The nature of solutions to linear passive complementarity systems

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The nature of solutions to linear passive complementarity systems

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Abstract

Linear passive systems with complementarity conditions (as an application, one may consider linear passive networks with ideal diodes) are studied. For these systems contained in the linear complementarity class of hybrid systems, existence and uniqueness of solutions are established. Moreover, the nature of the solutions is characterized. In particular, it is shown that derivatives of Dirac impulses cannot occur and Dirac impulses and jumps in the state variable can only occur at $t = 0$. These facts reduce the 'complexity' of the solution in a sense. Finally, we give an explicit characterization of the set of initial states from which no Dirac impulses or discontinuities in the state variable occur. This set of 'regular states' turns out to be invariant under the dynamics.

1 Introduction

Nowadays switches like thyristors and diodes are used in electrical networks for a great variety of applications in both power engineering and signal processing. For the analysis and simulation of the transient behaviour of such networks the switches are often modelled ideally [14, 7, 9]. It is well-known that ideal modelling causes the network model to be of mixed discrete and continuous nature. In particular, the circuit evolves through multiple topologies (modes) depending on the (discrete) states of the diodes. The mode transitions are triggered by inequalities and may result in discontinuities and Dirac impulses in the network's variables, see e.g. [11, 9, 7, 14]. From this point of view these switched electrical circuits can be seen as *hybrid systems* [8].

In this paper we consider linear passive networks with ideal diodes, which are studied in the framework of *linear complementarity systems LCS* [13, 6, 12, 3, 5]. *LCS* can be seen as dynamical extensions of the *linear complementarity problem* of mathematical programming [1]. Starting from this background, the main objectives of the paper are the following.

(i) Define a mathematically precise solution concept for linear passive networks with diodes.
(ii) Prove (global) existence and uniqueness of solutions (well-posedness).
(iii) Establish regularity properties of the solutions. In particular, it will be rigorously proven that derivatives of Dirac impulses do not occur (even for inconsistent initial states) and Dirac impulses occur only at the initial time. Moreover, it will turn out that the set of switching times is a right-isolated set, meaning that for all time instants there exists a positive length time interval in which the diodes do not change their state. This excludes a certain type of accumulation of event times (Zenoness).

Besides the motivation of well-posedness as a fundamental issue in the theory of dynamical systems, it plays also an important role in simulation methods. Indeed, to answer the question if and in what sense the approximations converge to the true solution(s) of the network model (i.e. consistency of the numerical method), one has to establish what is meant by a transient "true solution," one needs insight in the nature of solutions and the number of solutions.

Throughout the paper, $\mathbb{R}$ denotes the real numbers, $\mathbb{R}_+$ the nonnegative real numbers, $\mathbb{C}$ the complex numbers, $\mathbb{R}(s)$ the set of all rational functions with real coefficients, $\mathcal{L}_2(t_0, t_1)$ the square integrable functions on (t_0, t_1), and $\mathcal{B}$ the Bohl functions (i.e. functions having rational Laplace transforms) defined on $(0, \infty)$. The function $u \in \mathcal{L}_2(t_0, t_1)$ is called nonnegative, denoted by $u \geq 0$, if $u(t) \geq 0$ for almost all $t \in (t_0, t_1)$. For vector valued functions, inequalities are understood to hold componentwise. The inner product of the Hilbert space $\mathcal{L}_2(0, T)$ is denoted by $\langle \cdot, \cdot \rangle$. The distribution $\delta_t^{(i)}$ stands for the i-th distributional derivative of the Dirac impulse supported at t.

The dual cone of a set $\mathcal{Q} \subseteq \mathbb{R}^n$ is defined by $\mathcal{Q}^* = \{x \in \mathbb{R}^n | x^\top y \geq 0 \text{ for all } y \in \mathcal{Q}\}$. For a positive integer k, the set $\bar{k}$ is defined as $\{1, 2, \ldots, k\}$. Given a matrix $A \in \mathbb{R}^{nxn}$ and index sets $J \subseteq \bar{n}$ and $K \subseteq \bar{n}$, the submatrix A_{JK} of A is defined as the matrix whose entries lie in the rows of A indexed by J and the columns indexed by K. If $J = \bar{n}$, we denote the submatrix A_{JK} also by $A_{\bullet K}$. Similarly, if $K = \bar{n}$, we write $A_{J\bullet}$ for the submatrix A_{JK}. $\mathcal{I}$ denotes the identity matrix of any dimension. As usual, we say that a triple (A, B, C) is minimal, when (A, B) is controllable and (C, A) is observable.

For any proposition $P(\sigma)$ depending on the parameter σ, we say that '$P(\sigma)$ holds for all sufficiently large σ,' if there exists a $\sigma_0 \in \mathbb{R}$ such that $P(\sigma)$ holds for all $\sigma > \sigma_0$.

2 Preliminaries

In this section, we recall some definitions and results that are needed in the sequel.

2.1 Linear complementarity problem

First, we define the linear complementarity problem.

Problem 2.1 (Linear Complementarity Problem) $LCP(q, M)$: Given $q \in \mathbb{R}^n$ and $M \in \mathbb{R}^{nxn}$ find $z \in \mathbb{R}^n$ such that

(a) $z \geq 0$ and $q + Mz \geq 0$

(b) $z^\top(q + Mz) = 0$

For an extensive survey on *LCP*, we refer to [1]. The solution set of $LCP(q, M)$ will be denoted by $SOL(q, M)$.

Remark 2.2 We shall employ the following standard result of *LCP* theory several times. If $z_i \in SOL(q_i, M_i)$ with

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$i \in \{1, 2\}$ then

$$(z_1 - z_2)^T((q_1 + M_1 z_1) - (q_2 - M_2 z_2)) \\ = -z_1^T(q_2 + M_2 z_2) - z_2^T(q_1 + M_1 z_1) \leq 0$$

2.2 Passive systems

Next, we recall the definition and the characterization of passivity in order to be self-contained.

Definition 2.3 [15] Consider a system (A, B, C, D) described by the equations

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$
$$y(t) = Cx(t) + Du(t) \tag{2}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^k$, $y(t) \in \mathbb{R}^k$ and A, B, C, and D are matrices of appropriate dimensions. The quadruple (A, B, C, D) is called *passive*, or *dissipative* with respect to the supply rate $u^\top y$, if there exists a nonnegative function $V : \mathbb{R}^n \to \mathbb{R}_+$, called a *storage function*, such that for all $t_0 \leq t_1$ and all time functions $(u, x, y) \in \mathcal{L}_2^{k+n+k}(t_0, t_1)$ satisfying (1)-(2) the following inequality holds:

$$V(x(t_0)) + \int_{t_0}^{t_1} u^\top(t)y(t)dt \geq V(x(t_1)).$$

The above inequality is called the *dissipation inequality*.

Next, we state a well-known theorem on passive systems which is sometimes called the positive real lemma.

Theorem 2.4 [15] *Assume that (A, B, C) is minimal. Let $G(s) = C(sI - A)^{-1}B + D$ be the transfer function of (A, B, C, D). Then the following statements are equivalent:*

(a) (A, B, C, D) is passive.

(b) The matrix inequalities

$$K = K^\top > 0 \text{ and } \begin{bmatrix} A^\top K + KA & KB - C^\top \\ B^\top K - C & -(D + D^\top) \end{bmatrix} \leq 0$$

have a solution.

(c) $G(s)$ is positive real, i.e., $G(\lambda) + G^\top(\bar{\lambda}) \geq 0$ for all $\lambda \in \mathbb{C}$ with nonnegative real parts.

Moreover, $V(x) = \frac{1}{2}x^\top Kx$ defines a quadratic storage function if and only if K satisfies the linear matrix inequalities above.

3 Linear complementarity systems

In this section, we shall briefly review linear complementarity systems. For a detailed treatment, see [12, 13, 3, 5].

The linear complementarity system $LCS(A, B, C, D)$ is given by the system of differential/algebraic equations and inequalities

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{3a}$$
$$y(t) = Cx(t) + Du(t) \tag{3b}$$
$$0 \leq u \perp y \geq 0, \tag{3c}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^k$, $y(t) \in \mathbb{R}^k$ and A, B, C, and D are matrices of appropriate dimensions.

Note that (3) is a nonlinear system in which u and y are not input and output variables due to the complementarity conditions imposed by (3c).

Linear electrical networks consisting of (linear) resistors, inductors, capacitors, gyrators, transformers (RLCGT) and ideal diodes are obvious examples of linear complementarity systems. For an overview of the applications of complementarity systems, see [4].

The complete specification of the solution concept for linear complementarity systems can be found in [3]. To define such a general concept, it is natural to employ the distributional theory, since the abrupt changes in the trajectories can be adequately modelled by impulses. We illustrate this by the following 'standard' example of complementarity systems.

Example 3.1 [12, 3] Consider the mechanical system depicted in figure 1. For simplicity, we assume that the stop is placed at the equilibrium position of the left cart and purely inelastic, and that the masses of the carts and constants of the linear springs are equal to one. The equations of the motion are given as follows

$$\begin{aligned} \dot{x}_1 &= x_3 \\ \dot{x}_2 &= x_4 \\ \dot{x}_3 &= -2x_1 + x_2 + u \\ \dot{x}_4 &= x_1 - x_2 \\ y &= x_1 \\ 0 \leq u &\perp y \geq 0, \end{aligned}$$

where u is the reaction force exerted by the stop. The deviations of the left and right cart from their equilibrium positions are denoted by x_1 and x_2, respectively. For the initial state

$$x_0 = e^{-A\frac{\pi}{2}}(0 \quad -1 \quad -1 \quad 0)^\top \cong (0\; 0.26\; 0.62\; -1.24)^\top,$$

the triple (u, x, y)

$$u(t) = \delta_{\frac{\pi}{2}} + \begin{cases} 0 & \text{if } 0 \leq t < \frac{\pi}{2} \\ \cos(t - \frac{\pi}{2}) & \text{if } \frac{\pi}{2} < t \leq \pi \end{cases}$$

$$x(t) = \begin{cases} e^{At}x_0 & \text{if } 0 \leq t < \frac{\pi}{2} \\ (0 \;\; -\cos(t - \frac{\pi}{2}) \;\; 0 \;\; \sin(t - \frac{\pi}{2}))^\top & \text{if } \frac{\pi}{2} < t \leq \pi \end{cases}$$

$$y(t) = \begin{cases} Ce^{At}x_0 & \text{if } 0 \leq t < \frac{\pi}{2} \\ 0 & \text{if } \frac{\pi}{2} < t \leq \pi \end{cases}$$

is a *solution* on $[0, \pi]$ in the sense of [3] with an impulsive force at time $t = \frac{\pi}{2}$. Note that the Dirac pulse corresponds to the collision of the left cart to the stop with nonzero velocity.

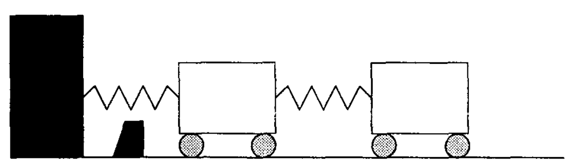

Figure 1: Two-carts system.

In this example, we saw that Dirac impulses can occur at a time instant $t > 0$. However, as we shall see below, the passivity assumption limits the occurrence of impulses. Motivated by this fact, we shall give another solution concept for (3) instead of using the one described in [3]. To do so, we need to recall the definition of a *Bohl distribution* and an *initial solution*.

Definition 3.2 We call u a *Bohl distribution*, if $u = u_{imp} + u_{reg}$ with $u_{imp} = \sum_{i=0}^{l} u^{-i}\delta_0^{(i)}$ for $u^{-i} \in \mathbb{R}$ and $u_{reg} \in \mathcal{B}$. We call u_{imp} the impulsive part of u and u_{reg} the regular part of u. The space of all Bohl distributions is denoted by $\mathcal{B}_{imp}$.

For any Bohl distribution u, the leading coefficient of its impulsive part is defined as $\mathrm{lead}(u) = 0$, if $u_{imp} = 0$, and defined as $\mathrm{lead}(u) = u^{-l}$, if $u_{imp} = u^{-l}\delta_0^{(l)} + u^{-l+1}\delta_0^{(l-1)} + \cdots + u^0\delta_0^{(0)}$ with $u^{-l} \neq 0$.

Given $u \in \mathcal{B}_{imp}$, we say that $u \succcurlyeq 0$ (initially nonnegative), if either

(a) $\mathrm{lead}(u) > 0$, or

(b) $\mathrm{lead}(u) = 0$ and there exists an $\epsilon > 0$ such that $u_{reg}(t) \geq 0$ for all $t \in (0, \epsilon)$.

Inequalities are understood componentwise for distributions in $\mathcal{B}_{imp}^{k}$.

Definition 3.3 The distribution $(u, x, y) \in \mathcal{B}_{imp}^{k+n+k}$ is said to be an *initial solution to (3) with initial state* x_0 if

(a) $\dot{x} = Ax + Bu + x_0\delta$ and $y = Cx + Du$ as equalities of distributions

(b) $u \succcurlyeq 0$ and $y \succcurlyeq 0$

(c) for all $i \in \bar{k}$, either $u_i = 0$ or $y_i = 0$ as equalities of distributions.

Definition 3.4 The distribution space $\mathcal{L}_{2,imp}(0, T)$ is defined as the set of all $u = u_{imp} + u_{reg}$ with $u_{imp} = \sum_{i=0}^{l} u^{-i}\delta_0^{(i)}$ for $u^{-i} \in \mathbb{R}$ and $u_{reg} \in \mathcal{L}_2(0, T)$

Now, we can give the following solution concept for (3).

Definition 3.5 $(u, x, y) \in \mathcal{L}_{2,imp}^{k+n+k}(0, T)$ is said to be a *solution on* $(0, T)$ *to (3) with initial state* x_0 if

(a) $\dot{x} = Ax + Bu + x_0\delta$ and $y = Cx + Du$ as equalities of distributions

(b) $(u_{imp}, x_{imp}, y_{imp})$ is the impulsive part of an initial solution to (3) with initial state x_0

(c) $u \geqslant 0$, $y \geqslant 0$, and $\langle u_{reg}, y_{reg}\rangle = 0$.

The solution concept of definition 3.5 is more restrictive (in the sense that it does only allow Dirac impulses (and its derivatives) supported at $t = 0$) than the one defined in [3]. To illustrate this, the mechanical system considered in example 3.1 has no solution on $(0, \pi)$ for the given initial state in the sense of definition 3.5, but does have a solution in the sense of [3], which is based on concatenation of initial solutions.

Initial solutions play an important role for LCS and can be characterized by the solutions of a generalization of the linear complementarity problem called the *Rational Complementarity Problem* [5].

Problem 3.6 $RCP(q(s), M(s))$: Given $q(s) \in \mathbb{R}^k(s)$ and $M(s) \in \mathbb{R}^{k\times k}(s)$, find $u(s) \in \mathbb{R}^k(s)$ such that

(a) $u(\sigma) \geq 0$ and $q(\sigma)+M(\sigma)u(\sigma) \geq 0$ for all sufficiently large $\sigma \in \mathbb{R}$.

(b) $u^\top(s)(q(s) + M(s)u(s)) = 0$

Existence and uniqueness of the solutions of the RCP are studied in [5] and necessary and sufficient conditions are presented in terms of a series of LCPs. To be complete, we recall the relation between RCP and initial solutions.

Theorem 3.7 [5] *The following statements are equivalent.*

(a) The linear complementarity system (3) has an initial solution for initial state x_0

(b) $RCP(C(s\mathbb{1} - A)^{-1}x_0, C(s\mathbb{1} - A)^{-1}B + D)$ *has a solution.*

Moreover, there is a one-to-one correspondence (given by the Laplace transform and its inverse) between solutions to the RCP and the u-part of initial solutions to LCS for initial state x_0.

We use the notation $RCP(x_0)$ rather than $RCP(C(s\mathbb{1} - A)^{-1}x_0, C(s\mathbb{1}-A)^{-1}B+D)$ whenever (A, B, C, D) is clear from the context.

Example 3.8 Consider the system described in example 3.1. *(a)* The triple(u, x, y) given by

$$u(t) = \cos\, t$$
$$x(t) = \begin{pmatrix}0 & -\cos\, t & 0 & \sin\, t\end{pmatrix}^\top$$
$$y(t) = 0.$$

is an initial solution of the system with initial state $x_0 = \begin{pmatrix}0 & -1 & 0 & 0\end{pmatrix}^\top$. According to theorem 3.7, we can compute this initial solution by solving the corresponding $RCP(x_0)$ given by

$$y(s) := \frac{-s}{s^4 + 3s^2 + 1} + \frac{s^2 + 1}{s^4 + 3s^2 + 1}u(s)$$
$$u^\top(s)y(s) = 0$$

for all $s \in \mathbb{C}$ and $u(\sigma) \geq 0$, $y(\sigma) \geq 0$ for all sufficiently large σ. From the complementarity conditions and rationality of $u(s)$ and $y(s)$, it can be concluded that $y(s) \equiv 0$ or $u(s) \equiv 0$. Together with nonnegativity conditions, this implies that the solution of $RCP(x_0)$ is equal to $u(s) = \frac{s}{s^2+1}$ (and $y(s) = 0$). By taking the inverse Laplace transform of this pair, we have $u(t) = \cos\, t$ (and $y(t) = 0$), as expected.

(b) Consider $RCP(\begin{pmatrix}0 & -1 & -1 & 0\end{pmatrix}^\top)$. It can be verified that the solution of this problem is given by $u(s) = 1+\frac{s}{s^2+1}$(and $y(s) = 0$). Note that $u(s)$ is not strictly proper, which indicates that the corresponding initial solution (with $u(t) = \delta_0(t) + \cos\, t$ and $y(t) = 0$) has a nontrivial impulsive part.

In contrast with example 3.8 *(a)*, the solution of the RCP is not strictly proper in *(b)*. Strict properness of the solution means that there exists a smooth continuation with the given initial condition. Indeed, smooth continuation is possible for the initial state given in *(a)*, but not for the initial state given in

(b). Explicit characterization of the initial states from which smooth continuation is of particular interest. In the most general case, it seems difficult to achieve this. However, by utilizing the passivity of the system, we shall be able to establish such a characterization in the next section.

The following lemma justifies the use of Bohl functions for linear passive complementarity systems. For an index set $I \subseteq \bar{k}$, I^c denotes the set $\{i \in \bar{k} \mid i \notin \bar{k}\}$.

Lemma 3.9 *Assume that (A, B, C, D) is passive, (A, B, C) is minimal and B has full column rank. Then, for each $I \subseteq \bar{k}$ there exists an $F^I \in \mathbb{R}^{n \times n}$ such that $\dot{x} = F^I x$ for any $(u, x, y) \in \mathcal{L}_2^{k+n+k}(0, \infty)$ satisfying (1)-(2) with $y_I(t) = 0$ and $u_{I^c}(t) = 0$ for all $t \in [0, \infty)$.*

Proof

According to [2, Theorem 3.10] and [3, Lemma 3.3], it suffices to show that the transfer matrix $G_{II} := C_{I\bullet}(s\mathcal{I} - A)^{-1}B_{\bullet I} + D_{II}$ is invertible as a rational matrix for any $I \subseteq \bar{k}$. Suppose that $\det G_{II}(s) \equiv 0$. Then there exists a rational vector function $v(s) \not\equiv 0$ such that $G_{II}(s)v(s) = 0$. Take $\sigma > 0$ such that $v(\sigma) \neq 0$ and $\sigma\mathcal{I} - A$ is invertible. Define $\bar{u}$ as

$$\bar{u}_i := \begin{cases} 0 & \text{if } i \notin I \\ v_i(\sigma) & \text{if } i \in I \end{cases}$$

The triple

$$\bar{u}(t) = \bar{u}e^{\sigma t}$$
$$\bar{x}(t) = (\sigma\mathcal{I} - A)^{-1}B\bar{u}e^{\sigma t}$$
$$\bar{y}(t) = G(\sigma)\bar{u}e^{\sigma t}$$

satisfies the system equations (1)-(2). Since (A, B, C, D) is passive, there exists $K > 0$ such that the dissipation inequality

$$\bar{x}^\top(t_0)K\bar{x}(t_0) + \int_{t_0}^{t_1} \bar{u}^\top(t)\bar{y}(t)dt \geq \bar{x}^\top(t_1)K\bar{x}(t_1) \quad (4)$$

holds for all t_0 and t_1 with $t_1 \geq t_0$. By construction, it holds that $\bar{u}^\top(t)\bar{y}(t) = 0$ for all t. Applying this and $\lim_{t_0 \to -\infty} \bar{x}(t_0) = 0$ (note that $\sigma > 0$), to (4) yields (by letting t_0 tend to $-\infty$)

$$0 \geq \bar{x}^\top(t_1)K\bar{x}(t_1)$$

for all t_1. Hence, due to $K > 0$, $\bar{x}(t) = 0$ for all t and thus $B\bar{u} = B_{\bullet I}v(\sigma) = 0$. Since B has full column rank, $v(\sigma) = 0$. This is a contradiction. □

4 Main Results

In this section, we shall present existence and uniqueness (in the sense of definition 3.5) results for linear passive complementarity systems. First, we state the following theorem which guarantees existence and uniqueness of the solutions of RCP (and consequently, of initial solutions to the LCS given an initial state).

Theorem 4.1 [5] *If (A, B, C, D) is passive, (A, B, C) is minimal and B has full column rank, then $RCP(x_0)$ has a unique solution for each $x_0 \in \mathbb{R}^n$.*

Now, we can state the following theorem which plays a crucial role in the sequel.

Theorem 4.2 *Assume that (A, B, C, D) is passive, (A, B, C) is minimal and B has full column rank. Define $\mathcal{Q} := SOL(0, D)$, $G(s) := C(s\mathcal{I} - A)^{-1}B + D$ and let $u_{x_0}(s)$ be the solution to $RCP(x_0)$. The following assertions hold:*

(a) For each $x_0 \in \mathbb{R}^n$, $u_{x_0}(s)$ is proper.

(b) For all $x_0 \in \mathbb{R}^n$, $C(x_0 + Bu^0) \in \mathcal{Q}^*$ where $u^0 = \lim_{s\to\infty} u_{x_0}(s)$.

(c) $u_{x_0}(s)$ is strictly proper if and only if $Cx_0 \in \mathcal{Q}^*$.

(d) $\lim_{s\to\infty} u_{x_0}(s) \in \mathcal{Q}$.

Proof

(a) The triple

$$\bar{u}(t) = u_{x_0}(\sigma)e^{\sigma t} \quad (5)$$
$$\bar{x}(t) = (\sigma\mathcal{I} - A)^{-1}Bu_{x_0}(\sigma)e^{\sigma t} \quad (6)$$
$$\bar{y}(t) = G(\sigma)u_{x_0}(\sigma)e^{\sigma t} \quad (7)$$

satisfies the system equations for all $\sigma \in \mathbb{R}$ with $\sigma\mathcal{I} - A$ nonsingular. It follows from passivity that there exists a $K > 0$ such that for all t_1 and t_0 with $t_1 \geq t_0$

$$\bar{x}^\top(t_1)K\bar{x}(t_1) - \bar{x}^\top(t_0)K\bar{x}(t_0) \leq \int_{t_0}^{t_1} \bar{u}^\top(t)\bar{y}(t)dt \quad (8)$$

By substituting (5)-(7) into the dissipation inequality (8), one obtains

$$u_{x_0}^\top(\sigma)B^\top(\sigma\mathcal{I} - A)^{-\top}K(\sigma\mathcal{I} - A)^{-1}Bu_{x_0}(\sigma) \leq \frac{1}{2\sigma}u_{x_0}^\top(\sigma)G(\sigma)u_{x_0}(\sigma) \quad (9)$$

Since $K > 0$, B has full column rank and $(\sigma\mathcal{I} - A)^{-1}$ is strictly proper, there exists an $\alpha > 0$ such that

$$\frac{\alpha}{\sigma^2}\|u_{x_0}(\sigma)\|^2 \leq u_{x_0}^\top(\sigma)B^\top(\sigma\mathcal{I} - A)^{-\top}K(\sigma\mathcal{I} - A)^{-1}Bu_{x_0}(\sigma) \quad (10)$$

for all sufficiently large σ. Since $u_{x_0}(s)$ is a solution of $RCP(x_0)$, we have

$$\frac{1}{2\sigma}u_{x_0}^\top(\sigma)G(\sigma)u_{x_0}(\sigma) = -\frac{1}{2\sigma}u_{x_0}^\top(\sigma)C(\sigma\mathcal{I} - A)^{-1}x_0 \leq \frac{1}{2\sigma}\|C(\sigma\mathcal{I} - A)^{-1}x_0\|\|u_{x_0}(\sigma)\| \quad (11)$$

It is not difficult to see that there exists $\beta > 0$ such that $\|C(\sigma\mathcal{I} - A)^{-1}\| \leq \frac{\beta}{\sigma}$ for all sufficiently large σ. Thus, (9), (10) and (11) yield now that

$$\|u_{x_0}(\sigma)\| \leq \frac{\beta}{2\alpha\sigma}\|x_0\| \quad (12)$$

for all sufficiently large σ. Hence, $u_{x_0}(s)$ is proper.

(b) Note that the solution $u_{x_0}(s)$ to $RCP(x_0)$ forms a solution to $LCP(C(\sigma\mathcal{I} - A)^{-1}x_0, G(\sigma))$ for sufficiently large σ. In view of remark 2.2, we have for each $v \in \mathcal{Q}$ that

$$(u_{x_0}(\sigma) - v)^\top(C(\sigma\mathcal{I} - A)^{-1}x_0 + G(\sigma)u_{x_0}(\sigma) - Dv) \leq 0$$

for all sufficiently large σ. Since $D \geq 0$ (according to theorem 2.4 statement (b)) and $G(\sigma) = C(\sigma\mathcal{I} - A)^{-1}B + D$, we obtain

$$(u_{x_0}(\sigma) - v)^\top(C(\sigma\mathcal{I} - A)^{-1}x_0 + C(\sigma\mathcal{I} - A)^{-1}Bu_{x_0}(\sigma)) \leq 0 \quad (13)$$

for all sufficiently large σ. Multiplying this relation by σ and letting σ tend to infinity give

$$(u^0 - v)^\top(Cx_0 + CBu^0) \leq 0$$

Since $\mathcal{Q}$ is a cone, we have for all $\lambda \geq 0$ and all $v \in \mathcal{Q}$

$$(u_0 - \lambda v)^\top (Cx_0 + CBu^0) \leq 0$$

and hence,

$$\lambda v^\top (Cx_0 + CBu^0) \geq u_0^\top (Cx_0 + CBu^0).$$

It follows that $v^\top (Cx_0 + CBu^0) \geq 0$ for all $v \in \mathcal{Q}$ and thus $C(x_0 + Bu^0) \in \mathcal{Q}^*$.

(*c*) 'only if' Suppose that the solution $u_{x_0}(s)$ of $RCP(x_0)$ is strictly proper. According to statement (*b*), $Cx_0 \in \mathcal{Q}^*$, because $u^0 = 0$.

'if' Suppose that $Cx_0 \in \mathcal{Q}^*$. We know that $u_{x_0}(s)$ is proper. Take the power series expansion of $u_{x_0}(s)$ around infinity as

$$u_{x_0}(s) = u^0 + u^1 s^{-1} + u^2 s^{-2} + \ldots \qquad (14)$$

By substituting (14) into

$$u_{x_0}^\top(s)(C(s\mathbf{1} - A)^{-1} x_0 + G(s) u_{x_0}(s)) = 0,$$

we obtain by considering the coefficients corresponding to s^0 and s^{-1}

$$u^{0\top} D u^0 = 0 \qquad (15)$$

$$u^{0\top} C x_0 + u^{0\top} D u^1 + u^{1\top} D u^0 + u^{0\top} C B u^0 = 0 \qquad (16)$$

Since $u_{x_0}(s)$ is the solution of $RCP(x_0)$, $u^0 \geq 0$ and $Du^0 \geq 0$. Together with (15), these give $u^0 = \lim_{s\to\infty} u_{x_0}(s) \in \mathcal{Q}$ (this proves statement *d*). The relation (15) also implies

$$(D + D^\top) u^0 = 0 \qquad (17)$$

Now, (16) and (17) give

$$u^{0\top} C x_0 + u^{0\top} C B u^0 = 0. \qquad (18)$$

According to theorem 2.4, passivity of (A, B, C, D) implies the existence of a symmetric $K > 0$ such that

$$\begin{bmatrix} A^\top K + KA & KB - C^\top \\ B^\top K - C & -(D + D^\top) \end{bmatrix} \leq 0 \qquad (19)$$

Premultiplying (19) by $(\gamma z^\top \; u^{0\top})$ and postmultiplying by $(\gamma z^\top \; u^{0\top})^\top$ for arbitrary $z \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}$, yields (use (17))

$$\gamma^2 z^\top (A^\top K + KA) z + 2\gamma z^\top (KB - C^\top) u^0 \leq 0$$

Considering this expression as an inequality for a quadratic form in γ, yields that $z^\top (KB - C^\top) u^0 \leq 0$. Taking $z = (KB - C^\top) u^0$ results in

$$(KB - C^\top) u^0 = 0 \qquad (20)$$

From (20), we obtain $u^{0\top} C B u^0 = u^{0\top} B^\top K B u^0$. Since $u^0 \in \mathcal{Q}$ and $Cx_0 \in \mathcal{Q}^*$, (18) gives

$$0 \geq -u^{0\top} C x_0 = u^{0\top} C B u^0 = u^{0\top} B^\top K B u^0 \geq 0$$

Finally, positive definiteness of K and the full column rank of B imply $u^0 = 0$, i.e., $u_{x_0}(s)$ is strictly proper.

(*d*) This has already been shown in the proof of (c). □

Due to the one-to-one relation between initial solutions and solutions to RCP via the Laplace transform, the properness of solutions to the RCP implies that initial solutions can only have an impulsive part consisting of the Dirac distribution (and not its derivatives, i.e. $u_{imp} = u^0\delta$). Hence, in linear passive electrical networks with ideal diodes derivatives of Dirac impulses do not occur. This fact is widely believed true, but the authors are not aware of any rigorous proof. The framework proposed here makes it possible to prove this intuition.

Definition 4.3 A state x_0 is called *regular* for $LCS(A, B, C, D)$, if the corresponding initial solution is smooth (i.e. has a zero impulsive part). The collection of regular states is denoted by $\mathcal{R}$.

Since strictly proper RCP-solutions correspond to smooth initial solutions of LCS, statement *c* in Theorem 4.2 gives a characterization of the regular states: $x_0 \in \mathcal{R}$ if and only if $Cx_0 \in \mathcal{Q}^*$ with $\mathcal{Q} = SOL(0, D)$. As we shall see, this characterization plays a key role in the proof of global existence of solutions as the set of such initial states will be proven to be invariant under the dynamics.

According to [1, Cor. 3.8.10 and Thm 3.1.7 (c)] and because $D \geq 0$ one has $Cx_0 \in \mathcal{Q}^*$ if and only if $LCP(Cx_0, D)$ is solvable. Hence, a test for deciding the regularity of an initial state consist of determining whether or not a certain LCP has a solution.

To give an idea about the structure of the set of regular states $\mathcal{R}$ and the cone $\mathcal{Q}^*$, a few examples are in order.

Example 4.4 (a) If $D = 0$, then $\mathcal{Q} = \mathbb{R}^k_+$ and $\mathcal{Q}^* = \mathbb{R}^k_+$. Hence, $\mathcal{R} = \{x_0 \in \mathbb{R}^n \mid Cx_0 \geq 0\}$.

(b) If $D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then $\mathcal{Q} = \{\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mid u_1 \geq 0 \text{ and } u_2 = 0\}$. Consequently, $\mathcal{Q}^* = \{\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mid y_1 \geq 0\}$.

(c) If D is positive definite, it follows that $\mathcal{Q} = \{0\}$, which implies that $\mathcal{Q}^* = \mathbb{R}^k$ and thus $\mathcal{R} = \mathbb{R}^n$.

A direct implication of the statements *b* and *c* in Theorem 4.2 is that, if smooth continuation is not possible for x_0, it is possible after one re-initialization (jump) of the state variable. Indeed, if the impulsive part of the (unique) initial solution is equal to $u^0\delta$, the state after re-initialization is equal to $x_0 + Bu^0$ [2, 3]. Since $C(x_0 + Bu^0) \in \mathcal{Q}^*$, it follows from statement *(c)* that $x_0 + Bu^0 \in \mathcal{R}$. Hence, from $x_0 + Bu^0$ there exists a smooth initial solution. This means that we have proven local existence of a solution in the sense of definition 3.5 on $[0, \varepsilon)$ with $\varepsilon > 0$ by concatenating initial solutions.

Now, we can state the main result of the paper.

Theorem 4.5 *Suppose that (A, B, C, D) is passive, (A, B, C) is minimal and B is of full column rank. Then, for all x_0 and $T > 0$, (3) has a unique solution in the sense of definition 3.5 on $(0, T)$ with initial state x_0.*

Proof

Existence: The construction of a solution will be based on concatenation of initial solutions. Since after at most one jump of the state variable the state $x(0+) = x_0 + Bu^0$ is contained in $\mathcal{R}$, a solution (u, x, y) exists on $(0, \tau_1)$ for some $\tau_1 > 0$ as argued before. Note that $x(0+) \in \mathcal{R}$ and that (u, x, y) is part of a smooth initial solution with initial state $x(0+)$. Since $t \to (u, x, y)(t + \rho)$ forms a smooth initial solution for any $\rho \in (0, \varepsilon)$, we have that $x(\rho) \in \mathcal{R}$ for all $\rho \in (0, \varepsilon)$. Since (u, x, y) is a Bohl function, the limit $\lim_{t\uparrow\varepsilon} x(t) = x(\varepsilon)$ exists. The closedness of $\mathcal{R}$ (due to the closedness of $\mathcal{Q}^*$) implies that $x(\varepsilon) \in \mathcal{R}$. Hence, there exists a smooth continuation (a smooth initial solution) from $x(\varepsilon)$ that defines a solution on $(0, \tau_2)$ with $\tau_2 > \tau_1$. This construction can be repeated as long as the limit $\lim_{t \uparrow \tau} x(t)$ exists, where $(0, \tau)$ is the time-interval on which a solution has been generated so far. A reason that a global solution (on $(0, T)$) does not exist can be that the intervals of continuation $[\tau_i, \tau_{i+1})$ are getting smaller and smaller such that $\lim_{i \to \infty} \tau_i = \tau^* < T$ and $\lim_{t \uparrow \tau^*} x(t)$ does not exist. To complete the proof we will show the existence of the latter limit in all circumstances.

Suppose the maximal interval on which a solution (u, x, y) can be defined is $[0, \tau^*)$, $\tau^* < T$. According to Lemma 3.9 there is at most exponential growth ($\dot{x} = F^I x$) between mode changes. Since x is continuous on $(0, \tau^*)$, this implies that x is bounded (say $\|x(t)\| \leq M$ for all $t \in (0, \tau^*)$). On an interval $(s, t) \subseteq (0, \tau^*)$ where (u, x, y) is governed by the dynamics $\dot{x} = F^I x$ of mode I, the following estimate holds

$$\begin{aligned} \|x(t) - x(s)\| &= \|e^{F^I(t-s)}x(s) - x(s)\| \\ &\leq c_I \mid t - s \mid \|x(s)\| \leq c_I M \mid t - s \mid \end{aligned} \tag{21}$$

Note that the matrix function $t \to \frac{e^{tF^I} - I}{t}$ is bounded (by c_I) on $(0, \tau^*)$. Hence, for $(s, t) \subseteq (0, \tau^*)$ with x possibly evolving through several modes we get from (21) that

$$\|x(t) - x(s)\| \leq M \max_{I \subseteq \bar{k}} c_I \mid t - s \mid .$$

This implies that x is Lipschitz continuous on $(0, \tau^*)$ and thus also uniformly continuous. A standard result in mathematical analysis [10, ex. 4.13] states that $x^* := \lim_{t \uparrow \tau^*} x(t)$ exists. From the construction above it can be derived that $x(t) \in \mathcal{R}$ for all $t \in (0, \tau^*)$ and hence, $x^* \in \mathcal{R}$ (due to closedness of $\mathcal{R}$), which implies that smooth continuation is possible (local existence) from x^* beyond τ^*. This contradicts the definition of τ^*. Hence, existence of a solution on $(0, T)$ is guaranteed.

Uniqueness: Suppose that two solutions (u, x, y) and (u', x', y') exist in the sense of Definition 3.5 for initial state x_0. Since there exists exactly one initial solution from the initial state x_0, the re-initialization from x_0 must be unique and thus $x(0+) = x'(0+)$. Clearly, $(u - u', x - x', y - y')$ satisfies (1)-(2) with initial state 0. The dissipation inequality yields

$$\int_0^t [u(\tau) - u'(\tau)]^\top [y(\tau) - y'(\tau)] d\tau \geq [x(t) - x'(t)]^\top K [x(t) - x'(t)]$$

for all $t \in (0, T)$. From the fact that u, u', y, y' are nonnegative almost everywhere and the complementarity of (u, y) and (u', y'), we obtain $\int_0^t [u(\tau) - u'(\tau)]^\top [y(\tau) - y'(\tau)] d\tau \leq 0$. Hence, $[x(t) - x'(t)]^\top K [x(t) - x'(t)] \leq 0$ for all $t \in (0, T)$. Since $K > 0$, we obtain $x(t) = x'(t)$ for all t. Since B is of full column rank, this gives $u = u'$ and $y = y'$ almost everywhere. □

Note that it follows that the set of regular states $\mathcal{R}$ is invariant under the dynamics. Indeed, the constructed solution in the proof is the only solution and lies in $\mathcal{R}$ for all $t > 0$.

5 Conclusions

In this paper we studied linear passive complementarity systems, which describe linear electrical networks with ideal diodes. The main results of the paper show global existence and uniqueness of solutions. Moreover, it is proven that in linear electrical networks with ideal diodes only Dirac impulses occur (and not its derivatives) and that impulses can occur only at time $t = 0$. These properties are widely believed true in the circuit theory community, but as far as the authors are aware this paper gives the first rigorous proof of these facts. Finally, we explicitly characterized the set of initial states from which no Dirac impulses and/or state jumps occur (so-called regular states) in terms of the dual cone of the solution set of a particular linear complementarity problem. The set of regular states turned out to be invariant under the dynamics.

Current work is concerned with the analysis of a time-stepping method used for the transient simulation of switched electrical networks. Based on the rigorous foundation given here, the goal is to prove that the approximations obtained by the time-stepping method converge to the actual solution of the network model.

Needless to say, generalization to nonlinear systems is an interesting open problem. The main difficulty is the absence of a similar tool to RCP in the nonlinear context.

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