The Nature of Option Interactions and the Valuation of Investments with Multiple Real Options

Lenos Trigeorgis*

Abstract

This paper deals with the nature of option interactions and the valuation of capital budgeting projects possessing flexibility in the form of multiple real options. It identifies situations where option interactions can be small or large, negative or positive. Interactions generally depend on the type, separation, degree of being "in the money," and the order of the options involved. The paper illustrates, through a generic example, the importance of properly accounting for interactions among the options to defer, abandon, contract or expand investment, and switch use. It is shown that the incremental value of an additional option, in the presence of other options, is generally less than its value in isolation, and declines as more options are present. Therefore, valuation errors from ignoring a particular option may be small. However, configurations of real options exhibiting precisely the opposite behavior are identified. Comparative statics results confirm that the value of flexibility, despite interactions, manifests familiar option properties.

I. Introduction

Academics and practitioners alike now recognize that standard discounted cash flow (DCF) techniques when applied improperly often undervalue projects with real operating options and other strategic interactions. In practice, many corporate managers overrule passive net present value (NPV) analysis and use intuition and executive judgment to value future managerial flexibility.

Recently, Myers (1987), Kester (1984), Mason and Merton (1985), and Trigeorgis and Mason (1987), among others, suggest the use of option-based techniques to value the managerial flexibility implicit in investment opportunities. Managerial flexibility is a set of "real options," for example, the options to defer, abandon, contract, or expand investment, or switch investment to an alternative use.[1]

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[1]The option to defer investment has been examined by Tourinho (1979) in valuing reserves of natural resources, by McDonald and Siegel (1986), and by Paddock, Siegel, and Smith (1988) in

The real options literature to date has tended to focus on valuing individual options (i.e., one type of operating option at a time).[2] However, managerial flexibility embedded in investment projects typically takes the form of a *collection* of real options. This paper demonstrates that interactions among real options present in combination generally make their individual values nonadditive. Although many readers may intuit that certain options do in fact interact, the nature of such interactions and the conditions under which they may be small or large, as well as negative or positive, may not be trivial. In particular, the paper illustrates through a generic project the size and type of interactions among the options to defer, abandon, contract, expand, and switch use.

The combined value of operating options can have a large impact on the value of a project. However, the incremental value of an additional option often tends to be lower the greater the number of other options already present. Neglecting a particular option while including others may not necessarily cause significant valuation errors. However, valuing each option individually and summing these separate option values can substantially overstate the value of a project. Configurations of real options that can exhibit precisely the opposite behavior are also identified. Sensitivity analysis shows that, despite interactions, projects seen as collections of real options preserve a number of the familiar option properties.

The remainder of the paper is organized as follows. Section II describes a generic investment project with multiple operating options, along with the model specification and assumptions. The nature of option interactions and option value (non)additivity are examined in Section III. Section IV demonstrates interactions among options by first valuing various real options separately, then in combination. A summary and concluding remarks are provided in Section V.

II. An Investment Opportunity with Multiple Real Options

A. Project Description

Consider a generic investment opportunity with multiple real options. Construction of the project requires a series of investment outlays at specific times during a "building stage." For example, an initial outlay of I_1, followed by subsequent outlays of I_2 and I_3. The project generates its first cash flows during the "operating stage" that follows the last investment outlay, I_3.

The investment opportunity allows management the flexibility to:

a) *defer* undertaking the project;

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valuing offshore petroleum leases. Majd and Pindyck (1987) value the option to delay sequential construction (time to build). The option to temporarily shut down operations has been analyzed by McDonald and Siegel (1985), and by Brennan and Schwartz (1985). Myers and Majd (1990) analyze the option to abandon for salvage value. Stulz (1982) values options on the maximum (and minimum) of risky assets, which may be useful in analyzing the option to switch between alternative uses. Baldwin and Ruback (1986) show that future asset price uncertainty creates a valuable switching option that benefits short-lived assets.

[2]A notable exception is Brennan and Schwartz (1985) who utilize the convenience yield derived from futures and spot prices of a commodity to determine the combined value of the options to temporarily shut down (and open) a mine, and to abandon it for salvage, but do not address the interactions among individual option values.

b) permanently *abandon* construction, with no recovery, by foregoing subsequent planned investment outlays;
c) *contract* the scale of the project by reducing planned investment outlays;
d) *expand* the project's scale by making an additional investment outlay;
e) *switch* the investment from the current to its best alternative use, here modeled as a specified salvage value.

The above generic investment with its collection of real options is summarized in Figure 1. This project could describe many practical situations. For example, a large company engaged in the exploitation of natural resources could be offered the opportunity to purchase a lease on undeveloped land with potential mineral resources. The lease, expiring in T_1 years, would give management the right to start the project within that period by making an investment outlay, I_1, for construction of roads and other infrastructure. This would be followed by a second outlay of I_2 for excavation, and a third outlay of I_3 for the construction of a processing plant. Reducing this outlay to I_3' would result in a c-percent contraction in the operation scale of this plant. If the mineral is later found to enjoy a stronger demand than initially expected, the rate of production could be enhanced by x percent by expanding the processing plant at a cost of I_4. All along, management retains the option to salvage a percentage of its investment.

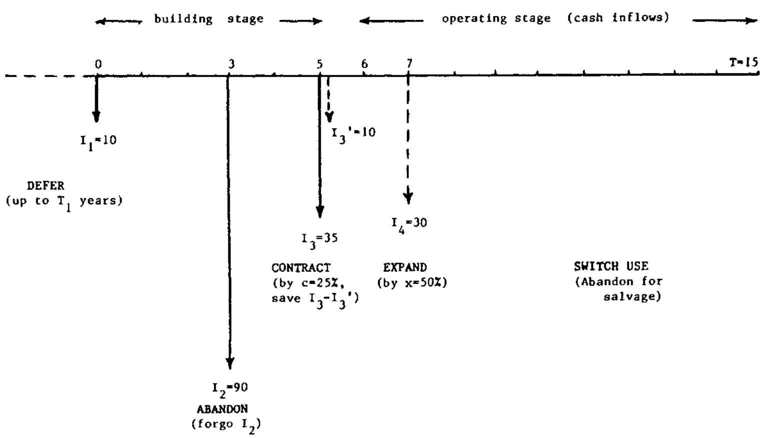

FIGURE 1

A generic project requiring a series of outlays (vertical arrows, *I*s), allowing management the flexibility (collection of real options) to defer, abandon, contract, or expand investment, and switch use.

As another example, a firm is considering the introduction of one of its existing patented products into a new geographic market. Management can delay introduction up to the time the patent expires, in T_1 years. Initiating the project requires an outlay of I_1 to purchase land, to be followed by an outlay of I_2 to build a plant in the new area. Upon the plant's completion, management plans a large one-time advertising expenditure of I_3, which, if the product's prospects at that time seem limited, could be reduced to I_3' with a c-percent market share loss. If, a year after introduction, the product is more enthusiastically received in the new market than originally expected, management can expand the project by x percent by adding to plant capacity at a cost of I_4. If at any time market conditions deteriorate, management can salvage a portion of the investment by selling the plant and equipment. The next part illustrates traditional valuation of this project.

B. Traditional NPV and Managerial Flexibility

Assume that the generic project is expected to generate annual cash flows during the operating stage, starting in year 6. Under traditional NPV analysis, these expected cash flows and the terminal project value would be discounted at an appropriate risk-adjusted rate. Assume that this calculation results in a "gross" project value of $V = 100$. This is simply the present value of expected cash flows from immediately undertaking the project, not including any required investment outlays or embedded real options. Following standard practice in the real options literature, this V (or its modified scale) will serve as the underlying asset value for the project's various real options.

Assuming the particular values shown in Figure 1 and subtracting the present value of the planned investment outlays, $I = 114.7$,[3] the passive NPV of immediately undertaking the above project, in the absence of managerial flexibility, is

$$\text{NPV} = V - I = 100 - 114.7 = -14.7.$$

The project would be rejected because its NPV is negative. The presence of managerial flexibility, however, can make the investment opportunity economically attractive.

Management's flexibility, or collection of options, to revise its future actions, contingent on uncertain future developments, introduces an asymmetry or skewness in the probability distribution of NPV. This asymmetry expands the opportunity's true value relative to passive NPV by improving its profit potential while limiting losses. Correct valuation thus requires an expanded NPV rule encompassing both sources of a real investment opportunity's value, the passive NPV of expected cash flows, and a value component for the combined value of the flexibility represented by the project's real options,

$$\text{Expanded NPV} = \text{Passive NPV} + \text{Combined Option Value}.$$

Traditional valuation approaches that either ignore these options altogether (passive NPV) or attempt to value such investment opportunities using a constant discount rate can lead to significant valuation errors. This is so because asymmetric claims on an asset do not generally have the same discount rate as the asset itself. This asymmetry can be properly analyzed by viewing flexibility in an options framework.[4]

An options-based approach to this problem, however, must recognize that flexibility seldom takes the form of a single option but instead typically is present as

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[3]The investment costs ($I_1 = 10$ in year 0, $I_2 = 90$ in year 3, and $I_3 = 35$ in year 5) are assumed known in advance and placed in an "escrow account" earning the riskless rate. Discounted continuously at the assumed risk-free rate of $r = 5$ percent, they yield a present value of $I = 114.7$.

[4]For an options approach to capital budgeting, see, for example, Trigeorgis (1986), (1988), (1990), and (1991a).

a combination of options. Therefore, proper analysis must account for the possible interactions among multiple options and the extent to which option values are not strictly additive. The rest of this paper values flexibility in the form of various combinations of options written on project value, and demonstrates the degree of interaction among different option combinations. The model specification and assumptions used in the option valuation of the above project are described next.

C. Model Specification and Assumptions

The valuation of operating options in this paper is based on the log-transformed version of binomial numerical analysis described in Trigeorgis (1991b).[5] Following standard practice in the real options literature, the gross project value (V_t) is assumed to follow a standard diffusion Wiener process given by[6]

$$dV/V = (\alpha - \delta)dt + \sigma dz, \tag{1}$$

where α is the instantaneous actual expected return on the project, σ is the instantaneous standard deviation of project value, dz is a standard Wiener process, and δ is the rate of return shortfall between the equilibrium total expected rate of return of a similar-risk traded financial asset, α^*, and the actual expected return of a nontraded real asset, α (see McDonald and Siegel (1984), (1985)). (δ may also capture any proportional cash flow (dividend-like) payout on the operating project, or even the net convenience yield in the case of commodities.)

Current option values can be determined by discounting certainty-equivalent or risk-neutral expectations of future payoffs at the riskless interest rate, r. In general, any contingent claim on an asset (traded or not) can be priced in a world with systematic risk by replacing the actual growth rate, α, with a certainty-equivalent rate, $\hat{\alpha} \equiv \alpha - \text{RP}$, where RP represents an appropriate risk premium, and then behaving as if the world were risk neutral (e.g., see Constantinides (1978), Cox, Ingersoll, and Ross (1985), Lemma 4, or Hull (1989), Ch. 7). (In general, $\text{RP} = S\sigma$, where $S \equiv (\alpha - r)/\sigma$ is the asset's market price of risk or reward-to-variability ratio. Note that if the market price of risk, S, is zero, investors are neutral to the asset's risk. If the CAPM holds, then $S = S_M \rho_M$, or $\text{RP} = S_M \rho_M \sigma$, where S_M is the market price of risk of the market portfolio (M) and ρ_M is the asset's correlation with the market.) Given that $\alpha = \alpha^* - \delta$, then $\alpha - \text{RP} = (\alpha^* - \text{RP}) - \delta \equiv r - \delta$. This is equivalent to a risk-neutral valuation (e.g., see Cox and Ross (1976), Harrison and Kreps (1979)), where the actual drift (α) would be replaced by the risk-neutral equivalent drift, $\hat{\alpha} = r - \delta$. (Such a world, where expected growth rates are adjusted from α to $\hat{\alpha} \equiv \alpha - \text{RP} = r - \delta$, will be referred to as a "risk-neutral" world. In this world, instead of using actual probabilities, expectations are formulated using equivalent "risk-neutral" probabilities (Harrison and Kreps (1979), Cox and Ross (1976), Trigeorgis and Mason (1987)).) For traded assets (in equilibrium) or for those real assets with no systematic risk (e.g., R & D projects, oil exploration, etc.), $\hat{\alpha} = r$ or $\delta = 0$.

[5]For other numerical work on options, see, for example, Brennan (1979) and Geske and Shastri (1985); for applications to real options, see Myers and Majd (1990), and Majd and Pindyck (1987).

[6]Although the precise numerical results may be somewhat different, the basic interaction effects would hold under alternative specifications of the underlying stochastic process (e.g., a mean-reverting process).

In discrete time, Log V follows an arithmetic Brownian motion, which can be approximated, over successively smaller intervals, by an equivalent binomial Markov random walk progressing in a triangular lattice as in Figure 2, A and B. Adjustments for cash flows (dividends) and for the asymmetries introduced by real options (the discrete-time equivalent of specifying boundary conditions in continuous-time models with partial differential equations) are made at appropriate times in a backward risk-neutral iterative process as in Trigeorgis (1991b).

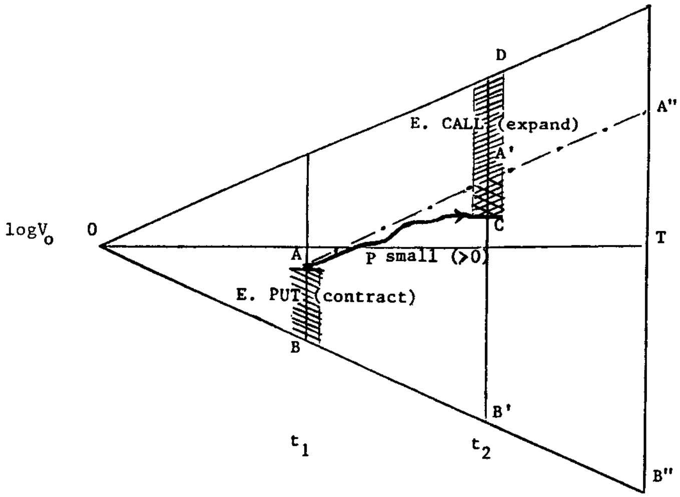

FIGURE 2A

Options of different type, approximately additive. Both European out-of-the-money options (a put and a call) with nonoverlapping exercise boundaries and small probability of joint exercise ($P > 0$); interactions (proxied by double-shaded area $A'C$) are small, and separate option values can be approximately added.

The subsequent option analysis is based on the following base case input assumptions (see Figure 1):

a. Initial gross project value, $V = 100$;[7]
b. Annual risk-free interest rate, $r = 5$ percent;
c. Variance of project value, Var = 0.25;
d. Expected project life, $T = 15$ years;
e. Opportunity is deferrable for $T_1 = 2$ years; the project begins with first investment outlay of $I_1 = 10$;
f. Construction can be abandoned, with no recovery, by foregoing the second investment outlay, I_2, of 90 in year 3;

[7] V is determined from discounting expected cash flows at the opportunity cost of capital. A proportional cash flow (e.g., 10 percent of current gross project value) is treated similarly to a dividend payout (see Myers and Majd (1990) for a good discussion), although here for simplicity no dividends are assumed. In general, one can incorporate any opportunity cost in delaying investment—which would be subtracted from the drift of the original project stochastic process—resulting either from a) intermediate cash flows missed by holding an option on the project (i.e., by waiting) rather than operating it immediately, or b) competitive erosion (see Trigeorgis (1986), Chapter 6).

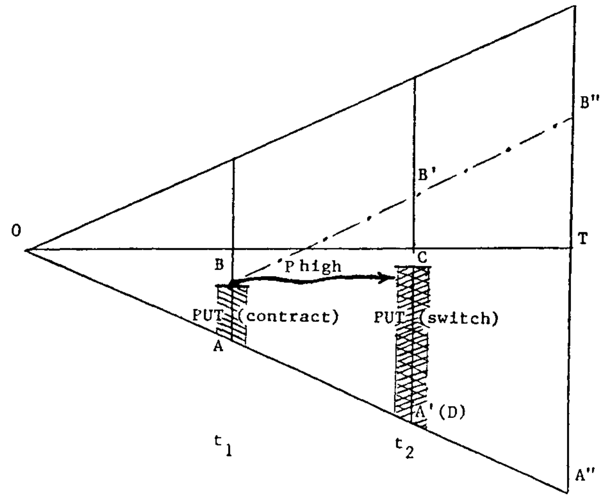

FIGURE 2B
Options of similar type, nonadditive. Both out-of-the-money puts with high overlap (area $A'C$) and high probability of joint exercise, interactions are high (and negative if prior put, as here; positive in case of calls)

g. The scale of the project can be contracted by $c = 25$ percent in year 5 by reducing the third investment outlay, I_3, to $I_3' = 10$;
h. The scale of the project can be expanded by $x = 50$ percent by making an additional investment outlay, I_4, of 30 in year 7;
i. The project's salvage value is $S = 50$ percent of cumulative investment costs (i.e., it "jumps" upward by 50 percent of each new cost outlay incurred), while declining exponentially at a rate of 10 percent per year between cost outlays.

Before proceeding with the presentation and discussion of the results for valuing the above investment opportunity with its collection of embedded real options, however, it would be useful to next examine the nature of real option interactions.

III. The Nature of Option Interactions and Option Value (Non)Additivity

Additivity of individual option values is trivial when options are written on distinct assets (e.g., calls or puts on shares of IBM stock). However, option additivity is not trivial when the options are written on the same unique underlying asset. Examples of interacting financial options include putable convertible bonds, callable extendable bonds, or simply securities callable by the issuer at two distinct times. Real options also typically come as an inseparable package with a single underlying asset (the gross project value, V). In situations such as these, options can interact.

First, the mere presence of subsequent options increases the value of the effective underlying asset for earlier options. In essence, prior real options have as their underlying asset the whole portfolio of gross project value plus the then value of any future options. At an extreme, the inseparability of real options from their underlying asset allows also the possibility that exercise of a prior put option on the asset, such as the option to abandon early, may eliminate or "kill" that asset. Due to the real asset's uniqueness and unavailability of other identical assets, this may preclude exercising future options on it (e.g., later to contract the project or switch between uses).

More generally, however, exercise of a prior real option may alter the underlying asset itself and, hence, the value of subsequent options on it, causing a second-order interaction. For example, the option to contract would decrease, while the option to expand would increase the project scale, affecting the value of other options on it. Further, the (conditional) probability of exercising a latter option, in the presence of an earlier option, would be higher or lower than the (marginal) probability of its exercise as a separate option, depending on whether the prior option is of the same or of opposite type, respectively. Real options may thus interact for various reasons and to varying degrees, depending on the probability of their joint exercise during the investment's life.

To illustrate the nature of these interactions, consider having a package of two options on the same asset. The degree of interaction and (non)additivity of option values—and the extent to which the underlying asset for a prior or subsequent option is altered—will be seen to depend on a) whether the options are of the same type (e.g., two puts or two calls) or are opposites (i.e., a put and a call), b) the separation of their exercise times (influenced by whether they are European or American options), c) their relative degree of being "in or out of the money," and d) their order or sequence. All these factors affect the degree of overlap between their exercise regions and the probability of their joint exercise.

The underlying principle in this analysis is that the value of an option in the presence of other options may differ from its value in isolation. Alternatively, the combined value of two options in the presence of each other may differ from the alternative of evaluating each option separately and adding the results. Two effects may be at work here, each affecting the direction or sign of the interaction as well as its magnitude.

First, recall that the value of a prior option would be altered if followed by a subsequent option because it would effectively be written on a higher underlying asset, V' (equal to the gross project value, V, plus the then expected value of the subsequent option). Specifically, in terms of sign, if the *first* option is a put, *its* value would be lower (giving a negative interaction), and if a call, higher (exhibiting a positive interaction), *relative to its value as a separate option.* (The magnitude of alteration in the *prior* option's value or the degree of its interaction would be larger the greater the joint probability of exercising both options, P, which depends on the similarity of the options involved.)

Second, the effective underlying asset for the *latter* option may be lower conditional on prior exercise of an earlier put option (e.g., to contract project scale), V'', than if the prior option were not exercised (i.e., maintaining project value, V). This may lead to a double negative effect if the prior option is a put.[8] If,

[8]This result holds unambiguously in the case of a subsequent call. If the latter option is also a put, the second effect would still be negative if exercise of the prior put reduces proportionately the scale of

instead, the prior option were a call (e.g., to expand project scale), the interaction can be positive, with the incremental value of both the prior and the latter option possibly being greater than their separate values.[9] In either situation, the degree of interaction between the two options would again be directly proportional to the probability of joint exercise.

If the two options are of opposite type (e.g., a pair of a put and a call) so that they are optimally exercisable under opposite (negatively correlated) circumstances, then the conditional probability of exercising the latter option given prior exercise of the former would be small—smaller than the marginal probability of exercising the latter option alone. The degree of interaction would then also be small and the options approximately additive. If the two options are of the same type (either a pair of puts or a pair of calls), then the conditional probability of exercise would be higher, and so would be the magnitude of interaction (deviation from option value additivity). Again, the sign of the interaction would depend on whether the prior option is a put (negative) or a call (positive).

One can examine further, with the aid of graphical illustrations, the possible variations in the magnitude and sign of interaction between two options, starting first from situations where option value additivity holds and extending into cases with higher degrees of interaction. First, as noted, interactions are small and the separate option values are approximately additive when the two options are of opposite type: a put (e.g., the option to contract) and a call (e.g., to expand), and are both out of the money.[10] As noted, two such European options would in fact be purely additive (i.e., their interaction would be precisely zero) only if both matured at the same time (i.e., $t_1 = t_2$). In this extreme case, although the marginal probabilities that either option may be independently exercised at (their common) maturity are positive, the joint (or conditional) probability of exercising both options is precisely zero ($P = 0$); with no interaction, each option retains its full, undistorted value as if evaluated independently, and thus their separate values are exactly additive.

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the latter put. However, in cases where the exercise price of the latter put is not reduced in proportion to project value, the second effect may be positive, although the net overall interaction may still be negative.

[9]If exercising the prior call (e.g., in a compound call option) could expand the underlying asset or project scale (i.e., $V'' > V$), a subsequent option on that higher asset may be more valuable and interactions can be positive.

The option to defer a project—a call whose exercise does not alter the "underlying asset" for subsequent options—is more complex. First, as the cash flows and future options are pushed back allowing more time for crucial variables to change, the increased variability may make subsequent options somewhat more valuable. However, if project initiation is delayed, for example, because the project is not yet good enough, a subsequent call option to expand may be less valuable and exhibit a negative interaction, though mitigated by the above positive side effect. More important, since the option to defer is written on the portfolio of gross project value plus the value of subsequent options, it would, at first glance, appear to be more valuable, other factors being the same. At the same time, however, the presence of subsequent options may enable management to adjust better to changing circumstances, increasing the value of early investment compared with a similar situation without such flexibility. Thus, the incremental value of the option to wait would tend to decrease, relative to immediate investment. This effect typically would dominate and lead to negative overall interactions between the flexibility to defer and other subsequent real options.

[10]Ironically, it is a better approximation to add up their separate values, other factors being the same, when the options are small (out of the money). To turn this around, it is least appropriate simply to add up separate option values precisely when they are most needed, that is, when they are most valuable (in the money).

The situation depicted in Figure 2A is similar, except that there is a separation in the exercise times of the two opposite-type, out-of-the-money (European) options, with the put maturing at an earlier time. Although there is again a high positive marginal probability that the put option will be exercised at time t_1 or that the call option may later be independently exercised at time t_2, the conditional probability of exercising the latter call option, given a prior exercise of the first put ($P_{L|F}$), is nevertheless small (> 0)—smaller than the marginal probability of exercising the latter option alone, P_L. For the put option to be optimally exercised at t_1, the state variable (the log of asset value) must drop below the "exercise boundary" into the "exercise region" (shaded area) *AB*. Following exercise of the first option, its subsequent movement would then be constrained within the trapezoidal area $ABB'A'$ by t_2 (or $ABB''A''$ by maturity, T), which only partially overlaps with the exercise region (*CD*) of the subsequent option (double-shaded area $A'C$)—with only a small chance of reaching the second exercise boundary necessary for triggering exercise of the latter call option. The smaller (greater) this overlap ($A'C$), the smaller (greater) the conditional probability of joint exercise and the smaller (greater) the degree of interaction between the two option values. If it is small enough, as in this case of opposite types of options, the separate option values would still be approximately additive. If the order were reversed so that the call option preceded the put, the options would still be of opposite type with nonoverlapping exercise regions and low conditional probability of joint exercise (related to the double-shaded area $A'C$), so that their interaction would again be small—though it may be of opposite (i.e., positive) sign—and the separate options could still be approximately added.[11]

If the prior option were also a put instead of a call, as shown in Figure 2B, the separate option values would be far from additive. As the options would then be of similar type, in this case both puts, their exercise regions would overlap significantly and the conditional probability of exercising one put, given earlier exercise of the other (as indicated by the increased double-shaded area $A'C$), would be high (< 1). Because exercise of the prior put (e.g., to contract) would reduce the project's scale and value and, hence, the other put option's (e.g., to switch between uses) with high probability, P, the expected incremental value of the latter option would be smaller. As noted, the prior put's value may also be somewhat smaller—a double negative effect—than if evaluated separately, because it is written on the project's portfolio with the future put, even though the latter may be reduced by the first-order interaction. Similarly, interactions would again be high, though *positive*, if the similar-type options were both calls (e.g., to expand the project at two distinct times) instead of puts. Of course, interactions can get more complicated if more than two options are considered. For example, if the pair of European calls (a

[11]The options would still be approximately additive, though less so, if one of the two European options (e.g., the put) were replaced with its American counterpart, extending the possible exercise times on the same side relative to the other European option's maturity. But, the conditional probability of joint exercise, here proxied by a double-shaded trapezoidal area, and, hence, the degree of interaction would be somewhat higher.

If the American put option (e.g., to switch use) extends its potential exercise times both before and after the other (European call) option's maturity, a hybrid situation is possible. That is, negative interaction in the first part (where part of the put precedes the call) and positive in the latter part (where the call precedes part of the put). Both interactions would have small magnitude and partially cancel each other out, leading to better additive approximation.

compound European call)—or, by extension, an American call—were preceded by another put option, (potentially dominating) negative interactions could arise between the positively interacting pair and the prior put.

As a final observation, it is possible that exercising a prior real put option (e.g., to abandon the project by simply foregoing an upcoming investment outlay) may "kill" other future options. In the special but extreme case that the exercise regions of two put options overlap fully and the first option can kill the latter one with certainty (with $P_{L|F} = 1$), the expected incremental value of the latter option would be negligible. The combined value would then simply be the full separate value of the first option (essentially written only on the base-scale project, because the latter option is valueless). More generally, however, if the latter option—for example, being instead a call with nonoverlapping exercise boundaries similar to the situation in Figure 2A—were not completely within the "shooting range" of the prior killing put so that the conditional probability is less than 1, it would not be completely "dead"; it would still retain some value as long as there were some chance it could be exercised without prior exercise of the first killing put (i.e., $P_{L|\bar{F}} > 0$).

Alternatively, if the condition for optimally exercising the one put (e.g., to abandon) also simultaneously satisfies, or is a subset of, the condition for optimally exercising the other (e.g., switching between uses), the combined value of the two options would then simply be the higher of the two separate values, an extreme case of full negative interaction. Such may also be the case when the separation between the exercise times of two similar-type options is negligible. More generally, the nature of interaction and, hence, the extent to which the values of two separate options may or may not approximately add up can be summarized as follows.

There is no (small) interaction and, hence, the separate option values would be (approximately) additive (i.e., option value additivity holds), if the conditional probability of exercising both options before maturity is zero (small).[12] Conversely, the interaction would be highest (high), making it most inappropriate to add up the separate option values, if it is certain (likely) both options will be exercised jointly (or the conditional probability of a joint exercise, $P_{L|F}$, is 1 (high)). The interaction would typically be positive if the prior option is a call and negative if a put. In the latter case (as when the separation between two similar-type options is negligible), the combined option value may be only (somewhat higher than) the higher of the separate individual values, that is, the incremental value of the lesser option may be negligible (small). Supportive numerical results based on the fairly rich generic project example described earlier in Section II are presented next.

IV. Presentation and Discussion of Results

This section presents the numerical valuation results for the generic project's multiple real options, first in isolation (i.e., one option at a time) and later in combination.

[12]In the continuous-time analogue, of course, the conditional probability is not precisely zero. Option value additivity would still approximately hold, however, if it is small enough.

A. The Value of Separate Options

The option to *defer* alone is basically valued as an American call option on the project, with an exercise price equal to the necessary investment outlays. As shown in Table 1, this option increases the project's Expanded NPV to 26.3, in contrast to the no-flexibility base case NPV of −14.7. Alternatively, the value of this option is 41,

$$\text{Option Value} \quad = \quad \text{Expanded NPV} - \text{Passive NPV} \quad = \quad 26.3 - (-14.7) \quad = \quad 41.$$

The option to permanently *abandon during construction* is valued as a compound call option on the project. If management has only this option, then the project has an Expanded NPV of 22.1. Thus, the value of the option to abandon during construction amounts to 37 percent of V.

TABLE 1

Interactions among Multiple Real Options

(Base case: $V = 100$; $r = 5\%$, Var = 0 25, Life $T = 15$; Defer $T_1 = 2$ yrs.)

NPV of project without real options: −14.7				
Value with one real option:				
DEFER (*D*)	ABANDON (*A*)	CONTRACT (*C*)	EXPAND (*E*)	SWITCH USE (*S*)
26 3* (41)**	22.1 (36.8)	−7.8 (6.9)	20.3 (35)	24.6 (39.3)
Value with two real options:				
D & A	*D & C*	*D & E*	*D & S*	*A & C*
36.4 (51 1)	27.7 (42.4)	54.7 (69.4)	38.2 (52.9)	22.6 (37.3)
A & E	*A & S*	*C & E*	*C & S*	*E & S*
50.6 (65.3)	24.6 (39.3)	27.1 (41 8)	25.5 (40 2)	54 7 (69.4)
Value with three real options:				
D & A & C	*D & A & E*	*D & A & S*	*D & C & E*	*D & C & S*
36.8 (51 5)	68.2 (82.9)	38.2 (52.9)	57.1 (71.8)	38 7 (53.4)
D & E & S	*A & C & E*	*A & C & S*	*A & E & S*	*C & E & S*
71 (85.7)	51.9 (66.6)	25.5 (40.2)	54.7 (69.4)	55.9 (70 6)
Value with four real options:				
D & A & C & E	*D & A & C & S*	*D & A & E & S*	*D & C & E & S*	*A & C & E & S*
69.3 (84)	38.7 (53.4)	71 (85.7)	71.9 (86.6)	55 9 (70.6)
Value with five real options:				
D & A & C & E & S (ALL)				
71.9 (86.6)				

* Project value including option(s), i.e., Expanded NPV.
**Value of option(s).

The option to *contract* the scale of operations is valued as a European put on part of the project, with exercise price equal to the potential cost savings. Including just this option increases the opportunity's value by 7 percent of V (from −14.7 to −7.8). Similarly, the option to *expand* in the base case project is worth 35 percent of the gross project value. This option is valued analogous to a European call to acquire part of the project by paying an extra outlay as the exercise price.

The option to *switch use* is valued as an American put on the project, with an exercise price equal to the value in its best alternative use, here assumed to be its salvage value. As shown in Table 1, its value in isolation is 40 percent of V.

B. The Value of Option Combinations with Interactions

As explained in the previous section, the value of an option in the presence of others may differ from its value in isolation. The presence of subsequent options increases the effective underlying asset for prior options. Moreover, exercise of a prior real option (e.g., expanding or contracting a project) may alter the underlying asset and value of subsequent options on it. The valuation results for the generic project, when particular real options are valued in the presence of others, illustrate option configurations where interactions can be small or large, as well as negative or positive.

Table 1 shows the value of the project with different *combinations* of operating options. For example, the value of the project increases from -14.7 (without any options) to 26.3, with the option to defer only; to 36.4, with the options to defer and abandon (*D* & *A*); to 36.8, with the options to defer, abandon, and contract (*D* & *A* & *C*); to 69.3, with the options to defer, abandon, contract, and expand (*D* & *A* & *C* & *E*); and, finally, to 71.9, with all five options. Thus, the combined option values (shown in parentheses in Table 1) increase from 41, with only the option to defer, to 51 (*D* & *A*), to 52 (*D* & *A* & *C*), to 84 (*D* & *A* & *C* & *E*), and finally to 86.6 (ALL), with all five options. These results confirm that real option values in the presence of each other are not generally additive. For example, although the value of the option to defer alone is 41 and the value of the option to abandon in isolation is 37, the value of both options present simultaneously is only 51, showing substantial negative interaction.

As noted, the degree of interaction is related to option type and the degree of overlap of exercise regions. Specifically, recall that options tend to be more additive when a) the options involved are of opposite type, that is, a call option optimally exercised when circumstances become favorable and a put option optimally exercised when circumstances become unfavorable, b) the times of possible exercise of the two options are close together, for example two European options maturing at the same time, as opposed to having distinctly different maturities or being American options, and c) the options are more out of the money, that is, having relatively high exercise prices for calls and low exercise prices for puts (leading to lower overlap of exercise regions). As an example, because the options to contract in year 5 and to expand in year 7 are of different type (i.e., a prior put and a latter call) and have no overlap between their exercise regions, their separate values ($C = 6.9$, $E = 35$) are basically additive, i.e., $C \,\&\, E = 41.8$. (As noted, interaction becomes precisely zero and the two European opposite-type options are purely additive if they are exercisable at exactly the same time as well as being out of the money.)

Table 2 shows the magnitude of interaction as the separation of exercise times varies between the options to abandon construction (in year 3) and to expand (by $x = 50$ percent if invest $I_E \equiv I_4 = 30$). The maturity of the option to expand, t, is allowed to vary from year 3 to year 13 (so that separation varies from 0 to 10 years). The size of interaction for a given separation is the difference between the combined option value ($A3 \,\&\, Et$) and the sum of separate values ($A3{+}Et$). Figure 3 shows that for these (opposite-type) options, (negative) interaction increases with separation at a decreasing rate.

TABLE 2

Interaction between the Options to Abandon and to Expand vs. Separation of Exercise Times (as Maturity of Option to Expand Increases)

OPTION TO ABANDON in yr. 3 (*A*3)	36.8						
	*E*3	*E*4	*E*5	*E*7	*E*9	*E*11	*E*13
OPTION TO EXPAND in yr. *t* (*Et*)	28.1	30.2	32.0	35.0	37.4	39.4	41.0
COMBINED OPTION VALUE (*A*3 & *Et*)	64.9	63.0	63.8	65.3	66.5	67.5	68.4
SUM OF SEPARATE VALUES (*A*3 + *Et*)	64.9	67.0	68.8	71.8	74.2	76.1	77.7
INTERACTION [(*A*3 & *Et*) − (*A*3 + *Et*)]	0	−4.0	−5.0	−6.5	−7.7	−8.6	−9.3
SEPARATION in yrs. (= *t* − 3)	0	1	2	4	6	8	10

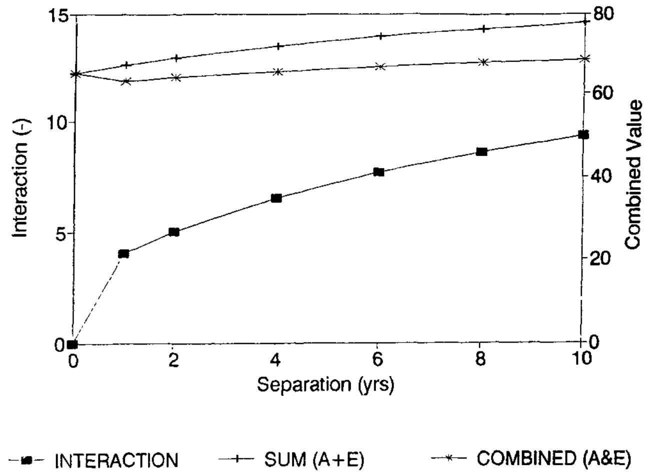

FIGURE 3

Interaction vs. Separation

The degree of (negative) interaction between the opposite-type options to abandon (*A*) and to later expand (*E*), i.e., the difference between the combined option value (*A* & *E*) and the sum of separate values (*A* + *E*), increases with the separation of exercise times.

Table 3 illustrates how the size of interaction between the option to abandon construction (in year 3) and the option to expand (by x = 50 percent if invest I_E) in year 7 varies as the exercise price (I_E) declines (or, alternatively, the scale of expansion, x, increases) and the option to expand gets relatively more deep in the money (proxied by xV/I_E). Figure 4 confirms that the magnitude of (negative) interaction between these (opposite-type) options increases with the relative degree of being in the money.

Furthermore, when considering operating put options with extensively overlapping exercise regions, there is high negative interaction and the combined value is slightly higher than the separate values. Two examples are the options to contract (C = 6.9) and switch use (S = 39.3) where C & S = 40.2, and the options to abandon (A = 37) and switch use (S = 39.3), where A & S = 39.3.[13] Similarly,

[13]The reader should be cautioned that, by design, the option to abandon construction with no recovery has no incremental value in the presence of the option to switch use.

TABLE 3

Interaction between the Options to Abandon and to Expand vs. Depth In the Money (as Exercise Price of Option to Expand Declines)

OPTION TO ABANDON in yr. 3 (*A3*)	36.8						
EXERCISE PRICE (I_E) (assuming x = 50% scale expansion in yr 7)	∞	150	75	50	30	20	15
[or % of scale expansion (x) (assuming I_E = 30)	0	0 1	0 2	0 3	0.5	0.75	1.0]
DEPTH IN THE MONEY (xV/I_E)	0 (out of money)	0 333	0.667	1 0 (at the money)	1.667	2.5 (in the money)	3.333
OPTION TO EXPAND in yr 7 (*E7*)	0	3 2	9 7	17 6	35.0	58.3	82 2
COMBINED OPTION VALUE (*A3* & *E7*)	36 8	39 5	45 1	51.3	65 3	84.3	104.4
SUM OF SEPARATE VALUES (*A3* + *E7*)	36 8	39 9	46.5	54.3	71 8	95.1	119 0
INTERACTION [(*A3* & *E7*) − (*A3* + *E7*)]	0	−0 4	−1.4	−3.0	−6.5	−10.8	−14.6

there is heavy negative interaction among the values of the options to abandon, contract, and switch use ($A \& C \& S = 40.2$ vs. $A + C + S = 83$).

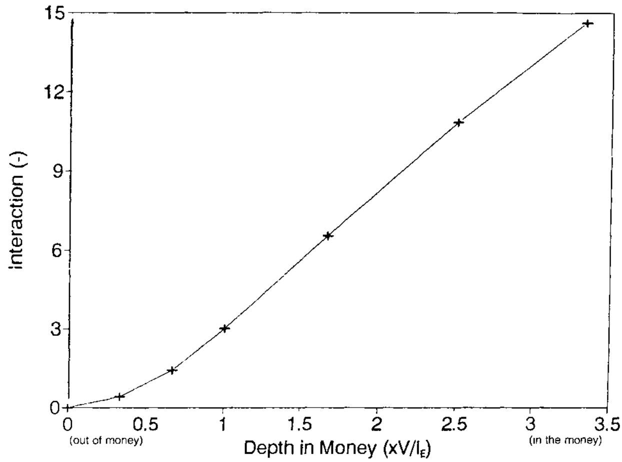

FIGURE 4

Interaction vs. Depth in the Money

The degree of negative interaction between the option to abandon and to expand increases with the depth in the money (or as the exercise price of the option to expand, I_E, declines)

In addition to option type and the degree of overlap of exercise regions, affected by separation, and depth in the money (or exercise price), the order or sequence of the options is seen to significantly affect option additivity. As mentioned, if a put precedes a latter option, the combined option value will typically exhibit negative interaction. However, if a put follows a prior call, there can be a positive interaction. Finally, the interaction will be positive if a call follows a prior call.

To illustrate the possibility of positive interactions, a second (European call) option to expand the project (by 35 percent with another optional investment of

15) is added to the basic example. Table 4 shows the total project value (including options) and the option values in a number of interesting cases. Note first that if the second option to expand has the same exercise date (i.e., year 5) as the opposite-type option to contract, the purely additive case with precisely zero interaction is obtained, i.e., $E5$ & $C5$ = 32.1 = $E5 + C5$.[14] But with two options to expand, a compound call situation, positive interactions are large. As expected, the presence of each call option enhances the other's value, leading to substantial positive interaction.[15] Specifically, the combined value of the options to expand in year 4 and in year 7 exceeds the sum of their individual values, that is, $E4$ & $E7$ = 65.3 > $E4 + E7$ = 59.2. This substantial positive interaction effect is maintained in the presence of the option to contract ($E4$ & $C5$ & $E7$ = 76.7 > $E4+C5+E7$ = 66.1) and when all other options are also jointly considered (ALL & $E4$ = 121.7 vs. ALL + $E4$ = 111). Having illustrated the case of significant positive interactions,[16] the paper returns to the basic generic project, with just one expansion option in year 7, to examine the marginal effect on project value of having increasingly more options.

TABLE 4

Adding a Second Option to Expand: A Case of Positive Interactions

NPV of project without real options −14.7			
Value with one real option:			
EXPAND yr 5 (*E5*)	EXPAND yr 4 (*E4*)	EXPAND yr 7 (*E7*)	CONTRACT yr. 5 (*C5*)
10.5* (25.2)**	9.6 (24.2)	20.3 (35)	−7.8 (6.9)
Value with two real options:			
E5 & *C5*	*E4* & *C5*	*C5* & *E7*	*E4* & *E7*
17.4 (32.1)	16.5 (31.2)	27.1 (41.8)	50.6 (65.3)
Value with three real options:			
E4 & *C5* & *E7*			
62 (76.7)			
Value with six real options:			
ALL & *E4*			
107 (121.7)			

* Project value including option(s), i.e., Expanded NPV
**Value of option(s).

As a result of predominantly negative option interactions, the combined value of all five options (87) in the basic generic example is slightly more than half the

[14] If the additional option to expand were instead shifted to mature in year 4 so that it precedes the option to contract, then there is a slight positive interaction, $E4$ & $C5$ = 31.2 vs. $E4+C5$ = 31.1. If the order is reversed, the options to contract in year 5 and to expand later in year 7 are still about additive, but with a slight negative interaction instead, specifically $C5$ & $E7$ = 41.8 vs. $C5 + E7$ = 41.9.

[15] Assuming a proportional nature for the call options to expand, the presence of the latter call option increases the value of the first call option. This is clear since the first call option is written on the total project value, which increases in the presence of the second call option. Similarly, the possibility of exercising the first call option to expand project scale increases the value of the second proportionate call option, leading to a double positive or super-additive effect.

[16] Although, in principle, there can be significant positive interactions, such as when expanding more than once, for the sequence of real options dealt with in this paper, interactions are typically negative. Positive interactions are more prevalent, however, *between* interdependent projects such as in R & D, in investments designed to gain a positioning in a new market, and in other so-called strategic investment commitments.

sum of the values of each separate option (159).[17] The value of the combined flexibility is, however, of the same order of magnitude as the value of the project's expected cash inflows (87 vs. 100). Thus, ignoring the value of all real options would significantly understate the true economic value of such projects (here, by about half). But measuring the value of this flexibility by simply adding the individual option values would seriously overstate (almost double) true worth.

Ignoring certain options (typically, puts), however, would not necessarily lead to significant valuation errors because the incremental value of an additional option tends to become lower as the overlap of its exercise region with those of other included options becomes greater.[18] As can be seen from Table 1, having only one option may be nearly half as valuable (e.g., to switch use = 39) as having all five (87); having two may be three-quarters as valuable (e.g., to abandon and expand = 65, or to defer and expand = 69); and having three may be as valuable (e.g., D & E & S = 86 or D & A & E = 83) as four (e.g., D & C & E & S = 87 or D & A & E & S = 86) and as all five (ALL = 87). Because of this diminishing marginal option value effect, although a few particular options may have been neglected in the treatment of the generic project, the valuation results may still represent a close approximation to the true value, especially if those options that were included were appropriately selected to minimize their overlap.[19]

Finally, interacting real options do maintain a number of the usual option properties. Figure 5A–C examines the sensitivity of the total generic project value, including all interacting real options (determined to be 72 in the base case example above), to changes in various factors that affect option values. With other factors held constant, the total project value a) increases with V as shown in Figure 5A; b) increases as project variance rises as seen in Figure 5B; and c) increases with more years to defer as shown in Figure 5C. In aggregate, this particular configuration manifests call option-like properties. Thus, the project valued as a collection of real options does preserve, despite option interactions, a number of the familiar option properties.

V. Summary and Conclusion

The paper deviates from the current real options literature that tends to focus on valuing one type of operating option at a time. Instead, this paper is concerned with valuing firm projects with collections of real options and quantifying the interactions among these options. Although the values of real options may not be additive, the combined flexibility that they afford management may be as economically significant as the value of the project's expected cash flows. The

[17]With the second option to expand in year 4 included, the joint value of all six options increases to 122, or 67 percent of their added separate values. Although the particular numbers would change when both expansion opportunities are considered, the net aggregate behavior of these options remains essentially the same.

[18]The exact size of the error would, of course, depend on the type of neglected options, that is, whether they are puts or calls similar to the other options present, and the extent to which they are in the money, depending on the relative size of their exercise costs. It would also depend on the overlap of their exercise regions with those of the options included, as indicated earlier.

[19]A simple selection rule is to eliminate those (usually put) options that are of similar type and are exercisable under similar circumstances as other included real options, particularly if their exercise costs are such that they are in-the-money options.

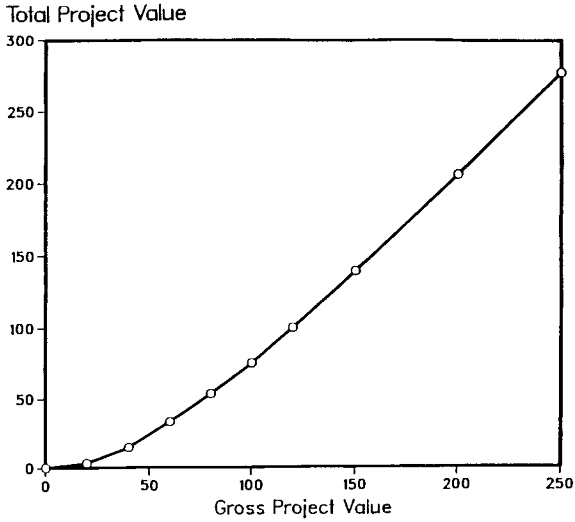

FIGURE 5A
Sensitivity analysis of the impact of gross project value (V) on the total project value (including the value of all real options)

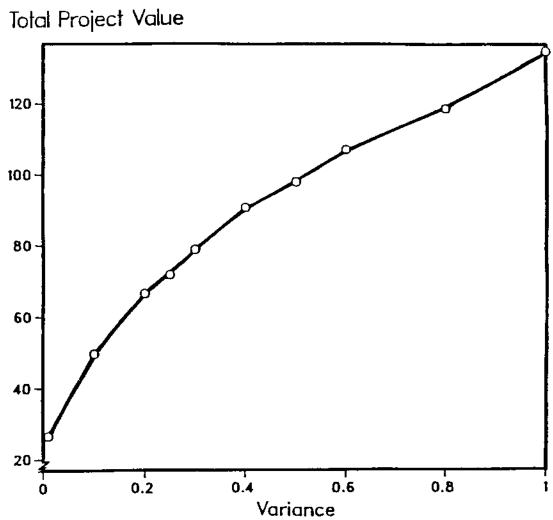

FIGURE 5B
Sensitivity analysis of the impact of project variance on the total project value (including the value of real options).

paper examines the nature of option interactions, and illustrates situations where option interactions are small and, therefore, simple option additivity is a good approximation. Other situations where high interactions seriously invalidate option additivity are also identified. Interactions are seen to depend on the type, separation, degree of being in or out of the money, and order of the options involved, factors that impact on the joint probability of exercise. In principle, interactions

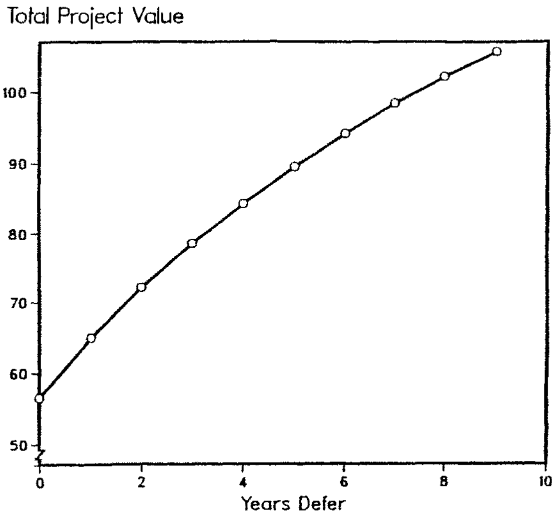

FIGURE 5C
Sensitivity analysis of the impact of years to defer (i.e., the expiration period of the option to defer) on total project value (including the value of all real options)

between pairs of options may be positive as well as negative. In practice, where negative interactions are more prevalent within a given project, they may so reduce the incremental value of certain options that simply ignoring them may not create any significant valuation errors. Sensitivity analysis results also confirm that projects with a variety of such interacting real options do preserve familiar option properties.

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