omega t + \alpha_i)$, and $z_i = (\rho_i^2 + 1)^{1/2}$, with ω, α_i, and $\rho_i,\ i = \overline{1, n}$, constants.

Definition 4 A hyperbolic relative equilibrium in $\mathbf{H}^2$ is a solution of system (9) with $\sigma = -1$ of the form $\mathbf{q}_i = (x_i, y_i, z_i),\ i = \overline{1, n}$, where

$$x_i = \text{constant}, \qquad y_i = \rho_i \sinh(\omega t + \alpha_i), \quad \text{and} \quad z_i = \rho_i \cosh(\omega t + \alpha_i), \tag{19}$$

and ω, α_i, and $\rho_i = (1 + x_i^2)^{1/2} \geq 1,\ i = \overline{1, n}$, are constants.

Remark 7 We could also define parabolic relative equilibria in terms of parabolic rotations (see the Appendix), but it is easy to show that such orbits do not exist (Diacu 2011, 2012).

Remark 8 $\mathbf{H}^2$ is free of fixed points, i.e. there are no solutions of (9) with $\sigma = -1$ such that $\overline{\nabla}_{\mathbf{q}_i} U(\mathbf{q})(t) = \mathbf{p}_i(t) = \mathbf{0},\ t \in \mathbb{R},\ i = \overline{1, n}$.

4.1 Elliptic Relative Equilibria in $\mathbf{H}^2$

We now consider elliptic relative equilibria. The proof of the following result is similar to the one we gave for Theorem 3.

Theorem 6 *Consider the n-body problem with equal masses in $\mathbf{H}^2$. Then, for any $m > 0$ and $z > 1$, there are a positive and a negative ω that produce elliptic relative equilibria in which the bodies are at the vertices of an n-gon rotating in the plane $z = constant$.*

Corollary 2 *Consider the 3-body problem with equal masses, $m := m_1 = m_2 = m_3$, in $\mathbf{H}^2$. Then for any $m > 0$ and $z > 1$, there are a positive and a negative ω that produce relative elliptic equilibria in which the bodies are at the vertices of an equilateral triangle that rotates in the plane $z = constant$. Moreover, for every $\omega^2/m > 0$ there is a unique $z > 1$ as above.*

Proof Substituting in system (9) with $\sigma = -1$ a solution of the form

$$x_i = \rho\cos(\omega t + \alpha_i), \qquad y_i = \rho\sin(\omega t + \alpha_i), \qquad z_i = z, \tag{20}$$

with $z = (\rho^2 + 1)^{1/2}$, $\alpha_1 = 0, \alpha_2 = 2\pi/3, \alpha_3 = 4\pi/3$, we are led to the equation

$$\frac{8}{\sqrt{3}(3z^4 - 2z^2 - 1)^{3/2}} = \frac{\omega^2}{m}. \tag{21}$$

The left hand side is positive for $z > 1$, tends to infinity when $z \to 1$, and tends to zero when $z \to \infty$. So for any z in this interval and $m > 0$, there are a positive and a negative ω for which the above equation is satisfied. □

As we already proved in the previous section, an equilateral triangle rotating in its own plane forms an elliptic relative equilibrium in $\mathbf{S}^2$ only if the three masses lying at its vertices are equal. The same result is true in $\mathbf{H}^2$, as we will further show.

Proposition 3 *In the* 3*-body problem in* $\mathbf{H}^2$, *if the bodies* m_1, m_2, m_3 *are initially at the vertices of an equilateral triangle in the plane* $z =$ *constant for some* $z > 1$, *then there are initial velocities that lead to an elliptic relative equilibrium in which the triangle rotates in its own plane if and only if* $m_1 = m_2 = m_3$.

Proof It follows from Theorem 6 that if $m_1 = m_2 = m_3$ then the rotating equilateral triangle is an elliptic relative equilibrium. To prove the converse, substitute into system (9) with $\sigma = -1$ a solution of the form (20) with $i = 1, 2, 3$, $\rho := \rho_1, \rho_2, \rho_3$, $z := z_1 = z_2 = z_3 = (\rho^2 + 1)^{1/2}$, and $\alpha_1 = 0, \alpha_2 = 2\pi/3, \alpha_3 = 4\pi/3$. The computations then lead to the system

$$m_1 + m_2 = \zeta\omega^2, \qquad m_2 + m_3 = \zeta\omega^2, \qquad m_3 + m_1 = \zeta\omega^2, \tag{22}$$

where $\zeta = \sqrt{3}(3z^4 - 2z^2 - 1)^{3/2}/4$. But for any $z =$ constant with $z > 1$, the above system has a solution only for $m_1 = m_2 = m_3 = \zeta\omega^2/2$. □

The following result resembles Theorem 4. The proof works the same way, by taking $\sigma = -1$ and replacing the elliptical trigonometric functions with hyperbolic ones.

Theorem 7 *For all Lagrangian solutions in* $\mathbf{H}^2$, *the masses* m_1, m_2 *and* m_3 *have to rotate on the same circle*, *whose plane must be orthogonal to the rotation axis*, *and therefore* $m_1 = m_2 = m_3$.

We can further prove an analogue of Theorem 5.

Theorem 8 *Consider the* 3*-body problem in* $\mathbf{H}^2$ *with masses*, $m_1 = m_2 = m_3 =: m$. *Fix the body* m_1 *at* $(0, 0, 1)$ *and the bodies* m_2 *and* m_3 *at the opposite ends of a diameter on the circle* $z =$ *constant*. *Then*, *for any* $m > 0$ *and* $z > 1$, *there are a positive and a negative* ω, *which produce elliptic relative equilibria that rotate around the* z *axis*.

Proof Substituting into system (9) with $\sigma = -1$ a solution of the form

$$x_1 = 0, \qquad y_1 = 0, \qquad z_1 = 1,$$
$$x_2 = \rho \cos \omega t, \qquad y_2 = \rho \sin \omega t, \qquad z_2 = z,$$
$$x_3 = \rho \cos(\omega t + \pi), \qquad y_3 = \rho \sin(\omega t + \pi), \qquad z_3 = z,$$

where $\rho \geq 0$ and $z \geq 1$ are constants satisfying $z^2 = \rho^2 + 1$, we are led either to identities or to the algebraic equation

$$\frac{4z^2 + 1}{4z^3(z^2 - 1)^{3/2}} = \frac{\omega^2}{m}. \tag{23}$$

The function on the left hand side is positive for $z > 1$. Therefore, for every $m > 0$ and $z > 1$, there are a positive and a negative ω that lead to a geodesic elliptic relative equilibrium. Moreover, for every $\omega^2/m > 0$, there is one $z > 1$ that satisfies (23). □

4.2 Hyperbolic Relative Equilibria in $\mathbf{H}^2$

Since parabolic relative equilibria do not exist, it is natural to ask whether hyperbolic relative equilibria show up. For three equal masses, the answer is given by the following result, which shows that, in $\mathbf{H}^2$, three bodies can move along hyperbolas lying in parallel planes of $\mathbb{R}^3$, maintaining the initial distances among themselves and remaining on the same geodesic, which rotates hyperbolically.

Theorem 9 *In the* 3*-body problem with* $m_1 = m_2 = m_3 =: m$ *in* $\mathbf{H}^2$, *for any given* $m > 0$ *and* $x \neq 0$, *there exist a positive and a negative* ω *that lead to hyperbolic relative equilibria.*

Proof We will show that $\mathbf{q}_i = (x_i, y_i, z_i)$, $i = 1, 2, 3$, is a hyperbolic relative equilibrium of system (9) with $\sigma = -1$ for

$$x_1 = 0, \qquad y_1 = \sinh \omega t, \qquad z_1 = \cosh \omega t,$$
$$x_2 = x, \qquad y_2 = \rho \sinh \omega t, \qquad z_2 = \rho \cosh \omega t,$$
$$x_3 = -x, \qquad y_3 = \rho \sinh \omega t, \qquad z_3 = \rho \cosh \omega t,$$

where $\rho = (1 + x^2)^{1/2}$. Notice first that

$$x_1x_2 + y_1y_2 - z_1z_2 = x_1x_3 + y_1y_3 - z_1z_3 = -\rho,$$
$$x_2x_3 + y_2y_3 - z_2z_3 = -2x^2 - 1,$$
$$\dot{x}_1^2 + \dot{y}_1^2 - \dot{z}_1^2 = \omega^2, \qquad \dot{x}_2^2 + \dot{y}_2^2 - \dot{z}_2^2 = \dot{x}_3^2 + \dot{y}_3^2 - \dot{z}_3^2 = \rho^2\omega^2.$$

Substituting the above coordinates and expressions into system (9) with $\sigma = -1$, we are led either to identities or to the equation

$$\frac{4x^2 + 5}{4x^2|x|(x^2 + 1)^{3/2}} = \frac{\omega^2}{m}, \tag{24}$$

from which the statement of the theorem follows. □

Remark 9 Theorem 9 is also true if, say, $m := m_1$ and $M := m_2 = m_3$. Then the analogue of (24) is

$$\frac{m}{x^2|x|(x^2+1)^{1/2}} + \frac{M}{4x^2|x|(x^2+1)^{3/2}} = \omega^2,$$

and it is obvious that for any $m, M > 0$ and $x \neq 0$, there are a positive and negative ω satisfying the above equation.

Remark 10 Theorem 9 also works for two bodies of equal masses, $m := m_1 = m_2$, of coordinates $x_1 = -x_2 = x$, $y_1 = y_2 = \rho \sinh \omega t$, $z_1 = z_2 = \rho \cosh \omega t$, where x is a positive constant and $\rho = (x^2+1)^{3/2}$. Then the analogue of (24) is

$$\frac{1}{4x^2|x|(x^2+1)^{3/2}} = \frac{\omega^2}{m},$$

which supports a statement similar to the one in Theorem 9.

5 Saari's Conjecture

In 1970, Don Saari conjectured that solutions of the classical n-body problem with constant moment of inertia are relative equilibria (Saari 1970, 2005). The moment of inertia is defined in classical Newtonian celestial mechanics as $\frac{1}{2}\sum_{i=1}^n m_i \mathbf{q}_i \cdot \mathbf{q}_i$, a function that gives a crude measure of the bodies' distribution in space. But this definition makes no sense in $\mathbf{S}^2$ and $\mathbf{H}^2$ because $\mathbf{q}_i \odot \mathbf{q}_i = \sigma$ for every $i = \overline{1,n}$. To avoid this problem, we adopt the standard point of view used in physics, and define the moment of inertia about the direction of the angular momentum. But while fixing an axis in $\mathbf{S}^2$ does not restrain generality, the symmetry of $\mathbf{H}^2$ makes us distinguish between two cases. Indeed, in $\mathbf{S}^2$ we can assume that the rotation takes place around the z axis, and thus define the moment of inertia as

$$\mathbf{I} := \sum_{i=1}^n m_i \left(x_i^2 + y_i^2\right). \tag{25}$$

In $\mathbf{H}^2$, all possibilities can be reduced via suitable isometric transformations (see the Appendix) to: (i) the symmetry about the z axis, when the moment of inertia takes the same form (25), and (ii) the symmetry about the x axis, which corresponds to hyperbolic rotations, when in agreement with the definition of the Lorentz product we define the moment of inertia as

$$\mathbf{J} := \sum_{i=1}^n m_i \left(y_i^2 - z_i^2\right). \tag{26}$$

The parabolic rotations will not be considered because there are no parabolic relative equilibria. These definitions allow us to formulate the following conjecture.

Saari's Conjecture in $\mathbf{S}^2$ and $\mathbf{H}^2$ *For the gravitational n-body problem in $\mathbf{S}^2$ and $\mathbf{H}^2$, every solution that has a constant moment of inertia about the direction of the angular momentum is either an elliptic relative equilibrium in $\mathbf{S}^2$ or $\mathbf{H}^2$, or a hyperbolic relative equilibrium in $\mathbf{H}^2$.*

By generalizing an idea we used in the Euclidean case (Diacu et al. 2005, 2008), we can now settle this conjecture when the bodies undergo another constraint. More precisely, we will prove the following result.

Theorem 10 *For the gravitational n-body problem in $\mathbf{S}^2$ and $\mathbf{H}^2$, every solution with constant moment of inertia about the direction of the angular momentum for which the bodies remain aligned along a geodesic that rotates elliptically in $\mathbf{S}^2$ or $\mathbf{H}^2$, or hyperbolically in $\mathbf{H}^2$, is either an elliptic relative equilibrium in $\mathbf{S}^2$ or $\mathbf{H}^2$, or a hyperbolic relative equilibrium in $\mathbf{H}^2$.*

Proof Let us first prove the case in which $\mathbf{I}$ is constant in $\mathbf{S}^2$ and $\mathbf{H}^2$, i.e. when the geodesic rotates elliptically. According to the above definition of $\mathbf{I}$, we can assume without loss of generality that the geodesic passes through the point $(0,0,1)$ and rotates about the z-axis with angular velocity $\omega(t) \neq 0$. The angular momentum of each body is $\mathbf{L}_i = m_i \mathbf{q}_i \otimes \dot{\mathbf{q}}_i$, so its derivative with respect to t takes the form

$$\begin{aligned}\dot{\mathbf{L}}_i &= m_i \dot{\mathbf{q}}_i \otimes \dot{\mathbf{q}}_i + m_i \mathbf{q}_i \otimes \ddot{\mathbf{q}}_i = m_i \mathbf{q}_i \otimes \widetilde{\nabla}_{\mathbf{q}_i} U_\kappa(\mathbf{q}) - m_i \dot{\mathbf{q}}_i^2 \mathbf{q}_i \otimes \mathbf{q}_i \\ &= m_i \mathbf{q}_i \otimes \widetilde{\nabla}_{\mathbf{q}_i} U_\kappa(\mathbf{q}),\end{aligned}$$

with $\kappa = 1$ in $\mathbf{S}^2$ and $\kappa = -1$ in $\mathbf{H}^2$. Since $\mathbf{q}_i \odot \widetilde{\nabla}_{\mathbf{q}_i} U_\kappa(\mathbf{q}) = 0$, it follows that $\widetilde{\nabla}_{\mathbf{q}_i} U_\kappa(\mathbf{q})$ is either zero or orthogonal to $\mathbf{q}_i$. (Recall that orthogonality here is meant in terms of the standard inner product because, both in $\mathbf{S}^2$ and $\mathbf{H}^2$, $\mathbf{q}_i \odot \widetilde{\nabla}_{\mathbf{q}_i} U_\kappa(\mathbf{q}) = \mathbf{q}_i \cdot \nabla_{\mathbf{q}_i} U_\kappa(\mathbf{q})$.) If $\widetilde{\nabla}_{\mathbf{q}_i} U_\kappa(\mathbf{q}) = \mathbf{0}$, then $\dot{\mathbf{L}}_i = \mathbf{0}$, so $\dot{L}_i^z = 0$.

Assume now that $\widetilde{\nabla}_{\mathbf{q}_i} U_\kappa(\mathbf{q})$ is orthogonal to $\mathbf{q}_i$. Since all the particles are on a geodesic, their corresponding position vectors are in the same plane, therefore any linear combination of them is in this plane, so $\widetilde{\nabla}_{\mathbf{q}_i} U_\kappa(\mathbf{q})$ is in the same plane. Thus $\widetilde{\nabla}_{\mathbf{q}_i} U_\kappa(\mathbf{q})$ and $\mathbf{q}_i$ are in a plane orthogonal to the xy plane. It follows that $\dot{\mathbf{L}}_i$ is parallel to the xy plane and orthogonal to the z-axis. Thus the z-component, $\dot{L}_i^z$, of $\dot{\mathbf{L}}_i$ is 0, the same conclusion we obtained in the case $\widetilde{\nabla}_{\mathbf{q}_i} U_\kappa(\mathbf{q}) = \mathbf{0}$. Consequently, $L_i^z = c_i$, where c_i is a constant.

Let us also remark that since the angular momentum and angular velocity vectors are parallel to the z-axis, $L_i^z = \mathbf{I}_i \omega(t)$, where $\mathbf{I}_i = m_i(x_i^2 + y_i^2)$ is the moment of inertia of the body m_i about the z-axis. Since the total moment of inertia, $\mathbf{I}$, is constant, and $\omega(t)$ is the same for all bodies because they belong to the same rotating geodesic, it follows that $\sum_{i=1}^n \mathbf{I}_i \omega(t) = \mathbf{I}\omega(t) = c$, where c is a constant. Consequently, ω is a constant vector.

Moreover, since $L_i^z = c_i$, it follows that $\mathbf{I}_i \omega(t) = c_i$. Then every $\mathbf{I}_i$ is constant, and so is every z_i, $i = \overline{1,n}$. Hence each body of mass m_i has a constant z_i-coordinate, and all bodies rotate with the same constant angular velocity around the z-axis, properties that agree with our definition of an elliptic relative equilibrium.

We now prove the case $\mathbf{J}=$ constant, i.e. when the geodesic rotates hyperbolically in $\mathbf{H}^2$. According to the definition of $\mathbf{J}$, we can assume that the bodies are on a moving geodesic whose plane contains the x axis for all time and whose vertex slides along the geodesic hyperbola $x=0$. (This moving geodesic hyperbola can be also visualized as the intersection between the sheet $z>0$ of the hyperboloid and the plane containing the x axis and rotating about it. For an instant, this plane also contains the z axis.)

The angular momentum of each body is $\mathbf{L}_i = m_i \mathbf{q}_i \boxtimes \dot{\mathbf{q}}_i$, so we can show as before that its derivative takes the form $\dot{\mathbf{L}}_i = m_i \mathbf{q}_i \boxtimes \overline{\nabla}_{\mathbf{q}_i} U(\mathbf{q})$. Again, $\overline{\nabla}_{\mathbf{q}_i} U(\mathbf{q})$ is either zero or orthogonal to $\mathbf{q}_i$. In the former case we can draw the same conclusion as earlier: that $\dot{\mathbf{L}}_i = \mathbf{0}$, so in particular $\dot{L}_i^x = 0$. In the latter case, $\mathbf{q}_i$ and $\overline{\nabla}_{\mathbf{q}_i} U(\mathbf{q})$ are in the plane of the moving hyperbola, so their cross product, $\mathbf{q}_i \boxtimes \overline{\nabla}_{\mathbf{q}_i} U(\mathbf{q})$ (which differs from the standard cross product only by its opposite z component), is orthogonal to the x axis, and therefore $\dot{L}_i^x = 0$. Thus $\dot{L}_i^x = 0$ in either case.

From here the proof proceeds as before by replacing $\mathbf{I}$ with $\mathbf{J}$ and the z axis with the x axis, and noticing that $L_i^x = \mathbf{J}_i \omega(t)$, to show that every m_i has a constant x_i coordinate. In other words, each body is moving along a (in general non-geodesic) hyperbola given by the intersection of the hyperboloid with a plane orthogonal to the x-axis. These facts, in combination with the sliding of the moving geodesic hyperbola along the fixed geodesic hyperbola $x=0$, are in agreement with our definition of a hyperbolic relative equilibrium. □

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Appendix

Since the Weierstrass model of the hyperbolic (Bolyai–Lobachevsky) plane is little known among nonlinear analysts or experts in differential equations, we present here its basic properties. This model is appealing for at least two reasons: (i) it allows an obvious comparison with the sphere, both from the geometric and analytic point of view; (ii) it emphasizes the differences between the Bolyai–Lobachevsky and the Euclidean plane as clearly as the well-known differences between the Euclidean plane and the sphere. As far as we are concerned, this model was the key for obtaining the results we proved for the n-body problem for $\kappa < 0$.

The Weierstrass model is constructed on one of the sheets of the hyperboloid $x^2+y^2-z^2=-1$ in the 3-dimensional Minkowski space $\mathbb{M}^3 := (\mathbb{R}^3, \boxdot)$, in which $\mathbf{a} \boxdot \mathbf{b} = a_x b_x + a_y b_y - a_z b_z$, with $\mathbf{a}=(a_x, a_y, a_z)$ and $\mathbf{b}=(b_x, b_y, b_z)$, represents the Lorentz inner product. We choose the $z>0$ sheet of the hyperboloid, which we identify with the Bolyai–Lobachevsky plane $\mathbf{H}^2$.

A linear transformation $T: \mathbb{M}^3 \to \mathbb{M}^3$ is orthogonal if $T(\mathbf{a}) \boxdot T(\mathbf{a}) = \mathbf{a} \boxdot \mathbf{a}$ for any $\mathbf{a} \in \mathbb{M}^3$. The set of these transformations, together with the Lorentz inner product, forms the orthogonal group $O(\mathbb{M}^3)$, given by matrices of determinant ± 1. Therefore the group $SO(\mathbb{M}^3)$ of orthogonal transformations of determinant 1 is a subgroup of $O(\mathbb{M}^3)$. Another subgroup of $O(\mathbb{M}^3)$ is $G(\mathbb{M}^3)$, which is formed by the transformations T that leave $\mathbf{H}^2$ invariant. Furthermore, $G(\mathbb{M}^3)$ has the closed Lorentz subgroup, $\mathrm{Lor}(\mathbb{M}^3) := G(\mathbb{M}^3) \cap SO(\mathbb{M}^3)$.

An important result is the Principal Axis Theorem for $\mathrm{Lor}(\mathbb{M}^3)$ (Hano and Nomizu 1983). Let us define the Lorentzian rotations about an axis as the 1-parameter subgroups of $\mathrm{Lor}(\mathbb{M}^3)$ that leave the axis pointwise fixed. Then the Principal Axis Theorem states that every Lorentzian transformation has one of the forms:

$$A = P\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}P^{-1}, \qquad B = P\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh s & \sinh s \\ 0 & \sinh s & \cosh s \end{bmatrix}P^{-1}, \quad \text{or}$$

$$C = P\begin{bmatrix} 1 & -t & t \\ t & 1-t^2/2 & t^2/2 \\ t & -t^2/2 & 1+t^2/2 \end{bmatrix}P^{-1},$$

where $\theta \in [0, 2\pi)$, $s, t \in \mathbb{R}$, and $P \in \mathrm{Lor}(\mathbb{M}^3)$. They are called elliptic, hyperbolic, and parabolic, respectively. The elliptic transformations are rotations about the z axis; the hyperbolic transformations are rotations about the x axis; and the parabolic transformations are rotations about the line $x = 0$, $y = z$. This result resembles Euler's Principal Axis Theorem, which states that any element of $SO(3)$ can be written, in some orthonormal basis, as a rotation about the z axis.

The geodesics of $\mathbf{H}^2$ are the hyperbolas obtained by intersecting the hyperboloid with planes passing through the origin of the coordinate system. For any two distinct points $\mathbf{a}$ and $\mathbf{b}$ of $\mathbf{H}^2$, there is a unique geodesic that connects them, and the distance between these points is given by $d(\mathbf{a}, \mathbf{b}) = \cosh^{-1}(-\mathbf{a} \boxdot \mathbf{b})$.

In the framework of Weierstrass's model, the parallels' postulate of hyperbolic geometry can be translated as follows. Take a geodesic γ, i.e. a hyperbola obtained by intersecting a plane through the origin, O, of the coordinate system with the upper sheet, $z > 0$, of the hyperboloid. This hyperbola has two asymptotes in its plane: the straight lines a and b, intersecting at O. Take a point, P, on the upper sheet of the hyperboloid but not on the chosen hyperbola. The plane aP produces the geodesic hyperbola α, whereas bP produces β. These two hyperbolas intersect at P. Then α and γ are parallel geodesics meeting at infinity along a, while β and γ are parallel geodesics meeting at infinity along b. All the hyperbolas between α and β (also obtained from planes through O) are non-secant with γ.

Like the Euclidean plane, the abstract Bolyai–Lobachevsky plane has no privileged points or geodesics. But the Weierstrass model has convenient points and geodesics, such as $(0, 0, 1)$ and the geodesics passing through it. The elements of $\mathrm{Lor}(\mathbb{M}^3)$ allow us to move the geodesics of $\mathbf{H}^2$ to convenient positions, a property we frequently use in this paper to simplify our arguments. Other properties of the Weierstrass model can be found in Faber (1983) and Reynolds (1993). The Lorentz group is treated in some detail in Baker (2002), but the Principal Axis Theorems for the Lorentz group contained there fails to include parabolic rotations, and is therefore incomplete. Weierstrass's model of hyperbolic geometry was first mentioned in Killing (1880) with more details in Killing (1885).

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