

The Navier–Stokes-alpha model of fluid turbulence

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Dedicated to V.E. Zakharov on the occasion of his 60th birthday

Abstract

We review the properties of the nonlinearly dispersive Navier–Stokes-alpha (NS- α) model of incompressible fluid turbulence — also called the viscous Camassa–Holm equations in the literature. We first re-derive the NS- α model by filtering the velocity of the fluid loop in Kelvin’s circulation theorem for the Navier–Stokes equations. Then we show that this filtering causes the wavenumber spectrum of the translational kinetic energy for the NS- α model to roll off as k^{-3} for $k\alpha > 1$ in three dimensions, instead of continuing along the slower Kolmogorov scaling law, $k^{-5/3}$, that it follows for $k\alpha < 1$. This roll off at higher wavenumbers shortens the inertial range for the NS- α model and thereby makes it more computable. We also explain how the NS- α model is related to large eddy simulation (LES) turbulence modeling and to the stress tensor for second-grade fluids. We close by surveying recent results in the literature for the NS- α model and its inviscid limit (the Euler- α model). © 2001 Published by Elsevier Science B.V.

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1. Introduction

The energy in a turbulent fluid flow cascades toward ever smaller scales until it reaches the dissipation scale, where it can be transformed into heat. This cascade — creating fluid motions at ever smaller scales — is a characteristic feature of turbulence. This feature is also the main difficulty in simulating turbulence numerically, because all numerical simulations will have finite resolution and will eventually be unable to keep up with the cascade all the way to the dissipation scale, especially for complex flows, e.g., near walls and interfaces.

The effects of subgrid-scale fluid motions occurring below the available resolution of numerical simulations must be modeled. One way of modeling these effects is simply to discard the energy that reaches such subgrid scales. This is clearly unacceptable, though, and many creative alternatives have been offered. A prominent example is the large eddy simulation (LES) approach, see, e.g., [1,2]. The LES approach is based on applying a spatial filter to the Navier–Stokes equations. The reduction of flow complexity and information content achieved in the LES approach depends on the characteristics of the filter that one uses, its type and width. In particular, the LES approach introduces a length scale into the description of fluid dynamics, namely, the width of the filter used. Note that the

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LES approach is conceptually different from the Reynolds averaged Navier–Stokes (or, RANS) approach, which is based on statistical arguments and exact ensemble averages, rather than spatial and temporal filtering. After filtering, however, just as in the RANS approach, one faces the classic turbulence closure problem: how to model the filtered-out subgrid scales in terms of the remaining resolved fields? In practice, this problem is compounded by the requirement that the solution be simulated numerically, thereby introducing further approximations.

This paper begins by reviewing a modeling scheme — called here the Navier–Stokes-alpha model, or NS- α model (also called the viscous Camassa–Holm equations in [3–6]) — that introduces an energy “penalty” inhibiting the creation of smaller and smaller excitations below a certain length scale (denoted alpha). This energy penalty results in a *nonlinearly dispersive modification* of the Navier–Stokes equations. The alpha-modification appears in the nonlinearity, it depends on length scale and we emphasize that it is dispersive, not dissipative. We shall re-derive the modified equations in Section 2 from the viewpoint of Kelvin’s circulation theorem. As we shall show in Section 3, this modification causes the translational kinetic energy wavenumber spectrum of the NS- α model to roll off rapidly below the length scale alpha as k^{-3} in three dimensions, instead of continuing to follow the slower Kolmogorov scaling law, $k^{-5/3}$. This roll off shortens the inertial range of the NS- α model and thus makes it more computable. This is the main new result of the paper, so we shall comment now on its main implications.

Since the energy spectrum rolls off faster, the wavenumber $k = \kappa_\alpha$ at which viscous dissipation takes over in the NS- α model must be *lower* than for the original Navier–Stokes equations. Hence, for a given driving force and viscosity, the number of active degrees of freedom N_{dof} for the NS- α model must be *smaller* than for Navier–Stokes. In Section 3, we give a heuristic estimate of the number N_{dof} and compare it with the rigorous estimate derived in [6] for the fractal dimension D_{frac} of the global attractor for the NS- α model. Namely,

$$D_{\text{frac}} \leq (N_{\text{dof}})^{3/2}, \quad N_{\text{dof}} \equiv (L\kappa_\alpha)^3 \simeq \frac{L}{\alpha} Re^{3/2}, \quad (1.1)$$

where L is the integral scale (or domain size), κ_α the end of the NS- α inertial range and $Re = L^{4/3}\epsilon_\alpha^{1/3}/\nu$ the Reynolds number (with NS- α energy dissipation rate ϵ_α and viscosity ν). The corresponding number of degrees of freedom for a Navier–Stokes flow with the *same* parameters is

$$N_{\text{dof}}^{\text{NS}} \equiv \left(\frac{L}{\ell_{Ko}} \right)^3 \simeq Re^{9/4}, \quad (1.2)$$

where ℓ_{Ko} denotes the Kolmogorov dissipation length scale. The implication of these estimates of degrees of freedom for numerical simulations that access a significant number of them using the NS- α model would be an increase in computational speed relative to Navier–Stokes of

$$\left(\frac{N_{\text{dof}}^{\text{NS}}}{N_{\text{dof}}} \right)^{4/3} = \left(\frac{\alpha}{L} \right)^{4/3} Re. \quad (1.3)$$

Thus, if α tends to a constant value, say $L/100$, when the Reynolds number increases — as found in [3–5] by comparing steady NS- α solutions with experimental data for turbulent flows in pipes and channels — then one could expect to obtain a substantial increase in computability by using the NS- α model at high Reynolds numbers. An early indication of the reliability of using these estimates to gain a relative increase in computational speed in direct numerical simulations (DNSs) of homogeneous turbulence in a periodic domain is given in [7]. There, a computational speed up was achieved by using the NS- α model in DNS of turbulence in a periodic domain by a factor about equal to the fluctuation Reynolds number (≈ 250). In the case considered in [7], this factor happens to be about $Re/100$.

In Section 4, we discuss the relation of the NS- α model to similar equations which are derived in a different physical context, namely in the context of non-Newtonian fluids. The main difference between the NS- α model and

the non-Newtonian second-grade fluids which it resembles lies in the choice of dissipation. The NS- α model uses the standard Navier–Stokes viscosity, while the second-grade fluid uses a weaker form of dissipation — namely, wavenumber independent damping of the fluid velocity. Discussions of the relative advantages of the two forms of dissipation (e.g., in terms of boundary data requirements and well-posedness) are beyond the scope of this review. However, we remark that the steady solutions of the NS- α model with the standard Navier–Stokes viscosity were found in [3–5] to agree with experimental mean velocity profile data for turbulent flows in pipes and channels. The corresponding steady solutions of the second-grade fluid with its weaker form of dissipation were not found to agree with this experimental data. Finally, in Section 5, we provide a brief guide to the recent literature for those who might be interested in the mathematical context in which the alpha model was originally derived, as well as in its potential applications in turbulence modeling. With regard to the latter, the NS- α model was recently shown to be transformable to a generalized similarity model for LES turbulence modeling [8].

2. Kelvin-filtered turbulence models

Although it was first derived from energetic considerations using the Euler–Poincaré variational framework in [9,10], the Navier–Stokes-alpha model may be motivated by an equivalent argument based on Kelvin’s circulation theorem. The original Navier–Stokes (NS) equations are

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla p = \nu \Delta \mathbf{v} + \mathbf{f}, \quad \text{with } \nabla \cdot \mathbf{v} = 0 \quad (2.1)$$

for a forcing \mathbf{f} and constant kinematic viscosity ν . These equations satisfy *Kelvin’s circulation theorem*,

$$\frac{d}{dt} \oint_{\gamma(\mathbf{v})} \mathbf{v} \cdot d\mathbf{x} = \oint_{\gamma(\mathbf{v})} (\nu \Delta \mathbf{v} + \mathbf{f}) \cdot d\mathbf{x} \quad (2.2)$$

for a fluid loop $\gamma(\mathbf{v})$ that moves with velocity $\mathbf{v}(\mathbf{x}, t)$, the Eulerian fluid velocity.

2.1. Kelvin-filtering the Navier–Stokes equations

The equations for the NS- α model emerge from a modification of the Kelvin circulation theorem (2.2) to integrate around a loop $\gamma(\mathbf{u})$ that moves with a *spatially filtered Eulerian fluid velocity* given by $\mathbf{u} = g * \mathbf{v}$, where $*$ denotes the convolution,

$$\mathbf{u} = g * \mathbf{v} = \int g(\mathbf{x} - \mathbf{y}) \mathbf{v} d^3 y. \quad (2.3)$$

The “inverse” is denoted as

$$\mathbf{v} = \mathcal{O} \mathbf{u}, \quad (2.4)$$

thereby defining an operator \mathcal{O} whose Green’s function is the filter g and which we shall assume is positive, symmetric, isotropic and time-independent. Under these assumptions the quantity (kinetic energy)

$$E = \frac{1}{2} \int \mathbf{u} \cdot \mathbf{v} d^3 x = \frac{1}{2} \int \mathbf{v} \cdot g * \mathbf{v} d^3 x = \frac{1}{2} \int \mathbf{u} \cdot \mathcal{O} \mathbf{u} d^3 x \quad (2.5)$$

defines a norm.

We obtain a modification to the Navier–Stokes equations (2.1) by replacing in their Kelvin’s circulation theorem (2.2) the loop $\gamma(\mathbf{v})$ with another loop $\gamma(\mathbf{u})$ moving with the spatially filtered velocity, \mathbf{u} . Then we have

$$\frac{d}{dt} \oint_{\gamma(\mathbf{v})} \mathbf{v} \cdot d\mathbf{x} = \oint_{\gamma(\mathbf{u})} (\nu \Delta \mathbf{v} + \mathbf{f}) \cdot d\mathbf{x}. \quad (2.6)$$

After taking the time derivative inside the Kelvin loop integral moving with filtered velocity \mathbf{u} and reconstructing the gradient of pressure, we find the *Kelvin-filtered Navier–Stokes equation*

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{v} + \nabla \mathbf{u}^T \cdot \mathbf{v} + \nabla p = \nu \Delta \mathbf{v} + \mathbf{f} \quad (2.7)$$

with

$$\nabla \cdot \mathbf{u} = 0, \quad \text{and} \quad \mathbf{v} = \mathcal{O}\mathbf{u}. \quad (2.8)$$

The velocity $\mathbf{u}(\mathbf{x}, t)$ is the spatially filtered Eulerian fluid velocity in Eq. (2.3). Note that continuity equation is now imposed as $\nabla \cdot \mathbf{u} = \nabla \cdot (g * \mathbf{v}) = 0$. The energy balance relation derived from the Navier–Stokes-alpha equations (2.7) is

$$\frac{d}{dt} \int \frac{1}{2} \mathbf{u} \cdot \mathbf{v} d^3x = \int \mathbf{u} \cdot \mathbf{f} d^3x - \int \nu |\nabla \mathcal{O}^{1/2} \mathbf{u}|^2 d^3x, \quad (2.9)$$

where for the moment we have dropped the boundary terms that appear upon integrating by parts, i.e., for the moment we ignore the boundary effects and consider either the case of the whole space with solutions vanishing sufficiently rapidly at infinity, or the case of periodic boundary conditions.

2.2. Similarities with previous work

1. Except for the term $(\nabla \mathbf{u}) \cdot \mathbf{v}$, the Kelvin-filtered Navier–Stokes equation (2.7) is otherwise quite similar to Leray’s regularization of the Navier–Stokes equations proposed in 1934 [11]. Extension of the Leray regularization to satisfy the Kelvin circulation theorem was cited as an outstanding problem in Gallavotti’s review.¹ Looking at the equation of motion for the vorticity $\mathbf{q} = \nabla \times \mathbf{v}$ reveals even more similarity with Leray’s regularization of the Navier–Stokes equations. (See the section about the vorticity below.)
2. At first glance, the Kelvin-filtered equations (2.7) in the absence of dissipation and forcing may seem reminiscent of a form of the Euler equations that was discussed in [13,14] (see also [15]), namely,

$$\frac{\partial \gamma_i}{\partial t} + u^j \frac{\partial \gamma_i}{\partial x^j} = -\gamma_j \frac{\partial u^j}{\partial x^i}, \quad \gamma_i = u_i + \frac{\partial \phi}{\partial x^i}, \quad \frac{\partial u_i}{\partial x^i} = 0. \quad (2.10)$$

In light of the third equation in this set, the second one is essentially the Hodge decomposition of the vector γ with Euclidean components γ_i . Comparing these equations with Euler’s equations for the incompressible motion of an ideal fluid,

$$\frac{\partial u_i}{\partial t} + u^j \frac{\partial u_i}{\partial x^j} + \nabla p = 0, \quad \frac{\partial u_i}{\partial x^i} = 0, \quad (2.11)$$

gives a relation between the pressure gradient and the “gauge function” ϕ that is reminiscent of (but different from) Bernoulli’s law,

$$\nabla p = \nabla \left(\frac{1}{2} |\mathbf{u}|^2 - \frac{\partial \phi}{\partial t} - u^j \frac{\partial \phi}{\partial x^j} \right). \quad (2.12)$$

¹ We are grateful to G. Eyink for pointing out [12] to us.

This type of relationship also arises in Hamilton's principle when one uses Clebsch variables [16], in which case ϕ is a Lagrange multiplier that enforces the continuity equation. In contrast, Eqs. (2.7) in the ideal unforced case give the *Kelvin-filtered Euler model*,

$$\frac{\partial v_i}{\partial t} + u^j \frac{\partial v_i}{\partial x^j} = -v_j \frac{\partial u^j}{\partial x^i} - \frac{\partial p}{\partial x^i}, \quad v_i = \mathcal{O}u_i, \quad \frac{\partial u_i}{\partial x^i} = 0. \quad (2.13)$$

2.3. Conservation laws

References to the interesting geometrical properties of the Euler- α model equations — namely Eqs. (2.13) when \mathcal{O} is the Helmholtz operator — are cited in the last section. At this point, we only comment that in the case of periodic boundary conditions, or the case of the whole space with solutions vanishing sufficiently rapidly at infinity, these non-dissipative equations preserve the following two quadratic invariants, the *kinetic energy*,

$$E = \frac{1}{2} \int \mathbf{u} \cdot \mathbf{v} \, d^3x, \quad (2.14)$$

and the *\mathbf{v} -helicity*,

$$\Lambda = \int \mathbf{v} \cdot \text{curl } \mathbf{v} \, d^3x. \quad (2.15)$$

The kinetic energy, E_α , and the \mathbf{v} -helicity, Λ , defined above, are also preserved in bounded domains provided appropriate boundary conditions are imposed. Boundary conditions sufficient for energy conservation when \mathcal{O} is the Helmholtz operator $1 - \alpha^2 \Delta$ are

$$\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \cdot (\nabla \mathbf{u} + \nabla \mathbf{u}^T)) = 0, \quad (2.16)$$

where the vector $\hat{\mathbf{n}}$ is normal to the boundary and superscript $(\cdot)^T$ denotes matrix transpose. Boundary conditions sufficient for \mathbf{v} -helicity conservation in general are $\lambda \hat{\mathbf{n}} \cdot \mathbf{u} = 0$ and $p \hat{\mathbf{n}} \cdot \text{curl } \mathbf{v} = 0$, as seen from the helicity equation,

$$\frac{\partial \lambda}{\partial t} = -\text{div}(\lambda \mathbf{u} + p \text{curl } \mathbf{v}), \quad \text{where } \lambda \equiv \mathbf{v} \cdot \text{curl } \mathbf{v}. \quad (2.17)$$

This equation is obtained by using *only* the motion equation in (2.13) and its curl. Therefore, conservation of helicity holds with these boundary conditions simply because of the *form* of the Kelvin-filtered motion equation in (2.13), independent of the relation between \mathbf{v} and \mathbf{u} , and regardless of whether \mathbf{u} is incompressible.

2.4. Vortex transport and stretching

Let $\mathbf{q} = \nabla \times \mathbf{v}$ be the vorticity of the unfiltered Eulerian velocity. The curl of the Kelvin-filtered Navier–Stokes equation (2.7) gives the vortex transport and stretching equation

$$\frac{\partial \mathbf{q}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{q} - \mathbf{q} \cdot \nabla \mathbf{u} = \nu \Delta \mathbf{q} + \nabla \times \mathbf{f}. \quad (2.18)$$

We note that the coefficient $\nabla \mathbf{u}$ in the vortex stretching term is the gradient of the *spatially filtered* Eulerian velocity \mathbf{u} . Thus, Kelvin-filtering tempers the vortex stretching in the modified Navier–Stokes equations (2.7), while preserving the original form of the vortex dynamics. This tempered vorticity stretching is also reminiscent of Leray's approach [11] for regularizing the Navier–Stokes equations.

Indeed, when \mathcal{O} is the Helmholtz operator we take full advantage of this regularized vorticity stretching effect to prove in [6] the global existence and uniqueness of strong solutions for the three-dimensional NS- α model. In fact,

when the forcing is in a certain Gevrey class of regularity (real analytic), then the NS- α solutions are also Gevrey regular, so that the Fourier coefficients of the solution decay exponentially fast, with respect to the wavenumbers, at a rate that increases as α^2 increases. See [17,18] for the corresponding analysis of the Gevrey properties of the Navier–Stokes equations. This result suggests that the filtering or smoothing of the velocity \mathbf{u} due to the presence of α in the NS- α model could enhance the decay of its wavenumber spectrum and produce an earlier observation of this exponentially falling tail when the forcing is sufficiently smooth, even when the viscosity is small.

2.5. Specializing to the Navier–Stokes- α model

The special case of the NS- α model emerges from the Kelvin-filtered Navier–Stokes equation (2.7) when we choose the operator \mathcal{O} to be the Helmholtz operator, thereby introducing a constant α that has dimensions of length,

$$\mathcal{O} = 1 - \alpha^2 \Delta \quad \text{with } \alpha = \text{const.} \quad (2.19)$$

In this case, the filtered and unfiltered fluid velocities in Eq. (2.4) are related by

$$\mathbf{v} = (1 - \alpha^2 \Delta) \mathbf{u}. \quad (2.20)$$

The *Navier–Stokes- α model* is given by the Kelvin-filtered Navier–Stokes equation (2.7) with definition (2.20). The original derivation of the ideal *Euler- α model* (the $\nu = 0$ case of the NS- α model) obtained by using the Euler–Poincaré approach is given in [9,10]. The physical interpretations of \mathbf{u} and \mathbf{v} as the Eulerian and Lagrangian mean velocities are given in [19].

The corresponding kinetic energy norm (2.5) for the NS- α model is given by

$$E_\alpha = \int \left[\frac{1}{2} |\mathbf{u}|^2 + \frac{\alpha^2}{2} |\nabla \mathbf{u}|^2 \right] d^3x, \quad (2.21)$$

and we repeat that \mathbf{u} is the spatially filtered Eulerian fluid velocity. This kinetic energy is the sum of a translational kinetic energy based on the filtered velocity \mathbf{u} , and a gradient-velocity kinetic energy, multiplied by α^2 . Thus, by showing the global boundedness, in time, of the kinetic energy (2.21) one concludes that the coefficient $\nabla \mathbf{u}$ in the vortex stretching relation (2.18) is *bounded in L^2* for the NS- α model. The second term in the kinetic energy (2.21) imposes an energy penalty for creating small scales. The spatial integral of $|\nabla \mathbf{u}|^2$ in the second term has the same dimensions as filtered enstrophy (the spatial integral of $|\nabla \times \mathbf{u}|^2$, the squared filtered vorticity). For a domain with boundary, the spatial integral of $|\nabla \mathbf{u}|^2$ in the second term in the kinetic energy norm (2.21) is replaced by the integral of $\text{trace}(4\mathbf{D} \cdot \mathbf{D})$, where \mathbf{D} is the strain rate tensor, $\mathbf{D} = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$ in Euclidean coordinates [20]. For the case we treat here, in Euclidean coordinates and in the absence of boundaries, all of these norms are equivalent for an incompressible flow.

3. Spectral scaling for the NS- α model

3.1. A preliminary scaling argument

Scaling ideas originally due to Kraichnan [21] for the case of two-dimensional turbulence may be applied to estimate the effect of the second term in the NS- α model energy (2.21) on the energy spectrum in the present case. For sufficiently small wavenumbers ($k\alpha \ll 1$, but $kL \gg 1$) the first term in the energy E_α in (2.21) dominates and the second term may be neglected. In this wavenumber region, the standard Kolmogorov scaling argument for turbulence gives a $k^{-5/3}$ spectrum by the usual dimensional argument for the inertial range,

$$k = [\text{L}]^{-1}, \quad \epsilon_\alpha = [\text{L}]^2 [\text{T}]^{-3}, \quad E_\alpha(k) = [\text{L}]^3 [\text{T}]^{-2} = \epsilon_\alpha^a k^b \quad \text{if } k\alpha \ll 1, \quad (3.1)$$

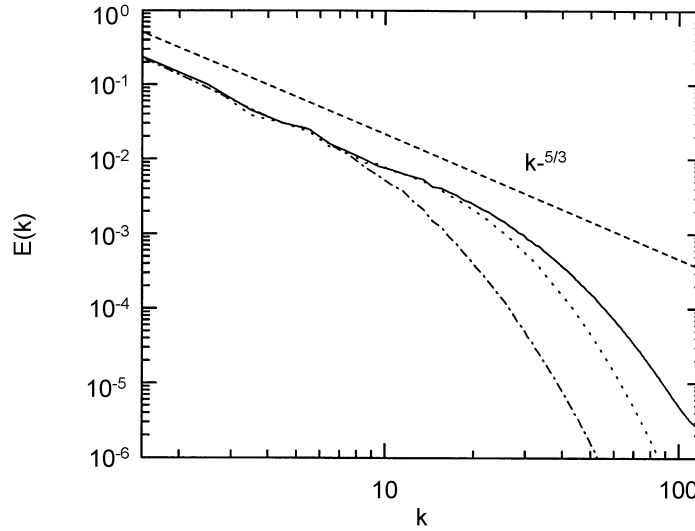


Fig. 1. The DNS energy spectrum, $E(k) = E_\alpha(k)$, versus the wavenumber k for three cases with the same viscosities, same forcings and mesh sizes of 256^3 for $\alpha = 0$ (solid line), $\frac{1}{32}$ (dotted line) and $\frac{1}{8}$ (dotted-dash line). In the inertial range ($k < 20$), a power spectrum with $k^{-5/3}$ can be identified. For finite α , this behavior is seen to roll off to a steeper spectrum for $k \geq 1/\alpha$.

so that $a = \frac{2}{3}$ and $b = -\frac{5}{3}$. Conversely, for sufficiently large wavenumbers ($k\alpha \gg 1$), the second term dominates in E_α . A preliminary scaling argument would then indicate a k^{-3} spectrum for E_α in the large wavenumber region, according to

$$k = [L]^{-1}, \quad \eta_\alpha = [T]^{-3}, \quad E_\alpha(k) = [L]^3 [T]^{-2} = \eta_\alpha^a k^b \quad \text{if } k\alpha \gg 1, \quad (3.2)$$

so that $a = \frac{2}{3}$ and $b = -3$ in this case. Here η_α is the rate of dissipation of the $\int |\nabla \mathbf{u}|^2$ part of the kinetic energy (with the same dimensions as enstrophy). Therefore, in the wavenumber region near $k\alpha \sim 1$, the kinetic energy spectrum may be expected to have a break in slope and roll off from $k^{-5/3}$ to k^{-3} scaling.²

The expectation from this preliminary scaling argument seems to be confirmed in DNSs of the NS- α model, as shown in Fig. 1 taken from [7]. Fig. 1 shows the energy spectra resulting from three DNSs of the NS- α model with mesh sizes of 256^3 for three cases: with $\alpha/L = 0$ (the Navier–Stokes equations), $\frac{1}{32}$ and $\frac{1}{8}$ for the same viscosity $\nu = 0.001$. The corresponding Taylor microscale Reynolds numbers R_λ are reported as 147, 182 and 279, respectively. The higher α (higher Reynolds number) flows are found numerically to have *more compact energy spectra*.

3.2. A more refined argument

A more refined argument for the spectral scaling of the NS- α model shall now be given that depends explicitly upon properties of the nonlinearity. Following [22], let u_κ denote the component of \mathbf{u} formed by the Fourier modes of fluid velocity \mathbf{u} with wavenumbers in $[\kappa, 2\kappa)$ and similarly let v_κ denote the corresponding component of \mathbf{v} , where κ is well beyond the active wavenumbers of the driving force. The NS- α energy balance for this component is then

$$\frac{1}{2} \frac{d}{dt} (u_\kappa, v_\kappa) + \nu (-\Delta u_\kappa, v_\kappa) = T_\kappa - T_{2\kappa}, \quad (3.3)$$

² A version of this scaling argument was suggested to one of the authors (DDH) by G. Eyink and D. Thomson.

where T_κ and $T_{2\kappa}$ denote, respectively, the energy transfer rate at κ from the low wavenumber modes $u_<$ to the high wavenumber modes $u_\kappa + u_>$, and at 2κ from $u_< + u_{2\kappa}$ to $u_>$, where $u_<$ and $u_>$ are defined as

$$u_< \equiv \sum_{j < \kappa} u_j, \quad u_> \equiv \sum_{j \geq 2\kappa} u_j. \quad (3.4)$$

In particular,

$$T_\kappa = -(\tilde{B}(u_<, v_<), u_\kappa) + (\tilde{B}(u_\kappa + u_>, v_\kappa + v_>), u_<), \quad (3.5)$$

where the bilinear operator $\tilde{B}(u, v)$ is given by

$$\tilde{B}(u, v) = -P_\sigma(u \times (\nabla \times v)), \quad (3.6)$$

and P_σ is the L^2 orthogonal projection (Leray projection), see, e.g., [6] for more mathematical details. For Eq. (3.3) to hold, it is essential that $\tilde{B}(u, v)$ has the property

$$(\tilde{B}(u, v), u) = 0. \quad (3.7)$$

Note that the property (3.7) does not hold for the *enstrophy* transfer rate of the NS- α model, so the NS- α model does not possess a Kraichnan type inertial range due to an enstrophy cascade. Time-averaging equation (3.3) gives

$$\nu \langle (-\Delta u_\kappa, v_\kappa) \rangle = \langle T_\kappa \rangle - \langle T_{2\kappa} \rangle. \quad (3.8)$$

Consequently, upon introducing the energy spectrum $E_\alpha(\kappa)$, this time-averaged energy transfer equation implies

$$\nu \kappa^3 E_\alpha(\kappa) \sim \nu \int_\kappa^{2\kappa} \kappa^2 E_\alpha(\kappa) d\kappa \sim \langle T_\kappa \rangle - \langle T_{2\kappa} \rangle. \quad (3.9)$$

Hence, as long as the energy dissipation rate is small compared to the energy transfer rate, i.e., provided

$$\nu \kappa^3 E_\alpha(\kappa) \ll \langle T_\kappa \rangle, \quad (3.10)$$

we have the inertial range condition

$$\langle T_\kappa \rangle \sim \langle T_{2\kappa} \rangle. \quad (3.11)$$

The total energy dissipation rate is estimated as

$$\epsilon_\alpha = \left\langle \frac{\nu}{L^3} \int_{[0,L]^3} u \cdot (-\Delta v) d^3x \right\rangle. \quad (3.12)$$

Kraichnan [21] posits the following mechanism for the turbulent cascade. In the inertial range, the eddies u_κ transfer their energy to the eddies $u_{2\kappa}$ in the time τ_κ it takes to travel their length $\sim 1/\kappa$. Their average velocity being

$$U_\kappa \equiv \left\langle \frac{1}{L^3} \int_{[0,L]^3} u_\kappa \cdot u_\kappa d^3x \right\rangle^{1/2} = \left(\int_\kappa^{2\kappa} \frac{E_\alpha(\kappa) d\kappa}{(1 + \alpha^2 \kappa^2)} \right)^{1/2} \sim \left(\frac{\kappa E_\alpha(\kappa)}{(1 + \alpha^2 \kappa^2)} \right)^{1/2}, \quad (3.13)$$

the eddy energy exchange, or turnover time τ_κ will be

$$\tau_\kappa \sim \frac{1}{(\kappa U_\kappa)}. \quad (3.14)$$

Consequently, the total energy dissipation rate is related to the κ spectral energy density by using (3.13) as

$$\epsilon_\alpha \sim \tau_\kappa^{-1} \int_\kappa^{2\kappa} E_\alpha(k) dk \sim \kappa U_\kappa \int_\kappa^{2\kappa} E_\alpha(k) dk \sim \frac{\kappa^{5/2}}{(1 + \alpha^2 \kappa^2)^{1/2}} E_\alpha(\kappa)^{3/2}, \quad (3.15)$$

which yields the following spectral scaling law for the NS- α inertial range,

$$E_\alpha(\kappa) \sim \frac{\epsilon_\alpha^{2/3} (1 + \alpha^2 \kappa^2)^{1/3}}{\kappa^{5/3}}. \quad (3.16)$$

Thus, the total energy present in the NS- α inertial range is actually enhanced by the presence of alpha. Notice, however, that in terms of the filtered velocity \mathbf{u} alone, the *translational kinetic energy spectrum* is given by

$$\frac{E_\alpha(\kappa)}{(1 + \alpha^2 \kappa^2)} \sim \frac{\epsilon_\alpha^{2/3}}{\kappa^{5/3}} \frac{1}{(1 + \alpha^2 \kappa^2)^{2/3}} = \begin{cases} \frac{\epsilon_\alpha^{2/3}}{\kappa^{5/3}} & \text{if } \kappa \alpha \ll 1, \\ \frac{\epsilon_\alpha^{2/3}}{\alpha^{4/3} \kappa^3} & \text{if } \kappa \alpha \gg 1. \end{cases} \quad (3.17)$$

Hence, the anticipated κ^{-3} behavior appears in the spectrum for the α -filtered velocity as a consequence of Eq. (3.17) arising from Kraichnan's argument [21] associating energy transfer rates and eddy turnover times for the filtered velocity. According to this argument, the presence of alpha reduces the energy associated with the higher wavenumbers in the L^2 norm of the filtered velocity \mathbf{u} . This conclusion from (3.17) agrees with the trends shown in Fig. 1 obtained from high resolution DNS studies of the NS- α equation in [7]. However, these numerical studies do not have sufficient dynamic range to confirm this conclusion entirely. Thus, uncertainty remains in claiming a numerical confirmation because the quantity $E_\alpha(\kappa)/(1 + \alpha^2 \kappa^2)$ in Eq. (3.17) is unaffected by the alpha-modification for $\kappa \alpha \ll 1$ and may already be out of the inertial range and into the dissipation range for $\kappa \alpha \gg 1$. Studies in progress using DNS of the limit $\alpha \rightarrow \infty$ NS- α equation hope to clarify this point [23]. Next we shall discuss the extent of the inertial range for the NS- α model.

Eq. (3.17) holds only in the inertial range, i.e., provided, cf. (3.10),

$$\nu \kappa^3 E_\alpha(\kappa) \ll \epsilon_\alpha. \quad (3.18)$$

This inertial-range inequality may be expressed equivalently as $\kappa \ll \kappa_\alpha$, where κ_α (the end of the NS- α inertial range) is given in terms of the NS- α Kolmogorov wavenumber $\kappa_{\alpha, Ko}$ by using (3.15) and (3.17) to find

$$\kappa_\alpha^4 (1 + \alpha^2 \kappa_\alpha^2) \sim \frac{\epsilon_\alpha}{\nu^3} = \kappa_{\alpha, Ko}^4. \quad (3.19)$$

Thus, as expected from the numerical simulations of [7], *the inertial range is shortened* to $\kappa < \kappa_\alpha$ for the NS- α model by its nonlinear dispersive filtering with lengthscale α . For sufficiently large NS- α Kolmogorov wavenumber $\kappa_{\alpha, Ko}$ and with α fixed, the wavenumber κ_α at the end of the NS- α inertial range is determined from (3.19) to be

$$\kappa_\alpha \sim \left(\frac{1}{\alpha} \right)^{1/3} \kappa_{\alpha, Ko}^{2/3}. \quad (3.20)$$

This is a relationship among the three progressively larger wavenumbers,

$$\frac{1}{\alpha} < \kappa_\alpha < \kappa_{\alpha, Ko}.$$

Shortening the inertial range for the NS- α model to $\kappa < \kappa_\alpha$ rather than $\kappa < \kappa_{\alpha, Ko}$ implies fewer active degrees of freedom in the solution.

3.3. Counting degrees of freedom

If one expects turbulence to be “extensive” in the thermodynamic sense, then one may expect that the number of “active degrees of freedom” N_{dof} for alpha-model turbulence should scale as

$$N_{\text{dof}} \sim (L\kappa_\alpha)^3 \sim \left(\frac{L}{\alpha}\right) (L\kappa_{\alpha,Ko})^2 \sim \frac{L}{\alpha} Re^{3/2}, \quad (3.21)$$

where L is the integral scale (or domain size), κ_α is the end of the NS- α inertial range and $Re = L^{4/3}\epsilon_\alpha^{1/3}/\nu$ is the Reynolds number (with dissipation rate ϵ_α and viscosity ν). The corresponding number of degrees of freedom for Navier–Stokes with the *same* parameters is

$$N_{\text{dof}}^{\text{NS}} \equiv (L\kappa_{\alpha,Ko})^3 \sim Re^{9/4}, \quad (3.22)$$

and one sees a possible trade-off in the relative Reynolds number scaling of the two models. Should these estimates of the number of degrees of freedom needed for numerical simulations using the NS- α model relative to Navier–Stokes not be overly optimistic, the implication would be one factor of $(N_{\text{dof}}^{\text{NS}}/N_{\text{dof}})^{1/3}$ in relative increased computational speed gained by the NS- α model for each spatial dimension and yet another factor for the accompanying lessened CFL time step restriction. Altogether, this would be a gain in speed of

$$\left(\frac{N_{\text{dof}}^{\text{NS}}}{N_{\text{dof}}}\right)^{4/3} = \left(\frac{\alpha}{L}\right)^{4/3} Re. \quad (3.23)$$

Since $\alpha/L \ll 1$ and $Re \gg 1$, the two factors in the last expression compete, but the Reynolds number should eventually win out, because Re can keep increasing while the number α/L is expected to tend to a constant value, say $\alpha/L = 1/100$, at high (but experimentally attainable) Reynolds numbers, at least for simple flow geometries. Empirical indications for this tendency were found in [3–5] by comparing steady NS- α solutions with experimental mean-velocity-profile data for turbulent flows in pipes and channels.

Thus, according to this scaling argument, a factor of 10^4 in increased computational speed for resolved scales greater than α could occur by using the NS- α model at the Reynolds number for which the ratio $\kappa_{\alpha,Ko}/\kappa_\alpha = 10$. An early indication of the feasibility of obtaining such factors in increased computational speed was realized in the DNSs of homogeneous turbulence reported in [7], in which $\kappa_{\alpha,Ko}/\kappa_\alpha \simeq 4$ and the full factor of $4^4 = 256$ in computational speed was obtained using spectral methods in a periodic domain at little or no cost of accuracy in the statistics of the resolved scales.

3.4. Related mathematical results

The paper [6] shows that strong solutions of the NS- α model exist globally, they are unique, and they lie on a global attractor whose fractal (Lyapunov) dimension is bounded above by

$$\mathcal{D}_{\text{frac}} \leq N_{\text{dof}}^{3/2}, \quad (3.24)$$

with N_{dof} defined as in Eq. (3.21). This rigorous mathematical bound exceeds the expected value obtained above by heuristically counting degrees of freedom assuming “extensive” turbulence. Thus, it would imply a smaller increase in computational speed than that of (3.23). However, this rigorous bound may have room for improvement.

3.5. Oboukov cascade rate

We have established in Eq. (3.14) that the eddy turnover rate based on the translational velocity for the NS- α model is

$$\tau_\kappa^{-1} \sim \kappa U_\kappa \sim \sqrt{\frac{\kappa^3 E_\alpha(\kappa)}{1 + \alpha^2 \kappa^2}} \sim \left(\frac{\epsilon_\alpha \kappa^2}{1 + \alpha^2 \kappa^2} \right)^{1/3}. \quad (3.25)$$

Therefore,

- for low wavenumbers ($\kappa\alpha \ll 1$), information (or error) propagates between scales κ and 2κ at a rate proportional to $\kappa^{2/3}$, in agreement with the classical *accelerated cascade* of Oboukov [24] as cited, e.g., in [25]; while
- for high wavenumbers ($\kappa\alpha \gg 1$), information propagates at a constant rate, *independently of wavenumber* for the NS- α model (as occurs also in the case of 2D turbulence discussed by Leith and Kraichnan [26]).

Without an accelerated cascade at high wavenumber, the NS- α model should tend to be *more predictable* than the original Navier–Stokes model. Of course, it makes sense that an averaged or filtered description of fluid flow would propagate high wavenumber information and errors at a slower rate than an instantaneous, unfiltered description does (e.g., think of climate versus weather). The κ^{-3} spectrum of the translational kinetic energy in the alpha models for $\alpha\kappa \gg 1$ is consistent with this interpretation of *reduced error propagation rate* as measured in the L^2 norm of the filtered velocity.

4. Rheology of NS- α turbulence: second-grade fluids

We rewrite the NS-alpha equations (2.7) with $\mathbf{v} = (1 - \alpha^2 \Delta)\mathbf{u}$ and α constant in their equivalent constitutive form,

$$\frac{d\mathbf{u}}{dt} = \nabla \cdot \mathbf{T}, \quad \text{where } \mathbf{T} = -p\mathbf{I} + 2\nu(1 - \alpha^2 \Delta)\mathbf{D} + 2\alpha^2 \mathring{\mathbf{D}}, \quad (4.1)$$

with $\nabla \cdot \mathbf{u} = 0$, strain rate $\mathbf{D} = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$, vorticity tensor $\mathbf{\Omega} = \frac{1}{2}(\nabla \mathbf{u} - \nabla \mathbf{u}^T)$, and co-rotational (Jaumann) derivative given by $\mathring{\mathbf{D}} = d\mathbf{D}/dt + \mathbf{D}\mathbf{\Omega} - \mathbf{\Omega}\mathbf{D}$.

In Eq. (4.1), one recognizes the constitutive relation for the NS- α model as a variant of the rate-dependent incompressible homogeneous fluid of second grade [27,28], in which the dissipation, however, is modified by composition with the Helmholtz operator $(1 - \alpha^2 \Delta)$. Thus, the NS- α model has nearly the same stress tensor as the second-grade fluid. However, it is not quite the same. The stress tensor for the NS- α model uses Navier–Stokes viscous dissipation instead of the weaker form of dissipation used for second-grade fluids that is independent of wavenumber. Note that despite first appearances, there is no hyperviscosity in the NS- α model, only the standard Navier–Stokes viscosity.

Equations for the second-grade fluid were treated recently in the mathematical literature [29–31]. Also, in [20] local well-posedness results for some initial value problems for differential type fluids arising in non-Newtonian fluids (second- and third-grade fluid) were obtained by transferring the problem from the Eulerian to the Lagrangian setting, thereby extending the Arnold [32] and Ebin–Marsden program [33] to the case of second-grade fluids. For the case of second-grade fluids, for $k\alpha > 1$ the wavenumber spectrum should still roll over to k^{-3} , provided the Kolmogorov/Oboukov argument still holds for the weaker dissipation.

The association of turbulence closure models with non-Newtonian fluids is natural. There is a tradition at least since Rivlin [34] of modeling turbulence by using continuum mechanics principles such as objectivity and material

frame indifference (see also [35]). For example, this sort of approach is taken in deriving Reynolds stress algebraic equation models [36]. Rate-dependent closure models of mean turbulence have also been obtained by the two-scale DIA approach [37] and by the renormalization group method [38].

Despite its similarity to the rheology of second-grade fluids, the alpha parameter in the NS- α model is actually not a material parameter. Rather, it is a *flow regime parameter*. Since the NS- α model describes mean quantities, it was proposed as a turbulence closure model and this ansatz was tested by comparing its steady solutions (using the standard Navier–Stokes viscosity) to mean velocity measurements in turbulent channel and pipe flows in [3–5]. These experimental tests show that alpha depends slightly on Reynolds number and varies slightly with distance from the wall for the low to moderate Reynolds numbers available in channel flow. However, at the high to very high Reynolds numbers available in pipe flows, alpha becomes independent of Reynolds number and takes a small value — about 1% of the pipe diameter.

5. Guide to recent NS- α model literature

In [3–5], the authors introduce random fluctuations into the description of the fluid parcel trajectories in the Lagrangian in Hamilton’s principle for ideal incompressible fluid dynamics. They then take its statistical average and use the Euler–Poincaré theory of Holm et al. [9,10,39] to derive Eulerian closure equations for the corresponding averaged ideal fluid motions. The Euler–Poincaré equation that is used in deriving the ideal dynamics of the NS- α model is equivalent in the Eulerian picture to the corresponding Euler–Lagrange equation for fluid parcel trajectories for Lagrangians that are invariant under the right-action of the diffeomorphism group. See [9,10,39], and references therein for more discussions of Euler–Poincaré equations. The Euler–Poincaré theory is applied for modeling fluctuation effects on 3D Lagrangian mean and Eulerian mean fluid motion in [19].

The NS- α model equations (also sometimes called in the literature the viscous Camassa–Holm equations) are proposed in [3–5] as a turbulence closure approximation for the Navier–Stokes equations. The analytic form of the velocity profiles based on the steady NS- α equations away from the viscous sublayer, but covering at least 95% of the channel, depends on two free parameters: the flux Reynolds number R , and the wall-stress Reynolds number R_0 . (Due to measurement limitations, most experimental data are contained in this region.) The authors further reduce the parameter dependence to a single free parameter by assuming a certain drag law for the wall friction $D \sim R_0^2/R^2$. For most of the channel the steady NS- α solution is shown to be compatible with empirical and numerical velocity profiles in this subregion. The NS- α steady velocity profiles agree well outside the viscous sublayer with data obtained from mean velocity measurements and simulations of turbulent channel and pipe flow over a wide range of Reynolds numbers [3–5].

5.1. DNS and comparisons with LES

In [7], DNSs of the NS- α equations are compared with Navier–Stokes DNS and interpreted as behaving like an LES model, i.e., the NS- α model is shown to produce an accurate dynamical description of the large scale features of turbulence driven at large scales, even at resolutions for which the fine scales are not resolved. The NS- α model may at first appear not to be an LES model, since it has rate dependence that is not admitted by LES models. However, the NS- α model was recently shown to transform to a generalized LES similarity model in [8]. This is a promising result for LES modeling, since the mathematical theorems available in [6] for existence, uniqueness and finite dimensional global attractor for the NS- α model will now be transferable to the continuous formulations of this class of generalized LES similarity models. As a basis for numerical schemes, the NS- α model also resembles vorticity methods, as observed in [20,40,41].

5.2. Geodesic motion

The completely integrable one-dimensional Camassa–Holm equation [42] is expressed on the real line as

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + 2v \frac{\partial u}{\partial x} = 0, \quad u(x, t) = \frac{1}{2} \int_{-\infty}^{\infty} e^{|x-y|} v(y, t) dy. \quad (5.1)$$

Thus, we have $v = u - \partial^2 u / \partial x^2$, cf. Eqs. (2.7) with definition (2.20). This one-dimensional equation is formally the Euler–Poincaré equation for geodesic motion on the diffeomorphism group with respect to the metric given by the mean kinetic energy Lagrangian, which is right-invariant under the action of the diffeomorphism group. See [9,10] for detailed discussions, applications and references to Euler–Poincaré equations of this type for ideal fluids and plasmas. See [9,10] for the original derivation of the n -dimensional Camassa–Holm, or Euler- α equation in Euclidean space. See [43,44] for discussions of its generalization to Riemannian manifolds, its existence and uniqueness on a finite time interval, and more about its relation to the theory of second-grade fluids. Additional properties of the Euler- α equations, such as smoothness of the geodesic spray (the Ebin–Marsden theorem), are also shown to hold in [44] and the limit of zero viscosity for the corresponding viscous equations is shown to be a regular limit, even in the presence of boundaries for either homogeneous (Dirichlet) boundary conditions, or for boundary conditions involving the second fundamental form of the boundary. Functional-analytic studies of the ideal Euler- α model are made in [20,45]. The methods introduced and applied in [20,45], while geometrical in nature, also address analytical issues and obtain results that were not previously available by traditional techniques for partial differential equations. For example, these methods produce local in time existence of C^∞ viscosity independent solutions for the Euler- α equations in n dimensions (and in particular 3D) for a fluid container that can be an arbitrary Riemannian manifold with boundary.³

5.3. Mathematical estimates for strong solutions with dissipation

Paper [6] provides the mathematical estimates that are needed to show that the solutions of the NS- α model exist globally, are unique and possess a global attractor with finite fractal dimension satisfying Eq. (3.24). The estimates in [6] do depend on the viscosity remaining positive.

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³ In [45] the Euler- α model is termed the “averaged Euler equations”.

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