# The Navier-Stokes limit of the Boltzmann equation for bounded collision kernels 

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#### Abstract

The present work establishes a Navier-Stokes limit for the Boltzmann equation considered over the infinite spatial domain $\mathbf{R}^{3}$. Appropriately scaled families of DiPerna-Lions renormalized solutions are shown to have fluctuations whose limit points (in the $w-L^{1}$ topology) are governed by Leray solutions of the limiting Navier-Stokes equations. This completes the arguments in Bardos-Golse-Levermore [Commun. Pure Appl. Math. 46(5), 667-753 (1993)] for the steady case, and in Lions-Masmoudi [Arch. Ration. Mech. Anal. 158(3), 173-193 (2001)] for the time-dependent case.


## Introduction

Hydrodynamic models such as the Euler or Navier-Stokes equations were first established by applying Newton's second law of motion to infinitesimal volume elements of the fluid under consideration. All equations from fluid dynamics can be obtained in this way - see Chap. I of [38]. However, this method fails to relate equations of state (expressing for example the pressure in terms of the density and temperature) or transport coefficients (like the heat conduction or the viscosity) to microscopic data (such as the laws governing molecular interactions). In the particular case of gas dynamics, kinetic theory allows one to express thermodynamic functions and transport coefficients for perfect gases in terms of purely mechanical data concerning collisions between the gas molecules.

In his 6th problem, Hilbert asked for a full mathematical justification of this procedure; in his own words [35]: " $[. .$.$] Boltzmann’s work on the$
principles of mechanics suggests the problem of developing mathematically the limiting processes [...] which lead from the atomistic view to the laws of motion of continua".

The present work answers one part of this question, namely the limiting process leading from the Boltzmann equation of the classical kinetic theory of gases to the Navier-Stokes equations of incompressible fluids. This limit was first discussed by Y. Sone in [59] in the steady case on the basis of formal asymptotic expansions, and later by C. Bardos, F. Golse and C.D. Levermore in [5], [6] in the time dependent case by a systematic moment-closure method. The first complete mathematical proof of this limit is due to C . Bardos and S . Ukai [11] in the case of small initial data leading to smooth solutions; K. Asano [3] studied independently the same limit for short times. A complete justification of the Hilbert expansion for the incompressible Navier-Stokes limit method as in [59] has been given by A. DeMasi, R. Esposito and J. Lebowitz in [21]. However these methods fail to encompass the generality of all physically admissible initial data for either the Boltzmann or the Navier-Stokes equations, at least as long as it remains unknown whether initially smooth solutions to these equations may develop singularities in finite time. What is worse, the proof based on Hilbert's expansion [21] - at least in its present formulation leads to solutions of the Boltzmann equation that are not a.e. nonnegative, which is incompatible with the original physical meaning of solutions of the Boltzmann equation as phase space densities. However, it seems ${ }^{1}$ that one can obtain nonnegative solutions by adding initial layers to the truncated expansion used in [21], following the method of M. Lachowicz [37] for the compressible Euler limit.

That the Navier-Stokes equation of incompressible fluids can be derived from the Boltzmann equation may seem somewhat surprising, since the latter models compressible fluids - more precisely, perfect gases. However, it is well known (see for instance [36]) that, for compressible fluids in the low Mach number limit, fluctuations about an equilibrium state are governed by the equations of incompressible fluids. In other words, the limit considered in the present paper concerns incompressible flows of a compressible fluid.

At present, the only known theorems giving the global existence of solutions to either equations in the spatial domain $\mathbf{R}^{3}$ for all physically admissible initial data are those of J. Leray [40] in the case of the NavierStokes equations, and of R. DiPerna and P.-L. Lions [23] in the case of the Boltzmann equation. Both results lead to weak solutions to which the methods of either [11] or [21] cannot be applied.

For that reason, a program concerning the derivation of Leray (weak) solutions of the Navier-Stokes equations from DiPerna-Lions renormalized solutions of the Boltzmann equation was discussed in detail by C. Bardos, F. Golse and C.D. Levermore in [7]. There, this derivation is established rigorously in the time-discretized case under two conditions bearing on the

[^0]sequence of renormalized solutions considered. This method was extended by P.-L. Lions and N. Masmoudi to the time dependent case in [50], under the same two conditions. That these conditions hold is not guaranteed by the theory of the Boltzmann equation in its present state, so that the derivations in [7] and [50] remained incomplete. We refer to [62] for a recent survey of these issues.

In the present work, we show how to circumvent the need for both assumptions left unverified in either [7] or [50], thereby proving the NavierStokes limit of the Boltzmann equation (including a convection-diffusion equation for the temperature field) for all physically admissible initial data. Our discussion is restricted so far to bounded collision kernels, as in the case of cutoff Maxwellian molecules. Yet our methods could apply to more general cases which we hope to analyze in subsequent publications.

An alternate approach, proposed by J. Quastel and H.-T. Yau in [54] consists in deriving the Navier-Stokes equations from some stochastic lattice gas. Some of the methods in [54] might eventually prove useful in the context of hydrodynamic limits. However, this result in itself is somewhat remote from Hilbert's original question: indeed the microscopic model in [54] is neither a fundamental principle of physics nor a consequence thereof. On the contrary, the Boltzmann equation has been widely accepted and used as a legitimate microscopic model. In fact, it has been rigorously derived by O. Lanford from the Newtonian dynamics of a large number of spheres interacting by elastic collisions [39] - see also Chap. 4 of [19].

Notation for spaces. The notations $L_{x}^{p}, L^{p}\left(\mathbf{R}^{D}\right)$ and $L^{p}(d x)$ all designate $L^{p}\left(\mathbf{R}^{D}, d x\right)$. In general, whenever a positive Borel measure $m$ is defined on a topological space $X$, the notation $L^{p}(d m)$ designates $L^{p}(X, d m)$. For any Banach space $E$, the notation $w-E$ designates $E$ endowed with its weak topology, while the notation $w^{*}-E^{\prime}$ designates $E^{\prime}$ (the topological dual of $E$ ) endowed with its weak-* topology.

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## 1. Statement of the problem and main results

1.1. The Boltzmann equation. In kinetic theory, a gas is described by a function $F \equiv F(t, x, v) \geq 0$ measuring the density of gas molecules which at time $t \in \mathbf{R}_{+}$are located at $x \in \mathbf{R}^{3}$ and have instantaneous velocity $v \in \mathbf{R}^{3}$. This function $F$, usually called the "distribution function" or the "number density", is governed by the Boltzmann equation

$$
\begin{equation*}
\partial_{t} F+v \cdot \nabla_{x} F=\mathscr{B}(F, F) \tag{1.1}
\end{equation*}
$$

where $\mathscr{B}(F, F)$ is the Boltzmann collision integral. This collision integral acts only on the $v$-argument of the number density $F$ and is given by the expression

$$
\begin{align*}
& \mathscr{B}(F, F)(t, x, v)= \\
& \quad \iint_{\mathbf{R}^{3} \times \mathbf{S}^{2}}\left(F^{\prime} F_{1}^{\prime}-F F_{1}\right) b\left(v-v_{1}, \omega\right)\left|\cos \left(v-v_{1}, \omega\right)\right| d \omega d v_{1} \tag{1.2}
\end{align*}
$$

where the notations $F_{1}, F^{\prime}$ and $F_{1}^{\prime}$ designate respectively the values $F\left(t, x, v_{1}\right)$, $F\left(t, x, v^{\prime}\right)$ and $F\left(t, x, v_{1}^{\prime}\right)$, with $v^{\prime}$ and $v_{1}^{\prime}$ given in terms of $v_{1} \in \mathbf{R}^{3}$ and $\omega \in \mathbf{S}^{2}$ by the formulas

$$
\begin{equation*}
v^{\prime}=v-\left(v-v_{1}\right) \cdot \omega \omega, \quad v_{1}^{\prime}=v_{1}+\left(v-v_{1}\right) \cdot \omega \omega \tag{1.3}
\end{equation*}
$$

These formulas give all possible solutions to the system with unknowns $v^{\prime}$ and $v_{1}^{\prime}$

$$
\begin{equation*}
v^{\prime}+v_{1}^{\prime}=v+v_{1}, \quad\left|v^{\prime}\right|^{2}+\left|v_{1}^{\prime}\right|^{2}=|v|^{2}+\left|v_{1}\right|^{2} \tag{1.4}
\end{equation*}
$$

in terms of the data $v$ and $v_{1}$ and of an arbitrary unit vector $\omega$. The relations (1.4) are the conservation of momentum and kinetic energy for each binary collision between gas molecules (of like mass). The notation $d \omega$ designates the uniform measure on the sphere $\mathbf{S}^{2}$ normalized so that

$$
\begin{equation*}
\int_{\mathbf{S}^{2}} d \omega=2, \text { which implies that } \int_{\mathbf{S}^{2}}|\cos (z, \omega)| d \omega=1 \text { for all } z \in \mathbf{R}^{3} \tag{1.5}
\end{equation*}
$$

below, we use the notation

$$
\begin{equation*}
d \sigma_{v, v_{1}}(\omega)=\left|\cos \left(v-v_{1}, \omega\right)\right| d \omega \tag{1.6}
\end{equation*}
$$

The geometrical interpretation of the measure $d \sigma_{v, v_{1}}(\omega)$ is as follows. Let ( $v, v_{1}, v^{\prime}, v_{1}^{\prime}$ ) be any quadruple satisfying (1.4); clearly

$$
\left|v_{1}^{\prime}-v^{\prime}\right|=\left|v_{1}-v\right|
$$

Whenever $v \neq v_{1}$, define $\sigma=\frac{v_{1}^{\prime}-v^{\prime}}{\left|v_{1}-v\right|}$. For $\omega \in \mathbf{S}^{2}$ as in the collision formulas (1.3), one has

$$
\left|v_{1}^{\prime}-v^{\prime}\right| \sigma=\left(v_{1}-v\right)-2\left(v_{1}-v\right) \cdot \omega \omega
$$



Fig. 1. The geometry of collisions in the center of mass reference frame

Set $V=v_{1}-v \neq 0$; the map

$$
\mathbf{S}^{2} \ni \omega \rightarrow \sigma=\frac{V}{|V|}-2 \frac{V}{|V|} \cdot \omega \omega \in \mathbf{S}^{2}
$$

is a double cover, and the image of the measure $|\cos (V, \omega)| d \omega$ under this map is precisely $2 d \sigma$ (i.e. twice the uniform measure on $\mathbf{S}^{2}$ ). In other words, given a pair of colliding particles with pre-collision velocities $v$ and $v_{1}, d \sigma_{v, v_{1}}(\omega)$ represents the element of solid angle around their post-collision velocities in the center of mass reference frame. In particular, the term $|\cos (V, \omega)|$ has an intrinsic meaning in the representation (1.3) of the collision relations in terms of the parameter $\omega$. We decided to use this representation in order to be consistent with most of the references quoted in the present work.

Observe that

$$
\begin{equation*}
d \sigma_{v, v_{1}}(\omega)=d \sigma_{v_{1}, v}(\omega) \tag{1.7}
\end{equation*}
$$

Since the map $\left(v, v_{1}\right) \mapsto\left(v^{\prime}, v_{1}^{\prime}\right)$ is a linear isometry of $\mathbf{R}^{3} \times \mathbf{R}^{3}$ for each $\omega \in \mathbf{S}^{2}$, one has
(1.8) $d \omega d v d v_{1}=d \omega d v^{\prime} d v_{1}^{\prime}$ and $d \sigma_{v, v_{1}}(\omega) d v d v_{1}=d \sigma_{v^{\prime}, v_{1}^{\prime}}(\omega) d v^{\prime} d v_{1}^{\prime}$.

The collision kernel $b \equiv b(z, \omega)$ is in general an a.e. positive function defined on $\mathbf{R}^{3} \times \mathbf{S}^{2}$ that encodes whichever features of the molecular interaction are relevant in kinetic theory; it satisfies the symmetries

$$
\begin{equation*}
b\left(v-v_{1}, \omega\right)=b\left(v_{1}-v, \omega\right)=b\left(v^{\prime}-v_{1}^{\prime}, \omega\right), \tag{1.9}
\end{equation*}
$$

for a.e. $\left(v, v_{1}, \omega\right) \in \mathbf{R}^{3} \times \mathbf{R}^{3} \times \mathbf{S}^{2}$. It is also assumed to satisfy the condition

$$
\begin{equation*}
(1+|v|)^{-2} \int_{\left|v_{1}\right|<R} \int_{\mathbf{S}^{2}} b\left(v-v_{1}, \omega\right) d \sigma_{v, v_{1}}(\omega) d v_{1} \rightarrow 0 \text { as }|v| \rightarrow+\infty \tag{1.1}
\end{equation*}
$$

for all $R>0$. This estimate holds for all physically relevant potentials satisfying Grad's angular cutoff assumption (see [32] and [18] pp. 74-79 for more details). These properties of the collision kernel $b$, especially the symmetries (1.8) and (1.9), imply that the relation

$$
\begin{align*}
\iiint\left(f^{\prime}\right. & \left.f_{1}^{\prime}-f f_{1}\right) \varphi\left(v, v_{1}, v^{\prime}, v_{1}^{\prime}\right) b\left(v-v_{1}, \omega\right) d \sigma_{v, v_{1}}(\omega) d v_{1} d v  \tag{1.11}\\
& =\iiint\left(f^{\prime} f_{1}^{\prime}-f f_{1}\right) \phi\left(v, v_{1}, v^{\prime}, v_{1}^{\prime}\right) b\left(v-v_{1}, \omega\right) d \sigma_{v, v_{1}}(\omega) d v_{1} d v \\
& =\iiint\left(f^{\prime} f_{1}^{\prime}-f f_{1}\right) \Phi\left(v, v_{1}, v^{\prime}, v_{1}^{\prime}\right) b\left(v-v_{1}, \omega\right) d \sigma_{v, v_{1}}(\omega) d v_{1} d v
\end{align*}
$$

with

$$
\begin{aligned}
\phi\left(v, v_{1}, v^{\prime}, v_{1}^{\prime}\right)= & \frac{1}{2}\left(\varphi\left(v, v_{1}, v^{\prime}, v_{1}^{\prime}\right)+\varphi\left(v_{1}, v, v_{1}^{\prime}, v^{\prime}\right)\right) \\
\Phi\left(v, v_{1}, v^{\prime}, v_{1}^{\prime}\right)= & \frac{1}{4}\left(\varphi\left(v, v_{1}, v^{\prime}, v_{1}^{\prime}\right)+\varphi\left(v_{1}, v, v_{1}^{\prime}, v^{\prime}\right)\right. \\
& \left.-\varphi\left(v^{\prime}, v_{1}^{\prime}, v, v_{1}\right)-\varphi\left(v_{1}^{\prime}, v^{\prime}, v_{1}, v\right)\right)
\end{aligned}
$$

holds whenever these integrals make sense, for example if $f \in C_{c}\left(\mathbf{R}^{3}\right)$ and if $\varphi \in C\left(\mathbf{R}^{3} \times \mathbf{R}^{3} \times \mathbf{R}^{3} \times \mathbf{R}^{3}\right)$.

The equilibrium states for the Boltzmann collision integral, in other words the number densities $E \equiv E(v)$ such that $\mathscr{B}(E, E)=0$, are the Maxwellians, i.e. the distribution functions of the form

$$
\begin{equation*}
M_{(\rho, u, \theta)}(v)=\frac{\rho}{(2 \pi \theta)^{3 / 2}} e^{-\frac{|v-u|^{2}}{2 \theta}} \tag{1.12}
\end{equation*}
$$

for some $\rho>0, \theta>0$ and $u \in \mathbf{R}^{3}$. Below, we shall always use the notation $M$ to designate $M_{(1,0,1)}$.

In the sequel, we are concerned with solutions to the Boltzmann equation which converge to some Maxwellian state as $|x| \rightarrow+\infty$; without loss of
generality, we assume that this Maxwellian is precisely $M$. Consider the following scaled variant of (1.1):

$$
\begin{align*}
\epsilon \partial_{t} F_{\epsilon}+v \cdot \nabla_{x} F_{\epsilon} & =\frac{1}{\epsilon} \mathscr{B}\left(F_{\epsilon}, F_{\epsilon}\right), \quad t>0, x, v \in \mathbf{R}^{3}, \\
F_{\epsilon}(t, x, v) & \rightarrow M(v), \text { as }|x| \rightarrow+\infty  \tag{1.1.}\\
F_{\epsilon}(0, x, v) & =F_{\epsilon}^{i n}(x, v), \quad x, v \in \mathbf{R}^{3},
\end{align*}
$$

where $\epsilon>0$ designates the common order of magnitude of the Knudsen and Mach numbers (see the introduction in [7] for a detailed discussion on these scalings), and where $F_{\epsilon}^{i n} \geq 0$ a.e. is a family of measurable functions such that

$$
\begin{equation*}
\sup _{\epsilon>0} \frac{1}{\epsilon^{2}} \iint\left[F_{\epsilon}^{i n} \log \left(\frac{F_{\epsilon}^{i n}}{M}\right)-F_{\epsilon}^{i n}+M\right] d v d x<+\infty . \tag{1.14}
\end{equation*}
$$

For any pair of measurable functions $f$ and $g$ defined a.e. on $\mathbf{R}^{3} \times \mathbf{R}^{3}$ and satisfying $f \geq 0$ and $g>0$ a.e., we use the following notation for the relative entropy

$$
\begin{equation*}
H(f \mid g)=\iint\left[f \log \left(\frac{f}{g}\right)-f+g\right] d v d x \in[0,+\infty] . \tag{1.15}
\end{equation*}
$$

A renormalized solution relative to $M$ of (1.13) is a nonnegative function $F_{\epsilon}$ that belongs to $C\left(\mathbf{R}_{+} ; L_{l o c}^{1}\left(\mathbf{R}^{3} ; L^{1}\left(\mathbf{R}^{3}\right)\right)\right.$ ), satisfies

$$
\begin{equation*}
\Gamma^{\prime}\left(\frac{F_{\epsilon}}{M}\right) \mathscr{B}\left(F_{\epsilon}, F_{\epsilon}\right) \in L_{l o c}^{1}\left(\mathbf{R}_{+} \times \mathbf{R}^{3} \times \mathbf{R}^{3}\right) \tag{1.16}
\end{equation*}
$$

for all $\Gamma \in C^{1}\left(\mathbf{R}_{+}\right)$such that

$$
\begin{equation*}
\Gamma(0)=0 \text { and } z \mapsto(1+z) \Gamma^{\prime}(z) \text { is bounded on } \mathbf{R}_{+}, \tag{1.17}
\end{equation*}
$$

has finite relative entropy for all positive times:

$$
\begin{equation*}
H\left(F_{\epsilon}(t, \cdot, \cdot) \mid M\right)<+\infty, \quad t>0 \tag{1.18}
\end{equation*}
$$

and finally satisfies

$$
\begin{align*}
& \int_{0}^{+\infty} \iint \Gamma\left(\frac{F_{\epsilon}}{M}\right)\left(\partial_{t} \chi+\frac{1}{\epsilon} v \cdot \nabla_{x} \chi\right) M d v d x d t \\
& \quad+\iint \Gamma\left(\frac{F_{\epsilon}{ }_{\epsilon}}{M}\right) \chi(0, x, v) M d v d x  \tag{1.19}\\
& \quad+\frac{1}{\epsilon^{2}} \int_{0}^{+\infty} \iint \Gamma^{\prime}\left(\frac{F_{\epsilon}}{M}\right) \mathscr{B}\left(F_{\epsilon}, F_{\epsilon}\right) \chi d v d x d t=0
\end{align*}
$$

for each test function $\chi \in C_{c}^{\infty}\left(\mathbf{R}_{+} \times \mathbf{R}^{3} \times \mathbf{R}^{3}\right)$.
The methods due to R. DiPerna and P.-L. Lions [23] - and their extension by P.-L. Lions [43] - lead to the following global existence result.

Theorem 1.1. Let $\epsilon>0$ and $F_{\epsilon}^{i n} \equiv F_{\epsilon}^{i n}(x, v)$ be an a.e. nonnegative, measurable function defined on $\mathbf{R}^{3} \times \mathbf{R}^{3}$ such that $H\left(F_{\epsilon}^{i n} \mid M\right)<+\infty$. Then there exists a renormalized solution of (1.13) relative to $M$ which satisfies

- the local conservation of mass in the sense of distributions

$$
\begin{equation*}
\partial_{t} \int F_{\epsilon} d v+\nabla_{x} \cdot \frac{1}{\epsilon} \int v F_{\epsilon} d v=0, \quad t>0, x \in \mathbf{R}^{3} \tag{1.20}
\end{equation*}
$$

- and the relative entropy inequality

$$
\begin{equation*}
H\left(F_{\epsilon}(t, \cdot, \cdot) \mid M\right)+\frac{1}{\epsilon^{2}} \int_{0}^{t} \iint D\left(F_{\epsilon}\right) d v d x d s \leq H\left(F_{\epsilon}^{i n} \mid M\right) \tag{1.21}
\end{equation*}
$$

for all $t>0$, where the dissipation term $D(f)$ is defined for all positive measurable functions $f \equiv f(v)$ by

$$
\begin{equation*}
D(f)=\frac{1}{4} \iint\left(f^{\prime} f_{1}^{\prime}-f f_{1}\right) \log \left(\frac{f^{\prime} f_{1}^{\prime}}{f f_{1}}\right) b\left(v-v_{1}, \omega\right) d \sigma_{v, v_{1}}(\omega) d v_{1} \tag{1.22}
\end{equation*}
$$

Whether the local conservation of momentum

$$
\begin{equation*}
\partial_{t} \int v F_{\epsilon} d v+\nabla_{x} \cdot \frac{1}{\epsilon} \int v \otimes v F_{\epsilon} d v=0 \tag{1.23}
\end{equation*}
$$

holds in the sense of distributions on $\mathbf{R}_{+}^{*} \times \mathbf{R}^{3}$ is still unknown, unless $F_{\epsilon}$ is a classical solution of the Boltzmann equation. (P.-L. Lions and N. Masmoudi made an interesting additional observation on this particular point in [51]). This is one of the difficulties in rigorously deriving hydrodynamic models from the Boltzmann equation.

Likewise, it is still unknown whether (1.21) is an equality unless $F_{\epsilon}$ is a classical solution; the relation (1.21) with an equal sign is one of the most important formal properties of the Boltzmann equation, known as Boltzmann's H Theorem.

Remark. The notion of renormalized solution relative to $M$ of the Boltzmann equation (1.13) slightly differs from the original notion of renormalized solution defined in [23] and [43]. By Theorem IV. 1 of [43], for each $\epsilon>0$ and each $F_{\epsilon}^{i n} \geq 0$ a.e. such that $H\left(F_{\epsilon}^{i n} \mid M\right)<+\infty$, there exists a renormalized solution ${ }^{2}$ of the Cauchy problem (1.13), i.e. a function $F_{\epsilon} \in C\left([0,+\infty) ; L_{l o c}^{1}\left(d x ; L^{1}(d v)\right)\right)$ such that

$$
\begin{equation*}
\frac{\mathscr{B}\left(F_{\epsilon}, F_{\epsilon}\right)}{1+F_{\epsilon}} \in L_{l o c}^{1}(d t d x d v) \tag{1.24}
\end{equation*}
$$

[^1]which satisfies (1.18) and is such that
$$
\left(\epsilon \partial_{t}+v \cdot \nabla_{x}\right) \log \left(1+F_{\epsilon}\right)=\frac{1}{\epsilon} \frac{\mathscr{B}\left(F_{\epsilon}, F_{\epsilon}\right)}{1+F_{\epsilon}}
$$
holds in the sense of distributions on $\mathbf{R}_{+} \times \mathbf{R}^{3} \times \mathbf{R}^{3}$ with initial condition
$$
\left.\log \left(1+F_{\epsilon}\right)\right|_{t=0}=\log \left(1+F_{\epsilon}^{i n}\right)
$$

Observe that

$$
0 \leq\left(1+F_{\epsilon}\right)\left|\Gamma^{\prime}\left(\frac{F_{\epsilon}}{M}\right)\right| \leq(1+M) \sup _{z \geq 0}\left((1+z)\left|\Gamma^{\prime}(z)\right|\right)
$$

hence the bound (1.24) implies the bound (1.16). Based on this bound, an elementary argument shows that the renormalized solutions in the usual sense constructed in Theorem IV. 1 of [43] are in fact renormalized solutions relative to $M$ as defined above.
1.2. The Navier-Stokes equations. The Navier-Stokes equations govern the velocity field $u \equiv u(t, x) \in \mathbf{R}^{3}$ of an incompressible fluid. They are

$$
\begin{array}{rlrl}
\nabla_{x} \cdot u & =0, & & t>0, x \in \mathbf{R}^{3}, \\
\partial_{t} u+u \cdot \nabla_{x} u+\nabla_{x} p & =v \Delta_{x} u, & & t>0,  \tag{1.25}\\
x \in \mathbf{R}^{3},
\end{array}
$$

where $v>0$ is the kinematic viscosity of the fluid. The first equality in (1.25) says that the fluid motion preserves the volume, and is referred to as the incompressibility condition; the second equality expresses Newton's second law of motion for any infinitesimal volume of fluid.

Consider the function spaces

$$
\mathscr{H}=\left\{u \in L^{2}\left(\mathbf{R}^{3} ; \mathbf{R}^{3}\right) \mid \nabla_{x} \cdot u=0\right\}, \quad \mathcal{V}=\mathscr{H} \cap H^{1}\left(\mathbf{R}^{3} ; \mathbf{R}^{3}\right) .
$$

In particular, $\mathscr{H}$ is the space of three-dimensional incompressible velocity fields with finite kinetic energy $\frac{1}{2} \int|u|^{2} d x$.

Let $u^{i n} \in \mathscr{H}$, and consider the Cauchy problem for (1.25) with initial data

$$
\begin{equation*}
u(0, x)=u^{i n}(x), \quad x \in \mathbf{R}^{3} \tag{1.26}
\end{equation*}
$$

A weak solution to the Cauchy problem (1.25)-(1.26) is an element $u \in$ $C\left(\mathbf{R}_{+} ; w-\mathcal{H}\right) \cap L^{2}\left(\mathbf{R}_{+} ; \mathcal{V}\right)$ that satisfies the relation

$$
\begin{align*}
\int_{0}^{+\infty} \int u(t, x) \cdot \partial_{t} \chi(t, x) d x d t & +\int_{0}^{+\infty} \int u^{\otimes 2}(t, x): \nabla_{x} \chi(t, x) d x d t  \tag{1.27}\\
& +\int u^{i n}(x) \cdot \chi(0, x)=v \int_{0}^{+\infty} \int \nabla_{x} u(s, x): \nabla_{x} \chi(s, x) d x d s
\end{align*}
$$

for each divergence free test vector field $\chi \in C_{c}^{\infty}\left(\mathbf{R}_{+} \times \mathbf{R}^{3} ; \mathbf{R}^{3}\right)$. We recall that the only global existence theorem known to hold for the Cauchy problem (1.25)-(1.26) without restriction on the size of the initial data in the class $\mathscr{H}$ of three-dimensional divergence-free velocity fields with finite kinetic energy is the following ${ }^{3}$

Theorem 1.2 (J. Leray [40]). For each $u^{i n} \in \mathscr{H}$, there exists at least one weak solution of (1.25)-(1.26) satisfying the energy inequality

$$
\begin{equation*}
\frac{1}{2} \int|u(t, x)|^{2} d x+v \int_{0}^{t} \int\left|\nabla_{x} u(s, x)\right|^{2} d x d s \leq \frac{1}{2} \int\left|u^{i n}(x)\right|^{2} d x \tag{1.28}
\end{equation*}
$$

for all $t>0$.
A weak solution of (1.25)-(1.26) that satisfies in addition the energy inequality (1.25) for all $t>0$ is referred to as a Leray solution. It is unknown whether there is a unique Leray solution of (1.25)-(1.26). However, if the system (1.25)-(1.26) has a classical solution with bounded $x$-derivatives, this solution is unique within the class of Leray solutions of (1.25)-(1.26). It remains unknown whether equality holds in (1.28), unless $u$ is a classical solution of (1.25)-(1.26), much in the same way that it is unknown whether equality holds in (1.21) unless $F_{\epsilon}$ is a classical solution of (1.13).

The Leray energy inequality (1.28) and the DiPerna-Lions entropy inequality (1.21) are similar objects. More precisely, it was proved by C. Bardos, F. Golse and C.D. Levermore in [7] that the Leray energy inequality (1.28) is the limiting form of the DiPerna-Lions entropy inequality (1.21). This confirms the view expressed by P.-L. Lions (see [46], p. 432): "[...] the global existence result of [renormalized] solutions [...] can be seen as the analogue for Boltzmann's equation to the pioneering work on the Navier-Stokes equations by J. Leray".
1.3. The Navier-Stokes-Fourier system. The Navier-Stokes-Fourier system is an extension of the Navier-Stokes equations which governs both the velocity field $u \equiv u(t, x) \in \mathbf{R}^{3}$ and the (fluctuations of) temperature field $\theta \equiv \theta(t, x) \in \mathbf{R}$ in an incompressible fluid. In the setting considered below, the temperature field is just advected by the velocity field $u$ and diffuses according to Fourier's law. More complicated effects such as viscous heating for example (see [10] or [44] p. 10, fla. (1.41)) do not appear in the scaling

[^2]considered in the present work. The Navier-Stokes-Fourier system reads
\[

$$
\begin{array}{rlrl}
\nabla_{x} \cdot u & =0, & & t>0, x \in \mathbf{R}^{3}, \\
\partial_{t} u+u \cdot \nabla_{x} u+\nabla_{x} p & =v \Delta_{x} u, & & t>0,  \tag{1.29}\\
\partial_{t} \theta+u \cdot \mathbf{R}^{3}
\end{array}
$$,
\]

where $\kappa>0$ designates the heat conduction coefficient.
Consider the Cauchy problem for (1.29) with initial data

$$
\begin{equation*}
u(0, x)=u^{i n}(x), \quad \theta(0, x)=\theta^{i n}(x), \quad x \in \mathbf{R}^{3} \tag{1.30}
\end{equation*}
$$

where $u^{i n} \in \mathscr{H}$ and $\theta^{i n} \in L^{2}\left(\mathbf{R}^{3}\right)$. A weak solution of the Cauchy problem (1.29)-(1.30) is a couple $(u, \theta)$ where $u$ is a weak solution of the Navier-Stokes equation and $\theta$ a solution in the sense of distributions of the drift-diffusion Cauchy problem

$$
\begin{align*}
& \partial_{t} \theta+\nabla_{x} \cdot(u \theta)=\kappa \Delta_{x} \theta, \quad t>0, x \in \mathbf{R}^{3}, \\
& \theta(0, x)=\theta^{i n}(x), \quad x \in \mathbf{R}^{3} . \tag{1.31}
\end{align*}
$$

Theorem 1.3. For each $u^{i n} \in \mathscr{H}$ and $\theta^{i n} \in L^{2}\left(\mathbf{R}^{3}\right)$, there exists at least one weak solution $(u, \theta)$ of (1.29)-(1.30) that satisfies the energy inequality

$$
\begin{align*}
\frac{1}{2} \int\left(|u(t, x)|^{2}\right. & \left.+\frac{5}{2} \theta(t, x)^{2}\right) d x \\
& +\int_{0}^{t} \int\left[\nu\left|\nabla_{x} u(s, x)\right|^{2}+\frac{5}{2} \kappa\left|\nabla_{x} \theta(s, x)\right|^{2}\right] d x d s  \tag{1.32}\\
& \leq \frac{1}{2} \int\left(\left|u^{i n}(x)\right|^{2}+\frac{5}{2} \theta^{i n}(x)^{2}\right) d x
\end{align*}
$$

for all $t>0$.
A weak solution of (1.29)-(1.30) that also satisfies (1.32) is also referred to as a Leray solution in the sequel (although J. Leray himself did not study thermal effects, the theorem above is a straightforward extension of his fundamental paper [40]).

However, there is a certain arbitrariness in considering the Lyapunov functional

$$
\begin{equation*}
\frac{1}{2} \int\left(|u(t, x)|^{2}+\frac{5}{2} \theta(t, x)^{2}\right) d x \tag{1.33}
\end{equation*}
$$

in the energy inequality (1.32). A similar existence theorem holds with the coefficient $\frac{5}{2}$ multiplying the temperature replaced by any positive number. The reason for using specifically the coefficient $\frac{5}{2}$ in the theorem above is that the quantity (1.33) is the leading order of the relative entropy (1.18) in the Navier-Stokes limit as $\epsilon \rightarrow 0$ in space dimension 3. In spite of the fact that this Lyapunov functional reduces to the kinetic energy in the case $\theta \equiv 0$, it does not coincide with the total (kinetic plus internal) energy in the general case: see [44] p. 110 and ff. for a detailed description of models involving the temperature in incompressible fluids.
1.4. Main results. The Navier-Stokes (or Navier-Stokes-Fourier) limit of the Boltzmann equation considers fluctuations of the number density about an absolute Maxwellian. The following notations for such fluctuations are taken from [7].
1.4.1. The Boltzmann equation near a uniform Maxwellian. First, the relative number density and the fluctuations of number density are denoted respectively by

$$
\begin{equation*}
G_{\epsilon}=\frac{F_{\epsilon}}{M}, \quad g_{\epsilon}=\frac{F_{\epsilon}-M}{\epsilon M} \tag{1.34}
\end{equation*}
$$

while the scaled collision integrand is

$$
\begin{equation*}
q_{\epsilon}=\frac{1}{\epsilon^{2}}\left(G_{\epsilon}^{\prime} G_{\epsilon 1}^{\prime}-G_{\epsilon} G_{\epsilon 1}\right) \tag{1.35}
\end{equation*}
$$

The integral for the unit measure $M d v$ is denoted

$$
\begin{equation*}
\langle g\rangle=\int g(v) M(v) d v, \quad \text { for all } g \in L^{1}(M d v) \tag{1.36}
\end{equation*}
$$

Without loss of generality - i.e. after normalization if needed - one can assume that the measure

$$
\begin{equation*}
d \mu\left(v, v_{1}, \omega\right)=b\left(v-v_{1}, \omega\right) d \sigma_{v, v_{1}}(\omega) M_{1} d v_{1} M d v \tag{1.37}
\end{equation*}
$$

also is a unit measure; the integral for this unit measure is denoted

$$
\begin{equation*}
\langle\langle q\rangle\rangle=\iiint q\left(v, v_{1}, \omega\right) d \mu\left(v, v_{1}, \omega\right), \quad \text { for all } q \in L^{1}(d \mu) \tag{1.38}
\end{equation*}
$$

We shall need (minus) the linearized collision operator

$$
\begin{equation*}
\mathcal{L} g=\iint\left(g+g_{1}-g^{\prime}-g_{1}^{\prime}\right) b\left(v-v_{1}, \omega\right) d \sigma_{v, v_{1}}(\omega) M_{1} d v_{1} \tag{1.39}
\end{equation*}
$$

as well as (half) the Hessian of the collision integral at the Maxwellian $M$ which is denoted by

$$
\begin{equation*}
\mathcal{Q}(g, g)=\iint\left(g^{\prime} g_{1}^{\prime}-g g_{1}\right) b\left(v-v_{1}, \omega\right) d \sigma_{v, v_{1}}(\omega) M_{1} d v_{1} \tag{1.40}
\end{equation*}
$$

Assume from now on that the Boltzmann collision kernel $b$ satisfies the assumption

$$
\begin{equation*}
\frac{1}{b_{\infty}} \leq b(z, \omega) \leq b_{\infty}, \quad z \in \mathbf{R}^{3}, \omega \in \mathbf{S}^{2}, \quad \text { for some } b_{\infty}>0 \tag{H1}
\end{equation*}
$$

The main properties of the linearized collision operator $\mathcal{L}$ were proved by H. Grad [32] and are recalled below.

Proposition 1.4. For any collision kernel bsatisfying (H1), $\mathcal{L}$ is a bounded nonnegative self-adjoint Fredholm operator on $L^{2}(M d v)$ with nullspace

$$
\begin{equation*}
\operatorname{ker} \mathcal{L}=\operatorname{span}\left\{1, v_{1}, v_{2}, v_{3},|v|^{2}\right\} \tag{1.41}
\end{equation*}
$$

Because each entry of the tensor $v^{\otimes 2}-\frac{1}{3}|v|^{2} I$ and of the vector $\frac{1}{2} v\left(|v|^{2}-5\right)$ is orthogonal to $\operatorname{ker} \mathcal{L}$, there exist a unique tensor $A$ and a unique vector $B$ such that

$$
\begin{array}{ll}
\mathcal{L} A=v^{\otimes 2}-\frac{1}{3}|v|^{2} I, & A \in(\operatorname{ker} \mathscr{L})^{\perp} \subset L^{2}(M d v)  \tag{1.42}\\
\mathcal{L} B=\frac{1}{2} v\left(|v|^{2}-5\right), & B \in(\operatorname{ker} \mathscr{L})^{\perp} \subset L^{2}(M d v)
\end{array}
$$

The main properties of the bilinear operator $\mathcal{Q}$ used in the present work are collected in the next proposition.

Proposition 1.5. For any collision kernel $b$ satisfying (H1) and all $p \in$ $[1, \infty], Q$ defines by polarization a continuous, symmetric bilinear operator (still denoted by $Q$ ) from $L^{p}(M d v) \times L^{p}(M d v)$ to $L^{p}(M d v)$. Further,

$$
\begin{equation*}
\mathcal{Q}(g, g)=\frac{1}{2} \mathcal{L}\left(g^{2}\right), \quad \text { for all } g \in \operatorname{ker} \mathcal{L} . \tag{1.43}
\end{equation*}
$$

Sketch of proof. The proof of (1.43) can be found in [6] or in [15] (Lemma 2.5 , p. 74). As for the continuity property, pick $p \in[1, \infty)$ and $f \in$ $L^{p}(M d v)$; by (H1), the symmetry property (1.9) and Jensen's inequality (observing that the measure $d \sigma_{v, v_{1}}(\omega) M_{1} d v_{1}$ has total mass 1)

$$
\begin{aligned}
& \int\left|\iint f^{\prime} f_{1}^{\prime} b d \sigma_{v, v_{1}}(\omega) M_{1} d v_{1}\right|^{p} M d v \\
& \leq b_{\infty}^{p} \iiint\left|f^{\prime}\right|^{p}\left|f_{1}^{\prime}\right|^{p} d \sigma_{v, v_{1}}(\omega) M_{1} d v_{1} M d v \\
&=b_{\infty}^{p} \iiint|f|^{p}\left|f_{1}\right|^{p} d \sigma_{v, v_{1}}(\omega) M_{1} d v_{1} M d v \\
&=b_{\infty}^{p} \iint|f|^{p}\left|f_{1}\right|^{p} M_{1} d v_{1} M d v=\|f\|_{L^{p}(M d v)}^{2 p}
\end{aligned}
$$

where the first equality follows from the change of variables $\left(v, v_{1}\right) \mapsto$ ( $v^{\prime}, v_{1}^{\prime}$ ), the second equality in (1.8), and the last relation in (1.9).

Within the class of collision kernels satisfying (H1), we further restrict our attention to those for which

$$
\begin{equation*}
\frac{|A(v)|+|B(v)|}{1+|v|^{p}} \in L_{v}^{\infty} \text { for some } p \geq 0 \tag{H2}
\end{equation*}
$$

The class of collision kernels satisfying both (H1) and (H2) is not empty since it contains at least all collision kernels of the form $b(z, \omega)=$ $\mathbf{b}(|\cos (z, \omega)|)$ satisfying (H1). These collision kernels correspond to cutoff Maxwellian molecules and satisfy (H2) with $p=3$ (see [18], pp. 82-87).
1.4.2. The limit theorems. From now on, we denote by $P$ the Leray projection, i.e. the orthogonal projection on the space of divergence free vector fields in $L^{2}\left(\mathbf{R}^{3}\right)$ - in particular, for any $p \in H^{1}\left(\mathbf{R}^{3}\right)$, one has $P\left(\nabla_{x} p\right)=0$. The operator $P$ so defined coincides with a classical pseudo-differential operator of order 0 on $\mathbf{R}^{3}$, and therefore has a natural extension to tempered distributions on $\mathbf{R}^{3}$.

Theorem 1.6 (Weak Navier-Stokes-Fourier limit). Let b satisfy (H1)(H2), and let $F_{\epsilon}^{i n}$ be a family of a.e. nonnegative, measurable functions on $\mathbf{R}^{3} \times \mathbf{R}^{3}$ satisfying the bound

$$
\begin{equation*}
H\left(F_{\epsilon}^{i n} \mid M\right) \leq C^{i n} \epsilon^{2} \tag{1.44}
\end{equation*}
$$

for some $C^{\text {in }}>0$ and all $\epsilon>0$, as well as the convergence properties

$$
\begin{align*}
& P\left(\frac{1}{\epsilon} \int v F_{\epsilon}^{i n} d v\right) \rightarrow u^{\text {in }} \text { in } w-L^{2}\left(\mathbf{R}^{3}\right) \text { as } \epsilon \rightarrow 0,  \tag{1.45}\\
& \frac{1}{\epsilon} \int\left(\frac{1}{5}|v|^{2}-1\right)\left(F_{\epsilon}^{i n}-M\right) d v \rightarrow \theta^{\text {in }} \text { in } w-L^{2}\left(\mathbf{R}^{3}\right) \text { as } \epsilon \rightarrow 0 .
\end{align*}
$$

Let $F_{\epsilon}$ be a family of renormalized solutions to (1.13). Then the family

$$
\left(\frac{1}{\epsilon} \int v F_{\epsilon} d v, \frac{1}{\epsilon} \int\left(\frac{1}{3}|v|^{2}-1\right) F_{\epsilon} d v\right)
$$

is relatively compact in $w-L_{l o c}^{1}(d t d x)$ and each of its limit points as $\epsilon \rightarrow 0$ is a weak solution of (1.29)-(1.30) with viscosity and heat diffusion coefficient given by the formulas

$$
\begin{equation*}
v=\frac{1}{10} \int A:(\mathscr{L} A) M d v, \quad \kappa=\frac{2}{15} \int B \cdot(\mathscr{L} B) M d v \tag{1.46}
\end{equation*}
$$

For well-prepared initial data, the weak-compactness result above leads to Leray instead of weak solutions. First we recall from [7] the following definition:

Definition 1.7. A family $g_{\epsilon} \equiv g_{\epsilon}(x, v)$ of $L_{l o c}^{1}(M d v d x)$ converges to $g \equiv$ $g(x, v)$ entropically at rate $\epsilon$ as $\epsilon \rightarrow 0$ if

- for each $\epsilon, 1+\epsilon g_{\epsilon} \geq 0$ a.e. on $\mathbf{R}^{3} \times \mathbf{R}^{3}$,
- $g_{\epsilon} \rightarrow g$ in $w-L_{l o c}^{1}(M d v d x)$ as $\epsilon \rightarrow 0$,
- and

$$
\frac{1}{\epsilon^{2}} H\left(M\left(1+\epsilon g_{\epsilon}\right) \mid M\right) \rightarrow \frac{1}{2} \int\left\langle g^{2}\right\rangle d x
$$

as $\epsilon \rightarrow 0$.

This notion of convergence is the natural one in the context of the Navier-Stokes(-Fourier) limit of the Boltzmann equation, as was shown in [7]. Specifically, this reference proved that the Leray energy inequality (1.32) is the limiting form of the DiPerna-Lions entropy inequality (1.21) by establishing the inequality recalled as (11.7) in Appendix B below. By using this inequality together with Theorem 1.6 , one obtains the following convergence statement relating DiPerna-Lions solutions of the Boltzmann equation to Leray solutions of the Navier-Stokes-Fourier system.

Corollary 1.8 (Well-prepared data). Under the same assumptions as in Theorem 1.6, assume that

$$
\frac{1}{\epsilon} \frac{F_{\epsilon}^{i n}(x, v)-M(v)}{M(v)} \rightarrow u^{i n}(x) \cdot v+\theta^{i n}(x) \frac{1}{2}\left(|v|^{2}-5\right)
$$

entropically at rate $\epsilon$ as $\epsilon \rightarrow 0$, where $u^{\text {in }} \in \mathscr{H}$. Then all limit points of the family

$$
\left(\frac{1}{\epsilon} \int v F_{\epsilon} d v, \frac{1}{\epsilon} \int\left(\frac{1}{3}|v|^{2}-1\right) F_{\epsilon} d v\right)
$$

as $\epsilon \rightarrow 0$ are Leray solutions of (1.29)-(1.30) with viscosity and heat diffusion coefficient given by the formulas (1.46).

Further, if the limiting initial data $u^{i n}$ is smooth and such that (1.29)(1.30) has a (unique) smooth solution $u$, the weak compactness result above can be strengthened into a strong convergence result as shown below.

Theorem 1.9 (Strong Navier-Stokes-Fourier limit). Under the same assumptions as in Theorem 1.6, assume that

$$
\frac{1}{\epsilon} \frac{F_{\epsilon}^{i n}(x, v)-M(v)}{M(v)} \rightarrow u^{i n}(x) \cdot v+\theta^{i n}(x) \frac{1}{2}\left(|v|^{2}-5\right)
$$

entropically at rate $\epsilon$ as $\epsilon \rightarrow 0$, where $u^{i n}$ is a divergence-free vector field such that the Navier-Stokes equations (1.25)-(1.26) with $v$ given by (1.46) have a strong solution u (see [20], Chaps. 9 and 10). Let $\theta$ be the solution of the drift-diffusion equation (1.31) with $\kappa$ given by (1.46).

Then, for all $t \geq 0$,

$$
\frac{1}{\epsilon} \frac{F_{\epsilon}(t, x, v)-M(v)}{M(v)} \rightarrow u(t, x) \cdot v+\theta(t, x) \frac{1}{2}\left(|v|^{2}-5\right)
$$

entropically at rate $\epsilon$ as $\epsilon \rightarrow 0$.
Theorem 1.9 is a straightforward consequence of Theorem 1.6 and of a squeezing argument based on the fact that the inequality (1.32) becomes an equality in the case of strong solutions. Its proof closely follows the proofs of Theorems 6.2 and 7.4 in [7], and of Proposition 6.1 in [25]. In fact Theorem 1.9 holds true provided that (1.29)-(1.30) has a unique weak
solution $(u, \theta)$ for which equality holds in (1.32), since its proof does not use consequences of the regularity of $(u, \theta)$ other than the uniqueness of the solution and the energy equality.

This discussion shows that the convergence proofs in the present paper would not become obsolete should one eventually prove that the NavierStokes equations posed in $\mathbf{R}^{3}$ have global smooth solutions for arbitrary smooth initial data. We shall return to this in Sect. 9.
1.5. The state of the art for the Navier-Stokes limit. The only previously existing results on the Navier-Stokes limit - without restrictions on the size or regularity of the initial data, i.e. starting from renormalized solutions of the Boltzmann equation - are due to C. Bardos, F. Golse and D. Levermore [7] for the steady problem and to P.-L. Lions and N. Masmoudi [50] for the time-dependent problem. Both are based on two assumptions recalled below:

- first, the family of renormalized solutions $F_{\epsilon}$ of (1.13) considered in the limit as $\epsilon \rightarrow 0$ satisfies local conservation of momentum, i.e.

$$
\begin{equation*}
\epsilon \partial_{t} \int v F_{\epsilon} d v+\nabla_{x} \cdot \int v^{\otimes 2} F_{\epsilon} d v=0 \tag{A1}
\end{equation*}
$$

in the sense of distributions on $\mathbf{R}_{+}^{*} \times \mathbf{R}^{3}$;

- in addition, the family $F_{\epsilon}$ is such that

$$
\begin{equation*}
\frac{\left(1+|v|^{2}\right)\left(F_{\epsilon}-M\right)^{2}}{\epsilon^{2} M\left(F_{\epsilon}+M\right)} \text { is relatively compact in } w-L_{l o c}^{1}\left(d t d x ; L^{1}(M d v)\right) \tag{A2}
\end{equation*}
$$

As mentioned above, whether renormalized solutions of the Boltzmann equation satisfy (A1) remains a major open problem. Likewise, the global conservation of energy is not guaranteed by the DiPerna-Lions theory in its present state; only the inequality

$$
\begin{equation*}
\iint \frac{1}{2}|v|^{2} F_{\epsilon}(t, x, v) d x d v \leq \iint \frac{1}{2}|v|^{2} F_{\epsilon}^{i n}(x, v) d x d v \tag{1.47}
\end{equation*}
$$

is known to hold for all $t>0$ in the case of bounded domains. For this reason, only the Navier-Stokes equations, and not the Navier-Stokes-Fourier system, were derived in both references [7] and [50] under assumptions (A1) and (A2).

On the other hand, whether assumption (A2) is satisfied by renormalized solutions of the Boltzmann equation (1.13) in the Navier-Stokes scaling also remains unknown. It was proved by C. Bardos, F. Golse and C.D. Levermore - see Proposition 3.3 of [7] - that the quantity considered in (A2) is of order $O(|\log \epsilon|)$ in $L_{t}^{\infty}\left(L^{1}(M d v d x)\right)$. Such a control suffices to establish
all hydrodynamic limits leading to linearized macroscopic models such as the acoustic limit in [8], [9] and [25], the Stokes limit in [7], [8] and [51] or the Stokes-Fourier limit in [25]). However, this control is not sufficient to obtain the Navier-Stokes limit.

It was noticed for the first time in $[8,9]$ that the local conservation laws of momentum and energy could be established in the limit as $\epsilon \rightarrow 0$ by using the $O(|\log \epsilon|)$ estimate proved in Proposition 3.3 of [7] for the quantity appearing in (A2) and in the Stokes scaling. The method used in these works applied only to bounded collision kernels; however these papers made it clear that local conservation laws needed to be established in the hydrodynamic limit only. This was done for the first time by F. Golse and C.D. Levermore in [25], for general collision kernels (including in particular all hard cutoff potentials and Maxwell molecules), using both the $v-v_{1}$ and the $\left(v, v_{1}\right)-\left(v^{\prime}, v_{1}^{\prime}\right)$ symmetries of the Boltzmann collision integral (1.8) and (1.9) with Young's inequality and some of its variants described in Appendix A below. More recently, C.D. Levermore and N. Masmoudi [41] announced a proof of these local conservation laws in the hydrodynamic limit and for the Navier-Stokes scaling, this time under an assumption which, although slightly weaker than (A2), also remained unverified.

Hence, verifying (A2) remained the main obstruction to deriving the Navier-Stokes equations from the Boltzmann equation.
1.6. Method of proof. In the present paper, we follow the method initiated in [25] and establish the local conservation laws of both momentum and energy in the limit as $\epsilon \rightarrow 0$ : see Sect. 4 below. This step is based on two nonlinear estimates weaker than (A2). These estimates are stated in Proposition 3.4 and Corollary 3.5 in Sect. 3. In doing so, we bypass assumptions (A1)-(A2) that remain unverified.

In addition, the nonlinear controls in Proposition 3.4 and Corollary 3.5, together with other controls stated in Propositions 2.7, 3.8 and Corollary 3.9 allow one to take limits in some appropriately renormalized form of the Boltzmann equation (1.13) integrated against $v$ and $\frac{1}{5}|v|^{2}-1$ : this is done in Sect. 5, closely following the methods initiated in [8], [50] and [25].

Hence, the key points in this work are the new nonlinear controls stated

- in Proposition 3.4 and Corollary 3.5,
- in Propositions 2.7, 3.8 and Corollary 3.9.

These new nonlinear controls are combined with earlier techniques, such as:

- the entropy controls for fluctuations and velocity averaging estimates leading to relative $L^{1}$ compactness of appropriate moments of the fluctuations of number density already established in [7],
- the control of standing acoustic oscillations as in [50],
- the vanishing of conservation defects proved along the lines of [25].

The tools leading to the new nonlinear controls mentioned above are:

- two decompositions of the fluctuations of number density (the first being based on relative entropy controls while the second is based on the entropy production controls): see Sect. 2, and
- a refined velocity averaging result in $L^{1}$ (based on improved $v$-regularity estimates): see Sect. 3.
As regards this last item, we recall the basic principle of velocity averaging: let $f_{n} \equiv f_{n}(x, v)$ be a bounded sequence in $L_{l o c}^{2}\left(\mathbf{R}^{D} \times \mathbf{R}^{D}\right)$ such that $v \cdot \nabla_{x} f_{n}$ is also bounded in $L_{l o c}^{2}\left(\mathbf{R}^{D} \times \mathbf{R}^{D}\right)$. Then, given any $\xi^{*} \neq 0$, $f_{n}(\cdot, v)$ is microlocally bounded in $H^{1}$ in the direction $\xi^{*}$ near each point $x \in \mathbf{R}^{D}$ and for each $v \in \mathbf{R}^{D} \backslash\left(\mathbf{R} \xi^{*}\right)^{\perp}$. Since $\left(\mathbf{R} \xi^{*}\right)^{\perp}$ is a set of (Lebesgue) measure zero, the integral of $f_{n}$ on a small conical neighborhood of $\left(\mathbf{R}_{\xi}^{*}\right)^{\perp}$ is shown to vanish uniformly in $n$ as the neighborhood shrinks to $\left(\mathbf{R}_{\xi}^{*}\right)^{\perp}$ by applying the Cauchy-Schwarz inequality. Following essentially this line of reasoning, F. Golse, B. Perthame and R. Sentis [28] proved ${ }^{4}$ that, for any test function $\phi \in C_{c}\left(\mathbf{R}^{D}\right)$, the sequence of $v$-averages

$$
\rho_{n}(x)=\int f_{n}(x, v) \phi(v) d v
$$

is relatively compact in $L_{l o c}^{2}\left(\mathbf{R}^{D}\right)$. However, if the sequences $f_{n}$ and $v \cdot \nabla_{x} f_{n}$ are bounded in $L_{l o c}^{1}\left(\mathbf{R}^{D} \times \mathbf{R}^{D}\right)$, it may happen that the sequence of $v$-averages $\rho_{n}$ is not even locally uniformly integrable in $\mathbf{R}^{D}$. The simple counterexample given in [27] (Example 1, pp. 123-124) is based on concentrations in $v$. Intuitively, if the $f_{n}$ 's concentrate in the variable $v$ in a single direction $v^{*} \neq 0$, the $v$-average $\rho_{n}$ simply reduces to $\phi\left(v^{*}\right) f_{n}$ - in other words, the benefit of averaging in $v$ is lost. Unless $D=1$, one cannot expect to prove compactness on $\rho_{n}$ in $L_{l o c}^{1}\left(\mathbf{R}^{D}\right)$ under the sole assumption that $f_{n}$ and $v \cdot \nabla_{x} f_{n}$ are bounded in $L_{l o c}^{1}\left(\mathbf{R}^{D} \times \mathbf{R}^{D}\right)$.

Yet if one excludes such concentrations by assuming additional regularity in the variable $v$ only on the $f_{n}$ 's - assuming for example that the sequence $f_{n}$ is also bounded in $L_{l o c}^{1}\left(d x ; L_{l o c}^{\infty}(d v)\right)$ - the sequence of $v$-averages $\rho_{n}$ is indeed locally uniformly integrable in $\mathbf{R}^{D}$. This was observed for the first time by L. Saint-Raymond [55]. (An earlier remark in the same direction appears in Lemma 8 of [27], stating the relative compactness of $\rho_{n}$ in $L_{l o c}^{1}\left(\mathbf{R}^{D}\right)$ under the additional assumption of slab symmetry $\left.{ }^{5}\right)$.

Various extensions of this new observation are described in detail (especially in the time-dependent setting needed in the present study) in the second half of Sect. 3 - especially in Lemma 3.6. It is based on a new interpolation mechanism involving the dispersion properties of the advection operator $v \cdot \nabla_{x}$ presented in Lemmas 3.2 and 3.3.

[^3]Thus, applying this new velocity averaging result to the Navier-Stokes limit requires additional $v$-regularity on (fluctuations of) the phase-space density. This was done by L. Saint-Raymond in [56] for the BGK model; doing the same for the Boltzmann equation is much more involved for reasons that will be described below. Modulo various truncations too technical to be discussed at this stage, this extra $v$-regularity is implied

- by the dissipation (i.e. the entropy production) estimate, and
- by a smoothing property in the $v$-variable for the gain part of the collision integral.
The derivations in [7] or [50] did not make much use of the dissipation controls. This was clearly seen as a major shortcoming in both articles. Indeed, when $F_{\epsilon}$ is a local Maxwellian, an easy argument (see [55]) shows that the ratio in (A2) is bounded but not necessarily weakly relatively compact in $L_{l o c}^{1}\left(d t d x ; L^{1}(M d v)\right)$. This observation suggests
- that the dissipation estimate implied by (1.21) might help to improve the $O(|\log \epsilon|)$ bound on the ratio in (A2) established in [7]; but
- that this dissipation estimate is not enough to prove the compactness statement in (A2), since the ratio in (A2) is in general not weakly relatively compact in $L_{l o c}^{1}\left(d t d x ; L^{1}(M d v)\right)$ when $F_{\epsilon}$ is a local Maxwellian, in other words when the dissipation term $D\left(F_{\epsilon}\right)$ in (1.22) vanishes.
Both these remarks have the merit of showing the importance of the dissipation estimate when trying to establish (A2). Yet they are somewhat misleading, especially as regards the precise manner in which this dissipation estimate is to be used for that purpose. Indeed, the local Maxwellian associated to $F_{\epsilon}-$ i.e. with the same macroscopic density, bulk velocity and temperature as $F_{\epsilon}$ - plays essentially no role in the present work.

For various reasons specific to the Boltzmann equation (see a more detailed discussion on this in Sect. 2), the distance from the phase-space density $F_{\epsilon}$ to its local Maxwellian is not generally well controlled by the dissipation estimate. The key to the extra-regularity in $v$ needed for the Navier-Stokes limit consists in choosing a substitute for the local Maxwellian of the phase space density $F_{\epsilon}-$ see (2.13) below. This local "pseudo-equilibrium" is not defined by a formula showing its regularity properties in $v$ explicitly such as (1.12). Instead, this local pseudo-equilibrium is defined in terms of the gain part of an artificial collision operator in such a way that its distance to $F_{\epsilon}$ is controlled by the dissipation integral. That the gain part of the collision integral is more regular in $v$ than $F_{\epsilon}$ itself was proved by P.-L. Lions in [45]. In fact, what is used is not exactly the main result in [45], but an earlier and linear variant of it due to H . Grad [32] and R. Caflisch [16].

The idea of using a fictitious collision operator is somewhat reminiscent of the argument used by P.-L. Lions in [45] p. 423 to prove that any function $F \in L_{v}^{1}$ such that $\mathscr{B}(F, F)=0$ is smooth - and therefore a Maxwellian.

It may also be interesting to compare the procedure described above for gaining compactness from the dissipation estimate with the work of L. Arkeryd and A. Nouri [2].

Both Sects. 2 and 3 describe the main steps leading to the nonlinear controls stated in Proposition 3.4, Corollary 3.5, and in Propositions 2.7, 3.8 and Corollary 3.9. However the complete proofs of these nonlinear controls are postponed to Sects. 6, 7 and 8 below.

Finally, Appendix B collects the main results established in [7] without using the unverified assumptions (A1) and (A2). Our proofs make occasional use of some of these results.

## 2. Analytical tools I: Decompositions of the number density based on entropy and dissipation

The present section is aimed at describing in detail the part of our argument that improves upon the use of dissipation controls that was made in [7] and [50].

Throughout the present section and the next, we consider $F_{\epsilon}$, a family of renormalized solutions to (1.13) with initial data $F_{\epsilon}^{\text {in }}$ such that (1.14) holds, as well as the relative number densities $G_{\epsilon}$ and fluctuations $g_{\epsilon}$ defined in (1.34). The Boltzmann collision kernel $b$ is assumed to satisfy (H1).

### 2.1. Entropy-based estimates

2.1.1. The Flat-Sharp decomposition. Most of the estimates in [7] were based on the relative entropy control

$$
\begin{equation*}
\frac{1}{\epsilon^{2}} \int\left\langle h\left(\epsilon g_{\epsilon}(t, x, \cdot)\right\rangle d x \leq C^{i n}\right. \tag{2.1}
\end{equation*}
$$

inferred from (1.44) and the entropy inequality (1.21). We keep here the notations from [7] (see also Appendix A below) and denote the nonlinearity involved in the relative entropy by

$$
\begin{equation*}
h(z)=(1+z) \log (1+z)-z, \quad z>-1 . \tag{2.2}
\end{equation*}
$$

Since $h(z) \sim \frac{1}{2} z^{2}$ near $z=0$, the entropy control (2.1) is as good as a $L^{2}$ control but only for the part of $g_{\epsilon}$ that does not exceed $1 / \epsilon$ in size. This suggests splitting the fluctuation of relative number density $g_{\epsilon}$ as follows. First, we consider the class of bump functions

$$
\begin{equation*}
\Upsilon=\left\{\gamma: \mathbf{R}_{+} \rightarrow[0,1] \mid \gamma \in C^{1}, \gamma\left(\left[\frac{3}{4}, \frac{5}{4}\right]\right)=\{1\}, \operatorname{supp} \gamma \subset\left[\frac{1}{2}, \frac{3}{2}\right]\right\} . \tag{2.3}
\end{equation*}
$$

We then present
The Flat-Sharp decomposition. Let $\gamma \in \Upsilon$; then

$$
\begin{equation*}
g_{\epsilon}=g_{\epsilon}+\epsilon^{\sharp} g_{\epsilon} \tag{2.4}
\end{equation*}
$$

with

$$
g_{\epsilon}=\frac{1}{\epsilon}\left(G_{\epsilon}-1\right) \gamma\left(G_{\epsilon}\right), \quad \sharp_{\epsilon}=\frac{1}{\epsilon^{2}}\left(G_{\epsilon}-1\right)\left(1-\gamma\left(G_{\epsilon}\right)\right) .
$$

While this new decomposition is not the same as the old one just before Corollary 3.2 of [7] (p. 696), it shares most of its key properties - and avoids one unpleasant feature that will be discussed after the statement of Proposition 3.4 below.

Proposition 2.1 (Entropy controls). Assume that the bump function $\gamma \in \Upsilon$ as in (2.3). The relative fluctuation $g_{\epsilon}$ of number density satisfies the following estimates:

- $g_{\epsilon}=O(1)$ in $L_{t}^{\infty}\left(L^{2}(M d v d x)\right)$;
- $\sharp g_{\epsilon}=O(1)$ in $L_{t}^{\infty}\left(L^{1}(M d v d x)\right)$;
- the family

$$
\frac{1}{\epsilon}\left\langle g_{\epsilon}\right\rangle\left(1-\gamma\left(\left\langle G_{\epsilon}\right\rangle\right)\right) \text { is of order } O(1) \text { in } L_{t}^{\infty}\left(L_{x}^{1}\right)
$$

- for each compact subset $E$ of $\mathbf{R}_{+}$,

$$
g_{\epsilon} \mathbf{1}_{E}\left(G_{\epsilon}\right)=O(1) \text { in } L_{t}^{\infty}\left(L^{2}(M d v d x)\right)
$$

- for each $\lambda>e$, and each $\epsilon>0$, one has

$$
\left\|\frac{1}{\epsilon}\left|g_{\epsilon}\right| \mathbf{1}_{G_{\epsilon} \geq \lambda}\right\|_{L_{t}^{\infty}\left(L^{1}(M d v d x)\right)} \leq C^{i n} \frac{1}{\log \lambda-1} .
$$

Proof. The control on ${ }^{b} g_{\epsilon}$ relies on the entropy inequality (2.1) and on the following elementary inequality: there exists $c>0$ such that

$$
\begin{equation*}
h(z) \geq c z^{2}, \quad|z| \leq \frac{1}{2} \tag{2.5}
\end{equation*}
$$

Likewise the fourth control relies on the entropy inequality (2.1) and on the fact that, for each compact subset $E$ of $\mathbf{R}_{+}$, there exists $c_{E}>0$ such that

$$
h(z) \geq c_{E} z^{2}, \quad z+1 \in E
$$

The control on $\sharp g_{\epsilon}$ relies on (2.1) and on the existence of $c^{\prime}>0$ such that

$$
\begin{equation*}
h(z) \geq c^{\prime}|z|, \quad z \in\left[-1,-\frac{1}{4}\right] \cup\left[\frac{1}{4},+\infty\right) \tag{2.6}
\end{equation*}
$$

By Jensen's inequality and the convexity of $h$, one has

$$
\int h\left(\epsilon\left\langle g_{\epsilon}\right\rangle\right) d x \leq \int\left\langle h\left(\epsilon g_{\epsilon}\right)\right\rangle d x \leq C^{i n} \epsilon^{2}
$$

because of (2.1). The third control follows from this estimate and (2.6).

Finally, observe that the function $z \mapsto \frac{h(z)}{z}$ is increasing on $\mathbf{R}_{+}$. Thus for each $\lambda>e$

$$
h\left(\epsilon\left|g_{\epsilon}\right|\right) \geq \frac{h(\lambda-1)}{\lambda-1} \epsilon\left|g_{\epsilon}\right| \text { wherever } G_{\epsilon} \geq \lambda
$$

(Notice that $G_{\epsilon} \geq \lambda$ implies that $g_{\epsilon} \geq e-1>0$ ). This and (10.1) imply that

$$
\begin{aligned}
\int\left\langle\frac{1}{\epsilon}\right| g_{\epsilon}\left|\mathbf{1}_{G_{\epsilon} \geq \lambda}\right\rangle d x & \leq \frac{\lambda-1}{h(\lambda-1)} \frac{1}{\epsilon^{2}} \int\left\langle h\left(\epsilon g_{\epsilon}\right)\right\rangle d x \\
& \leq C^{i n} \frac{\lambda-1}{h(\lambda-1)} \leq \frac{C^{i n}}{\log \lambda-1}
\end{aligned}
$$

which in turn establishes the last control.
2.1.2. Pointwise estimates implied by the Flat-Sharp decomposition. The old decomposition introduced on p. 696 of [7] had one feature used repeatedly there and in [50], namely the fact that the $L^{1}$ part in this decomposition $g_{\epsilon}^{2} / N_{\epsilon}$ controlled the square of the $L^{2}$ part $g_{\epsilon} / N_{\epsilon}$. The analogue with the Flat-Sharp decomposition (2.4) (namely, that $\left|g_{\epsilon}\right|^{2} \leq\left|\sharp g_{\epsilon}\right|$ ) is no longer true, but (2.4) leads to a precise localization of $g_{\epsilon}$ which has several useful implications.

Proposition 2.2 (Pointwise estimates). Assume that the bump function $\gamma \in \Upsilon$ as in (2.3). The relative number density fluctuation $g_{\epsilon}$ satisfies the following estimates:

- $\epsilon\left|{ }^{b} g_{\epsilon}\right| \leq \frac{1}{2}$;
- $\left(1-\gamma\left(G_{\epsilon}\right)\right) \leq\left. 4 \epsilon^{2}\right|^{\sharp} g_{\epsilon} \mid$, which implies that $\frac{1}{\epsilon}\left(1-\gamma\left(G_{\epsilon}\right)\right) \leq\left.\left. 2\right|^{\sharp} g_{\epsilon}\right|^{1 / 2}$;
- $\left(1-\gamma\left(G_{\epsilon}\right)\right) G_{\epsilon} \leq 5 \epsilon^{2}\left|\sharp g_{\epsilon}\right|$;
- for $k: \mathbf{R}_{+} \rightarrow[0,1]$, let $C_{k}=\|z \mapsto z k(z)\|_{L^{\infty}}$; then for any family $V_{\epsilon} \rightarrow+\infty$ as $\epsilon \rightarrow 0$, one has ${ }^{6}$

$$
\begin{aligned}
\int\left(k\left(G_{\epsilon}\right)\left|G_{\epsilon}-1\right|\right)^{\alpha} & M^{\beta}|v|^{p} \mathbf{1}_{|v|^{2} \geq V_{\epsilon}} d v \\
& \lesssim \frac{4 \pi}{\beta}(2 \pi)^{-\frac{3 \beta}{2}}\left(1+C_{k}\right)^{\alpha} V_{\epsilon}^{\frac{p+1}{2}} e^{-\frac{\beta}{2} V_{\epsilon}}
\end{aligned}
$$

as $\epsilon \rightarrow 0$.
Proof. Because $\gamma$ is supported in $\left[\frac{1}{2}, \frac{3}{2}\right]$,

$$
\left.\epsilon\right|^{\mathrm{b}} g_{\epsilon}\left|=\left|G_{\epsilon}-1\right| \gamma\left(G_{\epsilon}\right) \leq\left|G_{\epsilon}-1\right| \mathbf{1}_{\left|G_{\epsilon}-1\right| \leq \frac{1}{2}} \leq \frac{1}{2}\right.
$$

[^4]Because $(1-\gamma)$ vanishes identically on $\left[\frac{3}{4}, \frac{5}{4}\right]$,

$$
\begin{aligned}
\left.\epsilon^{2}\right|^{\sharp} g_{\epsilon} \mid & =\left|G_{\epsilon}-1\right|\left(1-\gamma\left(G_{\epsilon}\right)\right) \\
& \geq\left|G_{\epsilon}-1\right| \mathbf{1}_{\left|G_{\epsilon}-1\right| \geq \frac{1}{4}}\left(1-\gamma\left(G_{\epsilon}\right)\right) \\
& \geq \frac{1}{4} \mathbf{1}_{\left|G_{\epsilon}-1\right| \geq \frac{1}{4}}\left(1-\gamma\left(G_{\epsilon}\right)\right) \\
& =\frac{1}{4}\left(1-\gamma\left(G_{\epsilon}\right)\right) .
\end{aligned}
$$

Because $0 \leq 1-\gamma\left(G_{\epsilon}\right) \leq 1$, one also has

$$
\left.\epsilon^{2}\right|^{\sharp g_{\epsilon}} \left\lvert\, \geq \frac{1}{4}\left(1-\gamma\left(G_{\epsilon}\right)\right)^{2}\right. \text { and thus } \frac{1-\gamma\left(G_{\epsilon}\right)}{\epsilon} \leq\left.\left. 2\right|^{\sharp} g_{\epsilon}\right|^{1 / 2} .
$$

The third control is a direct consequence of the second because

$$
\left(1-\gamma\left(G_{\epsilon}\right)\right) G_{\epsilon}=\epsilon^{2}{ }^{\sharp} g_{\epsilon}+\left(1-\gamma\left(G_{\epsilon}\right)\right) .
$$

The last control relies on the obvious estimate $k\left(G_{\epsilon}\right)\left|G_{\epsilon}-1\right| \leq 1+C_{k}$ and on the standard tail estimate for the Gaussian distribution:

$$
\begin{aligned}
\int M^{\beta}|v|^{p} \mathbf{1}_{|v|^{2} \geq V_{\epsilon}} d v & =(2 \pi)^{-\frac{3 \beta}{2}} 4 \pi \int_{\sqrt{V_{\epsilon}}}^{+\infty} e^{-\frac{\beta}{2} r^{2}} r^{p+2} d r \\
& \sim(2 \pi)^{-\frac{3 \beta}{2}} 4 \pi \int_{\sqrt{V_{\epsilon}}}^{+\infty}-\frac{d}{d r}\left(\frac{1}{\beta} e^{-\frac{\beta}{2} r^{2}} r^{p+1}\right) d r \\
& =(2 \pi)^{-\frac{3 \beta}{2}} \frac{4 \pi}{\beta} e^{-\frac{\beta}{2} V_{\epsilon}} V_{\epsilon}^{\frac{p+1}{2}}
\end{aligned}
$$

as $\epsilon \rightarrow 0$, since $V_{\epsilon} \rightarrow+\infty$.

### 2.2. Dissipation-based estimates

2.2.1. Dissipation controls of the scaled collision integrand. While the fluctuations of relative number density $g_{\epsilon}$ satisfy the bound (2.1), the scaled collision integrand $q_{\epsilon}$ defined in (1.35) satisfies the dissipation bound

$$
\begin{align*}
\frac{1}{4 \epsilon^{4}} \int_{0}^{+\infty} \int\left\langle r\left(\frac{\epsilon^{2} q_{\epsilon}}{G_{\epsilon} G_{\epsilon 1}}\right) G_{\epsilon} G_{\epsilon 1}\right\rangle d x d t & =\frac{1}{\epsilon^{4}} \int_{0}^{+\infty} \iint D\left(F_{\epsilon}\right) d v d x d t  \tag{2.7}\\
& \leq \frac{1}{\epsilon^{2}} H\left(F_{\epsilon}^{i n} \mid M\right) \leq C^{i n}
\end{align*}
$$

inferred from (1.44) and the entropy inequality (1.21). Again we use the notations from [7] (see also Appendix A below) and denote the nonlinearity involved in the dissipation by

$$
\begin{equation*}
r(z)=z \log (1+z), \quad z>-1 . \tag{2.8}
\end{equation*}
$$

Since $r(z) \sim z^{2}$ near $z=0$, the dissipation control is also as good as an $L^{2}$ control but again this is only true of the part of $q_{\epsilon} /\left(G_{\epsilon} G_{\epsilon 1}\right)$ that does not exceed $1 / \epsilon^{2}$ in size. This suggests decomposing the scaled collision integrand as

$$
\begin{align*}
& q_{\epsilon}=\stackrel{b}{q}_{\epsilon}+\epsilon^{2} \ddot{q}_{\epsilon} \text { with } \\
& q_{\epsilon}=q_{\epsilon} \gamma\left(\frac{G_{\epsilon}^{\prime} G_{\epsilon 1}^{\prime}}{G_{\epsilon} G_{\epsilon 1}}\right) \text { and }  \tag{2.9}\\
& q_{\epsilon}=\frac{1}{\epsilon^{2}} q_{\epsilon}\left(1-\gamma\left(\frac{G_{\epsilon}^{\prime} G_{\epsilon 1}^{\prime}}{G_{\epsilon} G_{\epsilon 1}}\right)\right)
\end{align*}
$$

where $\gamma$ is any element of $\Upsilon$ as in (2.3).
Proposition 2.3 (Dissipation controls). The scaled collision integrand $q_{\epsilon}$ satisfies the following estimates

- for any function $\gamma \in \Upsilon$

$$
\iint \frac{\stackrel{b}{q}_{\epsilon}^{2}}{G_{\epsilon} G_{\epsilon 1}} b\left(v-v_{1}, \omega\right) d \sigma_{v, v_{1}}(\omega) M_{1} d v_{1} \leq \frac{4}{c} \frac{D\left(F_{\epsilon}\right)}{\epsilon^{4} M}
$$

where the constant $c$ is defined in (2.5), so that

$$
\left.\left\langle\frac{\stackrel{\rightharpoonup}{q}_{\epsilon}^{2}}{G_{\epsilon} G_{\epsilon 1}}\right\rangle\right\rangle=O(1) \text { in } L_{t, x}^{1} \text {; }
$$

- likewise

$$
\iint\left|{ }^{\sharp} q_{\epsilon}\right| b\left(v-v_{1}, \omega\right) d \sigma_{v, v_{1}}(\omega) M_{1} d v_{1} \leq \frac{4}{c^{\prime}} \frac{D\left(F_{\epsilon}\right)}{\epsilon^{4} M}
$$

where the constant $c$ is defined in (2.6), so that

$$
\left.\langle |\right|^{\sharp} q_{\epsilon}| \rangle=O(1) \text { in } L_{t, x}^{1}
$$

- more generally, for any compact subset $E$ of $[-1,+\infty)$,

$$
\left\langle\frac{q_{\epsilon}^{2}}{G_{\epsilon} G_{\epsilon 1}} \mathbf{1}_{E}\left(\frac{G_{\epsilon}^{\prime} G_{\epsilon 1}^{\prime}}{G_{\epsilon} G_{\epsilon 1}}-1\right)\right\rangle=O(1) \text { in } L_{t, x}^{1} .
$$

Proof. The elementary inequality $r(z) \geq h(z)$ for all $z>-1$ together with the inequalities (2.5) and (2.6) shows that, for the same positive constants $c$ and $c^{\prime}$ as in these inequalities, one has

$$
\begin{equation*}
r(z) \geq c z^{2}, \quad|z| \leq \frac{1}{2} \tag{2.10}
\end{equation*}
$$

and

$$
\begin{equation*}
r(z) \geq c^{\prime}|z|, \quad z \in\left(-1,-\frac{1}{4}\right] \cup\left[\frac{1}{4},+\infty\right) . \tag{2.11}
\end{equation*}
$$

Then, the condition

$$
\gamma\left(\frac{G_{\epsilon}^{\prime} G_{\epsilon 1}^{\prime}}{G_{\epsilon} G_{\epsilon 1}}\right) \neq 0 \text { implies that }\left|\frac{G_{\epsilon}^{\prime} G_{\epsilon 1}^{\prime}}{G_{\epsilon} G_{\epsilon 1}}-1\right|=\frac{\epsilon^{2}\left|q_{\epsilon}\right|}{G_{\epsilon} G_{\epsilon 1}} \leq \frac{1}{2}
$$

which shows that the first control follows from (2.10) and (2.7). Likewise, the condition

$$
1-\gamma\left(\frac{G_{\epsilon}^{\prime} G_{\epsilon 1}^{\prime}}{G_{\epsilon} G_{\epsilon 1}}\right) \neq 0 \text { implies that }\left|\frac{G_{\epsilon}^{\prime} G_{\epsilon 1}^{\prime}}{G_{\epsilon} G_{\epsilon 1}}-1\right|=\frac{\epsilon^{2}\left|q_{\epsilon}\right|}{G_{\epsilon} G_{\epsilon 1}} \geq \frac{1}{4}
$$

which shows that the second control follows from (2.11) and (2.7).
As for the third control, it is obtained in exactly the same way as the first, because, for any compact $E \subset[-1,+\infty)$, there exists $c_{E}>0$ such that $r(z) \geq c_{E} z^{2}$ for all $z \in E$.

These estimates were apparently used for the first time in proving that conservation defects vanish as $\epsilon \rightarrow 0$ in the proof of the Stokes or Acoustic limits [25].
2.2.2. A second decomposition of $g_{\epsilon}$. The Flat-Sharp decomposition (2.4) of $g_{\epsilon}$ is exclusively based on the level set of $g_{\epsilon}$, i.e. on the size of the values taken by $g_{\epsilon}$. Thus it cannot help in controlling the decay in $v$ of $g_{\epsilon}$, at least not beyond the $L^{1}\left(\left(1+|v|^{2}\right) M d v\right)$ control following from Young's inequality (see [7], Propositions 3.1 (1), 3.5 (1) and 3.3). This was the main reason for the unverified assumption (A2) from [7] recalled at the end of Sect. 1.

In the present work, we introduce a second decomposition of $g_{\epsilon}$, based in particular on the dissipation controls above. An obvious idea would be to adapt the arguments used in the case of the BGK model (see [55], [56]) and decompose the number density as

$$
\begin{equation*}
F_{\epsilon}=\left(F_{\epsilon}-M_{F_{\epsilon}}\right)+\left(M_{F_{\epsilon}}-M\right) \tag{2.12}
\end{equation*}
$$

where $M_{F_{\epsilon}}$ is the local Maxwellian with the same macroscopic density, bulk velocity and temperature as $F_{\epsilon}$ at every $(t, x)$.

However, instead of using the local Maxwellian of $F_{\epsilon}$ in the decomposition (2.12) above, we propose to replace it with the following quantity.

The local pseudo-equilibrium. The substitute for $M_{F_{\epsilon}}$ considered in this work is

$$
\begin{equation*}
\frac{\mathcal{A}^{+}\left(M \tilde{G}_{\epsilon}, M \tilde{G}_{\epsilon}\right)}{\left\langle\tilde{G}_{\epsilon}\right\rangle} \tag{2.13}
\end{equation*}
$$

with

$$
\begin{equation*}
\tilde{G}_{\epsilon}=1+\epsilon^{\mathrm{b}} g_{\epsilon}=\left(1-\gamma\left(G_{\epsilon}\right)\right)+\gamma\left(G_{\epsilon}\right) G_{\epsilon} \tag{2.14}
\end{equation*}
$$

and where $\mathcal{A}^{+}$is the gain part of a fictitious collision operator:

$$
\mathcal{A}^{+}(f, g)=\frac{1}{2} \iint\left(f^{\prime} g_{1}^{\prime}+f_{1}^{\prime} g^{\prime}\right) d \sigma_{v, v_{1}}(\omega) d v_{1}
$$

Using the gain term $\mathcal{A}^{+}$in a decomposition like (2.12) is suggested by the inequality below (satisfied by any measurable, a.e. positive function $F$ ):

$$
\begin{array}{r}
4 D(F) \geq \frac{R}{b_{\infty}}\left(\frac{\mathcal{A}^{+}(F, F)}{R}-F\right)\left(\log \left(\frac{\mathcal{A}^{+}(F, F)}{R}\right)-\log F\right)  \tag{2.15}\\
\text { with } R=\int F d v
\end{array}
$$

where $D(F)$ is the dissipation term defined in (1.22). This inequality (2.15) is obtained on account of (H1) by applying Jensen's inequality to the probability measure $F\left(v_{1}\right) d \sigma_{v, v_{1}}(\omega) d v_{1} / R$ and to the convex function $X \mapsto$ $(X-Y)(\log X-\log Y)$ with $Y>0$ fixed. It provides an explicit control of the distance between $F$ and $\mathcal{A}^{+}(F, F) / R$ in terms of $D(F)$. If $\mathcal{A}^{+}(F, F) / R$ is replaced by the local Maxwellian $M_{F}$, the same control is known to be false: see for example the work of G. Toscani and C. Villani [61], eq. (25) and the discussion pp. 671-672. Replacing then $\mathcal{A}^{+}\left(M G_{\epsilon}, M G_{\epsilon}\right) /\left\langle G_{\epsilon}\right\rangle$ by (2.13) has three further advantages that are fully exploited below in Sect. 6:

- $\left\langle\tilde{G}_{\epsilon}\right\rangle>0$,
- $\tilde{G}_{\epsilon}$ is $L^{\infty}$-bounded and $L^{1}(M d v)$-close to $G_{\epsilon}$,
- $\tilde{G}_{\epsilon}$ is $L^{2}(M d v)$-close to 1 .

Let $k: \mathbf{R}_{+} \rightarrow[0,1]$ and set $C_{k}=\|z \mapsto z k(z)\|_{L^{\infty}}$; then

$$
\begin{align*}
k\left(G_{\epsilon}\right) M g_{\epsilon} & =\frac{1}{\epsilon} k\left(G_{\epsilon}\right)\left(M G_{\epsilon}-\frac{\mathcal{A}^{+}\left(M \tilde{G}_{\epsilon}, M \tilde{G}_{\epsilon}\right)}{\left\langle\tilde{G}_{\epsilon}\right\rangle}\right) \\
& +\frac{1}{\epsilon} k\left(G_{\epsilon}\right)\left(\frac{\mathcal{A}^{+}\left(M \tilde{G}_{\epsilon}, M \tilde{G}_{\epsilon}\right)}{\left\langle\tilde{G}_{\epsilon}\right\rangle}-M\right)  \tag{2.16}\\
& =T_{1}+T_{2}
\end{align*}
$$

Using the last equality in (2.14) and the decomposition of the scaled collision integrand suggested by Proposition 2.3 to further decompose $T_{1}$ while decomposing $T_{2}$ as

$$
\begin{aligned}
T_{2} & =\frac{k\left(G_{\epsilon}\right)}{\epsilon\left\langle\tilde{G}_{\epsilon}\right\rangle}\left(\mathscr{A}^{+}\left(M \tilde{G}_{\epsilon}, M \tilde{G}_{\epsilon}\right)-\mathcal{A}^{+}(M, M)\right) \\
& +\frac{k\left(G_{\epsilon}\right)}{\epsilon} \mathscr{A}^{+}(M, M)\left(\frac{1}{\left\langle\tilde{G}_{\epsilon}\right\rangle}-1\right)
\end{aligned}
$$

leads to the following crucial inequality.

The Relaxation-based decomposition. For each $\epsilon>0$

$$
\begin{align*}
k\left(G_{\epsilon}\right) M\left|g_{\epsilon}\right| & \leq 4 \sqrt{\frac{3 b_{\infty}\left(c+c^{\prime}\right)}{c c^{\prime}}} \sqrt{C_{k}+2} \epsilon \sqrt{M} \sqrt{\frac{D\left(F_{\epsilon}\right)}{\epsilon^{4}}} \\
& +2 M\langle | g_{\epsilon}\left|+\left(C_{k}+3\right) \frac{1-\gamma\left(G_{\epsilon}\right)}{\epsilon}\right\rangle  \tag{2.17}\\
& +\mathscr{A}^{+}\left(M\left[5\left|g_{\epsilon}\right|+9\left(C_{k}+1\right) \frac{1-\gamma\left(G_{\epsilon}\right)}{\epsilon}\right], M\right),
\end{align*}
$$

where $b_{\infty}>0$ is the constant appearing in assumption (H1) on the collision kernel, while $c$ and $c^{\prime}$ are the positive constants appearing in (2.5) and (2.6).

Deriving inequality (2.17) from the basic decomposition (2.16) requires nontrivial computations that will occupy most of Sect. 6 below (see in particular the Subsect. 6.1 there).

The benefits of using this new decomposition may not seem obvious at first sight. Observe however that the first term on the right-hand side of (2.17) is $O(\epsilon)$ in $L^{2}\left(M^{-1} d v d x d t\right)$, while the second is of the form $M \times O(1)_{L_{t}^{\infty}\left(L_{x}^{2}\right)}$. This has the following implications on the decay in the $v$-variable:
Proposition 2.4. For any $p \geq 0$

$$
\int \epsilon \sqrt{M} \sqrt{\frac{D\left(F_{\epsilon}\right)}{\epsilon^{4}}}|v|^{p} d v=O(\epsilon) \text { in } L_{t, x}^{2}
$$

and

$$
\sup _{v \in \mathbf{R}^{3}}\left(|v|^{p} M\langle | g_{\epsilon}\left|+\left(C_{k}+3\right) \frac{1-\gamma\left(G_{\epsilon}\right)}{\epsilon}\right\rangle\right)=O(1) \text { in } L_{t}^{\infty}\left(L_{x}^{2}\right) .
$$

The first estimate follows from the dissipation estimate (2.7) and the Cauchy-Schwarz inequality. The second estimate follows from the first and second entropy controls in Proposition 2.1, and the second pointwise estimate in Proposition 2.2.

It remains to see that the third term enjoys similar decay properties; this will result from classical estimates recalled below.
2.2.3. The Caflisch-Grad estimates. For all $p \in[1,+\infty]$ and all $s \geq 0$, define $L^{p, s}$ as the space of a.e.-defined measurable functions $f$ on $\mathbf{R}^{3}$ such that $v \mapsto\left(1+|v|^{s}\right) f(v)$ belongs to $L^{p}\left(\mathbf{R}^{3}, d v\right)$ and set

$$
\|f\|_{L^{p, s}}=\left\|\left(1+|v|^{s}\right) f\right\|_{L^{p}} .
$$

Define further the linear operator

$$
\mathcal{K} f=\frac{1}{\sqrt{M}} \mathcal{A}^{+}(\sqrt{M} f, M) .
$$

The properties of the operator $\mathcal{K}$ were studied in detail by H. Grad in his fundamental paper [32]. His estimates, later improved by R. Caflisch [16], are recalled below.

Proposition 2.5 (Caflisch-Grad estimates). For all $s \geq 0$, the bilinear operator

$$
(f, g) \mapsto \mathcal{A}^{+}(f, g)
$$

is a continuous map from $L^{1, s} \times L^{1, s}$ to $L^{1, s}$. Further

- the operator $\mathcal{K}$ maps $L^{2}$ continuously into $L^{\infty, 3 / 2}$;
- for each $s>0, \mathcal{K}$ maps $L^{\infty, s}$ continuously into $L^{\infty, s+2}$;
- for each $s, \sigma>0, \mathcal{K}$ maps $L^{2, s}$ continuously into $L^{\infty, s+\frac{3}{2}}+L^{2, \sigma}$.

Proof. The second equality in (1.4) clearly implies that $\sup \left(|v|^{2},\left|v_{1}\right|^{2}\right) \leq$ $\left|v^{\prime}\right|^{2}+\left|v_{1}^{\prime}\right|^{2}$. Thus for each $s>0$ there exists $C_{s}>1$ such that $|v|^{s}+\left|v_{1}\right|^{s} \leq$ $C_{s}\left(\left|v^{\prime}\right|^{s}+\left|v_{1}^{\prime}\right|^{s}\right)$ for all $v$ and $v_{1} \in \mathbf{R}^{3}$ and all $\omega \in \mathbf{S}^{2}$. For each measurable $f$ one therefore has

$$
\begin{aligned}
\left\|\mathscr{A}^{+}(f, g)\right\|_{L^{1, s}} & \leq \frac{1}{2} \iiint\left(\left|f^{\prime}\right|\left|g_{1}^{\prime}\right|+\left|f_{1}^{\prime}\right|\left|g^{\prime}\right|\right)\left(1+|v|^{s}\right) d \sigma_{v, v_{1}}(\omega) d v_{1} d v \\
& =\frac{1}{2} \iiint\left|f^{\prime}\right|\left|g_{1}^{\prime}\right|\left(2+|v|^{s}+\left|v_{1}\right|^{s}\right) d \sigma_{v, v_{1}}(\omega) d v_{1} d v \\
& \leq \frac{1}{2} \iiint\left|f^{\prime}\right|\left|g_{1}^{\prime}\right| C_{s}\left(2+\left|v^{\prime}\right|^{s}+\left|v_{1}^{\prime}\right|^{s}\right) d \sigma_{v, v_{1}}(\omega) d v_{1} d v \\
& \leq C_{s} \iiint\left|f^{\prime}\right|\left|g_{1}^{\prime}\right|\left(1+\left|v^{\prime}\right|^{s}\right)\left(1+\left|v_{1}^{\prime}\right|^{s}\right) d \sigma_{v, v_{1}}(\omega) d v_{1} d v \\
& =C_{s} \iiint|f|\left|g_{1}\right|\left(1+|v|^{s}\right)\left(1+\left|v_{1}\right|^{s}\right) d \sigma_{v, v_{1}}(\omega) d v_{1} d v \\
& =C_{s} \int|f|\left(1+|v|^{s}\right) d v \int|g|\left(1+|v|^{s}\right) d v
\end{aligned}
$$

which establishes the first statement. The first equality above comes from the $v-v_{1}$ symmetry in (1.8) and (1.4), while the penultimate equality results from the $\left(v, v_{1}\right)-\left(v^{\prime}, v_{1}^{\prime}\right)$ symmetry in these same formulas (see (1.11)).

The second and third statements are much harder to prove: they are particular cases of estimates (6.1) and (6.2) of [16].

The fourth statement is an easy consequence of the first and second, as shown below. For all $f \in L^{2, s}$, write

$$
\mathcal{K} f=a_{1}+a_{2}
$$

with $a_{1}$ defined by

$$
\begin{aligned}
a_{1}=\frac{1}{\sqrt{M}} \iint \frac{1}{2} & \left(\sqrt{M^{\prime}} f^{\prime} \mathbf{1}_{|v|^{2} \leq 2\left|v^{\prime}\right|^{2}} M_{1}^{\prime}\right. \\
& \left.+\sqrt{M_{1}^{\prime}} f_{1}^{\prime} \mathbf{1}_{|v|^{2} \leq 2\left|v_{1}^{\prime}\right|^{2}} M^{\prime}\right) d \sigma_{v, v_{1}}(\omega) d v_{1}
\end{aligned}
$$

while $a_{2}$ is defined by

$$
\begin{aligned}
a_{2}=\frac{1}{\sqrt{M}} \iint \frac{1}{2} & \left(\sqrt{M^{\prime}} f^{\prime} M_{1}^{\prime}\left(1-\mathbf{1}_{|v|^{2} \leq 2\left|v^{\prime}\right|^{2}}\right)\right. \\
& \left.+\sqrt{M_{1}^{\prime}} f_{1}^{\prime} M^{\prime}\left(1-\mathbf{1}_{|v|^{2} \leq 2\left|v_{1}^{\prime}\right|^{2}}\right)\right) d \sigma_{v, v_{1}}(\omega) d v_{1}
\end{aligned}
$$

Then

$$
\begin{equation*}
|v|^{\frac{3}{2}+s}\left|a_{1}\right| \leq 2^{\frac{s}{2}}|v|^{\frac{3}{2}} \mathcal{K}\left(|v|^{s}|f|\right) . \tag{2.18}
\end{equation*}
$$

Using again the second equality in (1.4) implies that, for all $v, v_{1}$ and $\omega$, one has

$$
\left|v^{\prime}\right|^{2} \geq \frac{1}{2}|v|^{2} \quad \text { or } \quad\left|v_{1}^{\prime}\right|^{2} \geq \frac{1}{2}|v|^{2}
$$

In other words, for all $v, v_{1}$ and $\omega$,

$$
1 \leq \mathbf{1}_{|v|^{2} \leq 2\left|v^{\prime}\right|^{\prime}}+\mathbf{1}_{|v|^{2} \leq 2\left|v_{1}^{\prime}\right|^{2}}
$$

Therefore

$$
\begin{aligned}
|v|^{\sigma}\left|a_{2}\right| & \leq \frac{1}{\sqrt{M}} \mathcal{A}^{+}\left(\sqrt{M}|f|, 2^{\frac{\sigma}{2}}|v|^{\sigma} M\right) \\
& =\iint \frac{1}{2}\left(\left|f^{\prime}\right| \sqrt{M_{1}^{\prime}} 2^{\frac{\sigma}{2}}\left|v_{1}^{\prime}\right|^{\sigma}+\left|f_{1}^{\prime}\right| \sqrt{M^{\prime}} 2^{\frac{\sigma}{2}}\left|v^{\prime}\right|^{\sigma}\right) \sqrt{M_{1}} d \sigma_{v, v_{1}}(\omega) d v_{1} \\
& \leq\left(\iint d \sigma_{v, v_{1}}(\omega) M_{1} d v_{1}\right)^{\frac{1}{2}} \mathcal{A}^{+}\left(|f|^{2}, 2^{\sigma}|v|^{2 \sigma} M\right)^{\frac{1}{2}} \\
& =\mathcal{A}^{+}\left(|f|^{2}, 2^{\sigma}|v|^{2 \sigma} M\right)^{\frac{1}{2}} .
\end{aligned}
$$

The inequality (2.18) and the second continuity statement of Proposition 2.5 imply the existence of a positive constant $C$ such that

$$
\left\|a_{1}\right\|_{L^{\infty, \frac{3}{2}+s}} \leq C\|f\|_{L^{2, s}}
$$

The inequality (2.19) and the first continuity statement of Proposition 2.5 imply the existence of a positive constant $C^{\prime}$ such that

$$
\left\|a_{2}\right\|_{L^{2, \sigma}} \leq C^{\prime}\|f\|_{L^{2}} \leq C^{\prime}\|f\|_{L^{2, s}}
$$

These last two inequalities imply the fourth continuity statement of Proposition 2.5.
2.2.4. Conclusion. The Caflisch-Grad estimates summarized in Proposition 2.5 above imply that the third term on the right-hand side of the inequality (2.17) also is uniformly bounded a.e. pointwise in $v$ with good decay as $|v| \rightarrow+\infty$. Eventually this implies that, after truncating large values of $g_{\epsilon}$, whatever remains of the fluctuations of relative number density is uniformly bounded a.e. pointwise in $v$ and has good decay as $|v| \rightarrow+\infty$. Before going further, we need the following definition.
Definition 2.6. Let $f_{\epsilon}$ a bounded family of $L_{l o c}^{1}\left(\mathbf{R}_{x}^{m} \times \mathbf{R}_{y}^{n}\right)$. We say that $f_{\epsilon}$ is uniformly integrable in $y$ if and only if, for each $\eta>0$, there exists $\alpha>0$ such that, for each measurable family $\left(A_{x}\right)_{x \in \mathbf{R}^{m}}$ of measurable sets in $\mathbf{R}_{y}^{n}$ satisfying ${ }^{7} \sup _{x \in \mathbf{R}^{m}}\left|A_{x}\right|<\alpha$, one has

$$
\int\left(\int_{A_{x}}\left|f_{\epsilon}(x, y)\right| d y\right) d x<\eta, \quad \text { for each } \epsilon
$$

The family $f_{\epsilon}$ is said to be locally uniformly integrable in $y$ if and only if $\mathbf{1}_{K} f_{\epsilon}$ is uniformly integrable in y for each compact $K \subset \mathbf{R}_{x}^{m} \times \mathbf{R}_{y}^{n}$.

The main consequences of the two decompositions introduced in this section are stated in the next proposition. Obviously, the proof of the Navier-Stokes-Fourier limit will also use many of the results already proved in [7], mainly from the analogue of the Flat-Sharp decomposition there. We have chosen to summarize these results in Appendix B below so as to avoid a cluttered presentation of the material that is genuinely new in the present paper.

Proposition 2.7 ( $g_{\epsilon}$ controls I). For any $\gamma \in \Upsilon$ defined by (2.3), the family ${ }^{\mathrm{b}} \mathrm{g}_{\epsilon}$ has the following properties:

- for any sequence $\epsilon_{n} \rightarrow 0$, the associated sequence $M\left|{ }^{b} g_{\epsilon_{n}}\right|^{2}$ is locally uniformly integrable in $v$;
- $\left(1+|v|^{s}\right) g_{\epsilon}=O(1)$ in $L_{l o c}^{2}\left(d t d x ; L^{2}(M d v)\right)$ for all $s \geq 0$.

In addition, one has

$$
\left(1+|v|^{s}\right) \frac{1-\gamma\left(G_{\epsilon}\right)}{\epsilon^{2}}=O(1) \text { in } L_{l o c}^{1}\left(d t d x ; L^{1}(M d v)\right)
$$

for all $s \geq 0$.

## 3. Analytical tools II: A new limiting case of velocity averaging in $L^{1}$

While the ideas presented in the previous section certainly help in controlling the $v$ dependence of ${ }^{b} g_{\epsilon}$, more is needed to estimate the ${ }^{\sharp} g_{\epsilon}$ term. Indeed, in order to control ${ }^{\sharp} g_{\epsilon}$

[^5]- because of the last entropy control (the fifth estimate in Proposition 2.1), it suffices to control the truncated family ${ }^{\sharp} g_{\epsilon} \mathbf{1}_{G_{\epsilon} \leq e^{1 / \delta}}$ for some $\delta$ appropriately chosen in terms of $\epsilon$;
- because of the last pointwise estimate (the fourth statement in Proposition 2.2), one can further truncate in $v$ and consider finally the truncated family

$$
\sharp g_{\epsilon} \mathbf{1}_{G_{\epsilon} \leq e^{1 / \delta}} \mathbf{1}_{|v|^{2} \leq V_{\epsilon}}
$$

for some well-chosen $V_{\epsilon} \rightarrow+\infty$.
Next, we split the expression above as

$$
\begin{equation*}
\sharp g_{\epsilon} \mathbf{1}_{G_{\epsilon} \leq e^{1 / \delta}} \mathbf{1}_{|v|^{2} \leq V_{\epsilon}}=\left(\frac{1}{\epsilon} \mathbf{1}_{G_{\epsilon} \leq e^{1 / \delta}} g_{\epsilon}\right)\left(\left(1-\gamma\left(G_{\epsilon}\right)\right) \mathbf{1}_{|v|^{2} \leq V_{\epsilon}}\right) \text {. } \tag{3.1}
\end{equation*}
$$

To control the first factor, one can think of using the Relaxation-based decomposition (2.17) with $k(z)=\mathbf{1}_{z \leq e^{1 / \delta}}$. This would lead to estimating

$$
\begin{align*}
\frac{1}{\epsilon} \mathbf{1}_{G_{\epsilon} \leq e^{1 / \delta}} M g_{\epsilon} & =\sqrt{M} O\left(\frac{e^{1 / \delta}}{\epsilon}\right)_{L_{l o c}^{2}\left(d t d x ; L_{v}^{\infty}\right)}  \tag{3.2}\\
& +O\left(e^{1 / 2 \delta}\right)_{L^{2}\left(d t d x ; L^{2, s}\right)} \text { for all } s \geq 0
\end{align*}
$$

- see estimate (7.14) below. This control would match nicely with an estimate of the second factor in (3.1)

$$
\left(1-\gamma\left(G_{\epsilon}\right)\right) \mathbf{1}_{|v|^{2} \leq V_{\epsilon}} \text { in } L_{l o c}^{\infty}\left(d t d x ; L_{v}^{1}\right) ;
$$

- see Lemma 7.2. This term is obviously $O(1)$ in $L_{t, x, v}^{\infty}$; because of the second pointwise estimate in Proposition 2.2 and the second entropy control in Proposition 2.1, it is also $O\left(\epsilon^{2} e^{V_{\epsilon} / 2}\right)$ in $L_{t}^{\infty}\left(L_{x, v}^{1}\right)$, which can be taken as $o(\epsilon)$ by a suitable choice of $V_{\epsilon}$. This suggests trying the following classical argument.

Lemma 3.1. Let $\chi \in L^{1} \cap L^{\infty}\left(\mathbf{R}_{x}^{m} \times \mathbf{R}_{y}^{n}\right)$. There exists $\chi_{1} \in L_{x}^{\infty}\left(L_{y}^{1}\right)$ and $\chi_{2} \in L_{x}^{1}\left(L_{y}^{\infty}\right)$ such that $\chi=\chi_{1}+\chi_{2}$ with

$$
\left|\chi_{1}\right| \leq|\chi| \text { and }\left|\chi_{2}\right| \leq|\chi| \text { a.e., }
$$

and

$$
\left\|\chi_{1}\right\|_{L_{x}^{\infty}\left(L_{y}^{1}\right)} \leq\|\chi\|_{L^{1}}^{1 / 2}\|\chi\|_{L^{\infty}}^{1 / 2}, \quad\left\|\chi_{2}\right\|_{L_{x}^{1}\left(L_{y}^{\infty}\right)} \leq\|\chi\|_{L^{1}}^{1 / 2}\|\chi\|_{L^{\infty}}^{1 / 2} .
$$

Proof. The decomposition is obtained as

$$
\chi_{1}(x, y)=\mathbf{1}_{E}(x) \chi(x, y), \quad \chi_{2}(x, y)=\mathbf{1}_{E^{c}}(x) \chi(x, y),
$$

where, for some $\lambda>0$,

$$
E=\left\{x \in \mathbf{R}^{m}\left|\int\right| \chi(x, y) \mid d y \leq \lambda\|\chi\|_{L_{x, y}^{1}}^{1 / 2}\right\}
$$

Indeed

$$
\left\|\chi_{1}\right\|_{L_{x}^{\infty}\left(L_{y}^{1}\right)}=\sup _{x \in E} \int|\chi(x, y)| d y \leq \lambda\|\chi\|_{L_{x, y}^{1}}^{1 / 2}
$$

while

$$
\left|E^{c}\right| \leq \frac{1}{\lambda}\|\chi\|_{L_{x, y}^{1}}^{1 / 2} .
$$

Thus

$$
\left\|\chi_{2}\right\|_{L_{x}^{1}\left(L_{y}^{\infty}\right)} \leq\left|E^{c}\right| \sup _{x \in E^{c}}\|\chi(x, \cdot)\|_{L_{y}^{\infty}} \leq \frac{1}{\lambda}\|\chi\|_{L_{x, y}^{1}}^{1 / 2}\|\chi\|_{L_{x, y}^{\infty}} .
$$

Picking $\lambda=\|\chi\|_{L_{x, y}^{\infty}}^{1 / 2}$ leads to the desired result.
Applying the lemma to $\chi=\left(1-\gamma\left(G_{\epsilon}\right)\right) \mathbf{1}_{|v|^{2} \leq V_{\epsilon}}$ gives a $\chi_{1}$ that can be paired with (3.2) as described above. Handling the $\chi_{2}$ term requires a different idea presented below.
3.1. Advection/Dispersion bilinear interpolation. The Relaxation-based decomposition (2.17) improves the $L^{p}$-regularity in the variable $v$ of suitable truncations of the number density fluctuations $g_{\epsilon}$. It remains to transfer some of this extra $L^{p}$-regularity to the variable $x$. In the case where the spatial domain is the whole $\mathbf{R}^{3}$ space, doing so rests in particular on a dispersion argument originally due to C. Bardos and P. Degond [4] - see also [17]. We first recall this argument in the setting best adapted for future use in the present paper: see Proposition 1.11 of [15].
Lemma 3.2. Let $\phi^{0} \equiv \phi^{0}(t, x, v) \in L_{t}^{\infty}\left(L_{x}^{p}\left(L_{v}^{q}\right)\right)$ for some $1 \leq p<q \leq$ $+\infty$, and let $\phi \equiv \phi(\tau, t, x, v)$ be the solution of the Cauchy problem

$$
\begin{align*}
\partial_{\tau} \phi+\epsilon \partial_{t} \phi+v \cdot \nabla_{x} \phi & =0, \quad \tau>0, & t \in \mathbf{R}, \quad x, v \in \mathbf{R}^{3},  \tag{3.3}\\
\phi(0, t, x, v) & =\phi^{0}(t, x, v), & t \in \mathbf{R}, x, v \in \mathbf{R}^{3} .
\end{align*}
$$

Then, for all $\tau \in \mathbf{R}^{*}$,

$$
\|\phi(\tau, \cdot, \cdot, \cdot)\|_{L_{t}^{\infty}\left(L_{x}^{q}\left(L_{v}^{p}\right)\right)} \leq|\tau|^{-3\left(\frac{1}{p}-\frac{1}{q}\right)}\left\|\phi^{0}\right\|_{L_{t}^{\infty}\left(L_{x}^{p}\left(L_{v}^{q}\right)\right)}
$$

In the Cauchy problem (3.3) the variable $\tau$ is not the physical time variable, but an independent, fictitious time variable used below as an interpolation parameter. In other words, let

$$
U_{\tau}^{\epsilon}=e^{-\tau\left(\epsilon \partial_{t}+v \cdot \nabla_{x}\right)}
$$

be the group generated by $-\left(\epsilon \partial_{t}+v \cdot \nabla_{x}\right)$. Then, any function $\phi^{0} \equiv \phi^{0}(t, x, v)$ is decomposed into

$$
\begin{equation*}
\phi^{0}(t, x, v)=\left(U_{\tau^{*}}^{\epsilon} \phi^{0}\right)(t, x, v)+\int_{0}^{\tau^{*}}\left(\epsilon \partial_{t}+v \cdot \nabla_{x}\right)\left(U_{\tau}^{\epsilon} \phi^{0}\right)(t, x, v) d \tau \tag{3.4}
\end{equation*}
$$

In this decomposition, the first term has enhanced $L^{p}$-regularity in the variable $x$ while the second term is small with $\left|\tau^{*}\right|$ if $\phi^{0}$ belongs to the domain of $\epsilon \partial_{t}+v \cdot \nabla_{x}$. Introducing this fictitious time variable might seem somewhat unusual; yet this is in complete analogy with the classical procedure due to J.-L. Lions (see [42]) relating interpolation spaces with spaces of traces (in the sense of restrictions to some boundary) of functions taking their values in Banach spaces. These former methods would typically use elliptic operators and the domains of their fractional powers as the means of gauging the smoothness of the functions under consideration. In the present work, we use instead precisely that advection operator which appears in the scaled Boltzmann equation (1.13) and are concerned with dispersion rather than regularity properties.

Second, we write and prove a formula that will be used together with Lemma 3.2.

Lemma 3.3. Pick $\epsilon>0, t^{*}>0$, and $\tau^{*}>0$, and consider

$$
\Omega=\left\{(\tau, t, x, v) \in \mathbf{R} \times \mathbf{R} \times \mathbf{R}^{3} \times \mathbf{R}^{3} \mid 0<\tau<\tau^{*}, 0<t-\epsilon \tau<t^{*}\right\}
$$

Then, for all $f \in L_{l o c}^{1}(d t d x d v)$ such that $\left(\epsilon \partial_{t}+v \cdot \nabla_{x}\right) f$ belongs to $L_{l o c}^{1}(d t d x d v)$ and all compactly supported $\phi^{0} \in L^{\infty}(d t d x d v)$,

$$
\begin{align*}
\int_{0}^{t^{*}} \iint f \phi^{0} d t d x d v & =\int_{\epsilon \tau^{*}}^{t^{*}+\epsilon \tau^{*}} \iint f \phi\left(\tau^{*}, \cdot, \cdot, \cdot\right) d t d x d v  \tag{3.5}\\
& -\int_{\Omega} \phi(\tau, t, x, v)\left(\epsilon \partial_{t}+v \cdot \nabla_{x}\right) f(t, x, v) d \tau d t d x d v
\end{align*}
$$

where $\phi$ denotes the solution of (3.3).
The same identity (3.5) holds for any $\phi^{0}$ in $L^{\infty}(d t d x d v)$ (not necessarily compactly supported) and all $f$ in $L_{l o c}^{1}\left(d t ; L^{1}(d x d v)\right)$.

Proof. Apply Green's formula to the integral

$$
\int_{\Omega} f(t, x, v)\left(\partial_{\tau}+\epsilon \partial_{t}+v \cdot \nabla_{x}\right) \phi(\tau, t, x, v) d \tau d t d x d v=0 .
$$

Set $\phi^{0}$ to be the $\chi_{2}$ term obtained by applying Lemma 3.1 to $\chi=$ $\left(1-\gamma\left(G_{\epsilon}\right)\right) \mathbf{1}_{|v|^{2} \leq V_{\epsilon}}$; by Lemma 3.2, the resulting $\phi\left(\tau^{*}\right)$ enjoys for each $\tau^{*}>0$ the same properties as the $\chi_{1}$ term and is paired with (3.2) in the manner described above, while the streaming term is estimated by Young's inequality recalled in Appendix A and the third estimate in Proposition 2.1. This procedure eventually leads to the following crucial estimate.

Proposition 3.4 ( $\sharp g_{\epsilon}$ control). For each $\gamma \in \Upsilon$ defined by (2.3), the family $\sharp g_{\epsilon}$ satisfies the estimate

$$
\sharp g_{\epsilon}=O\left(\frac{1}{\log |\log \epsilon|}\right) \text { in } L_{l o c}^{1}\left(d t d x ; L^{1}(M d v)\right)
$$

as $\epsilon \rightarrow 0$.
This result shows a notable difference between the Flat-Sharp decomposition and the older decomposition in [7] (p. 696): here $\# g_{\epsilon} \rightarrow 0$, which means that the whole of the hydrodynamic limit is contained in the $L^{2}$ part of the Flat-Sharp decomposition - namely $g_{\epsilon}$.

Proposition 3.4 leads to an amplification of the last statement in Proposition 2.7, stated below.

Corollary 3.5. For each $\gamma \in \Upsilon$ defined by (2.3) and each $s \geq 0$, one has

$$
\left(1+|v|^{s}\right) \frac{1-\gamma\left(G_{\epsilon}\right)}{\epsilon^{2}}=O\left(\frac{1}{\sqrt{\log |\log \epsilon|}}\right) \text { in } L_{l o c}^{1}\left(d t d x ; L^{1}(M d v)\right)
$$

as $\epsilon \rightarrow 0$.
3.2. A limiting case of velocity averaging in $L^{1}$. The Advection/Dispersion bilinear interpolation procedure has another important application, leading to an improvement of the existing velocity averaging results in $L^{1}$ (namely, it extends the validity of Lemma 8 in [27] to any space dimension).

Lemma 3.6 (Local uniform integrability by velocity averaging). Consider a bounded family $f_{\epsilon}$ of $L_{l o c}^{\infty}\left(d t ; L_{l o c}^{1}(d x d v)\right)$ indexed by $\epsilon \in[0,1]$ such that $\left(\epsilon \partial_{t}+v \cdot \nabla_{x}\right) f_{\epsilon}$ is bounded in $L_{l o c}^{1}(d t d x d v)$. Suppose that $f_{\epsilon}$ is locally uniformly integrable in $v$. Then $f_{\epsilon}$ is locally uniformly integrable (in all variables $t, x$ and $v$ ).

Proof. Without loss of generality, one can assume that all the $f_{\epsilon}$ are supported in the ball of radius $R$ centered at the origin in $\mathbf{R}_{t} \times \mathbf{R}_{x}^{3} \times \mathbf{R}_{v}^{3}$. Let $B$ designate a measurable subset of that same ball.

For all $t \in \mathbf{R}$, call

$$
B_{t}=\left\{(x, v) \in \mathbf{R}^{3} \times \mathbf{R}^{3} \mid(t, x, v) \in B\right\}
$$

and

$$
B_{t, x}=\left\{v \in \mathbf{R}^{3} \mid(t, x, v) \in B\right\}
$$

Applying Lemma 3.1 twice to the indicator function of $B$ leads to the decomposition

$$
\mathbf{1}_{B}=\mathbf{1}_{B} \mathbf{1}_{B_{1}}+\mathbf{1}_{B} \mathbf{1}_{B_{2}} \mathbf{1}_{B_{21}^{t}}+\mathbf{1}_{B} \mathbf{1}_{B_{2}} \mathbf{1}_{B_{22}^{t}}
$$

where

$$
B_{1}=\left\{t \in \mathbf{R}| | B_{t}\left|>|B|^{1 / 2}\right\}, \quad B_{2}=\left\{t \in \mathbf{R}| | B_{t}\left|\leq|B|^{1 / 2}\right\}\right.\right.
$$

and, for all $t \in B_{2}$,

$$
B_{21}^{t}=\left\{x \in \mathbf{R}^{3}| | B_{t, x}\left|>|B|^{1 / 4}\right\}, \quad B_{22}^{t}=\left\{x \in \mathbf{R}^{3}| | B_{t, x}\left|\leq|B|^{1 / 4}\right\}\right.\right.
$$

Thus

$$
\begin{align*}
\left\|\mathbf{1}_{B} \mathbf{1}_{B_{1}}\right\|_{L_{t}^{1}\left(L_{x, v}^{\infty}\right)} & \leq|B|^{1 / 2}, \\
\left\|\mathbf{1}_{B} \mathbf{1}_{B_{2}} \mathbf{1}_{B_{21}^{t}}\right\|_{L_{t}^{\infty}\left(L_{x}^{1}\left(L_{v}^{\infty}\right)\right)} & \leq|B|^{1 / 4},  \tag{3.6}\\
\left\|\mathbf{1}_{B} \mathbf{1}_{B_{2}} \mathbf{1}_{B_{22}^{t}}\right\|_{L_{t, x}^{\infty}\left(L_{v}^{1}\right)} & \leq|B|^{1 / 4} .
\end{align*}
$$

Then

$$
\begin{equation*}
\iiint \mathbf{1}_{B} \mathbf{1}_{B_{1}}\left|f_{\epsilon}\right| d t d x d v \leq\left\|f_{\epsilon}\right\|_{L_{t}^{\infty}\left(L_{x, v}^{1}\right)}\left\|\mathbf{1}_{B} \mathbf{1}_{B_{1}}\right\|_{L_{t}^{1}\left(L_{x, v}^{\infty}\right)}=O\left(|B|^{1 / 2}\right) \tag{3.7}
\end{equation*}
$$

The definition of uniform integrability in $v$ can be equivalently recast as follows: for each $\eta>0$, there exists $\alpha>0$ such that, for each measurable $\chi: \mathbf{R}_{t} \times \mathbf{R}_{x}^{3} \times \mathbf{R}_{v}^{3} \rightarrow\{0,1\}$ such that $\|\chi\|_{L_{t, x}^{\infty}\left(L_{v}^{1}\right)}<\alpha$, one has

$$
\begin{equation*}
\iiint \chi\left|f_{\epsilon}\right| d t d x d v<\eta, \quad \text { uniformly in } \epsilon \tag{3.8}
\end{equation*}
$$

Let $\eta>0$ and $\alpha$ be so chosen; pick $0<\tau^{*}<1$ such that

$$
\begin{equation*}
\tau^{*}\left\|\left(\epsilon \partial_{t}+v \cdot \nabla_{x}\right) f\right\|_{L_{t, x, v}^{1}}<\eta . \tag{3.9}
\end{equation*}
$$

Assume that $B$ satisfies

$$
\begin{equation*}
|B|^{1 / 4}<\tau^{* 3} \alpha<\alpha \tag{3.10}
\end{equation*}
$$

then, by using (3.8) for $\chi=\mathbf{1}_{B} \mathbf{1}_{B_{2}} \mathbf{1}_{B_{22}^{t}}$ together with the third inequality in (3.6),

$$
\begin{equation*}
\iiint \mathbf{1}_{B} \mathbf{1}_{B_{2}} \mathbf{1}_{B_{22}^{t}}\left|f_{\epsilon}\right| d t d x d v<\eta \tag{3.11}
\end{equation*}
$$

It remains to estimate

$$
\iiint \mathbf{1}_{B} \mathbf{1}_{B_{2}} \mathbf{1}_{B_{21}^{t}}\left|f_{\epsilon}\right| d t d x d v
$$

Let $\phi^{0}=\mathbf{1}_{B} \mathbf{1}_{B_{2}} \mathbf{1}_{B_{21}^{t}}$ and $\phi$ be the solution of the Cauchy problem (3.3). Notice that $\phi$ takes its values in $\{0,1\}$. Lemma 3.2 and the second inequality in (3.6) imply that

$$
\begin{equation*}
\left\|\phi\left(\tau^{*}, \cdot, \cdot, \cdot\right)\right\|_{L_{t, x}^{\infty}\left(L_{v}^{1}\right)} \leq \frac{1}{\tau^{* 3}}\left\|\phi^{0}\right\|_{L_{t}^{\infty}\left(L_{x}^{1}\left(L_{v}^{\infty}\right)\right)} \leq \frac{|B|^{1 / 4}}{\tau^{* 3}}<\alpha \tag{3.12}
\end{equation*}
$$

by (3.10). Thus, the first term on the right-hand side of (3.5) satisfies

$$
\int_{\epsilon \tau^{*}}^{t^{*}+\epsilon \tau^{*}} \iint\left|f_{\epsilon}(t, x, v)\right| \phi\left(\tau^{*}, t, x, v\right) d t d x d v<\eta
$$

by (3.12) and (3.8). On account of this inequality, applying Lemma 3.3 with $\left|f_{\epsilon}\right|$ in place of $f$ and $\tau^{*}$ as in (3.9) shows that

$$
\begin{equation*}
\iiint \mathbf{1}_{B} \mathbf{1}_{B_{2}} \mathbf{1}_{B_{21}^{t}}\left|f_{\epsilon}\right| d t d x d v \leq \tau^{*}\left\|\left(\epsilon \partial_{t}+v \cdot \nabla_{x}\right) f\right\|_{L_{t, x, v}^{1}}+\eta<2 \eta \tag{3.13}
\end{equation*}
$$

Eventually, the three estimates (3.7), (3.11) and (3.13) show that

$$
\iiint_{B}\left|f_{\epsilon}\right| d t d x d v<O\left(\alpha^{2}\right)+3 \eta
$$

for each measurable $B \subset \mathbf{R}_{t} \times \mathbf{R}^{3} \times \mathbf{R}^{3}$ satisfying (3.10), once $\tau^{*}$ has been chosen as in (3.9). This implies that the family $f_{\epsilon}$ is locally uniformly integrable on $\mathbf{R}_{t} \times \mathbf{R}^{3} \times \mathbf{R}^{3}$ (in all variables $t, x$ and $v$ ).

One might object that the final result in Lemma 3.6 above is about $f_{\epsilon}$ itself and not about its moments in $v$, so that the reference to velocity averaging results as in [28] and [27] may seem inappropriate. However, the only nontrivial part in the proof of this result is the step appealing to Lemma 3.3, which essentially amounts to proving the local uniform integrability of $\int f_{\epsilon} d v$ (see [55] for the first observation in this direction). Besides, it has been previously noticed in various contexts that the velocity averaging method combined with additional regularity estimates in the velocity variable only gives compactness in all variables: see for instance [48], [22], and [13]. Our Lemma 3.6 can be viewed as an analogue in $L^{1}$ of these hypoellipticity results.

In fact, applying Lemma 3.3 a second time would show that, under the same assumptions on the family $f_{\epsilon}$, the family of moments $\int f_{\epsilon} \chi(v) d v$ is "strongly compact in $L_{l o c}^{1}(d t d x)$ relatively to the $x$ variable" for each $\chi \in C_{c}\left(\mathbf{R}^{3}\right)$. We refer to [30] for a precise statement and a complete proof. However, in order to establish the strong compactness of moments, we use here a slightly different argument.
Lemma 3.7. Let $f_{\epsilon}$ be a bounded family of $L_{l o c}^{2}\left(d t d x ; L^{2}(M d v)\right)$ indexed by $\epsilon \in[0,1]$ such that both families $\left|f_{\epsilon}\right|^{2}$ and $\left(\epsilon \partial_{t}+v \cdot \nabla_{x}\right) f_{\epsilon}$ are locally uniformly integrable with respect to the measure Mdvdxdt. Then, for each function $\phi \equiv \phi(v)$ in $L^{2}(M d v)$, each $t^{*}>0$ and each compact $K \subset \mathbf{R}^{3}$, there exists a function $\eta: \mathbf{R}_{+} \rightarrow \mathbf{R}_{+}$such that $\lim _{z \rightarrow 0^{+}} \eta(z)=0$ and

$$
\left\|\int f_{\epsilon}(t, x+y, v) \phi(v) M(v) d v-\int f_{\epsilon}(t, x, v) \phi(v) M(v) d v\right\|_{L^{2}\left(\left[0, t^{*}\right] \times K\right)}^{2} .
$$

for each $y \in \mathbf{R}^{3}$ such that $|y| \leq 1$, uniformly in $\epsilon \in[0,1]$.

This lemma is a minor amplification of Theorem 3 in [27] and its proof is deferred to Appendix C.

Finally, the first ${ }^{b} g_{\epsilon}$ controls (Proposition 2.7), the $\sharp g_{\epsilon}$ controls (Proposition 3.4) and both Lemmas 3.6 and 3.7 lead to the last piece of information needed about the $g_{\epsilon}$ family, stated below.

Proposition 3.8 ( $g_{\epsilon}$ controls II). Let $\gamma \in \Upsilon$ as in (2.3). Then the family ${ }_{g} g_{\epsilon}$ has the following properties

- for each compact $Q \subset \mathbf{R}_{+} \times \mathbf{R}^{3}$ and each sequence $\epsilon_{n} \rightarrow 0$, the extracted sequence $\left.\left.(t, x, v) \mapsto \mathbf{1}_{Q}(t, x)\right|^{b} g_{\epsilon_{n}}(t, x, v)\right|^{2}$ is uniformly integrable in $\mathbf{R}_{+} \times \mathbf{R}^{3} \times \mathbf{R}^{3}$ with respect to the measure dtdxMdv;
- for each sequence $\epsilon_{n} \rightarrow 0$, each function $\phi \equiv \phi(v)$ in $L^{2}(M d v)$, each $t^{*}>0$, and each compact $K \subset \mathbf{R}^{3}$, there exists a function $\eta: \mathbf{R}_{+} \rightarrow \mathbf{R}_{+}$ such that $\lim _{z \rightarrow 0^{+}} \eta(z)=0$ and

$$
\int_{0}^{t^{*}} \int_{K}\left|\left\langle{ }^{\mathrm{b}} g_{\epsilon_{n}} \phi\right\rangle(t, x+y)-\left\langle{ }^{\mathrm{b}} g_{\epsilon_{n}} \phi\right\rangle(t, x)\right|^{2} d x d t \leq \eta(|y|)
$$

for each $y \in \mathbf{R}^{3}$ such that $|y| \leq 1$, uniformly in $n$.
We close this section with an amplification of the first statement of Proposition 3.8.

Corollary 3.9. Let $\gamma \in \Upsilon$ as in (2.3). Then the family ${ }^{b} g_{\epsilon}$ has the following properties

- for each compact $Q \subset \mathbf{R}_{+} \times \mathbf{R}^{3}$ and each sequence $\epsilon_{n} \rightarrow 0$, the extracted sequence $(t, x, v) \mapsto \mathbf{1}_{Q}(t, x)\left(g_{\epsilon_{n}} \mathcal{L}^{b} g_{\epsilon_{n}}\right)(t, x, v)$ is uniformly integrable in $\mathbf{R}_{+} \times \mathbf{R}^{3} \times \mathbf{R}^{3}$ with respect to the measure $d t d x M d v$;
- for each compact $Q \subset \mathbf{R}_{+} \times \mathbf{R}^{3}$ and each sequence $\epsilon_{n} \rightarrow 0$, the extracted sequence $(t, x, v) \mapsto \mathbf{1}_{Q}(t, x) \mathcal{Q}\left({ }^{b} g_{\epsilon_{n}},{ }^{b} g_{\epsilon_{n}}\right)(t, x, v)$ is uniformly integrable in $\mathbf{R}_{+} \times \mathbf{R}^{3} \times \mathbf{R}^{3}$ with respect to the measure dtdxMdv.


## 4. Estimating conservation defects

The controls stated in Sect. 2 and proved below establish the local conservation laws of momentum and energy in the limit as $\epsilon \rightarrow 0$, by essentially the same method as in the proof of the Stokes-Fourier limit by F. Golse and C.D. Levermore [25].

Before stating the main result of the present section, we need to introduce a new class of bump functions. For each $C>0$, set

$$
\Upsilon_{C}=\left\{\gamma \in \Upsilon \mid\left\|\gamma^{\prime}\right\|_{L^{\infty}} \leq C\right\}
$$

Consider the transformation $\mathcal{T}$ defined by $\mathcal{T} \gamma=1-(1-\gamma)^{2}$; clearly $\mathcal{T}$ maps $\Upsilon_{C}$ into $\Upsilon_{2 C}$. Define

$$
\begin{equation*}
\tilde{\Upsilon}=\mathcal{T} \Upsilon_{8} \subset \Upsilon_{16} ; \quad \text { notice that } \tilde{\Upsilon} \neq \emptyset \text { since } \Upsilon_{8} \neq \emptyset \tag{4.1}
\end{equation*}
$$

For each $\gamma \in \tilde{\Upsilon}$, define

$$
\begin{equation*}
\hat{\gamma}(z)=\gamma(z)+(z-1) \frac{d \gamma}{d z}(z) \tag{4.2}
\end{equation*}
$$

Notice that

$$
\begin{equation*}
\operatorname{supp} \hat{\gamma} \subset\left[\frac{1}{2}, \frac{3}{2}\right], \quad \hat{\gamma}\left(\left[\frac{3}{4}, \frac{5}{4}\right]\right)=\{1\} \tag{4.3}
\end{equation*}
$$

On the other hand, let $\tilde{\gamma} \in \Upsilon_{8}$ be such that $\gamma=\mathcal{T} \tilde{\gamma}$ (the existence of $\tilde{\gamma}$ being guaranteed by the fact that $\gamma \in \tilde{\Upsilon}$ ). One has

$$
1-\hat{\gamma}(z)=(1-\tilde{\gamma}(z))\left[(1-\tilde{\gamma}(z))-2(z-1) \frac{d \tilde{\gamma}}{d z}(z)\right], \quad z \geq 0
$$

so that

$$
\begin{equation*}
|1-\hat{\gamma}(z)| \leq 9(1-\tilde{\gamma}(z)), \quad z \geq 0 . \tag{4.4}
\end{equation*}
$$

Proposition 4.1 (Vanishing of conservation defects). Let $\gamma \in \tilde{\Upsilon}$, and denote by $\xi \equiv \xi(v)$ any collision invariant (i.e. $\xi(v)=1$ or $\xi(v)=v_{1}, \ldots, v_{3}$ or else $\xi(v)=|v|^{2}$ ) or any linear combination thereof. Then

$$
\partial_{t}\left\langle g_{\epsilon} \xi\right\rangle+\frac{1}{\epsilon} \nabla_{x} \cdot\left\langle v{ }^{\mathrm{b}}{ }_{\epsilon} \xi\right\rangle \rightarrow 0, \text { in } L_{l o c}^{1}\left(\mathbf{R}^{+} \times \mathbf{R}^{3}\right)
$$

as $\epsilon \rightarrow 0$.
Proof. We start from the renormalized form (1.19) of the Boltzmann equation (1.13) with $\Gamma(Z)=(Z-1) \gamma(Z)$

$$
\begin{aligned}
& \left(\partial_{t}+\frac{1}{\epsilon} v \cdot \nabla_{x}\right)\left(M g_{\epsilon}\right) \\
& =\frac{1}{\epsilon^{3}} \iint\left(F_{\epsilon}^{\prime} F_{\epsilon 1}^{\prime}-F_{\epsilon} F_{\epsilon 1}\right)\left(\gamma\left(G_{\epsilon}\right)+\left(G_{\epsilon}-1\right) \frac{d \gamma}{d z}\left(G_{\epsilon}\right)\right) b d \sigma_{v, v_{1}}(\omega) d v_{1}
\end{aligned}
$$

from which we deduce that

$$
\partial_{t}\left\langle g_{\epsilon} \xi\right\rangle+\frac{1}{\epsilon} \nabla_{x} \cdot\left\langle v{ }^{\mathrm{b}}{ }_{\epsilon} \xi\right\rangle=\frac{1}{\epsilon}\left\langle\left\langle q_{\epsilon} \hat{\gamma}_{\epsilon} \xi\right\rangle\right.
$$

with the notation

$$
\hat{\gamma}_{\epsilon}=\hat{\gamma}\left(G_{\epsilon}\right),
$$

the function $\hat{\gamma}$ being defined in terms of $\gamma$ by (4.2).
In order to estimate the $L^{1}$-norm of the conservation defects, we consider the decomposition

$$
\begin{equation*}
\frac{1}{\epsilon} \|\left\langle q_{\epsilon} \hat{\gamma}_{\epsilon} \xi\right\rangle=\frac{1}{\epsilon}\left\langle\left\langle q_{\epsilon} \hat{\gamma}_{\epsilon}\left(1-\hat{\gamma}_{\epsilon 1}\right) \xi\right\rangle+\frac{1}{\epsilon}\left\langle\left\langle q_{\epsilon} \hat{\gamma}_{\epsilon} \hat{\gamma}_{\epsilon} \xi\right\rangle\right.\right. \tag{4.5}
\end{equation*}
$$

Because our choice of $\xi$ satisfies $\xi+\xi_{1}=\xi^{\prime}+\xi_{1}^{\prime}$, using the definition (1.35) and the collisional symmetry (1.11) with $f=M G_{\epsilon}$ and $\varphi=\hat{\gamma}_{\epsilon} \hat{\gamma}_{\epsilon 1} \xi \mathbf{1}_{|v|^{2}+\left|v_{1}\right|^{2}>16|\log \epsilon|}$ or $\varphi=\hat{\gamma}_{\epsilon} \hat{\gamma}_{\epsilon 1} \xi \mathbf{1}_{|v|^{2}+\left|v_{1}\right|^{2} \leq 16|\log \epsilon|}$ implies that

$$
\begin{aligned}
& \frac{1}{\epsilon}\left\langle\left\langle q_{\epsilon} \hat{\gamma}_{\epsilon} \hat{\gamma}_{\epsilon 1} \xi\right\rangle\right. \\
& \quad=\frac{1}{\epsilon}\left\langle\left\langle q_{\epsilon} \hat{\gamma}_{\epsilon} \hat{\gamma}_{\epsilon 1} \xi \mathbf{1}_{|v|^{2}+\left|v_{1}\right|^{2}>16|\log \epsilon|}\right\rangle\right\rangle+\frac{1}{\epsilon}\left\langle\left\langle q_{\epsilon} \hat{\gamma}_{\epsilon} \hat{\gamma}_{\epsilon 1} \xi \mathbf{1}_{|v|^{2}+\left|v_{1}\right|^{2} \leq 16|\log \epsilon|}\right\rangle\right\rangle \\
& \left.\quad=\frac{1}{2 \epsilon^{3}}\left\langle\left(G_{\epsilon}^{\prime} G_{\epsilon 1}^{\prime}-G_{\epsilon} G_{\epsilon 1}\right) \hat{\gamma}_{\epsilon} \hat{\gamma}_{\epsilon 1}\left(\xi+\xi_{1}\right) \mathbf{1}_{|v|^{2}+\left|v_{1}\right|^{2}>16|\log \epsilon|}\right\rangle\right\rangle \\
& \quad+\frac{1}{4 \epsilon^{3}}\left\langle\left\langle\left(G_{\epsilon}^{\prime} G_{\epsilon 1}^{\prime}-G_{\epsilon} G_{\epsilon 1}\right)\left(\hat{\gamma}_{\epsilon} \hat{\gamma}_{\epsilon 1}-\hat{\gamma}_{\epsilon}^{\prime} \hat{\gamma}_{\epsilon 1}^{\prime}\right)\left(\xi+\xi_{1}\right) \mathbf{1}_{|v|^{2}+\left|v_{1}\right|^{2} \leq 16|\log \epsilon|}\right\rangle\right\rangle .
\end{aligned}
$$

On the other hand, observe that

$$
\left(\hat{\gamma}_{\epsilon} \hat{\gamma}_{\epsilon 1}-\hat{\gamma}_{\epsilon}^{\prime} \hat{\gamma}_{\epsilon 1}^{\prime}\right)=\left(\hat{\gamma}_{\epsilon} \hat{\gamma}_{\epsilon 1}-1\right) \hat{\gamma}_{\epsilon}^{\prime} \hat{\gamma}_{\epsilon 1}^{\prime}-\left(\hat{\gamma}_{\epsilon}^{\prime} \hat{\gamma}_{\epsilon 1}^{\prime}-1\right) \hat{\gamma}_{\epsilon} \hat{\gamma}_{\epsilon 1}
$$

so that, using again (1.11) with $f=M G_{\epsilon}$ and this time with $\varphi=$ $\left(\hat{\gamma}_{\epsilon} \hat{\gamma}_{\epsilon 1}-1\right) \hat{\gamma}_{\epsilon}^{\prime} \hat{\gamma}_{\epsilon 1}^{\prime}\left(\xi+\xi_{1}\right) \mathbf{1}_{|v|^{2}+\left|v_{1}\right|^{2} \leq 16|\log \epsilon|}$, we have

$$
\begin{aligned}
& \frac{1}{4 \epsilon^{3}}\left\langle\left\langle\left(G_{\epsilon}^{\prime} G_{\epsilon 1}^{\prime}-G_{\epsilon} G_{\epsilon 1}\right)\left(\hat{\gamma}_{\epsilon} \hat{\gamma}_{\epsilon 1}-\hat{\gamma}_{\epsilon}^{\prime} \hat{\gamma}_{\epsilon 1}^{\prime}\right)\left(\xi+\xi_{1}\right) \mathbf{1}_{|v|^{2}+\left|v_{1}\right|^{2} \leq 16|\log \epsilon|}\right\rangle\right. \\
& \left.=\frac{1}{2 \epsilon^{3}}\left\langle\left(G_{\epsilon}^{\prime} G_{\epsilon 1}^{\prime}-G_{\epsilon} G_{\epsilon 1}\right)\left(\hat{\gamma}_{\epsilon} \hat{\gamma}_{\epsilon 1}-1\right) \hat{\gamma}_{\epsilon}^{\prime} \hat{\gamma}_{\epsilon 1}^{\prime}\left(\xi+\xi_{1}\right) \mathbf{1}_{|v|^{2}+\left|v_{1}\right|^{2} \leq 16|\log \epsilon|}\right\rangle\right\rangle .
\end{aligned}
$$

Therefore

$$
\begin{align*}
& \left.\left\lvert\, \partial_{t}\left\langle\left.\left\langle_{g_{\epsilon}} \xi\right\rangle+\frac{1}{\epsilon} \nabla_{x} \cdot\left\langle v v_{\epsilon} g_{\epsilon} \xi\right\rangle \right\rvert\, \leq \frac{1}{\epsilon}\langle |\right| q_{\epsilon}\left\|\hat{\gamma}_{\epsilon}\right\| 1-\hat{\gamma}_{\epsilon 1}| | \xi| \rangle\right.\right\rangle \\
& \left.\quad+\frac{1}{2 \epsilon}\left\langle\langle | q_{\epsilon}\right|\left|\hat{\gamma}_{\epsilon} \hat{\gamma}_{\epsilon 1}\right|\left(|\xi|+\left|\xi_{1}\right|\right) \mathbf{1}_{|v|^{2}+\left|v_{1}\right|^{2}>16|\log \epsilon|}\right\rangle  \tag{4.6}\\
& \left.\left.\quad+\frac{1}{2 \epsilon}\langle |\left|q_{\epsilon}\right|\left|\hat{\gamma}_{\epsilon} \hat{\gamma}_{\epsilon 1}-1\right|\left|\hat{\gamma}_{\epsilon}^{\prime} \hat{\gamma}_{\epsilon 1}^{\prime}\right|\left(|\xi|+\left|\xi_{1}\right|\right) \mathbf{1}_{|v|^{2}+\left|v_{1}\right|^{2} \leq 16|\log \epsilon|}\right\rangle\right\rangle \\
& \quad=I_{1}+I_{2}+I_{3} .
\end{align*}
$$

For any $\eta \geq \epsilon$, Young's inequality (10.4) together with the elementary properties of $h$ and $r$ recalled in (10.1) and (10.3) (see Appendix A below) imply that

$$
\begin{align*}
\frac{1}{\epsilon}\left|q_{\epsilon}\right||\xi(v)| & \leq \frac{4 \eta}{\epsilon^{4}} G_{\epsilon} G_{\epsilon 1}\left[h\left(\frac{\epsilon^{2} q_{\epsilon}}{G_{\epsilon} G_{\epsilon 1}}\right)+h^{*}\left(\frac{\epsilon}{4 \eta}|\xi(v)|\right)\right]  \tag{4.7}\\
& \leq \frac{4 \eta}{\epsilon^{4}} G_{\epsilon} G_{\epsilon 1} r\left(\frac{\epsilon^{2} q_{\epsilon}}{G_{\epsilon} G_{\epsilon 1}}\right)+\frac{4}{\epsilon^{2} \eta} G_{\epsilon} G_{\epsilon 1} e^{|\xi(v)| / 4}
\end{align*}
$$

Then, using (2.7) and the elementary bounds

$$
\begin{equation*}
\left|\hat{\gamma}_{\epsilon}\right| \leq 9, \quad\left|1-\hat{\gamma}_{\epsilon}\right| \leq 9, \quad 0 \leq G_{\epsilon}\left|\hat{\gamma}_{\epsilon}\right| \leq \frac{27}{2} \tag{4.8}
\end{equation*}
$$

leads to

$$
\begin{aligned}
& \left\|I_{1}\right\|_{L^{1}\left(\left[0, t^{*}\right] \times Q\right)} \leq 16 \cdot 9^{2} \cdot \eta \int_{0}^{t^{*}} \int_{Q} \int \frac{D\left(F_{\epsilon}\right)}{\epsilon^{4}} d v d x d t \\
& \left.\quad+\frac{4}{\eta} \int_{0}^{t^{*}} \int_{Q} \|\left\langle G_{\epsilon} G_{\epsilon 1}\right| \hat{\gamma}_{\epsilon}\left|\frac{\left|1-\hat{\gamma}_{\epsilon 1}\right|}{\epsilon^{2}} e^{|\xi(v)| / 4}\right\rangle\right\rangle d x d t \\
& \quad \leq 16 \cdot 9^{2} \cdot C^{i n} \eta+\frac{4 \cdot \frac{27}{2} \cdot b_{\infty}\left\langle e^{|\xi| / 4}\right\rangle}{\eta} \int_{0}^{t^{*}} \int_{Q}\left\langle\frac{\left|1-\hat{\gamma}_{\epsilon}\right|}{\epsilon^{2}} G_{\epsilon}\right\rangle d x d t \\
& \quad \leq 16 \cdot 9^{2} \cdot C^{i n} \eta+\frac{4 \cdot \frac{27}{2} \cdot 5 \cdot b_{\infty}\left\langle e^{|\xi| / 4}\right\rangle}{\eta} \int_{0}^{t^{*}} \int_{Q}\left\langle\frac{1}{\epsilon^{2}}\right| 1-\hat{\gamma}_{\epsilon}| | G_{\epsilon}-1| \rangle d x d t
\end{aligned}
$$

where the last inequality follows from the same argument as in the proof of the third pointwise control in Proposition 2.2. Because of the inequality (4.4), one has

$$
\left\langle\frac{1}{\epsilon^{2}}\right| 1-\hat{\gamma}_{\epsilon}| | G_{\epsilon}-1| \rangle \leq 9\left\langle\frac{1}{\epsilon^{2}}\left(1-\tilde{\gamma}_{\epsilon}\right)\right| G_{\epsilon}-1| \rangle
$$

so that, by applying Proposition 3.4 with the bump function $\tilde{\gamma} \in \Upsilon_{8}$ and by choosing $\eta=1 / \sqrt{\log |\log \epsilon|}$ we get

$$
\begin{equation*}
\left\|I_{1}\right\|_{L^{1}\left(\left[0, t^{*}\right] \times Q\right)} \leq \frac{C}{\sqrt{\log |\log \epsilon|}} \tag{4.9}
\end{equation*}
$$

Next we estimate the term $I_{2}$ defined in (4.6). Replacing $|\xi(v)|$ with $|\xi(v)|+\left|\xi\left(v_{1}\right)\right|$ in Young's inequality (4.7) above and using again the bounds (4.8) implies, by integrating first in $\omega$, that

$$
\begin{align*}
\left\|I_{2}\right\|_{L^{1}\left(\left[0, t^{*}\right] \times Q\right)} & \leq 8 \cdot 9^{2} \cdot C^{i n} \eta+\frac{2}{\eta \epsilon^{2}}  \tag{4.10}\\
& \left.\times \int_{0}^{t^{*}} \int_{Q}\left\langle\left\langle G_{\epsilon} G_{\epsilon 1}\right| \hat{\gamma}_{\epsilon} \| \hat{\gamma}_{\epsilon 1} \mid e^{\left(|\xi|+\left|\xi_{1}\right|\right) / 4} \mathbf{1}_{|v|^{2}+\left|v_{1}\right|^{2} \geq 16|\log \epsilon|}\right\rangle\right) d x d t \\
& \leq 8 \cdot 9^{2} \cdot C^{i n} \eta+\frac{2 \cdot\left(\frac{27}{2}\right)^{2} \cdot b_{\infty}}{\eta \epsilon^{2}} \\
& \times \int_{0}^{t^{*}} \int_{Q} \iint e^{\left(|\xi|+\left|\xi_{1}\right|\right) / 4} \mathbf{1}_{|v|^{2}+\left|v_{1}\right|^{2} \geq 16|\log \epsilon|^{*} M M_{1} d v d v_{1} d x d t} \\
& \leq O(\eta)+\frac{1}{\eta \epsilon^{2}} O\left(\epsilon^{4}|\log \epsilon|^{2}\right)=O\left(\epsilon|\log \epsilon|^{2}\right)
\end{align*}
$$

by choosing this time $\eta=\epsilon$.

It remains to estimate the term $I_{3}$ in (4.6). Using the decomposition (2.9) of the scaled collision integrand and once again the bounds (4.8) shows that

$$
\begin{align*}
& \left.\left.I_{3} \leq \frac{9^{2} \cdot\left(1+9^{2}\right)}{2 \epsilon^{3}} \|\left\langle\epsilon^{4}\right|{ }^{\sharp} q_{\epsilon}\left|\left(\left|\xi^{\prime}\right|+\mid \xi_{1}^{\prime}\right)\right|\right) \mathbf{1}_{\left|v^{\prime}\right|^{2}+\left|v_{1}^{\prime}\right|^{2} \leq 16|\log \epsilon|}\right\rangle  \tag{4.11}\\
& \left.\left.+\frac{1}{2 \epsilon^{3}}\left\langle\left.\epsilon^{2}\right|^{b} q_{\epsilon} \| \hat{\gamma}_{\epsilon} \hat{\gamma}_{\epsilon 1}\right|\left|1-\hat{\gamma}_{\epsilon}^{\prime} \hat{\gamma}_{\epsilon 1}^{\prime}\right|\left(\left|\xi^{\prime}\right|+\left|\xi_{1}^{\prime}\right|\right) \mathbf{1}_{\left|v^{\prime}\right|^{2}+\left|v_{1}^{\prime}\right|^{2} \leq 16|\log \epsilon|}\right\rangle\right) .
\end{align*}
$$

(Observe that we have exchanged the primed and unprimed variables in the integral defining $I_{3}$, on account of (1.9) and the second equality in (1.8)). Denote by $I_{3}^{1}$ and $I_{3}^{2}$ the two terms on the right-hand side of (4.11). The second statement in Proposition 2.3 shows that, for each $t^{*}>0$ and each compact subset $Q$ of $\mathbf{R}^{3}$, one has

$$
\begin{equation*}
\left\|I_{3}^{1}\right\|_{L^{1}\left(\left[0, t^{*}\right] \times Q\right)} \leq C \epsilon|\log \epsilon| \tag{4.12}
\end{equation*}
$$

By the Cauchy-Schwarz inequality,

$$
\begin{aligned}
& \left\|I_{3}^{2}\right\|_{L^{1}\left(\left[0, t^{*}\right] \times Q\right)} \leq\| \|\left\langle\frac{\left|{ }^{b} q_{\epsilon}\right|^{2}}{G_{\epsilon} G_{\epsilon 1}}\right\rangle\| \|_{L^{1}\left(\left[0, t^{*}\right] \times Q\right)}^{1 / 2}\left\|\frac{1}{4 \epsilon^{2}}\right\| G_{\epsilon} G_{\epsilon 1}\left|\hat{\gamma}_{\epsilon} \hat{\gamma}_{\epsilon 1}\right|^{2} \\
& \left.\left.\quad \times\left|1-\hat{\gamma}_{\epsilon}^{\prime} \hat{\gamma}_{\epsilon 1}^{\prime}\right|^{2}\left(\left|\xi^{\prime}\right|+\left|\xi_{1}^{\prime}\right|\right)^{2} \mathbf{1}_{\left|v^{\prime}\right|^{2}+\left|v_{1}^{\prime}\right|^{2} \leq 16|\log \epsilon|}\right\rangle\right\rangle \|_{L^{1}\left(\left[0, t^{*}\right] \times Q\right)}^{1 / 2}
\end{aligned}
$$

This inequality, together with the first estimate in Proposition 2.3, the third inequality in (4.8), and the formula

$$
1-\hat{\gamma}^{\prime} \hat{\gamma}_{1}^{\prime}=1-\hat{\gamma}^{\prime}+\hat{\gamma}^{\prime}\left(1-\hat{\gamma}_{1}^{\prime}\right)
$$

imply that

$$
\begin{align*}
& \left\|I_{3}^{2}\right\|_{L^{1}\left(\left[0, t^{*}\right] \times Q\right)} \\
& \quad \leq 9^{2} \cdot\left(\frac{27}{2}\right)^{2} \cdot \frac{2 \sqrt{C^{\text {in }}}}{\sqrt{c}} \| \frac{1}{4 \epsilon^{2}}\left\langle\left(\left(\left|1-\hat{\gamma}_{\epsilon}^{\prime}\right|+9\left|1-\hat{\gamma}_{\epsilon 1}^{\prime}\right|\right)^{2}\left(\left|\xi^{\prime}\right|+\left|\xi_{1}^{\prime}\right|\right)^{2}\right\rangle \|_{L^{1}\left(\left[0, t^{*}\right] \times Q\right)}^{1 / 2}\right. \tag{4.13}
\end{align*}
$$

$$
\begin{aligned}
& \leq C\left\|\frac{1}{\epsilon^{2}}\langle | 1-\hat{\gamma}_{\epsilon}\left|\left(1+|v|^{4}\right)\right\rangle\right\|_{L^{1}\left(\left[0, t^{*}\right] \times Q\right)}^{1 / 2} \\
& \leq C\left\|\frac{9}{\epsilon^{2}}\left\langle\left(1-\tilde{\gamma}\left(G_{\epsilon}\right)\right)\left(1+|v|^{4}\right)\right\rangle\right\|_{L^{1}\left(\left[0, t^{*}\right] \times Q\right)}^{1 / 2} \leq \frac{C}{(\log |\log \epsilon|)^{1 / 4}}
\end{aligned}
$$

for some constant $C>0$, where the penultimate inequality is based on (4.4) and the last inequality follows from Corollary 3.5 applied to the bump function $\tilde{\gamma} \in \Upsilon_{8}$. (The constant $C^{i n}$ is that which appears in (1.44), while the constant $c$ is defined in (2.5)).

Combining estimates (4.9), (4.10), (4.12) and (4.13) gives the expected convergence.

## 5. Proof of the weak Navier-Stokes limit theorem

Throughout this section, it is assumed that the bump function $\gamma$ belongs to $\tilde{\Upsilon}$ (defined by (4.1)). Using Proposition 4.1, the classical Sobolev embedding theorems, and the continuity of pseudo-differential operators of order 0 on $W^{s, p}$ for $1<p<+\infty$, one sees that, for all $s>0$

$$
\begin{equation*}
\partial_{t} P\left\langle v g_{\epsilon}^{\mathrm{b}}\right\rangle+P \nabla_{x} \cdot \frac{1}{\epsilon}\left\langle\left(v^{\otimes 2}-\frac{1}{3}|v|^{2} I\right)^{\mathrm{b}} g_{\epsilon}\right\rangle \rightarrow 0 \text { in } L_{l o c}^{1}\left(d t ; W_{l o c}^{-s, 1}\left(\mathbf{R}^{3}\right)\right), \tag{5.1}
\end{equation*}
$$

and

$$
\begin{equation*}
\left.\partial_{t}\left(\left(\frac{1}{5}|v|^{2}-1\right){ }^{\mathrm{b}} g_{\epsilon}\right\rangle+\nabla_{x} \cdot \frac{1}{\epsilon}\left\langle v\left(\frac{1}{5}|v|^{2}-1\right)\right)^{\mathrm{b}} g_{\epsilon}\right\rangle \rightarrow 0 \text { in } L_{l o c}^{1}(d t d x), \tag{5.2}
\end{equation*}
$$

as $\epsilon \rightarrow 0$. (We recall that $P$ is the Leray projection, i.e. the $L^{2}(d x)-$ orthogonal projection on the space of divergence-free vector fields).

By Proposition 2.1 and Theorem 11.1 in Appendix B below, pick any sequence $\epsilon_{n} \rightarrow 0$ such that

$$
\begin{align*}
b_{\epsilon_{n}} & \rightarrow g \text { in } w^{*}-L_{t}^{\infty}\left(L^{2}(M d v d x)\right) \\
\gamma_{\epsilon_{n}} q_{\epsilon_{n}} & \rightarrow q \text { in } w-L_{l o c}^{1}\left(d t d x ; L^{1}\left(\left(1+|v|^{2}\right) d \mu\right)\right) \tag{5.3}
\end{align*}
$$

In this section, we deal exclusively with such extracted sequences, drop the index $n$ and abuse the notations $g_{\epsilon},{ }^{b} g_{\epsilon}, \sharp g_{\epsilon}, q_{\epsilon}$ and so on to designate the subsequences $g_{\epsilon_{n}}, g_{\epsilon_{n}}, \sharp g_{\epsilon_{n}}$ and $q_{\epsilon_{n}}$. Call $u$ and $\theta$ the limiting (fluctuations of) velocity and temperature fields defined by

$$
\begin{equation*}
\left\langle v^{\mathrm{b}} g_{\epsilon}\right\rangle \rightarrow u, \quad\left\langle\left(\frac{1}{3}|v|^{2}-1\right)^{\mathrm{b}} g_{\epsilon}\right\rangle \rightarrow \theta \text { in } w^{*}-L_{t}^{\infty}\left(L_{x}^{2}\right) . \tag{5.4}
\end{equation*}
$$

The second entropy control in Proposition 2.1 implies that ${ }^{b} g_{\epsilon}$ and $g_{\epsilon}$ have the same limit $g$ in $w-L_{l o c}^{1}\left(d t d x ; L^{1}(M d v)\right)$; hence the Boussinesq relation (11.3) and the incompressibility condition (11.2) hold:

$$
\begin{equation*}
\nabla_{x} \cdot u=0, \quad \theta+\langle g\rangle=0 \tag{5.5}
\end{equation*}
$$

(see Theorem 11.1 in Appendix B).
Denote by $\zeta$ either the tensor $A$ or the vector $B$ defined in (1.42). By Proposition $1.4, \mathcal{L}$ is self-adjoint on $L^{2}(M d v)$ so that

$$
\begin{align*}
\frac{1}{\epsilon}\left\langle(\mathscr{L} \zeta){ }^{\mathrm{b}} \mathrm{~g}_{\epsilon}\right\rangle & =\frac{1}{\epsilon}\left\langle\zeta\left(\mathcal{L}^{\mathrm{b}} g_{\epsilon}\right)\right\rangle=\frac{1}{\epsilon}\left\langle\left\langle\zeta\left({ }^{\mathrm{b}} g_{\epsilon}+{ }^{\mathrm{b}} g_{\epsilon 1}-{ }^{\mathrm{b}} g_{\epsilon}^{\prime}-{ }^{\mathrm{b}} g_{\epsilon 1}^{\prime}\right)\right\rangle\right. \\
& =\left\langle\left\langle\zeta\left[\frac{1}{\epsilon}\left({ }^{b} g_{\epsilon}+{ }^{\mathrm{b}} g_{\epsilon 1}-{ }^{\mathrm{b}} g_{\epsilon}^{\prime}-{ }^{\mathrm{b}} g_{\epsilon 1}^{\prime}\right)+\left({ }^{\mathrm{b}} g_{\epsilon}{ }^{\mathrm{b}} g_{\epsilon 1}-{ }^{\mathrm{b}} g_{\epsilon}^{\prime}{ }^{\mathrm{b}} g_{\epsilon 1}^{\prime}\right)\right]\right\rangle\right.  \tag{5.6}\\
& +\left\langle\zeta \mathcal{Q}\left({ }^{\mathrm{b}} g_{\epsilon},{ }^{\mathrm{b}} g_{\epsilon}\right)\right\rangle .
\end{align*}
$$

The first term in the (last) right hand side of (5.6) converges to the diffusion term while the second term converges to the convection term in the Navier-Stokes-Fourier system. These limits are analyzed in detail in the next subsections.
5.1. The diffusion term. The convergence to the diffusion term is obtained by an argument that closely follows [25], except that the present work uses the Flat-Sharp decomposition (2.4) instead of the decomposition introduced on p. 696 of [7] as in [25]. This apparently minor difference makes our analysis slightly more difficult than that in [25].

## Proposition 5.1. Define

$$
v=\frac{1}{10}\langle A: \mathscr{L} A\rangle, \quad \kappa=\frac{2}{15}\langle B \cdot \mathscr{L} B\rangle .
$$

Then, as $\epsilon \rightarrow 0$,

$$
\begin{aligned}
& \left.\frac{1}{\epsilon}\langle(\mathcal{L} A))_{\epsilon}\right\rangle-\left\langle A Q\left(g_{\epsilon}, g_{\epsilon}\right)\right\rangle \rightarrow-v\left(\nabla_{x} u+\left(\nabla_{x} u\right)^{T}\right) ; \\
& \frac{1}{\epsilon}\left\langle(\mathscr{L} B) g_{\epsilon}\right\rangle-\left\langle B Q\left(g_{\epsilon}, g_{\epsilon}\right)\right\rangle \rightarrow-\frac{5}{2} \kappa \nabla_{x} \theta
\end{aligned}
$$

in $w-L_{l o c}^{1}(d t d x)$.
Proof. The convergence to the diffusion term depends upon a careful analysis of the integrand appearing in the first term of the (last) right-hand side in (5.6) that involves the scaled collision integrand

$$
\begin{aligned}
q_{\epsilon} & =\frac{1}{\epsilon^{2}}\left(G_{\epsilon}^{\prime} G_{\epsilon 1}^{\prime}-G_{\epsilon} G_{\epsilon 1}\right) \\
& =\frac{1}{\epsilon}\left(g_{\epsilon}^{\prime}+g_{\epsilon 1}^{\prime}-g_{\epsilon}-g_{\epsilon 1}\right)+\left(g_{\epsilon}^{\prime} g_{\epsilon 1}^{\prime}-g_{\epsilon} g_{\epsilon 1}\right) .
\end{aligned}
$$

Indeed consider the decomposition of the integrand in (5.6) as

$$
\begin{align*}
& {\left[\frac{1}{\epsilon}\left(g_{\epsilon}+g_{\epsilon 1}-g_{\epsilon}^{\prime}-g_{\epsilon 1}^{\prime}\right)+\left(g_{\epsilon}{ }^{b} g_{\epsilon 1}-g_{\epsilon}^{\prime} g_{\epsilon 1}^{\prime} g_{\epsilon 1}^{\prime}\right)\right]} \\
& \quad=-q_{\epsilon} \gamma_{\epsilon} \gamma_{\epsilon 1} \gamma_{\epsilon}^{\prime} \gamma_{\epsilon 1}^{\prime} \\
& \quad+\left[g_{\epsilon} g_{\epsilon 1}\left(1-\gamma_{\epsilon}^{\prime} \gamma_{\epsilon 1}^{\prime}\right)-g_{\epsilon}^{\prime}{ }_{\epsilon}^{\prime} g_{\epsilon 1}^{\prime}\left(1-\gamma_{\epsilon} \gamma_{\epsilon 1}\right)\right]  \tag{5.7}\\
& \quad+\frac{1}{\epsilon}\left[g_{\epsilon}+g_{\epsilon 1}-g_{g_{\epsilon}^{\prime}}-g_{\epsilon 1}^{\prime}-\gamma_{\epsilon} \gamma_{\epsilon 1} \gamma_{\epsilon}^{\prime} \gamma_{\epsilon 1}^{\prime}\left(g_{\epsilon}+g_{\epsilon 1}-g_{\epsilon}^{\prime}-g_{\epsilon 1}^{\prime}\right)\right] \\
& \quad=I_{1}+I_{2}+I_{3} .
\end{align*}
$$

Notice that this decomposition is slightly more complicated than its analogue in [25] (formula (10.6) there). In particular, the analogue of the normalizing factor $\gamma_{\epsilon}$ in that work is an affine function of $g_{\epsilon}$. This accounts for additional cancellations in (5.7).

Step 1: controlling $I_{1}$. By Theorem 11.1 in Appendix B below,

$$
q_{\epsilon} \gamma_{\epsilon} \rightarrow q \text { in } w-L_{l o c}^{1}\left(d t d x ; L^{1}(d \mu)\right) .
$$

On the other hand, each term in the product $\gamma_{\epsilon}^{\prime} \gamma_{\epsilon 1} \gamma_{\epsilon 1}^{\prime}$ is bounded by 1 in $L^{\infty}(d t d x d \mu)$. Hence the decomposition

$$
1-\gamma_{\epsilon}^{\prime} \gamma_{\epsilon 1} \gamma_{\epsilon 1}^{\prime}=1-\gamma_{\epsilon}^{\prime}+\gamma_{\epsilon}^{\prime}\left(1-\gamma_{\epsilon 1}^{\prime}\right)+\gamma_{\epsilon}^{\prime} \gamma_{\epsilon 1}^{\prime}\left(1-\gamma_{\epsilon 1}\right)
$$

implies that
$0 \leq 1-\gamma_{\epsilon}^{\prime} \gamma_{\epsilon 1} \gamma_{\epsilon 1}^{\prime} \leq\left(1-\gamma_{\epsilon}^{\prime}\right)+\left(1-\gamma_{\epsilon 1}^{\prime}\right)+\left(1-\gamma_{\epsilon 1}\right) \leq 4 \epsilon^{2}\left(\left.\right|^{\sharp} g_{\epsilon}^{\prime}\left|+\left.\right|^{\sharp} g_{\epsilon 1}^{\prime}\right|+\left.\right|^{\sharp} g_{\epsilon 1} \mid\right)$,
because of the second pointwise estimate in Proposition 2.2. Using the second entropy control in Proposition 2.1 with the symmetries (1.8) and (1.9) shows that
$\left.\left.\left.\left.\langle |\left|1-\gamma_{\epsilon}^{\prime} \gamma_{\epsilon 1} \gamma_{\epsilon 1}^{\prime}\right|\right\rangle\right\rangle \leq 4 \epsilon^{2}\langle |\left|g_{\epsilon}^{\prime}\right|+\left.\right|^{\sharp} g_{\epsilon 1}^{\prime}\left|+\left.\right|^{\sharp} g_{\epsilon \epsilon}\right|\right\rangle\right\rangle \leq 12 \epsilon^{2} b_{\infty}\| \| g_{\epsilon} \|_{L^{1}(M d v)} \rightarrow 0$
in $L_{l o c}^{1}(d t d x)$ as $\epsilon \rightarrow 0$. Hence the family $\gamma_{\epsilon}^{\prime} \gamma_{\epsilon 1} \gamma_{\epsilon 1}^{\prime}$ is bounded (by 1 ) in $L^{\infty}(d t d x d \mu)$ and converges in measure to 1 on compacts sets in $\mathbf{R}_{+} \times \mathbf{R}^{3} \times$ $\mathbf{R}^{3} \times \mathbf{R}^{3} \times \mathbf{S}^{2}$ as $\epsilon \rightarrow 0$. By Lemma 14.1 in Appendix E below,

$$
\begin{equation*}
q_{\epsilon} \gamma_{\epsilon} \gamma_{\epsilon 1} \gamma_{\epsilon}^{\prime} \gamma_{\epsilon 1}^{\prime} \rightarrow q \text { in } w-L_{l o c}^{1}\left(d t d x ; L^{1}(d \mu)\right) . \tag{5.8}
\end{equation*}
$$

We claim that in fact

$$
\begin{equation*}
q_{\epsilon} \gamma_{\epsilon} \gamma_{\epsilon 1} \gamma_{\epsilon}^{\prime} \gamma_{\epsilon 1}^{\prime} \rightarrow q \text { in } w-L_{l o c}^{2}\left(d t d x ; L^{2}(d \mu)\right) . \tag{5.9}
\end{equation*}
$$

Indeed, unless $\gamma_{\epsilon} \gamma_{\epsilon 1} \gamma_{\epsilon}^{\prime} \gamma_{\epsilon 1}^{\prime}=0$, one has

$$
\frac{1}{4} \leq G_{\epsilon} G_{\epsilon 1} \leq \frac{9}{4}, \quad-\frac{8}{9} \leq \frac{G_{\epsilon}^{\prime} G_{\epsilon 1}^{\prime}}{G_{\epsilon} G_{\epsilon 1}}-1 \leq 8 ;
$$

then, the third statement of Proposition 2.3 with $E=\left[-\frac{8}{9}, 8\right]$ implies that

$$
\left.\|\left(q_{\epsilon} \gamma_{\epsilon} \gamma_{\epsilon}^{\prime} \gamma_{\epsilon 1} \gamma_{\epsilon 1}^{\prime}\right)^{2}\right\rangle \leq \frac{9}{4}\left\langle\left\langle\frac{q_{\epsilon}^{2}}{G_{\epsilon} G_{\epsilon 1}} \mathbf{1}_{E}\left(\frac{G_{\epsilon}^{\prime} G_{\epsilon 1}^{\prime}}{G_{\epsilon} G_{\epsilon 1}}-1\right)\right\rangle=O(1)_{L_{t, x}^{1}} .\right.
$$

This bound together with the $w-L^{1}$ convergence (5.8) imply (5.9). In particular, $\zeta=A$ or $\zeta=B$ belongs to $L^{2}(M d v)$ in either case (see (1.42) above), so that the $w-L^{2}$ convergence (5.9) implies that

$$
\begin{equation*}
\left\langle\zeta q_{\epsilon} \gamma_{\epsilon} \gamma_{\epsilon}^{\prime} \gamma_{\epsilon 1} \gamma_{\epsilon 1}^{\prime}\right\rangle \rightarrow\left\langle\langle\zeta q\rangle=\left\langle\zeta v \cdot \nabla_{x} g\right\rangle \text { in } w-L_{l o c}^{2}(d t d x) .\right. \tag{5.10}
\end{equation*}
$$

The last equality in (5.10) follows from the limiting Boltzmann equation (11.6) in Theorem 11.1 (see Appendix B).
Step 2: controlling $I_{2}$ and $I_{3}$. The first pointwise estimate in Proposition 2.2 and the obvious formula

$$
\begin{aligned}
1-\gamma_{\epsilon 1}^{\prime} \gamma_{\epsilon}^{\prime} & =\left(1-\gamma_{\epsilon}^{\prime}\right)+\gamma_{\epsilon}^{\prime}\left(1-\gamma_{\epsilon 1}^{\prime}\right) \\
& \leq\left(1-\gamma_{\epsilon}^{\prime}\right)+\left(1-\gamma_{\epsilon 1}^{\prime}\right)
\end{aligned}
$$

imply that

$$
\left.\left.\left.\left.\langle\langle | \zeta|\left|{ }^{\mathrm{b}}{ }_{\epsilon}{ }^{\mathrm{b}} g_{\epsilon 1}\right|\left(1-\gamma_{\epsilon}^{\prime} \gamma_{\epsilon 1}^{\prime}\right)\right\rangle\right\rangle \leq \frac{1}{4 \epsilon^{2}}\langle ||\zeta|\left(1-\gamma_{\epsilon}^{\prime}+1-\gamma_{\epsilon 1}^{\prime}\right)\right\rangle\right\rangle .
$$

By assumption (H2) and relation (1.4), there exists $p \geq 0$ and $C_{p}>0$ such that, for all $\left(v, v_{1}, \omega\right)$ in $\mathbf{R}^{3} \times \mathbf{R}^{3} \times \mathbf{S}^{2}$

$$
|\zeta(v)| \leq C_{p}\left(1+\left|v^{\prime}\right|^{p}+\left|v_{1}^{\prime}\right|^{p}\right) \text { for each }\left(v, v_{1}, \omega\right) \in \mathbf{R}^{3} \times \mathbf{R}^{3} \times \mathbf{S}^{2}
$$

Thus, by using the $\left(v, v_{1}\right)-\left(v^{\prime}, v_{1}^{\prime}\right)$ symmetry (1.8) and (1.9), we have

$$
\begin{align*}
\left.\langle | \zeta\left|\left.\right|^{b} g_{\epsilon}{ }^{b} g_{\epsilon 1}\right|\left(1-\gamma_{\epsilon}^{\prime} \gamma_{\epsilon 1}^{\prime}\right)\right\rangle & \leq \frac{C_{p}}{4 \epsilon^{2}}\left\langle\left\langle\left(1+\left|v^{\prime}\right|^{p}+\left|v_{1}^{\prime}\right|^{p}\right)\left(1-\gamma_{\epsilon}^{\prime}+1-\gamma_{\epsilon 1}^{\prime}\right)\right\rangle\right.  \tag{5.11}\\
& \left.\leq\left.\frac{b_{\infty} C_{p}}{2 \epsilon^{2}}\langle 1+| v\right|^{p}\right\rangle\left\langle\left(1+|v|^{p}\right)\left(1-\gamma_{\epsilon}\right)\right\rangle
\end{align*}
$$

The term $I_{3}$ in the right-hand side of (5.7) can be recast as

$$
\begin{aligned}
\frac{1}{\epsilon}\left({ }^{b_{\epsilon}}\left(1-\gamma_{\epsilon 1} \gamma_{\epsilon}^{\prime} \gamma_{\epsilon 1}^{\prime}\right)\right. & +{ }^{b} g_{\epsilon 1}\left(1-\gamma_{\epsilon} \gamma_{\epsilon}^{\prime} \gamma_{\epsilon 1}^{\prime}\right) \\
& \left.-g_{\epsilon}^{\prime}\left(1-\gamma_{\epsilon} \gamma_{\epsilon 1} \gamma_{\epsilon 1}^{\prime}\right)-{ }^{\mathrm{b}} g_{\epsilon 1}^{\prime}\left(1-\gamma_{\epsilon} \gamma_{\epsilon 1} \gamma_{\epsilon}^{\prime}\right)\right)
\end{aligned}
$$

Then, proceeding as in (5.11) leads to
(5.12)

$$
\begin{aligned}
& \left.\left|\langle\langle | \zeta| \frac{1}{\epsilon}\right|{ }^{p} g_{\epsilon}\left|\left(1-\gamma_{\epsilon 1} \gamma_{\epsilon}^{\prime} \gamma_{\epsilon 1}^{\prime}\right)\right\rangle\right\rangle \mid \\
& \quad \leq \frac{C_{p}}{2 \epsilon^{2}}\left\langle\left\langle\left(1+\left|v^{\prime}\right|^{p}+\left|v_{1}^{\prime}\right|^{p}\right)\left(1-\gamma_{\epsilon}^{\prime}+1-\gamma_{\epsilon 1}^{\prime}\right)\right\rangle\right\rangle+\frac{1}{2 \epsilon^{2}}\left\langle\langle | \zeta \mid\left(1-\gamma_{\epsilon 1}\right)\right\rangle \\
& \left.\quad \leq\left.\frac{b_{\infty} C_{p}}{\epsilon^{2}}\langle 1+| v\right|^{p}\right\rangle\left\langle\left(1+|v|^{p}\right)\left(1-\gamma_{\epsilon}\right)\right\rangle+\frac{b_{\infty}}{2 \epsilon^{2}}\langle | \zeta| \rangle\left\langle 1-\gamma_{\epsilon}\right\rangle .
\end{aligned}
$$

Both inequalities (5.11) and (5.12) (and those deduced from (5.11)(5.12) by the $v-v_{1}$ and the $\left(v, v_{1}\right)-\left(v^{\prime}, v_{1}^{\prime}\right)$ symmetries (1.7), (1.8) and (1.9)) with Corollary 3.5 imply that

$$
\begin{equation*}
\left.\left.\left\langle\langle | I_{2} \mid\right\rangle\right\rangle \rightarrow 0 \text { and }\left\langle\langle | I_{3} \mid\right\rangle\right\rangle \rightarrow 0 \text { in } L_{l o c}^{1}(d t d x) \text { as } \epsilon \rightarrow 0 . \tag{5.13}
\end{equation*}
$$

Conclusion. Using both convergences (5.10) and (5.13) in formula (5.7) implies that

$$
\left\langle\left\langle\zeta\left[\frac{1}{\epsilon}\left({ }^{b} g_{\epsilon}+{ }^{b} g_{\epsilon 1}-{ }^{b} g_{\epsilon}^{\prime}-{ }^{g_{\epsilon 1}^{\prime}}\right)+\left({ }^{b} g_{\epsilon} g_{\epsilon 1}-{ }^{b} g_{\epsilon}^{\prime}{ }^{\mathrm{g}} g_{\epsilon 1}^{\prime}\right)\right] \| \rightarrow-\left\langle\zeta v \cdot \nabla_{x} g\right\rangle\right.\right.
$$

in $w-L_{l o c}^{1}(d t d x)$ as $\epsilon \rightarrow 0$. Because of (5.6) this is equivalent to

$$
\frac{1}{\epsilon}\left\langle(\mathscr{L} \zeta) \stackrel{b}{g}_{\epsilon}\right\rangle-\left\langle\zeta \mathcal{Q}\left(\stackrel{b}{g}_{\epsilon}, \stackrel{b}{g}_{\epsilon}\right)\right\rangle \rightarrow-\left\langle\zeta v \cdot \nabla_{x} g\right\rangle
$$

in $w-L_{l o c}^{1}(d t d x)$ as $\epsilon \rightarrow 0$. Using the limiting form of the number density fluctuation (see (11.1) below in Appendix B) leads precisely to the statement of Proposition 5.1.
5.2. The convection term. The convection term is the nonlinear part of the limiting system and its convergence is therefore the most difficult to establish. The analysis below rests not only on all the previous sections but also on the arguments in [7] and [50].

Proposition 5.2. The following convergences hold in the sense of distributions on $\mathbf{R}_{+}^{*} \times \mathbf{R}^{3}$ :

$$
\begin{aligned}
P \nabla_{x} \cdot\left\langle A Q\left({ }^{\mathrm{b}} g_{\epsilon},{ }_{\mathrm{b}}^{g}\right)\right. & \rightarrow P \nabla_{x} \cdot u^{\otimes 2}, \\
\nabla_{x} \cdot\left\langle B Q\left(\mathrm{~b}_{\epsilon}, \mathrm{b}_{\epsilon}\right)\right\rangle & \rightarrow \frac{5}{2} \nabla_{x} \cdot(u \theta),
\end{aligned}
$$

as $\epsilon \rightarrow 0$. (We recall that $P$ is the Leray projection, i.e. the $L^{2}$-orthogonal projection on the space of divergence-free vector fields).

Define

$$
{ }^{b} \rho_{\epsilon}=\left\langle{ }^{b} g_{\epsilon}\right\rangle, \quad{ }^{b} u_{\epsilon}=\left\langle v^{b} g_{\epsilon}\right\rangle, \quad{ }^{b} \theta_{\epsilon}=\left\langle\left(\frac{1}{3}|v|^{2}-1\right){ }^{b} g_{\epsilon}\right\rangle .
$$

- First, we establish that

$$
\begin{aligned}
& \nabla_{x} \cdot\left\langle A Q\left({ }^{b} g_{\epsilon}, g_{\epsilon}\right)\right\rangle-\nabla_{x} \cdot\left\langle A Q\left(\Pi^{b} g_{\epsilon}, \Pi^{b} g_{\epsilon}\right)\right\rangle \rightarrow 0 \\
& \nabla_{x} \cdot\left\langle B \mathcal{Q}\left({ }^{b} g_{\epsilon}, g_{\epsilon}\right)\right\rangle-\nabla_{x} \cdot\left\langle B \mathcal{Q}\left(\Pi^{b} g_{\epsilon}, \Pi^{b} g_{\epsilon}\right)\right\rangle \rightarrow 0
\end{aligned}
$$

in some appropriate sense as $\epsilon \rightarrow 0$, where

$$
\Pi^{b} g_{\epsilon}={ }^{b} \rho_{\epsilon}+{ }^{b} u_{\epsilon} \cdot v+{ }^{b} \theta_{\epsilon} \frac{1}{2}\left(|v|^{2}-3\right) ;
$$

in other words, $\Pi$ designates the orthogonal projection on the nullspace of $\mathcal{L}$ in $L^{2}(M d v)$;

- Then we show that

$$
\begin{gathered}
P \nabla_{x} \cdot\left\langle A Q\left(\Pi^{b} g_{\epsilon}, \Pi^{b} g_{\epsilon}\right)\right\rangle=P \nabla_{x} \cdot{ }^{b} u_{\epsilon}^{\otimes 2} \rightarrow P \nabla_{x} \cdot u^{\otimes 2}, \\
\nabla_{x} \cdot\left\langle B Q\left(\Pi^{b} g_{\epsilon}, \Pi^{b} g_{\epsilon}\right)\right\rangle=\frac{5}{2} \nabla_{x} \cdot\left({ }^{b} u_{\epsilon}{ }^{b} \theta_{\epsilon}\right) \rightarrow \frac{5}{2} \nabla_{x} \cdot(u \theta),
\end{gathered}
$$

in the sense of distributions on $\mathbf{R}_{+}^{*} \times \mathbf{R}^{3}$ as $\epsilon \rightarrow 0$.

## Step 1. Relaxation to the local infinitesimal Maxwellian

The main result in this first step expresses that the distance from $g_{\epsilon}$ to the space of local infinitesimal Maxwellians - i.e. elements of $L_{l o c}^{2}(d t d x) \otimes \operatorname{ker} \mathcal{L}$ - vanishes in the limit as $\epsilon \rightarrow 0$.

Lemma 5.3. As $\in \rightarrow 0$

$$
\begin{equation*}
g_{\epsilon}-\Pi^{b} g_{\epsilon} \rightarrow 0 \quad \text { in } L_{l o c}^{2}\left(d t d x ; L^{2}(M d v)\right) \tag{5.14}
\end{equation*}
$$

Proof. By Proposition 1.4, $\mathcal{L}$ is a nonnegative bounded Fredholm operator on $L^{2}(M d v)$, and therefore is coercive on $(\operatorname{ker} \mathcal{L})^{\perp}$. Thus, in order to establish (5.14), it suffices to prove that

$$
\begin{equation*}
\left\langle g_{\epsilon} \mathcal{L}^{b} g_{\epsilon}\right\rangle \rightarrow 0 \text { in } L_{l o c}^{1}(d t d x) . \tag{5.15}
\end{equation*}
$$

In order to do so, we pick an arbitrary compact $Q \subset \mathbf{R}_{+} \times \mathbf{R}^{3}$ and split $\mathbf{1}_{Q}\left\langle{ }^{b} g_{\epsilon} \mathcal{L} g_{\epsilon}\right\rangle$ as

$$
\begin{equation*}
\mathbf{1}_{Q}\left\langle g_{\epsilon} \mathcal{L}^{b} g_{\epsilon}\right\rangle=\left(\mathbf{1}_{Q}-\chi_{\epsilon}\right)\left\langle{ }^{b} g_{\epsilon} \mathcal{L}^{\mathrm{b}} g_{\epsilon}\right\rangle+\chi_{\epsilon}\left\langle{ }^{b} g_{\epsilon} \mathcal{L}^{\mathrm{b}} g_{\epsilon}\right\rangle \tag{5.16}
\end{equation*}
$$

with

$$
\begin{equation*}
\chi_{\epsilon}=\frac{\mathbf{1}_{Q}}{1+\frac{1}{3} \epsilon\langle | g_{\epsilon}| \rangle} . \tag{5.17}
\end{equation*}
$$

By Corollary 3.9, the sequence of functions $\mathbf{1}_{Q}{ }^{b} g_{\epsilon} \mathcal{L} g_{\epsilon}$ is uniformly integrable with respect to the measure $M d v d x d t$ and therefore relatively compact in $w-L^{1}(M d v d x d t)$ by Dunford-Pettis' theorem. On the other hand, $0 \leq \mathbf{1}_{Q}-\chi_{\epsilon} \leq 1$ and

$$
\mathbf{1}_{Q}-\chi_{\epsilon}=\frac{\frac{1}{3} \epsilon\langle | g_{\epsilon}| \rangle \mathbf{1}_{Q}}{1+\frac{1}{3} \epsilon\langle | g_{\epsilon}| \rangle} \leq \frac{1}{3} \epsilon\langle | g_{\epsilon}| \rangle \mathbf{1}_{Q} .
$$

By the first and second entropy controls in Proposition $2.1 \mathbf{1}_{Q}-\chi_{\epsilon}$ converges to 0 in $L_{l o c}^{1}(d t d x)$ and therefore in ( $M d v d x d t$-)measure on compact subsets of $\mathbf{R}_{+} \times \mathbf{R}^{3} \times \mathbf{R}^{3}$. Lemma 14.1 in Appendix E implies that

$$
\begin{equation*}
\left(\mathbf{1}_{Q}-\chi_{\epsilon}\right)\left\langle{ }^{\mathrm{b}} g_{\epsilon} \mathcal{L}^{\mathrm{b}} g_{\epsilon}\right\rangle=\left(\mathbf{1}_{Q}-\chi_{\epsilon}\right) \mathbf{1}_{Q}\left\langle{ }^{\mathrm{b}} g_{\epsilon} \mathcal{L}^{\mathrm{b}} g_{\epsilon}\right\rangle \rightarrow 0 \text { in } L^{1}(d t d x) . \tag{5.18}
\end{equation*}
$$

Then we prove that

$$
\begin{equation*}
\chi_{\epsilon}\left\langle{ }^{b} g_{\epsilon} \mathcal{L}^{b} g_{\epsilon}\right\rangle \rightarrow 0 \text { in } L^{1}(d t d x) . \tag{5.19}
\end{equation*}
$$

This is done in two steps. In the first one, we consider the quantity

$$
\begin{align*}
\epsilon \chi_{\epsilon}{ }^{\mathrm{b}} g_{\epsilon} & \left(\iint q_{\epsilon} b d \sigma_{v, v_{1}}(\omega) M_{1} d v_{1}+\frac{1}{\epsilon} \mathcal{L}^{\mathrm{b}} g_{\epsilon}\right)  \tag{5.20}\\
& =\epsilon g_{\epsilon}\left[-\chi_{\epsilon} \mathcal{L}^{\sharp} g_{\epsilon}+\chi_{\epsilon} \mathcal{Q}\left({ }^{b} g_{\epsilon},{ }^{b} g_{\epsilon}\right)+\mathcal{Q}\left({ }^{\sharp} g_{\epsilon}, \chi_{\epsilon}\left(2 \epsilon{ }^{\mathrm{b}} g_{\epsilon}+\epsilon^{2}{ }^{\sharp} g_{\epsilon}\right)\right)\right] .
\end{align*}
$$

Because $\mathcal{L}=-2 \mathcal{Q}(1, \cdot)$, the continuity property stated in Proposition 1.5 implies that $\mathcal{L}$ is a bounded operator on $L^{1}(M d v)$. Then, as $\epsilon \rightarrow 0$,

$$
\begin{equation*}
\left\|\epsilon g_{\epsilon} \chi_{\epsilon} \mathcal{L}^{\sharp} g_{\epsilon}\right\|_{L^{1}\left(d t d x ; L^{1}(M d v)\right)} \leq \frac{1}{2}\|\mathcal{L}\|_{L^{1}}\left\|\mathbf{1}_{Q} \sharp g_{\epsilon}\right\|_{L^{1}\left(d t d x ; L^{1}(M d v)\right)} \rightarrow 0 \tag{5.21}
\end{equation*}
$$

because of Proposition 3.4 and the first pointwise estimate in Proposition 2.2. Likewise the $L^{1}$-continuity of $\mathcal{Q}$ in Proposition 1.5 implies that

$$
\begin{align*}
& \left\|\epsilon^{b} g_{\epsilon} Q\left({ }^{\sharp} g_{\epsilon}, \chi_{\epsilon}\left(2 \epsilon^{b} g_{\epsilon}+\epsilon^{2} g_{\epsilon}\right)\right)\right\|_{L^{1}\left(d t d x ; L^{1}(M d v)\right)} \\
& \quad=\left\|\epsilon^{b} g_{\epsilon} Q\left(\mathbf{1}_{Q}{ }^{\sharp} g_{\epsilon}, \chi_{\epsilon}\left(2 \epsilon^{b} g_{\epsilon}+\epsilon^{2 \sharp} g_{\epsilon}\right)\right)\right\|_{L^{1}\left(d t d x ; L^{1}(M d v)\right)}  \tag{5.22}\\
& \quad \leq \frac{1}{2}\|\mathcal{Q}\|_{L^{1}}\left\|\mathbf{1}_{Q}{ }^{\sharp} g_{\epsilon}\right\|_{L^{1}\left(d t d x ; L^{1}(M d v)\right)} \\
& \quad \times\left(2\left\|\epsilon g_{\epsilon}\right\|_{L^{\infty}\left(d t d x ; L^{1}(M d v)\right)}+\left\|\epsilon^{2} \chi_{\epsilon}{ }^{\sharp} g_{\epsilon}\right\|_{L^{\infty}\left(d t d x ; L^{1}(M d v)\right)}\right) \rightarrow 0
\end{align*}
$$

again by Proposition 3.4, the first pointwise estimate in Proposition 2.2 and the elementary inequality

$$
\left|\epsilon^{2} \chi_{\epsilon}^{\sharp} g_{\epsilon}\right| \leq \frac{\epsilon\left|g_{\epsilon}\right|}{1+\frac{1}{3} \epsilon\langle | g_{\epsilon}| \rangle} \text { implying that }\left\|\epsilon^{2} \chi_{\epsilon}{ }^{\sharp} g_{\epsilon}\right\|_{L^{1}(M d v)} \leq 3 .
$$

Finally, by Corollary $3.9 \mathbf{1}_{Q} \mathcal{Q}\left({ }_{g} g_{\epsilon}, g_{\epsilon}\right)$ is uniformly integrable for the measure $M d v d x d t$ and therefore relatively compact in $w-L^{1}(M d v d x d t)$ (by Dunford-Pettis' theorem), while $\epsilon^{b} g_{\epsilon}$ is bounded in $L_{t, x, v}^{\infty}$ and $\epsilon g_{\epsilon} \rightarrow 0$ in $L_{l o c}^{2}(M d v d x d t)$ as $\epsilon \rightarrow 0$. Thus $\epsilon^{b} g_{\epsilon}$ converges to 0 in (Mdvdxdt-)measure on compact subsets of $\mathbf{R}_{+} \times \mathbf{R}^{3} \times \mathbf{R}^{3}$. By Lemma 14.1 in Appendix E

$$
\begin{equation*}
\epsilon \stackrel{b}{g}_{\epsilon} Q\left({ }^{b} g_{\epsilon}, \stackrel{b}{g}_{\epsilon}\right) \rightarrow 0 \text { in } L_{l o c}^{1}\left(d t d x ; L^{1}(M d v)\right) . \tag{5.23}
\end{equation*}
$$

The convergence statements (5.21), (5.22), (5.23), above show that

$$
\begin{equation*}
\left\|\epsilon \chi_{\epsilon}{ }^{\mathrm{b}} g_{\epsilon}\left(\iint q_{\epsilon} b d \sigma_{v, v_{1}}(\omega) M_{1} d v_{1} d \omega+\frac{1}{\epsilon} \mathcal{L}^{\mathrm{b}} g_{\epsilon}\right)\right\|_{L^{1}(M d v d x d t)} \rightarrow 0 . \tag{5.24}
\end{equation*}
$$

The second step in the proof of (5.19) is the following lemma that will also be used elsewhere.

Lemma 5.4. Assume that $\gamma \in \Upsilon$. The sequence

$$
\begin{equation*}
\left.\left\langle\langle | q_{\epsilon}{ }^{\mathrm{b}} g_{\epsilon} \mid\right\rangle\right\rangle \text { is bounded in } L_{l o c}^{1}\left(d t ; L_{x}^{1}\right) \text {. } \tag{5.25}
\end{equation*}
$$

Postponing the proof of Lemma 5.4, observe that both estimates (5.25) and (5.24) imply that (5.19) holds. The limit (5.18) being already established, this proves that (5.15) holds, as announced.

Proof of Lemma 5.4. Young's inequality (10.4) from Appendix A implies that, for all $t^{*}>0$ and $\epsilon \in[0,1]$

$$
\begin{aligned}
& \left.\int_{0}^{t^{*}} \int\left\langle\langle | q_{\epsilon}{ }^{\mathrm{b}} g_{\epsilon} \mid\right\rangle\right\rangle d x d t \\
& \leq \int_{0}^{t^{*}} \int \frac{1}{\epsilon^{4}}\left\langle\left\langle G_{\epsilon} G_{\epsilon 1}\left[r\left(\frac{\epsilon^{2} q_{\epsilon}}{G_{\epsilon} G_{\epsilon 1}}\right)+h^{*}\left(\epsilon^{2} g_{\epsilon}\right)\right]\right\rangle d x d t\right. \\
& \left.\leq 4 \int_{0}^{t^{*}} \iint \frac{D\left(F_{\epsilon}\right)}{\epsilon^{4}} d v d x d t+h^{*}(1) \int_{0}^{t^{*}} \int\left\langle G_{\epsilon} G_{\epsilon 1}{ }^{\triangleright} g_{\epsilon}^{2}\right\rangle\right\rangle d x d t \text {. }
\end{aligned}
$$

The first term on the right-hand side of this inequality is controlled by the entropy dissipation control (2.7). In order to control the second term, we first decompose it as

$$
\left\langle\left\langle G_{\epsilon} G_{\epsilon 1}{ }^{\mathrm{b}} \mathrm{~g}_{\epsilon}^{2}\right\rangle\right\rangle=\left\langle\left\langle\epsilon^{2} G_{\epsilon}{ }^{\mathrm{b}} g_{\epsilon}^{2} G_{\epsilon 1} \frac{1-\gamma_{\epsilon 1}}{\epsilon^{2}}\right\rangle\right\rangle+\left\langle\left\langle G_{\epsilon} \stackrel{\mathrm{b}}{ }_{\mathrm{g}}^{\epsilon} G_{\epsilon 1} \gamma_{\epsilon 1}\right\rangle\right\rangle .
$$

Using the bound (H1) on $b$, the first and the third pointwise controls in Proposition 2.2 lead to

$$
\left|\left\langle\left\langle\epsilon^{2} G_{\epsilon}{ }^{b} g_{\epsilon}^{2} G_{\epsilon 1} \frac{1-\gamma_{\epsilon 1}}{\epsilon^{2}}\right\rangle\right\rangle\right| \leq \frac{15}{8} b_{\infty}\langle | \not g_{\epsilon}| \rangle
$$

and

$$
\left|\left\langle G_{\epsilon}{ }^{\mathrm{b}} g_{\epsilon}^{2} G_{\epsilon 1} \gamma_{\epsilon 1}\right\rangle\right\rangle \left\lvert\, \leq \frac{9}{4} b_{\infty}\left\langle\left.\left.\right|^{\mathrm{b}} g_{\epsilon}\right|^{2}\right\rangle .\right.
$$

Eventually, the last three inequalities lead to

$$
\begin{align*}
& \int_{0}^{t^{*}} \int\left\langle\langle | q_{\epsilon}{ }^{\mathrm{b}} g_{\epsilon}\right| \geqslant d x d t \leq 4\left\|\frac{D\left(F_{\epsilon}\right)}{\epsilon^{4}}\right\|_{L_{t, x, v}^{1}}  \tag{5.26}\\
& \quad+t^{*} h^{*}(1) b_{\infty}\left(\frac{15}{8}\| \|^{\sharp} g_{\epsilon}\left\|_{L_{t}^{\infty}\left(L^{1}(M d v d x)\right)}+\frac{9}{4}\right\|\left\|_{\epsilon}^{\mathrm{b}}\right\|_{L_{t}^{\infty}\left(L^{2}(M d v d x)\right)}^{2}\right) .
\end{align*}
$$

Therefore (5.25) holds (see Proposition 2.1).
The following convergences are easy consequences of Lemma 5.3.
Corollary 5.5. One has

$$
\begin{align*}
\left\langle A Q\left({ }^{b} g_{\epsilon}, g_{\epsilon}\right)\right\rangle-{ }^{b} u_{\epsilon}^{\otimes 2}+\left.\left.\frac{1}{3}\right|^{b} u_{\epsilon}\right|^{2} I & \rightarrow 0  \tag{5.27}\\
\left\langle B Q\left({ }^{b} g_{\epsilon},{ }^{b} g_{\epsilon}\right)\right\rangle-\frac{5}{2}{ }^{b} u_{\epsilon}{ }^{b} \theta_{\epsilon} & \rightarrow 0
\end{align*}
$$

in $L_{l o c}^{1}(d t d x)$ as $\epsilon \rightarrow 0$.
Proof. We first recall from (1.42) that $\zeta \in L^{2}(M d v)$. By the CauchySchwarz inequality and the $L^{2}$ continuity of $Q$ stated in Proposition 1.5, one has, for some constant $C>0$

$$
\begin{align*}
& \left|\left\langle\zeta Q\left({ }^{b} g_{\epsilon},{ }^{b} g_{\epsilon}\right)\right\rangle-\left\langle\zeta Q\left(\Pi{ }^{b} g_{\epsilon}, \Pi{ }^{b} g_{\epsilon}\right)\right\rangle\right|=\left|\left\langle\zeta Q\left({ }^{b} g_{\epsilon}-\Pi{ }^{b} g_{\epsilon},{ }^{b} g_{\epsilon}+\Pi{ }^{b} g_{\epsilon}\right)\right\rangle\right|  \tag{5.28}\\
& \quad \leq\|\zeta\|_{L^{2}(M d v)}\left\|Q\left({ }^{b} g_{\epsilon}-\Pi{ }^{b} g_{\epsilon},{ }^{b} g_{\epsilon}+\Pi{ }^{b} g_{\epsilon}\right)\right\|_{L^{2}(M d v)} \\
& \quad \leq C\|\zeta\|_{L^{2}(M d v)}\left\|{ }^{b} g_{\epsilon}+\Pi{ }^{b} g_{\epsilon}\right\|_{L^{2}(M d v)}\left\|^{b} g_{\epsilon}-\Pi^{b} g_{\epsilon}\right\|_{L^{2}(M d v)} \\
& \quad \leq 2 C\|\zeta\|_{L^{2}(M d v)}\| \|_{\epsilon \epsilon}\left\|_{L^{2}(M d v)}\right\|^{b} g_{\epsilon}-\Pi{ }^{b} g_{\epsilon} \|_{L^{2}(M d v)} \\
& \quad=O(1)_{L_{t}^{\infty}\left(L_{x}^{2}\right)}\| \|_{g_{\epsilon}}-\Pi{ }^{b} g_{\epsilon} \|_{L^{2}(M d v)} \rightarrow 0
\end{align*}
$$

in $L_{l o c}^{1}(d t d x)$ as $\epsilon \rightarrow 0$, by the first entropy control in Proposition 2.1 and Lemma 5.3.

The symmetry relations

$$
\begin{aligned}
& \langle(\mathscr{L} A) f(v)\rangle=\left\langle(\mathscr{L} A) p\left(|v|^{2}\right)\right\rangle=0 \\
& \langle(\mathscr{L} B) f(v)\rangle=\left\langle(\mathscr{L} B) p\left(|v|^{2}\right)\right\rangle=0
\end{aligned}
$$

which hold for any $f \in \operatorname{ker} \mathcal{L}$ and any polynomial $p$, and formula (1.43) imply that

$$
\begin{align*}
\left\langle\zeta \mathcal{Q}\left(\Pi^{b} g_{\epsilon}, \Pi^{b} g_{\epsilon}\right)\right\rangle & =\left\langle\zeta \frac{1}{2} \mathcal{L}\left(\left(\Pi^{b} g_{\epsilon}\right)^{2}\right)\right\rangle \\
& =\frac{1}{2}\left\langle(\mathscr{L} \zeta)\left(\Pi^{b} g_{\epsilon}\right)^{2}\right\rangle  \tag{5.29}\\
& =\frac{1}{2}\left\langle(\mathscr{L} \zeta)\left[{ }^{b} u_{\epsilon}^{\otimes 2}: v^{\otimes 2}+{ }^{\mathrm{b}} \theta_{\epsilon}{ }^{\mathrm{b}} u_{\epsilon} \cdot v\left(|v|^{2}-3\right)\right]\right\rangle
\end{align*}
$$

Collecting the results from (5.28) and (5.29) and using the convergence (11.1) proves that

$$
\left\langle\zeta Q\left({ }^{b} g_{\epsilon},{ }^{b} g_{\epsilon}\right)\right\rangle-\frac{1}{2}\left\langle(\mathscr{L} \zeta)\left[{ }^{b} u_{\epsilon}^{\otimes 2}: v^{\otimes 2}+{ }^{b} \theta_{\epsilon}{ }^{b} u_{\epsilon} \cdot v\left(|v|^{2}-3\right)\right]\right\rangle \rightarrow 0
$$

which in turn implies the convergences stated in (5.27).
Step 2. Convergence of the nonlinear convection terms
Because the Navier-Stokes(-Fourier) system is nonlinear, weak convergences are not enough to take limits in the convection terms. First, in the next lemma we identify two quantities that converge strongly.

Lemma 5.6. As $\epsilon \rightarrow 0$,

$$
\begin{equation*}
P^{b} u_{\epsilon} \rightarrow u, \text { and } \frac{1}{5}\left(3^{\mathrm{b}} \theta_{\epsilon}-2^{\mathrm{b}} \rho_{\epsilon}\right) \rightarrow \theta \tag{5.30}
\end{equation*}
$$

in $L_{l o c}^{2}(d t d x)$.
Proof. Step 1: a priori estimates. By Proposition 4.1

$$
\begin{align*}
\partial_{t} P^{\mathrm{b}} u_{\epsilon}+P\left(\nabla_{x} \cdot \frac{1}{\epsilon}\left\langle(\mathcal{L} A)^{\mathrm{b}} g_{\epsilon}\right\rangle\right) & \rightarrow 0  \tag{5.31}\\
\partial_{t}\left(\frac{3}{2}^{\mathrm{b}} \theta_{\epsilon}-\stackrel{ }{\mathrm{b}}_{\epsilon}\right)+\nabla_{x} \cdot \frac{1}{\epsilon}\left\langle(\mathcal{L} B)^{\mathrm{b}} g_{\epsilon}\right\rangle & \rightarrow 0
\end{align*}
$$

in $L_{l o c}^{1}\left(d t ; W_{l o c}^{-s, 1}(d x)\right)$ for all $s>0$ and in $L_{l o c}^{1}(d t d x)$ respectively. By Proposition 5.1, for $\zeta=A$ or $\zeta=B$,

$$
\begin{equation*}
\frac{1}{\epsilon}\left\langle(\mathscr{L} \zeta){ }^{b} g_{\epsilon}\right\rangle-\left\langle\zeta \mathcal{Q}\left({ }^{b} g_{\epsilon}, \stackrel{b}{g}_{\epsilon}\right)\right\rangle \text { is bounded in } L_{l o c}^{1}(d t d x) . \tag{5.32}
\end{equation*}
$$

Because $\zeta \in L^{2}(M d v)$ (see (1.42)), by the Cauchy-Schwarz inequality and the $L^{2}$ continuity of $\mathcal{Q}$ stated in Proposition 1.5 , there exists $C>0$ such that

$$
\begin{equation*}
\left|\left\langle\zeta Q\left({ }^{\mathrm{b}} g_{\epsilon}, \stackrel{\mathrm{b}}{\epsilon}^{g_{\epsilon}}\right)\right\rangle\right| \leq C\|\zeta\|_{L^{2}(M d v)}\left\|^{\mathrm{b}} g_{\epsilon}\right\|_{L^{2}(M d v)}^{2}=O(1)_{L_{t}^{\infty}\left(L_{x}^{1}\right)} \tag{5.33}
\end{equation*}
$$

because of the first entropy control in Proposition 2.1. These last two controls imply that the family

$$
\begin{equation*}
\frac{1}{\epsilon}\left\langle(\mathscr{L} \zeta)^{\mathrm{b}} g_{\epsilon}\right\rangle \text { is bounded in } L_{\text {loc }}^{1}(d t d x) . \tag{5.34}
\end{equation*}
$$

Thus, the convergences (5.31) and the bound (5.34) imply that

$$
\begin{align*}
& \text { for all } s>1, \quad \partial_{t} P^{\mathrm{b}} u_{\epsilon} \text { is bounded in } L_{l o c}^{1}\left(d t ; W^{-s, 1}\left(\mathbf{R}^{3}\right)\right), \\
& \quad \text { and } \partial_{t}\left(\frac{3}{2}{ }^{\mathrm{b}} \theta_{\epsilon}-{ }^{\mathrm{b}} \rho_{\epsilon}\right) \text { is bounded in } L_{l o c}^{1}\left(d t ; W^{-1,1}\left(\mathbf{R}^{3}\right)\right) . \tag{5.35}
\end{align*}
$$

Further, the second statement of Proposition 3.8 with $\phi(v)=v$ or $\phi(v)=$ $\frac{1}{2}\left(|v|^{2}-5\right)$, and the $L^{2}$-continuity and translation invariance of $P$ imply that

$$
\begin{equation*}
\int_{0}^{t^{*}} \int_{Q}\left|P^{b} u_{\epsilon}(t, x+y)-P^{b} u_{\epsilon}(t, x)\right|^{2} d x d t \leq \eta(|y|) \tag{5.36}
\end{equation*}
$$

and $\int_{0}^{t^{*}} \int_{Q}\left|\left(\frac{3}{2}^{\mathrm{b}} \theta_{\epsilon}-\stackrel{b}{\rho}_{\epsilon}\right)(t, x+y)-\left(\frac{3}{2}^{\mathrm{b}} \theta_{\epsilon}-\stackrel{b}{\rho}_{\epsilon}\right)(t, x)\right|^{2} d x d t \leq \eta(|y|)$
for each $y \in \mathbf{R}^{3}$ such that $|y| \leq 1$, uniformly in $\epsilon$, where $\eta$ is the modulus of continuity in the statement of Proposition 3.8.
Step 2: convergence of $\frac{3}{2}{ }^{b} \theta_{\epsilon}-\rho_{\epsilon}$. The $L^{1}$ variant of Aubin's lemma (see [58], p. 84, Theorem 5), and both estimates (5.35) and (5.36) imply that the sequence $\frac{3}{2}{ }^{\mathrm{b}} \theta_{\epsilon}-{ }^{\mathfrak{b}} \rho_{\epsilon}$ is relatively compact in $L_{l o c}^{1}(d t d x)$. On the other hand, the sequence $\left(\frac{3}{2}{ }^{b} \theta_{\epsilon}-{ }^{b} \rho_{\epsilon}\right)^{2}$ is locally uniformly integrable on $\mathbf{R}_{+} \times \mathbf{R}^{3}$ by the first assertion in Proposition 3.8. Hence the sequence $\frac{3}{2}^{b} \theta_{\epsilon}-{ }^{b} \rho_{\epsilon}$ is relatively compact in $L_{l o c}^{2}(d t d x)$. (Indeed, if $a_{n}$ converges to $a$ in measure and if $a_{n}^{2}$ is locally uniformly integrable, then $a_{n} \rightarrow a$ in $L_{l o c}^{2}$ ). By (5.4), we already know that

$$
\frac{3}{2}^{b} \theta_{\epsilon}-\stackrel{b}{\rho}_{\epsilon} \rightarrow \frac{3}{2} \theta-\langle g\rangle
$$

in $w-L_{l o c}^{2}(d t d x)$ as $\epsilon \rightarrow 0$, and hence this convergence holds in the strong topology of $L_{l o c}^{2}(d t d x)$. The second relation in (5.5) (the Boussinesq relation) finally implies that

$$
\frac{3}{2}^{b} \theta_{\epsilon}-{ }^{b} \rho_{\epsilon} \rightarrow \frac{5}{2} \theta \text { strongly in } L_{l o c}^{2}(d t d x)
$$

as $\epsilon \rightarrow 0$, and this is precisely the second convergence in (5.30).
Step 3: convergence of $P^{b} u_{\epsilon}$. Let $\xi \in C_{c}^{\infty}\left(\mathbf{R}^{3}\right)$ be such that $\xi(x)=0$ whenever $|x|>1, \xi \geq 0$ and $\int \xi(x) d x=1$. For each $\delta>0$ we define
$\xi_{\delta}(x)=\delta^{-3} \xi(x / \delta)$. The first entropy control in Proposition 2.1 and the first estimate in (5.35) imply that, for all $s>0$ and all $\delta>0$,

$$
\begin{gathered}
P^{b} u_{\epsilon} \star \xi_{\delta} \text { is bounded in } L_{t}^{\infty}\left(H_{x}^{s}\right) \\
\partial_{t} P^{b} u_{\epsilon} \star \xi_{\delta} \text { is bounded in } L_{l o c}^{1}\left(d t ; H_{x}^{s}\right) .
\end{gathered}
$$

By Theorem 1, p. 71, and Lemma 4, p. 77 of [58], the family $P^{b} u_{\epsilon} \star \xi_{\delta}$ is relatively compact in $L_{l o c}^{2}(d t d x)$ for each $\delta>0$. The first convergence in (5.4) and the $L^{2}$-continuity of $P$ imply that

$$
\begin{align*}
P^{b} u_{\epsilon} & \rightarrow P u \in w-L_{l o c}^{2}(d t d x) \text { while } \\
P^{b} u_{\epsilon} \star \xi_{\delta} & \rightarrow P u \star \xi_{\delta} \text { strongly in } L_{l o c}^{2}(d t d x) . \tag{5.37}
\end{align*}
$$

Hence

$$
P^{b} u_{\epsilon} \cdot\left(P^{b} u_{\epsilon} \star \xi_{\delta}\right) \rightarrow P u \cdot\left(P u \star \xi_{\delta}\right) \text { in } L_{l o c}^{1}(d t d x)
$$

as $\epsilon \rightarrow 0$. On the other hand $P u \star \xi_{\delta} \rightarrow P u$ in $L_{l o c}^{2}(d t d x)$ and

$$
P^{b} u_{\epsilon}-P^{\triangleright} u_{\epsilon} \star \xi_{\delta} \rightarrow 0 \text { in } L_{l o c}^{2}(d t d x) \text { uniformly in } \epsilon \text { as } \delta \rightarrow 0
$$

because of the first estimate in (5.36). Therefore

$$
\left|P^{\mathrm{b}} u_{\epsilon}\right|^{2} \rightarrow|P u|^{2} \text { in } L_{l o c}^{1}(d t d x) \text { as } \epsilon \rightarrow 0
$$

With the first convergence in (5.37), this implies that

$$
\begin{equation*}
P^{\mathrm{b}} u_{\epsilon} \rightarrow P u \text { strongly in } L_{l o c}^{2}(d t d x) \tag{5.38}
\end{equation*}
$$

as $\epsilon \rightarrow 0$. The first relation in (5.5) (the incompressibility condition) implies that $P u=u$, so that (5.38) coincides with the first convergence in (5.30).

As indicated above, the convergences

$$
{ }^{\mathrm{b}} u_{\epsilon} \rightarrow u, \quad{ }^{\mathrm{b}} \theta_{\epsilon} \rightarrow \theta
$$

coming from (5.4) hold in $w-L_{l o c}^{2}\left(d t ; L_{x}^{2}\right)$ only, and even the strong limits (5.30) do not imply that

$$
{ }^{\mathrm{b}} u_{\epsilon}^{\otimes 2} \rightarrow u^{\otimes 2}, \quad{ }^{\mathrm{b}} u_{\epsilon}^{\mathrm{b}} \theta_{\epsilon} \rightarrow u \theta
$$

in the sense of distributions on $\mathbf{R}_{+}^{*} \times \mathbf{R}^{3}$. Instead, one has the following convergences.

Corollary 5.7. As $\epsilon \rightarrow 0$,

$$
\begin{align*}
& P \nabla_{x} \cdot{ }^{\mathrm{b}} u_{\epsilon}^{\otimes 2} \rightarrow P \nabla_{x} \cdot u^{\otimes 2}  \tag{5.39}\\
& \nabla_{x} \cdot\left({ }^{b} u_{\epsilon}{ }^{\mathrm{b}} \theta_{\epsilon}\right) \rightarrow \nabla_{x} \cdot(u \theta)
\end{align*}
$$

in the sense of distributions on $\mathbf{R}_{+}^{*} \times \mathbf{R}^{3}$.

Proof. Denote $\nabla_{x} \pi_{\epsilon}={ }^{b} u_{\epsilon}-P^{b} u_{\epsilon}$ and $\beta_{\epsilon}={ }^{b} \rho_{\epsilon}+{ }^{b} \theta_{\epsilon}$; one has

$$
\begin{equation*}
\nabla_{x} \pi_{\epsilon} \rightarrow 0 \text { and } \frac{2}{5} \beta_{\epsilon}={ }^{b} \theta_{\epsilon}-\frac{2}{5}\left(\frac{3}{2}^{b} \theta_{\epsilon}-{ }^{b} \rho_{\epsilon}\right) \rightarrow 0 \tag{5.40}
\end{equation*}
$$

in $w-L_{l o c}^{2}(d t d x)$, by the incompressibility and Boussinesq relations (see Theorem 11.1 in Appendix B). This and (5.30) imply that

$$
\begin{array}{r}
P \nabla_{x} \cdot\left({ }^{\mathrm{b}} u_{\epsilon}^{\otimes 2}\right)-P \nabla_{x} \cdot\left(\left(\nabla_{x} \pi_{\epsilon}\right)^{\otimes 2}\right)-P \nabla_{x} \cdot\left(u^{\otimes 2}\right) \rightarrow 0  \tag{5.41}\\
\nabla_{x} \cdot\left({ }^{b} u_{\epsilon}{ }^{\mathrm{b}} \theta_{\epsilon}\right)-\frac{2}{5} \nabla_{x} \cdot\left(\beta_{\epsilon} \nabla_{x} \pi_{\epsilon}\right)-\nabla_{x} \cdot(u \theta) \rightarrow 0
\end{array}
$$

in the sense of distributions on $\mathbf{R}_{+}^{*} \times \mathbf{R}^{3}$. (In other words, the cross-terms in the quadratic expressions $\nabla_{x} \cdot{ }^{\text {b }} u_{\epsilon}^{\otimes 2}$ and $\nabla_{x} \cdot\left({ }^{b} u_{\epsilon}{ }^{\mathrm{b}} \theta_{\epsilon}\right)$ vanish with $\epsilon$ ).

Because of (5.41), proving (5.39) reduces to proving that

$$
\begin{equation*}
P \nabla_{x} \cdot\left(\nabla_{x} \pi_{\epsilon}\right)^{\otimes 2} \rightarrow 0 \text { and } \nabla_{x} \cdot\left(\beta_{\epsilon} \nabla_{x} \pi_{\epsilon}\right) \rightarrow 0 \tag{5.42}
\end{equation*}
$$

in the sense of distributions on $\mathbf{R}_{+}^{*} \times \mathbf{R}^{3}$.
First we use a mollifier in the space variable, as follows: let $\xi \in C_{c}^{\infty}\left(\mathbf{R}^{3}\right)$ be such that $\xi(x)=0$ whenever $|x|>1, \xi \geq 0$ and $\int \xi(x) d x=1$. For each $\delta>0$ we define $\xi_{\delta}(x)=\delta^{-3} \xi(x / \delta)$ and the sequences $\pi_{\epsilon}^{\delta}=\pi_{\epsilon} * \xi_{\delta}$ and $\beta_{\epsilon}^{\delta}=\left({ }^{b} \rho_{\epsilon}+{ }^{\mathrm{b}} \theta_{\epsilon}\right) * \xi_{\delta}$.

Then, by Proposition 4.1 and (5.34)

$$
\begin{aligned}
\left.\epsilon \partial_{t}\left\langle v{ }^{b} g_{\epsilon}\right\rangle+\left.\nabla_{x}\left\langle\frac{1}{3}\right| v\right|^{2} g_{\epsilon}\right\rangle & \left.=-\nabla_{x} \cdot\left\langle(\mathscr{L} A)^{b} g_{\epsilon}\right\rangle+\left\langle v q_{\epsilon} \hat{\gamma}_{\epsilon}\right\rangle\right\rangle \\
\left.\left.\epsilon \partial_{t}\left\langle\frac{1}{3}\right| v\right|^{2} g_{\epsilon}\right\rangle+\nabla_{x} \cdot\left\langle\frac{5}{3} v v^{b} g_{\epsilon}\right\rangle & =-\nabla_{x} \cdot\left\langle\frac{2}{3}(\mathscr{L} B)^{b} g_{\epsilon}\right\rangle+\left\langle\left.\left\langle\frac{1}{3}\right| v\right|^{2} q_{\epsilon} \hat{\gamma}_{\epsilon}\right\rangle \rightarrow 0
\end{aligned}
$$

in $L_{l o c}^{1}\left(d t ; W_{l o c}^{-1,1}\left(\mathbf{R}^{3}\right)\right)$. (In fact Proposition 4.1 and (5.34) show that these vanishing terms are of order $O(\epsilon)$, but that much information is not needed here). In other words,

$$
\begin{aligned}
\epsilon \partial_{t}{ }^{b} u_{\epsilon}+\nabla_{x}\left({ }^{b} \rho_{\epsilon}+{ }^{b} \theta_{\epsilon}\right) & \rightarrow 0 \\
\epsilon \partial_{t}\left({ }^{b} \rho_{\epsilon}+{ }^{b} \theta_{\epsilon}\right)+\frac{5}{3} \nabla_{x} \cdot{ }^{b} u_{\epsilon} & \rightarrow 0
\end{aligned}
$$

in $L_{l o c}^{1}\left(d t ; W_{l o c}^{-1,1}\left(\mathbf{R}^{3}\right)\right)$. In particular, applying $I-P$ to the first equation results in

$$
\begin{aligned}
\epsilon \partial_{t} \nabla_{x} \pi_{\epsilon}+\nabla_{x} \beta_{\epsilon} & \rightarrow 0 \\
\epsilon \partial_{t} \beta_{\epsilon}+\frac{5}{3} \Delta_{x} \pi_{\epsilon} & \rightarrow 0
\end{aligned}
$$

in $L_{l o c}^{1}\left(d t ; W_{l o c}^{-1-s, 1}\left(\mathbf{R}^{3}\right)\right)$ for all $s>0$. Then applying the mollifier $\xi_{\delta}$ leads to

$$
\begin{aligned}
\epsilon \partial_{t} \nabla_{x} \pi_{\epsilon}^{\delta}+\nabla_{x} \beta_{\epsilon}^{\delta} & \rightarrow 0 \\
\epsilon \partial_{t} \beta_{\epsilon}^{\delta}+\frac{5}{3} \Delta_{x} \pi_{\epsilon}^{\delta} & \rightarrow 0
\end{aligned}
$$

in $L_{l o c}^{1}\left(d t ; H_{l o c}^{s}\left(\mathbf{R}^{3}\right)\right)$ for any $s>0$, while the families $\beta_{\epsilon}^{\delta}$ and $\nabla_{x} \pi_{\epsilon}^{\delta}$ are bounded in $L_{t}^{\infty}\left(L_{x}^{2}\right)$ by the first control of Proposition 2.1. The local argument for the incompressible limit due to P.-L. Lions and N. Masmoudi [49] (recalled in Lemma 13.1 of Appendix D below) shows that, for all $\delta>0$

$$
\begin{equation*}
P \nabla_{x} \cdot\left(\left(\nabla_{x} \pi_{\epsilon}^{\delta}\right)^{\otimes 2}\right) \rightarrow 0, \quad \nabla_{x} \cdot\left(\beta_{\epsilon}^{\delta} \nabla_{x} \pi_{\epsilon}^{\delta}\right) \rightarrow 0 \tag{5.43}
\end{equation*}
$$

in the sense of distributions on $\mathbf{R}_{+}^{*} \times \mathbf{R}^{3}$ as $\epsilon \rightarrow 0$.
It remains to remove the mollifier $\xi_{\delta}$ in (5.43). In order to do so, observe that by the last statement in Proposition 3.8, for each $t^{*}>0$ and each bounded $Q \subset \mathbf{R}^{3}$, there exists an increasing function $\eta: \mathbf{R}_{+} \rightarrow \mathbf{R}_{+}$such that $\lim _{z \rightarrow 0^{+}} \eta(z)=0$ and

$$
\begin{aligned}
&\left\|\nabla_{x} \pi_{\epsilon}(t, x+y)-\nabla_{x} \pi_{\epsilon}(t, x)\right\|_{L^{2}\left(\left[0, t^{*}\right] \times Q\right)} \leq \eta(|y|), \\
&\left\|\beta_{\epsilon}(t, x+y)-\beta_{\epsilon}(t, x)\right\|_{L^{2}\left(\left[0, t^{*}\right] \times Q\right)} \leq \eta(|y|),
\end{aligned}
$$

for all $y \in \mathbf{R}^{3}$ such that $|y| \leq 1$, uniformly in $\epsilon>0$. This implies that

$$
\begin{equation*}
\left\|\nabla_{x} \pi_{\epsilon}^{\delta}-\nabla_{x} \pi_{\epsilon}\right\|_{L^{2}\left(\left[0, t^{*}\right] \times Q\right)} \leq \eta(\delta), \quad\left\|\beta_{\epsilon}^{\delta}-\beta_{\epsilon}\right\|_{L^{2}\left(\left[0, t^{*}\right] \times Q\right)} \leq \eta(\delta) \tag{5.44}
\end{equation*}
$$

uniformly in $\epsilon$. Hence

$$
\begin{array}{r}
P \nabla_{x} \cdot\left(\left(\nabla_{x} \pi_{\epsilon}\right)^{\otimes 2}\right)-P \nabla_{x} \cdot\left(\left(\nabla_{x} \pi_{\epsilon}^{\delta}\right)^{\otimes 2}\right) \rightarrow 0,  \tag{5.45}\\
\frac{2}{5} \nabla_{x} \cdot\left(\left({ }^{b} \rho_{\epsilon}+{ }^{\mathrm{b}} \theta_{\epsilon}\right) \nabla_{x} \pi_{\epsilon}\right)-\frac{2}{5} \nabla_{x} \cdot\left(\beta_{\epsilon}^{\delta} \nabla_{x} \pi_{\epsilon}^{\delta}\right) \rightarrow 0
\end{array}
$$

in the sense of distributions on $\mathbf{R}_{+}^{*} \times \mathbf{R}^{3}$, uniformly in $\epsilon$ as $\delta \rightarrow 0$.
The limit (5.43) and the uniform convergence in (5.45) eventually imply (5.42), which in turn establishes (5.39).

Finally, Corollaries 5.5 and 5.7 imply Proposition 5.2.
5.3. Proof of the weak Navier-Stokes-Fourier limit. We conclude this section by showing how the results from the two previous subsections and those from [7] that are recalled in Appendix B eventually imply the Navier-Stokes-Fourier limit (Theorem 1.6).

We recall that $g_{\epsilon}$ and $q_{\epsilon}$ in fact designate subsequences satisfying (5.3) and (5.4). Then we take limits in all the terms appearing in (5.1) as well as in (5.2):

$$
P^{b} u_{\epsilon} \rightarrow u, \quad \frac{1}{5}\left(3^{b} \theta_{\epsilon}-2^{b} \rho_{\epsilon}\right) \rightarrow \theta \text { in } L_{l o c}^{2}(d t d x)
$$

by Lemma 5.6, while, by Propositions 5.1 and 5.2,

$$
\begin{aligned}
P\left(\nabla_{x} \cdot \frac{1}{\epsilon}\left\langle(\mathcal{L} A)^{b} g_{\epsilon}\right\rangle\right) & \rightarrow P \nabla_{x} \cdot(u \otimes u)-v \Delta_{x} u, \\
\nabla_{x} \cdot \frac{1}{\epsilon}\left\langle(\mathcal{L} B)^{b} g_{\epsilon}\right\rangle & \rightarrow \frac{5}{2} \nabla_{x} \cdot(u \theta)-\frac{5}{2} \kappa \Delta_{x} \theta,
\end{aligned}
$$

in the sense of distributions on $\mathbf{R}_{+}^{*} \times \mathbf{R}^{3}$. This shows that $(u, \theta)$ satisfies the Navier-Stokes-Fourier system (1.29) in the sense of distributions on $\mathbf{R}_{+}^{*} \times \mathbf{R}^{3}$.

On the other hand, the controls (5.35) and Appendix C of [44] imply that the quantities

$$
P^{b} u_{\epsilon} \rightarrow u, \quad \frac{1}{5}\left(3^{\mathrm{b}} \theta_{\epsilon}-2^{\mathrm{b}} \rho_{\epsilon}\right) \rightarrow \theta \text { in } C\left(\mathbf{R}_{+} ; w-L_{l o c}^{2}(d x)\right) \text { as } \epsilon \rightarrow 0 .
$$

In particular, at $t=0$ one has

$$
P\left\langle v g_{\epsilon}(0)\right\rangle \rightarrow u^{i n}, \quad \frac{1}{5}\left\langle\left(|v|^{2}-5\right){ }^{b} g_{\epsilon}(0)\right\rangle \rightarrow \theta^{\text {in }}
$$

in $w-L_{l o c}^{2}(d x)$. This proves that $(u, \theta)$ satisfies the initial condition (1.30). Finally

$$
\begin{aligned}
\left\langle v g_{\epsilon}\right\rangle={ }^{\mathrm{b}} u_{\epsilon}+\epsilon\left\langle v^{\sharp} g_{\epsilon}\right\rangle & \rightarrow u \\
\left\langle\left(\frac{1}{3}|v|^{2}-1\right) g_{\epsilon}\right\rangle=\left\langle\left(\frac{1}{3}|v|^{2}-1\right)^{\mathrm{b}} g_{\epsilon}\right\rangle+\epsilon\left(\left(\frac{1}{3}|v|^{2}-1\right){ }^{\sharp} g_{\epsilon}\right\rangle & \rightarrow \theta
\end{aligned}
$$

in $w-L_{l o c}^{1}(d t d x)$ because of (5.4) and the second entropy estimate in Proposition 2.1. This completes the proof of Theorem 1.6.

## 6. Proving Proposition 2.7

In this and the next two sections, we prove the results announced in Sects. 2 and 3 and subsequently used in Sects. 4 and 5.

We begin with the results stated in Sect. 2, where the Relaxation-based decomposition and Proposition 2.7 were stated without proof.
6.1. Proof of the Relaxation-based decomposition. Returning to the equality (2.16), we see that

$$
\begin{equation*}
k\left(G_{\epsilon}\right) M g_{\epsilon}=T_{1}+T_{2} \tag{6.1}
\end{equation*}
$$

with $k: \mathbf{R}_{+} \rightarrow[0,1]$ such that $\|z \mapsto z k(z)\|_{L^{\infty}}=C_{k}$ and

$$
\begin{aligned}
T_{1} & =\frac{k\left(G_{\epsilon}\right)}{\epsilon\left\langle\tilde{G}_{\epsilon}\right\rangle}\left[M G_{\epsilon}\left\langle\tilde{G}_{\epsilon}\right\rangle-\mathcal{A}^{+}\left(M \tilde{G}_{\epsilon}, M \tilde{G}_{\epsilon}\right)\right] \\
& =\frac{k\left(G_{\epsilon}\right)}{\epsilon\left\langle\tilde{G}_{\epsilon}\right\rangle} M \iint\left(G_{\epsilon} \tilde{G}_{\epsilon 1}-\tilde{G}_{\epsilon}^{\prime} \tilde{G}_{\epsilon 1}^{\prime}\right) d \sigma_{v, v_{1}}(\omega) M_{1} d v_{1}
\end{aligned}
$$

where the last equality uses the classical relation $M^{\prime} M_{1}^{\prime}=M M_{1}$ that follows from the formula (1.12) defining $M$ and the microscopic conservation
laws (1.4). Expanding each term of the form $\tilde{G}_{\epsilon}=\left(1-\gamma_{\epsilon}\right)+\gamma_{\epsilon} G_{\epsilon}$ in the formula above leads to

$$
\begin{align*}
T_{1} & =\frac{k\left(G_{\epsilon}\right)}{\epsilon\left\langle\tilde{G}_{\epsilon}\right\rangle} M \iint\left(1-\gamma_{\epsilon 1}\right)\left(G_{\epsilon}-\tilde{G}_{\epsilon}^{\prime} \tilde{G}_{\epsilon 1}^{\prime}\right) d \sigma_{v, v_{1}}(\omega) M_{1} d v_{1}  \tag{6.2}\\
& +\frac{k\left(G_{\epsilon}\right)}{\epsilon\left\langle\tilde{G}_{\epsilon}\right\rangle} M \iint \gamma_{\epsilon 1}\left(1-\gamma_{\epsilon}^{\prime}\right)\left(1-\gamma_{\epsilon 1}^{\prime}\right)\left(G_{\epsilon} G_{\epsilon 1}-1\right) d \sigma_{v, v_{1}}(\omega) M_{1} d v_{1} \\
& +\frac{k\left(G_{\epsilon}\right)}{\epsilon\left\langle\tilde{G}_{\epsilon}\right\rangle} M \iint \gamma_{\epsilon 1} \gamma_{\epsilon}^{\prime}\left(1-\gamma_{\epsilon 1}^{\prime}\right)\left(G_{\epsilon} G_{\epsilon 1}-G_{\epsilon}^{\prime}\right) d \sigma_{v, v_{1}}(\omega) M_{1} d v_{1} \\
& +\frac{k\left(G_{\epsilon}\right)}{\epsilon\left\langle\tilde{G}_{\epsilon}\right\rangle} M \iint \gamma_{\epsilon 1}\left(1-\gamma_{\epsilon}^{\prime}\right) \gamma_{\epsilon 1}^{\prime}\left(G_{\epsilon} G_{\epsilon 1}-G_{\epsilon 1}^{\prime}\right) d \sigma_{v, v_{1}}(\omega) M_{1} d v_{1} \\
& -\frac{\epsilon k\left(G_{\epsilon}\right)}{\left\langle\tilde{G}_{\epsilon}\right\rangle} M \iint \gamma_{\epsilon 1} \gamma_{\epsilon}^{\prime} \gamma_{\epsilon 1}^{\prime}{ }^{b} q_{\epsilon} d \sigma_{v, v_{1}}(\omega) M_{1} d v_{1} \\
& -\frac{\epsilon^{3} k\left(G_{\epsilon}\right)}{\left\langle\tilde{G}_{\epsilon}\right\rangle} M \iint \gamma_{\epsilon 1} \gamma_{\epsilon}^{\prime} \gamma_{\epsilon 1}^{\prime} q_{\epsilon} d \sigma_{v, v_{1}}(\omega) M_{1} d v_{1} \\
& =T_{11}+T_{12}+T_{13}+T_{14}+T_{15}+T_{16} .
\end{align*}
$$

In this decomposition of $T_{1}$, the first four terms result from using the truncated density $\tilde{G}_{\epsilon}$ (instead of $G_{\epsilon}$ ) in the definition of the local pseudoequilibrium (2.13).

The first statement in Proposition 2.2 implies that

$$
\begin{gather*}
0 \leq \frac{1}{2} \gamma\left(G_{\epsilon}\right) \leq \gamma\left(G_{\epsilon}\right) G_{\epsilon} \leq \frac{3}{2} \gamma\left(G_{\epsilon}\right) \leq \frac{3}{2} ;  \tag{6.3}\\
\text { thus } \frac{1}{2} \leq \tilde{G}_{\epsilon} \leq \frac{3}{2} \text { and } \frac{1}{2} \leq\left\langle\tilde{G}_{\epsilon}\right\rangle \leq \frac{3}{2} \text { a.e. }
\end{gather*}
$$

while the assumptions on $k$ imply that

$$
\begin{equation*}
0 \leq k\left(G_{\epsilon}\right) \leq 1 \text { and } 0 \leq k\left(G_{\epsilon}\right) G_{\epsilon} \leq C_{k} . \tag{6.4}
\end{equation*}
$$

These two bounds immediately imply that

$$
\begin{equation*}
\left|T_{11}\right| \leq 2\left(C_{k}+\frac{9}{4}\right) M\left\langle\frac{1-\gamma\left(G_{\epsilon}\right)}{\epsilon}\right\rangle \tag{6.5}
\end{equation*}
$$

Then

$$
\begin{align*}
\left|T_{13}\right|+\left|T_{14}\right| & \leq\left(3 C_{k}+3\right) M \iint\left(\frac{1-\gamma_{\epsilon 1}^{\prime}}{\epsilon}+\frac{1-\gamma_{\epsilon}^{\prime}}{\epsilon}\right) d \sigma_{v, v_{1}}(\omega) M_{1} d v_{1}  \tag{6.6}\\
& =\left(6 C_{k}+6\right) \mathcal{A}^{+}\left(M \frac{1-\gamma\left(G_{\epsilon}\right)}{\epsilon}, M\right)
\end{align*}
$$

using again the relation $M^{\prime} M_{1}^{\prime}=M M_{1}$ recalled above as well as the bounds (6.3) and (6.4). Likewise

$$
\begin{align*}
\left|T_{12}\right| & \leq\left(3 C_{k}+2\right) M \iint \frac{\left(1-\gamma_{\epsilon}^{\prime}\right)\left(1-\gamma_{\epsilon 1}^{\prime}\right)}{\epsilon} d \sigma_{v, v_{1}}(\omega) M_{1} d v_{1} \\
& \leq\left(3 C_{k}+2\right) M \iint\left(\frac{1-\gamma_{\epsilon}^{\prime}}{2 \epsilon}+\frac{1-\gamma_{\epsilon 1}^{\prime}}{2 \epsilon}\right) d \sigma_{v, v_{1}}(\omega) M_{1} d v_{1}  \tag{6.7}\\
& =\left(3 C_{k}+2\right) \mathscr{A}^{+}\left(M \frac{1-\gamma\left(G_{\epsilon}\right)}{\epsilon}, M\right),
\end{align*}
$$

where the second inequality in (6.7) come from the elementary estimate $\alpha \beta \leq \frac{1}{2}(\alpha+\beta)$ which holds for all $\alpha, \beta \in[0,1]$. Since $\gamma$ and $k$ are bounded by 1 while the collision kernel $b$ satisfies the assumption (H1), one has

$$
\begin{align*}
\left|T_{15}\right| & \leq 2 M \epsilon\left(\iint k\left(G_{\epsilon}\right) \gamma_{\epsilon 1} G_{\epsilon} G_{\epsilon 1} d \sigma_{v, v_{1}}(\omega) M_{1} d v_{1}\right)^{1 / 2} \\
& \times\left(\iint \frac{q_{\epsilon}^{2}}{G_{\epsilon} G_{\epsilon 1}} d \sigma_{v, v_{1}}(\omega) M_{1} d v_{1}\right)^{1 / 2}  \tag{6.8}\\
& \leq 2 \epsilon \sqrt{\frac{6 b_{\infty}}{c}} \sqrt{C_{k}} \sqrt{M} \sqrt{\frac{D\left(F_{\epsilon}\right)}{\epsilon^{4}}}
\end{align*}
$$

by the Cauchy-Schwarz inequality and the first statement in Proposition 2.3. Likewise

$$
\begin{equation*}
\left|T_{16}\right| \leq\left. 2 M \epsilon^{3} \iint\right|^{\sharp} q_{\epsilon} \left\lvert\, d \sigma_{v, v_{1}}(\omega) M_{1} d v_{1} \leq \frac{8 b_{\infty}}{c^{\prime}} \frac{D\left(F_{\epsilon}\right)}{\epsilon}\right. \tag{6.9}
\end{equation*}
$$

by the second statement in Proposition 2.3. On the other hand, the definition (2.9) implies that $\left|{ }^{\sharp} q_{\epsilon}\right| \leq \frac{1}{\epsilon^{2}}\left|q_{\epsilon}\right|$; thus

$$
\begin{equation*}
\left|T_{16}\right| \leq 2 M \epsilon^{3} \iint \frac{1}{\epsilon^{2}} k\left(G_{\epsilon}\right) \gamma_{\epsilon 1} \gamma_{\epsilon}^{\prime} \gamma_{\epsilon 1}^{\prime}\left|q_{\epsilon}\right| d \sigma_{v, v_{1}}(\omega) M_{1} d v_{1} \leq\left(3 C_{k}+\frac{9}{2}\right) \frac{1}{\epsilon} M, \tag{6.10}
\end{equation*}
$$

since the bounds (6.3) and (6.4) imply that

$$
\epsilon^{2} k\left(G_{\epsilon}\right) \gamma_{\epsilon 1} \gamma_{\epsilon}^{\prime} \gamma_{\epsilon 1}^{\prime}\left|q_{\epsilon}\right| \leq k\left(G_{\epsilon}\right) \gamma_{\epsilon 1} \gamma_{\epsilon}^{\prime} \gamma_{\epsilon 1}^{\prime}\left(G_{\epsilon} G_{\epsilon 1}+G_{\epsilon}^{\prime} G_{\epsilon 1}^{\prime}\right) \leq \frac{3}{2} C_{k}+\frac{9}{4} .
$$

Because of the elementary inequality $\min (a, b) \leq \sqrt{a b}$ which holds for all positive $a$ and $b$, the inequalities (6.9) and (6.10) imply

$$
\begin{equation*}
\left|T_{16}\right| \leq 2 \sqrt{\frac{6 b_{\infty}}{c^{\prime}}} \sqrt{C_{k}+\frac{3}{2}} \epsilon \sqrt{M} \sqrt{\frac{D\left(F_{\epsilon}\right)}{\epsilon^{4}}} . \tag{6.11}
\end{equation*}
$$

Estimating $T_{2}$ is simpler: the decomposition of it written in Sect. 2 and the obvious relation $\mathscr{A}^{+}(M, M)=M$ (that follows from the equality $M^{\prime} M_{1}^{\prime}=M M_{1}$ recalled above) lead to

$$
\begin{align*}
T_{2} & =\frac{k\left(G_{\epsilon}\right)}{\epsilon\left\langle\tilde{G}_{\epsilon}\right\rangle}\left(\mathscr{A}^{+}\left(M \tilde{G}_{\epsilon}, M \tilde{G}_{\epsilon}\right)-\mathscr{A}^{+}(M, M)\right) \\
& +\frac{k\left(G_{\epsilon}\right)}{\epsilon} \mathcal{A}^{+}(M, M)\left(\frac{1}{\left\langle\tilde{G}_{\epsilon}\right\rangle}-1\right)  \tag{6.12}\\
& =\frac{k\left(G_{\epsilon}\right)}{\left\langle\tilde{G}_{\epsilon}\right\rangle} \mathcal{A}^{+}\left(M^{b}{ }_{g}, M\left(\tilde{G}_{\epsilon}+1\right)\right)-\frac{k\left(G_{\epsilon}\right)}{\left\langle\tilde{G}_{\epsilon}\right\rangle} M\left\langle g_{\epsilon}\right\rangle \\
& =T_{21}+T_{22} .
\end{align*}
$$

Ву (6.3),

$$
\begin{equation*}
\left|T_{21}\right| \leq 2 \mathcal{A}^{+}\left(M\left|g_{\epsilon}\right|, \frac{5}{2} M\right)=5 \mathcal{A}^{+}\left(M\left|g_{\epsilon}\right|, M\right) ; \tag{6.13}
\end{equation*}
$$

for the same reason

$$
\begin{equation*}
\left|T_{22}\right| \leq 2 M\langle | g_{\epsilon}| \rangle . \tag{6.14}
\end{equation*}
$$

Using the decomposition (6.2) with the bounds (6.5), (6.6), (6.7), (6.8) and (6.11) and the decomposition (6.12) with the bounds (6.13) and (6.14) leads to the inequality (2.17).

### 6.2. Proof of Proposition 2.7

6.2.1. Proof of the first statement. Consider the inequality (2.17) with the choice $k=\gamma$, where $\gamma$ is the function involved in the Flat-Sharp decomposition (2.4). There, $C_{k}=\frac{3}{2}$ so that (2.17) and the second inequality in Proposition 2.2 imply that, for some positive constant $C$ (depending only on $b_{\infty}$ ),

$$
\begin{aligned}
\sqrt{M}\left|g_{\epsilon}\right| & \leq C \epsilon \sqrt{\frac{D\left(F_{\epsilon}\right)}{\epsilon^{4}}}+\left.C \sqrt{M}\langle |\right|_{g_{\epsilon}}\left|+\left|\sharp g_{\epsilon}\right|^{1 / 2}\right\rangle \\
& +\frac{C}{\sqrt{M}} A^{+}\left(M\left[\left|g_{\epsilon}\right|+\left|\left.\right|^{\sharp} g_{\epsilon}\right|^{1 / 2}\right], M\right) .
\end{aligned}
$$

Therefore, using the Caflisch-Grad estimates (particularly the second continuity statement in Proposition 2.5) shows that, for some positive constant $C^{\prime}$ (depending only on $b_{\infty}$ )

$$
\begin{aligned}
M\left|g_{\epsilon}\right|^{2} & \leq C^{\prime} \epsilon^{2} \frac{D\left(F_{\epsilon}\right)}{\epsilon^{4}} \\
& +C^{\prime}\left(\left\|g_{\epsilon}\right\|_{L^{2}(M d v)}^{2}+\| \|_{\epsilon} g_{\epsilon} \|_{L^{1}(M d v)}\right)\left(M+(1+|v|)^{-3}\right) .
\end{aligned}
$$

Because of the dissipation control (2.7) and the first and second statements in Proposition 2.1, this last inequality shows that

$$
\left.\left.M\right|^{\mathrm{b}} g_{\epsilon}\right|^{2}=a_{\epsilon}+b_{\epsilon} \text { with } a_{\epsilon}=O\left(\epsilon^{2}\right)_{L_{t, x, v}^{1}} \text { and } b_{\epsilon}=O(1)_{L_{t}^{\infty}\left(L_{x}^{1}\left(L_{v}^{\infty}\right)\right)} \text {. }
$$

Let $\epsilon_{n}$ be any sequence converging to 0 , let $Q=Q_{1} \times Q_{2}$ be a compact subset of $\mathbf{R}_{+} \times \mathbf{R}^{3}$ and let $\eta>0$ be chosen arbitrarily small. Pick $N>0$ such that, for all $n \geq N$,

$$
\begin{equation*}
\left\|a_{\epsilon_{n}}\right\|_{L_{t, x, v}^{1}}<\eta \tag{6.15}
\end{equation*}
$$

Then, pick $\alpha>0$ satisfying

$$
\begin{equation*}
\alpha\left|Q_{1}\right| \sup _{n \geq 0}\left\|b_{\epsilon_{n}}\right\|_{L_{t}^{\infty}\left(L_{x}^{1}\left(L_{v}^{\infty}\right)\right)}<\eta \tag{6.16}
\end{equation*}
$$

and such that, for each measurable set $B \subset \mathbf{R}_{+} \times \mathbf{R}^{3} \times \mathbf{R}^{3}$ satisfying $|B|<\alpha|Q|$,

$$
\begin{equation*}
\sup _{1 \leq n \leq N} \iiint_{B}\left|a_{\epsilon_{n}}\right| d t d x d v<\eta \tag{6.17}
\end{equation*}
$$

Consider a measurable family $\left(A_{t, x}\right)_{(t, x) \in Q}$ of measurable subsets of $\mathbf{R}_{v}^{3}$ such that

$$
\begin{equation*}
\sup _{(t, x) \in Q}\left|A_{t, x}\right|<\alpha \tag{6.18}
\end{equation*}
$$

Then, by using (6.15), (6.16), (6.18) and (6.17) with

$$
B=\left\{(t, x, v) \mid(t, x) \in Q, v \in A_{t, x}\right\},
$$

one has

$$
\iint \mathbf{1}_{Q}\left(\left.\left.\int_{A_{t, x}} M\right|^{b} g_{\epsilon_{n}}\right|^{2} d v\right) d t d x \leq \eta+\alpha\left|Q_{1}\right|\left\|b_{\epsilon}\right\|_{L_{t}^{\infty}\left(L_{x}^{1}\left(L_{v}^{\infty}\right)\right)}<2 \eta .
$$

This immediately entails the first statement in Proposition 2.7.
6.2.2. Proof of the second statement. Set $k(z)=\min \left(1, \frac{3}{z}\right)$ for $z \in \mathbf{R}_{+}^{*}$; then $C_{k}=3$. Observe that

$$
\begin{aligned}
k\left(G_{\epsilon}\right)\left|g_{\epsilon}\right| & =\left|g_{\epsilon}\right| \mathbf{1}_{G_{\epsilon} \leq 3}+\frac{3\left|g_{\epsilon}\right|}{G_{\epsilon}} \mathbf{1}_{G_{\epsilon}>3} \\
& \leq\left|g_{\epsilon}\right| \mathbf{1}_{G_{\epsilon} \leq 3}+\frac{3\left|g_{\epsilon}\right|}{\frac{2}{3}+\frac{1}{3} G_{\epsilon}}
\end{aligned}
$$

so that, by the fourth statement in Proposition 2.1 and the last statement of Theorem 11.1 (see below in Appendix B), the family

$$
\begin{equation*}
k\left(G_{\epsilon}\right) g_{\epsilon} \text { is bounded in } L_{t}^{\infty}\left(L^{2}(M d v d x)\right) \tag{6.19}
\end{equation*}
$$

With the choice of $k$ above, one sees immediately that, for each $z \in \mathbf{R}_{+}^{*}$,

$$
\begin{equation*}
\gamma(z) \leq k(z), \quad \frac{1}{4}(1-\gamma(z)) \leq k(z)|z-1|(1-\gamma(z)) \leq k(z)|z-1| . \tag{6.20}
\end{equation*}
$$

Therefore

$$
\left|g_{\epsilon}\right|+\frac{1-\gamma\left(G_{\epsilon}\right)}{\epsilon} \leq 5 k\left(G_{\epsilon}\right)\left|g_{\epsilon}\right|
$$

so that (2.17) and the second pointwise control in Proposition 2.2 imply that

$$
\begin{align*}
k\left(G_{\epsilon}\right) \sqrt{M}\left|g_{\epsilon}\right| & \leq k\left(G_{\epsilon}\right) \sqrt{M}\left|g_{\epsilon}\right| \mathbf{1}_{|v|^{2}>V_{\epsilon}}+C \epsilon \sqrt{\frac{D\left(F_{\epsilon}\right)}{\epsilon^{4}}} \mathbf{1}_{|v|^{2} \leq V_{\epsilon}} \\
& +C \sqrt{M}\langle | g_{\epsilon}\left|+\left|\sharp g_{\epsilon}\right|^{1 / 2}\right\rangle  \tag{6.21}\\
& +\frac{C}{\sqrt{M}} \mathcal{A}^{+}\left(\operatorname{Mk}\left(G_{\epsilon}\right)\left|g_{\epsilon}\right|, M\right)
\end{align*}
$$

for some constant $C>0$.
By the last statement in Proposition 2.2

$$
\begin{aligned}
k\left(G_{\epsilon}\right) \sqrt{M}\left|g_{\epsilon}\right| \mathbf{1}_{|v|^{2}>V_{\epsilon}} & =\frac{1}{\epsilon} k\left(G_{\epsilon}\right)\left|G_{\epsilon}-1\right|\left(\sqrt{M} \mathbf{1}_{|v|^{2}>V_{\epsilon}}\right) \\
& =O\left(\frac{1}{\epsilon} e^{-\frac{1}{4} V_{\epsilon}} V_{\epsilon}^{\frac{p}{2}+\frac{1}{4}}\right)_{L_{t, x}^{\infty}\left(L_{v}^{2, p}\right)}
\end{aligned}
$$

for all $p \geq 0$; next, the dissipation bound (2.7) implies that

$$
\begin{aligned}
\epsilon \sqrt{\frac{D\left(F_{\epsilon}\right)}{\epsilon^{4}}} \mathbf{1}_{|v|^{2} \leq V_{\epsilon}} & =\sqrt{\frac{D\left(F_{\epsilon}\right)}{\epsilon^{4}}}\left(\epsilon(1+|v|)^{p} \mathbf{1}_{|v|^{2} \leq V_{\epsilon}}\right)(1+|v|)^{-p} \\
& =O(1)_{L_{t, x, v}^{2}} O\left(\epsilon V_{\epsilon}^{\frac{p}{2}}\right)(1+|v|)^{-p}
\end{aligned}
$$

Finally, the first and second entropy controls in Proposition 2.1 imply that

$$
\sqrt{M}\langle | g_{\epsilon}\left|+\left|\sharp g_{\epsilon}\right|^{1 / 2}\right\rangle=O(1)_{\left.L_{t}^{\infty}\left(L_{x}^{2}\left(L_{v}^{2, p}\right)\right)\right)}
$$

for all $p \geq 0$.
Using the last three estimates with the choice $V_{\epsilon}=10|\log \epsilon|$ in the inequality (6.21) implies that

$$
\begin{equation*}
k\left(G_{\epsilon}\right) \sqrt{M}\left|g_{\epsilon}\right| \leq O(1)_{L_{l o c}^{2}\left(d t d x ; L_{v}^{2, p}\right)}+\frac{C}{\sqrt{M}} \mathcal{A}^{+}\left(M k\left(G_{\epsilon}\right)\left|g_{\epsilon}\right|, M\right) \tag{6.22}
\end{equation*}
$$

for all $p \geq 0$. Pick such a $p>1$; because of (6.19) and the second continuity statement in Proposition 2.5, one has

$$
k\left(G_{\epsilon}\right) \sqrt{M}\left|g_{\epsilon}\right|=O(1)_{L_{l o c}^{2}\left(d t d x ; L_{v}^{2, p}\right)}+O(1)_{L_{l o c}^{2}\left(d t d x ; L_{v}^{\infty, 3 / 2}\right)}
$$

In (6.22), using the third and fourth continuity statements in Proposition 2.5 with $\sigma=p$, one gets the control

$$
\begin{aligned}
k\left(G_{\epsilon}\right) \sqrt{M}\left|g_{\epsilon}\right| & =O(1)_{L_{l o c}^{2}\left(d t d x ; L_{v}^{2, p}\right)}+O(1)_{L_{l o c}^{2}\left(d t d x ; L_{v}^{\infty, p+3 / 2}\right)} \\
& +O(1)_{L_{l o c}^{2}\left(d t d x ; L_{v}^{\infty, 3 / 2+2 N}\right)}
\end{aligned}
$$

by induction on $N \geq 1$. For $N>p / 2$ this implies that

$$
\begin{aligned}
k\left(G_{\epsilon}\right) \sqrt{M}\left|g_{\epsilon}\right| & =O(1)_{L_{l o c}^{2}\left(d t d x ; L_{v}^{2, p}\right)}+O(1)_{L_{l o c}^{2}\left(d t d x ; L_{v}^{\infty, p+3 / 2}\right)} \\
& =O(1)_{L_{l o c}^{2}\left(d t d x ; L_{v}^{2, p-1}\right)}
\end{aligned}
$$

because $L^{\infty, p+3 / 2}\left(\mathbf{R}^{3}\right) \subset L^{2, p-1}\left(\mathbf{R}^{3}\right)$. Since $p>1$ was arbitrary, this implies that, for all $p \geq 0$

$$
\begin{equation*}
k\left(G_{\epsilon}\right) \sqrt{M}\left|g_{\epsilon}\right|=O(1)_{L_{l o c}^{2}\left(d t d x ; L_{v}^{2, p}\right)} . \tag{6.23}
\end{equation*}
$$

This control and the first inequality in (6.20) establish the second statement in Proposition 2.7.
6.2.3. Proof of the third statement. Let $t^{*}>0$ and $Q$ be a compact subset of $\mathbf{R}^{3}$. By the Cauchy-Schwarz inequality

$$
\begin{aligned}
& \left\|\frac{k\left(G_{\epsilon}\right)}{\epsilon} M\left|g_{\epsilon}\right|\left(1-\gamma\left(G_{\epsilon}\right)\right)(1+|v|)^{p}\right\|_{L^{1}\left(\left[0, t^{*}\right] \times Q ; L_{v}^{1}\right)} \\
& \leq\left\|k\left(G_{\epsilon}\right) \sqrt{M}\left|g_{\epsilon}\right|(1+|v|)^{p}\right\|_{L^{2}\left(\left[0, t^{*}\right] \times Q ; L_{v}^{2}\right)}\left\|\frac{1-\gamma\left(G_{\epsilon}\right)}{\epsilon}\right\|_{L^{2}\left(\left[0, t^{*}\right] \times Q ; L^{2}(M d v)\right)} \\
& \leq\left\|k\left(G_{\epsilon}\right) \sqrt{M}\left|g_{\epsilon}\right|(1+|v|)^{p}\right\|_{L^{2}\left(\left[0, t^{*}\right] \times Q ; L_{v}^{2}\right)}\left\|\left.\left.2\right|^{\sharp} g_{\epsilon}\right|^{1 / 2}\right\|_{L^{2}\left(\left[0, t^{*}\right] \times Q ; L^{2}(M d v)\right)},
\end{aligned}
$$

where the last inequality follows from the second pointwise estimate in Proposition 2.2. Thus, because of (6.23) and the second statement in Proposition 2.1

$$
\frac{k\left(G_{\epsilon}\right)}{\epsilon^{2}} M\left|G_{\epsilon}-1\right|\left(1-\gamma\left(G_{\epsilon}\right)\right)(1+|v|)^{p}=O(1)_{L_{l o c}^{1}\left(d t d x ; L_{v}^{1}\right)}
$$

this and the second inequality in (6.20) imply the third statement in Proposition 2.7.

## 7. Proving Proposition 3.4 and Corollary 3.5

The proof follows the argument sketched in Sect. 3.
7.1. Step 1: truncating large values of $G_{\epsilon}$. Choose $\left.\delta \in\right] 0,1\left[\right.$ and a $C^{1}$ function $k^{\delta}: \mathbf{R}_{+} \rightarrow[0,1]$ such that $k^{\delta} \equiv 1$ on $\left[0, e^{1 / \delta}\right], k^{\delta} \equiv 0$ on $\left[2 e^{1 / \delta}+\infty\right)$ and $\left\|\left(k^{\delta}\right)^{\prime}\right\|_{L^{\infty}} \leq 2 e^{-1 / \delta}$. Thus $C_{k^{\delta}} \leq 2 e^{1 / \delta}$. We use the notation $k_{\epsilon}^{\delta}$ below to designate $k^{\delta}\left(G_{\epsilon}\right)$.

We start from the decomposition

$$
\begin{equation*}
{ }^{\sharp} g_{\epsilon}=k_{\epsilon}^{\delta} \sharp g_{\epsilon}+\left(1-k_{\epsilon}^{\delta}\right) \sharp g_{\epsilon} \tag{7.1}
\end{equation*}
$$

and use the last statement of Proposition 2.1 which shows that

$$
\begin{align*}
\left\|\left(1-k_{\epsilon}^{\delta}\right) \sharp g_{\epsilon}\right\|_{L_{t}^{\infty}\left(L^{1}(M d v d x)\right)} & \leq\left\|\frac{1}{\epsilon} g_{\epsilon} \mathbf{1}_{G_{\epsilon} \geq e^{1 / \delta}}\right\|_{L_{t}^{\infty}\left(L^{1}(M d v d x)\right)}  \tag{7.2}\\
& \leq C^{i n} \frac{1}{\frac{1}{\delta}-1}=C^{i n} \frac{\delta}{1-\delta}
\end{align*}
$$

7.2. Step 2: truncating large velocities. For any family $V_{\epsilon}>0$ and each $p \geq 0$, one has

$$
\begin{aligned}
\left.k_{\epsilon}^{\delta} M\right|^{\sharp} g_{\epsilon} \mid \mathbf{1}_{|v|^{2}>V_{\epsilon}} & \leq k_{\epsilon}^{\delta}\left|G_{\epsilon}-1\right| V_{\epsilon}^{-p / 2}\left(M \frac{1-\gamma\left(G_{\epsilon}\right)}{\epsilon^{2}}|v|^{p} \mathbf{1}_{|v|^{2}>V_{\epsilon}}\right) \\
& \leq 2 e^{1 / \delta} V_{\epsilon}^{-p / 2} O(1)_{\left.L_{l o c}^{1}\left(d t d x ; L_{v}^{1}\right)\right)}
\end{aligned}
$$

because of the last statement in Proposition 2.7. Therefore, for each $p \geq 0$, each $t^{*}>0$, and each compact $Q \subset \mathbf{R}^{3}$

$$
\begin{equation*}
\left\|k_{\epsilon}^{\delta \sharp} g_{\epsilon} \mathbf{1}_{|v|^{2}>V_{\epsilon}}\right\|_{L^{1}\left(\left[0, t^{*}\right] \times Q ; L^{1}(M d v)\right)}=O\left(\frac{e^{1 / \delta}}{V_{\epsilon}^{p / 2}}\right) . \tag{7.3}
\end{equation*}
$$

7.3. Step 3: $L^{1}-L^{\infty}$ controls. First we recall that, because of the second pointwise estimate in Proposition 2.2 and the second entropy control in Proposition 2.1, one has

$$
\begin{equation*}
\left\|M\left(1-\gamma\left(G_{\epsilon}\right)\right)\right\|_{L_{t}^{\infty}\left(L_{x, v}^{1}\right)}=O\left(\epsilon^{2}\right), \text { while }\left\|1-\gamma\left(G_{\epsilon}\right)\right\|_{L_{t, x, v}^{\infty}} \leq 1 \tag{7.4}
\end{equation*}
$$

Therefore

$$
\begin{align*}
&\left\|\left(1-\gamma\left(G_{\epsilon}\right)\right) \mathbf{1}_{|v|^{2} \leq V_{\epsilon}}\right\|_{L_{t}^{\infty}\left(L_{x, v}^{1}\right)} \\
& \leq\left\|\mathbf{1}_{|v|^{2} \leq V_{\epsilon}} M^{-1}\right\|_{L_{v}^{\infty}}\left\|M\left(1-\gamma\left(G_{\epsilon}\right)\right)\right\|_{L_{t}^{\infty}\left(L_{x, v}^{1}\right)}  \tag{7.5}\\
&=O\left(e^{\frac{1}{2} V_{\epsilon}} \epsilon^{2}\right)
\end{align*}
$$

Applying Lemma 3.1 and taking into account the second equality in (7.4) and (7.5) implies the existence of $\phi_{\epsilon}^{0}$ and $\phi_{\epsilon}^{1}$ such that

$$
\begin{gather*}
0 \leq \phi_{\epsilon}^{j} \leq\left(1-\gamma\left(G_{\epsilon}\right)\right) \mathbf{1}_{|v|^{2} \leq V_{\epsilon}}, \quad j=0,1,  \tag{7.6}\\
\phi_{\epsilon}^{0}=O\left(\epsilon e^{\frac{1}{4} V_{\epsilon}}\right)_{L_{t}^{\infty}\left(L_{x}^{1}\left(L_{v}^{\infty}\right)\right)} \\
\phi_{\epsilon}^{1}=O\left(\epsilon e^{\frac{1}{4} V_{\epsilon}}\right)_{L_{t, x}^{\infty}\left(L_{v}^{1}\right)}
\end{gather*}
$$

and

$$
\begin{equation*}
\left(1-\gamma\left(G_{\epsilon}\right)\right) \mathbf{1}_{|v|^{2} \leq V_{\epsilon}}=\phi_{\epsilon}^{0}+\phi_{\epsilon}^{1} . \tag{7.8}
\end{equation*}
$$

### 7.4. Step 4: Applying the Advection/Dispersion interpolation

### 7.4.1. Controlling the advection term. Set

$$
\tilde{k}_{\delta}^{\epsilon}=\left(|Z-1| k^{\delta}(Z)\right)_{\mid Z=G_{\epsilon}}^{\prime}=\operatorname{sgn}\left(G_{\epsilon}-1\right) k^{\delta}\left(G_{\epsilon}\right)+\left|G_{\epsilon}-1\right|\left(k^{\delta}\right)^{\prime}\left(G_{\epsilon}\right)
$$

Lemma 7.1. The family of relative fluctuations $g_{\epsilon}$ satisfies the estimate

$$
\left|\left(\epsilon \partial_{t}+v \cdot \nabla_{x}\right)\left(\frac{k_{\delta}^{\epsilon}}{\epsilon}\left|g_{\epsilon}\right|\right)\right| \leq O\left(e^{1 / \delta}\right)_{L_{l o c}^{1}\left(d t ; L^{1}(M d v d x)\right)}+O\left(\frac{e^{1 / \delta}}{\epsilon^{2}}\right)_{L_{t, x, v}^{\infty}}
$$

Proof. Because $F_{\epsilon}$ is a renormalized solution relative to $M$ of (1.13), using the renormalized formulation (1.19) with $\Gamma(Z)=|Z-1| k^{\delta}(Z)$ leads to

$$
\left(\epsilon \partial_{t}+v \cdot \nabla_{x}\right)\left(\frac{k_{\delta}^{\epsilon}}{\epsilon}\left|g_{\epsilon}\right|\right)=\iint \frac{\tilde{k}_{\delta}^{\epsilon}}{\epsilon} q_{\epsilon} b d \sigma_{v, v_{1}}(\omega) M_{1} d v_{1} .
$$

Young's inequality (10.4) in Appendix A below implies that

$$
\begin{aligned}
\iint \frac{\left|\tilde{k}_{\delta}^{\epsilon}\right|}{\epsilon}\left|q_{\epsilon}\right| & b d \sigma_{v, v_{1}}(\omega) M_{1} d v_{1} \\
& =\iint \frac{\left|\tilde{k}_{\delta}^{\epsilon}\right|}{\epsilon^{4}} \frac{\epsilon^{2}\left|q_{\epsilon}\right|}{G_{\epsilon} G_{\epsilon 1}} \epsilon G_{\epsilon} G_{\epsilon 1} b d \sigma_{v, v_{1}}(\omega) M_{1} d v_{1} \\
& \leq \frac{\left|\tilde{k}_{\delta}^{\epsilon}\right|}{\epsilon^{4}} \iint r\left(\frac{\epsilon^{2}\left|q_{\epsilon}\right|}{G_{\epsilon} G_{\epsilon 1}}\right) G_{\epsilon} G_{\epsilon 1} b d \sigma_{v, v_{1}}(\omega) M_{1} d v_{1} \\
& +\frac{\left|\tilde{k}_{\delta}^{\epsilon}\right|}{\epsilon^{4}} h^{*}(\epsilon) G_{\epsilon} \iint G_{\epsilon 1} b d \sigma_{v, v_{1}}(\omega) M_{1} d v_{1} \\
& \leq \frac{4\left|\tilde{k}_{\delta}^{\epsilon}\right|}{\epsilon^{4}} \frac{D\left(F_{\epsilon}\right)}{M}+\frac{h^{*}(1) b_{\infty}\left|\tilde{k}_{\delta}^{\epsilon}\right|}{\epsilon^{2}} G_{\epsilon}\left\langle G_{\epsilon}\right\rangle
\end{aligned}
$$

where the last inequality comes from the superquadratic property of $h^{*}$ (see (10.3) in Appendix A below), the equality in (2.7) and the bound on the collision kernel implied by (H1). Since $\left|\tilde{k}_{\delta}^{\epsilon}\right| \leq 1+2 e^{-1 / \delta} \cdot 2 e^{1 / \delta} \leq 5$ and $\tilde{k}_{\delta}^{\epsilon} \equiv 0$ whenever $G_{\epsilon} \geq 2 e^{1 / \delta}$, one has

$$
\begin{aligned}
& \iint \frac{\left|\tilde{k}_{\delta}^{\epsilon}\right|}{\epsilon}\left|q_{\epsilon}\right| b M_{1} d v_{1} d \sigma_{v, v_{1}}(\omega)
\end{aligned} \begin{aligned}
& \text { (7.9) } \begin{aligned}
\epsilon^{4} & \frac{20}{M} \\
& +\frac{h^{*}(1) b_{\infty} 10 e^{1 / \delta}}{\epsilon^{2}}\left|\left\langle G_{\epsilon}-1\right\rangle\right| \mathbf{1}_{\left|\left\langle G_{\epsilon}\right\rangle-1\right|>1 / 2} \\
& +\frac{h^{*}(1) b_{\infty} 10 e^{1 / \delta}}{\epsilon^{2}}\left(1+\left|\left\langle G_{\epsilon}-1\right\rangle\right| \mathbf{1}_{\left|\left\langle G_{\epsilon}\right\rangle-1\right| \leq 1 / 2}\right) \\
& \leq I_{1}+I_{2}
\end{aligned}
\end{aligned}
$$

with

$$
\begin{equation*}
I_{1}=\frac{20}{\epsilon^{4}} \frac{D\left(F_{\epsilon}\right)}{M}+\frac{h^{*}(1) b_{\infty} 10 e^{1 / \delta}}{\epsilon^{2}}\left|\left\langle G_{\epsilon}-1\right\rangle\right| \mathbf{1}_{\left|\left\langle G_{\epsilon}\right\rangle-1\right|>1 / 2} \tag{7.10}
\end{equation*}
$$

and

$$
\begin{equation*}
I_{2}=\frac{h^{*}(1) b_{\infty} 10 e^{1 / \delta}}{\epsilon^{2}}\left(1+\left|\left\langle G_{\epsilon}-1\right\rangle\right| \mathbf{1}_{\left|\left\langle G_{\epsilon}\right\rangle-1\right| \leq 1 / 2}\right) \tag{7.11}
\end{equation*}
$$

Because of the elementary inequality

$$
\left|\left\langle G_{\epsilon}-1\right\rangle\right| \mathbf{1}_{\left|\left\langle G_{\epsilon}-1\right\rangle\right|>1 / 2} \leq \epsilon\left|\left\langle g_{\epsilon}\right\rangle\right|\left(1-\gamma\left(\left\langle G_{\epsilon}\right\rangle\right)\right)
$$

the dissipation bound (2.7) and the third entropy control in Proposition 2.1 imply that

$$
\begin{equation*}
I_{1}=O\left(e^{1 / \delta}\right)_{L_{l o c}^{1}\left(d t ; L^{1}(M d v d x)\right)} \tag{7.12}
\end{equation*}
$$

On the other hand

$$
\begin{equation*}
I_{2}=O\left(\frac{e^{1 / \delta}}{\epsilon^{2}}\right)_{L_{t, x, v}^{\infty}} \tag{7.13}
\end{equation*}
$$

Using the inequality (7.9) with (7.12) and (7.13) leads to the announced estimate.
7.4.2. Using the $L^{\infty}$ bound in $v$. We state in the following lemma an important auxiliary estimate which is used below in two different ways.

Lemma 7.2. Let $\phi \equiv \phi(t, x, v)$ be a measurable function defined a.e. on $\mathbf{R}_{+} \times \mathbf{R}^{3} \times \mathbf{R}^{3}$. Then, for each $t^{*}>0$ the family of relative number density fluctuations $g_{\epsilon}$ satisfies the bound

$$
\begin{aligned}
& \left\|\frac { k _ { \delta } ^ { \epsilon } } { \epsilon } M \left|g_{\epsilon}\|\phi \mid\|_{L^{1}\left(\left[0, t^{*}\right] ; L_{x, v}^{1}\right)} \leq C e^{1 / 2 \delta}\|M \phi\|_{L_{t}^{\infty}\left(L_{x, v}^{1}\right.}^{1 / 2}\|\phi\|_{L_{t, x, v}^{\infty}}^{1 / 2}\left\|\frac{D\left(F_{\epsilon}\right)}{\epsilon^{4}}\right\|_{L_{t, x, v}^{1}}^{1 / 2}\right.\right. \\
& +C \frac{e^{1 / \delta}}{\epsilon}\|M \phi\|_{L_{t}^{\infty}\left(L_{x, v}^{1}\right)}^{1 / 2}\|\phi\|_{L_{t, x}^{\infty}\left(L_{v}^{1}\right)}^{1 / 2}\left(\left\|{ }^{b} g_{\epsilon}\right\|_{L_{t}^{\infty}\left(L^{2}(M d v d x)\right)}+\left\|^{\sharp} g_{\epsilon}\right\|_{L_{t}^{\infty}\left(L^{1}(M d v d x)\right)}^{1 / 2}\right) .
\end{aligned}
$$

Proof. Because of the inequality (2.17) used here with $C_{k}=2 e^{1 / \delta}$

$$
\begin{aligned}
\frac{k_{\delta}^{\epsilon}}{\epsilon} M\left|g_{\epsilon}\right||\phi| & \leq 4 \sqrt{4 e^{1 / \delta \frac{3 b_{\infty}\left(c+c^{\prime}\right)}{c c^{\prime}}} \sqrt{M}|\phi| \sqrt{\frac{D\left(F_{\epsilon}\right)}{\epsilon^{4}}}} \\
& +2 \frac{M}{\epsilon}|\phi|\langle | g_{\epsilon}\left|+5 e^{1 / \delta} \frac{1-\gamma\left(G_{\epsilon}\right)}{\epsilon}\right\rangle \\
& +\frac{1}{\epsilon}|\phi| \mathcal{A}^{+}\left(M\left[\left.5\right|^{b} g_{\epsilon} \left\lvert\,+27 e^{1 / \delta} \frac{1-\gamma\left(G_{\epsilon}\right)}{\epsilon}\right.\right], M\right) .
\end{aligned}
$$

Using the second pointwise estimate in Proposition 2.2 and the second Caflisch-Grad estimate in Proposition 2.5 shows that, for some constant $C>0$

$$
\begin{align*}
& \frac{k_{\delta}^{\epsilon}}{\epsilon} M\left|g_{\epsilon}\right||\phi| \leq C e^{1 / 2 \delta} \sqrt{M}|\phi| \sqrt{\frac{D\left(F_{\epsilon}\right)}{\epsilon^{4}}}  \tag{7.14}\\
& \quad+C e^{1 / \delta} \frac{\sqrt{M}}{\epsilon}|\phi|\left((1+|v|)^{-3 / 2}+\sqrt{M}\right)\left(\left\|g_{\epsilon}\right\|_{L^{2}(M d v)}+\left\|^{\sharp} g_{\epsilon}\right\|_{L^{1}(M d v)}^{1 / 2}\right)
\end{align*}
$$

which implies the announced estimate by using the Cauchy-Schwarz inequality.
7.4.3. The Advection/Dispersion interpolation estimate. First apply Lemma 7.2 with $\phi=\phi_{\epsilon}^{1}$. Because of estimates (7.4), (7.6) and (7.7), this leads, for each $t^{*}>0$, to

$$
\begin{aligned}
\left\|\frac { k _ { \delta } ^ { \epsilon } } { \epsilon } M \left|g_{\epsilon}\left\|\phi_{\epsilon}^{1} \mid\right\|_{L^{1}\left(\left[0, t^{*}\right] ; L_{x, v}^{1}\right)}\right.\right. & \leq C e^{1 / 2 \delta} O(\epsilon)\left\|\frac{D\left(F_{\epsilon}\right)}{\epsilon^{4}}\right\|_{L_{t, x, v}^{1}}^{1 / 2} \\
& +C \frac{e^{1 / \delta}}{\epsilon} O(\epsilon) O\left(\sqrt{\epsilon} e^{\frac{1}{8} V_{\epsilon}}\right) \\
& \times\left(\left\|^{\mathrm{b}} g_{\epsilon}\right\|_{L_{t}^{\infty}\left(L^{2}(M d v d x)\right)}+\| \|^{\sharp} g_{\epsilon} \|_{L_{t}^{\infty}\left(L^{1}(M d v d x)\right)}^{1 / 2}\right) \\
& \leq C e^{1 / 2 \delta} O(\epsilon)+C e^{1 / \delta} O\left(\sqrt{\epsilon} e^{\frac{1}{8} V_{\epsilon}}\right) .
\end{aligned}
$$

Now we use the Advection/Dispersion interpolation technique, as explained in Lemma 3.3. Let $t^{*}>0$ and $\tau^{*}>0$. Below, $t^{*}$ is fixed while $\tau$ is the fictitious time used as interpolation parameter. Define $\phi_{\epsilon}=\phi_{\epsilon}(\tau, t, x, v)$ as the solution of the free transport equation (3.3) with initial data $\mathbf{1}_{\left[0, t^{*}\right]}(t) \phi_{\epsilon}^{0}(t, x, v)$. Observe that, for each $\tau \in\left[0, \tau^{*}\right]$,

$$
\begin{align*}
\left\|\phi_{\epsilon}(\tau, \cdot, \cdot, \cdot)\right\|_{L_{t, x, v}^{\infty}} & =1 \\
\left\|\phi_{\epsilon}(\tau, \cdot, \cdot, \cdot)\right\|_{L_{t}^{\infty}\left(L^{1}(M d v d x)\right)} & =O\left(\epsilon^{2}\right), \tag{7.16}
\end{align*}
$$

since the estimates on $\phi_{\epsilon}$ that follow from (7.4) and (7.6) are obviously propagated by the free transport equation (3.3). Because of Lemma 3.2 and the first estimate in (7.7)

$$
\begin{equation*}
\left\|\phi_{\epsilon}\left(\tau^{*}, \cdot, \cdot, \cdot\right)\right\|_{L_{t, x}^{\infty}\left(L_{v}^{1}\right)}=\frac{1}{\tau^{* 3}} O\left(\epsilon e^{\frac{1}{4} V_{\epsilon}}\right) \tag{7.17}
\end{equation*}
$$

Using (7.16), (7.17), and applying Lemma 7.2 with $\phi=\phi_{\epsilon}\left(\tau^{*}, \cdot, \cdot, \cdot\right)$ shows that

$$
\left.\begin{array}{l}
\left\|\frac { k _ { \delta } ^ { \epsilon } } { \epsilon } M \left|g_{\epsilon}\left\|\phi_{\epsilon}\left(\tau^{*}\right) \mid\right\|_{L^{1}\left(\left[0, t^{*}\right] ; L_{x, v}^{1}\right)}\right.\right.
\end{array} \quad \leq C e^{1 / 2 \delta} O(\epsilon)\left\|\frac{D\left(F_{\epsilon}\right)}{\epsilon^{4}}\right\|_{L_{t, x, v}^{1 / 2}}^{1 / 2}\right] \begin{aligned}
& +C \frac{e^{1 / \delta}}{\epsilon} O(\epsilon) \frac{1}{\tau^{* 3 / 2}} O\left(\sqrt{\epsilon} e^{\frac{1}{8} V_{\epsilon}}\right) \\
& \times\left(\| \|^{\mathrm{b}}\left\|_{L_{t}^{\infty}\left(L^{2}(M d v d x)\right)}+\right\|\left\|_{\epsilon}\right\|_{L_{t}^{\infty}\left(L^{1}(M d v d x)\right)}^{1 / 2}\right) \\
& \leq C e^{1 / 2 \delta} O(\epsilon)+\frac{C e^{1 / \delta}}{\tau^{* 3 / 2}} O\left(\sqrt{\epsilon} e^{\frac{1}{8} V_{\epsilon}}\right) .
\end{aligned}
$$

Finally, we apply Lemma 7.1 which, together with the first estimate in (7.16) implies that

$$
\begin{align*}
\int_{\Omega}\left|\phi_{\epsilon}\right| & \left|\left(\epsilon \partial_{t}+v \cdot \nabla_{x}\right)\left(\frac{k_{\delta}^{\epsilon}}{\epsilon}\left|g_{\epsilon}\right|\right)\right| M d v d x d t d \tau \\
& \leq \tau^{*} O\left(e^{1 / \delta}\right)\left\|\phi_{\epsilon}\right\|_{L_{\tau, t, x, v}^{\infty}}+\tau^{*} O\left(\frac{e^{1 / \delta}}{\epsilon^{2}}\right)\left\|\phi_{\epsilon}\right\|_{L_{\tau, t}^{\infty}\left(L^{1}(M d v d x)\right)}  \tag{7.19}\\
& \leq \tau^{*} O\left(e^{1 / \delta}\right)
\end{align*}
$$

where $\Omega$ is as in the statement of Lemma 3.3, i.e.

$$
\Omega=\left\{(\tau, t, x, v) \in \mathbf{R} \times \mathbf{R} \times \mathbf{R}^{3} \times \mathbf{R}^{3} \mid 0<\tau<\tau^{*}, 0<t-\epsilon \tau<t^{*}\right\}
$$

Eventually, we use all three estimates (7.15), (7.18) and (7.19), as explained in Sect. 3 before Proposition 3.4. This leads to

$$
\begin{align*}
& \int_{0}^{t^{*}} \int\left\langle\frac{k_{\delta}^{\epsilon}}{\epsilon}\right| g_{\epsilon}\left|\left(1-\gamma\left(G_{\epsilon}\right)\right) \mathbf{1}_{|v|^{2} \leq V_{\epsilon}}\right\rangle d x d t \\
& \leq \int_{0}^{t^{*}} \int\left\langle\frac{k_{\delta}^{\epsilon}}{\epsilon}\right| g_{\epsilon}\left|\phi_{\epsilon}^{1}\right\rangle d x d t+\int_{0}^{t^{*}} \int\left\langle\frac{k_{\delta}^{\epsilon}}{\epsilon}\right| g_{\epsilon}\left|\phi_{\epsilon}^{0}\right\rangle d x d t \\
&20) \leq \int_{0}^{t^{*}} \int\left\langle\frac{k_{\delta}^{\epsilon}}{\epsilon}\right| g_{\epsilon}\left|\phi_{\epsilon}^{1}\right\rangle d x d t+\int_{\epsilon \tau^{*}}^{t^{*}+\epsilon \tau^{*}} \int\left\langle\frac{k_{\delta}^{\epsilon}}{\epsilon}\right| g_{\epsilon}\left|\phi_{\epsilon}\left(\tau^{*}\right)\right\rangle d x d t  \tag{7.20}\\
&+\int_{\Omega}\left|\phi_{\epsilon}\right|\left|\left(\epsilon \partial_{t}+v \cdot \nabla_{x}\right)\left(\frac{k_{\delta}^{\epsilon}}{\epsilon}\left|g_{\epsilon}\right|\right)\right| M d v d x d t d \tau \\
& \leq C e^{1 / 2 \delta} O(\epsilon)+C e^{1 / \delta} O\left(\sqrt{\epsilon} e^{\frac{1}{8} V_{\epsilon}}\right) \\
&+C e^{1 / 2 \delta} O(\epsilon)+\frac{C e^{1 / \delta}}{\tau^{* 3 / 2}} O\left(\sqrt{\epsilon} e^{\frac{1}{8} V_{\epsilon}}\right)+\tau^{*} O\left(e^{1 / \delta}\right) .
\end{align*}
$$

The second inequality in (7.20) follows from applying Lemma 3.3 to $f=\frac{1}{\epsilon}\left|k_{\delta}^{\epsilon}\right| g_{\epsilon}$ and $\phi=\phi_{\epsilon}$.
7.5. Step 5: Final estimate. We conclude the proof of Proposition 3.4 by using the three estimates (7.2), (7.3) and finally (7.20). Let $t^{*}>0$ and $Q$ be a compact subset of $\mathbf{R}^{3}$; one has

$$
\begin{aligned}
\int_{0}^{t^{*}} \int_{Q}\langle | g_{\epsilon}| \rangle d x d t & \leq \int_{0}^{t^{*}} \int_{Q}\left\langle\frac{1-k_{\delta}^{\epsilon}}{\epsilon}\right| g_{\epsilon}\left|\left(1-\gamma\left(G_{\epsilon}\right)\right)\right\rangle d x d t \\
& +\int_{0}^{t^{*}} \int_{Q}\left\langle\frac{k_{\delta}^{\epsilon}}{\epsilon}\right| g_{\epsilon}\left|\left(1-\gamma\left(G_{\epsilon}\right)\right) \mathbf{1}_{|v|^{2}>V_{\epsilon}}\right\rangle d x d t \\
& +\int_{0}^{t^{*}} \int_{Q}\left\langle\frac{k_{\delta}^{\epsilon}}{\epsilon}\right| g_{\epsilon}\left|\left(1-\gamma\left(G_{\epsilon}\right)\right) \mathbf{1}_{|v|^{2} \leq V_{\epsilon}}\right\rangle d x d t \\
& \leq C^{i n} \frac{\delta t^{*}}{1-\delta}+O\left(\frac{e^{1 / \delta}}{V_{\epsilon}^{p / 2}}\right) \\
& +C e^{1 / 2 \delta} O(\epsilon)+C e^{1 / \delta} O\left(\sqrt{\epsilon} e^{\frac{1}{8} V_{\epsilon}}\right) \\
& +C e^{1 / 2 \delta} O(\epsilon)+\frac{C e^{1 / \delta}}{\tau^{* 3 / 2}} O\left(\sqrt{\epsilon} e^{\frac{1}{8} V_{\epsilon}}\right)+\tau^{*} O\left(e^{1 / \delta}\right)
\end{aligned}
$$

In this last estimate, pick

$$
\delta=\frac{1}{\log |\log \epsilon|} \text { so that } e^{1 / \delta}=|\log \epsilon|
$$

with

$$
V_{\epsilon}=2|\log \epsilon| \text { so that } e^{\frac{1}{8} V_{\epsilon}}=\left(\frac{1}{\epsilon}\right)^{1 / 4}
$$

finally pick

$$
\tau^{*}=\epsilon^{1 / 5} e^{\frac{1}{20} V_{\epsilon}}=\epsilon^{1 / 10}
$$

and set $p>2$. Substituting these values in (7.21) results in

$$
\left\|\sharp g_{\epsilon}\right\|_{L^{1}\left(\left[0, t^{*}\right] \times Q ; L^{1}(M d v)\right)} \leq C\left(\frac{1}{\log |\log \epsilon|}+|\log \epsilon|^{1-p / 2}+\epsilon^{1 / 10}|\log \epsilon|\right)
$$

which concludes the proof of Proposition 3.4.
7.6. Proof of Corollary 3.5. Because of Proposition 3.4 and the second pointwise estimate in Proposition 2.2, one has

$$
\frac{1-\gamma\left(G_{\epsilon}\right)}{\epsilon^{2}}=O\left(\frac{1}{\log |\log \epsilon|}\right) \text { in } L_{l o c}^{1}\left(d t d x ; L^{1}(M d v)\right)
$$

On the other hand, by the last statement of Proposition 2.7

$$
\frac{1-\gamma\left(G_{\epsilon}\right)}{\epsilon^{2}}(1+|v|)^{2 s}=O(1) \text { in } L_{l o c}^{1}\left(d t d x ; L^{1}(M d v)\right)
$$

for all $s>0$. The Cauchy-Schwarz inequality and these last two controls imply that

$$
\frac{1-\gamma\left(G_{\epsilon}\right)}{\epsilon^{2}}(1+|v|)^{s}=O\left(\frac{1}{\sqrt{\log |\log \epsilon|}}\right) \text { in } L_{l o c}^{1}\left(d t d x ; L^{1}(M d v)\right)
$$

as announced.

## 8. Proving Proposition 3.8

8.1. Uniform integrability of $\left|{ }^{\mathrm{b}} g_{\epsilon_{n}}\right|^{2}$. Pick $\gamma \in \Upsilon$. Since $F_{\epsilon}$ is a renormalized solution of (1.13) relatively to $M$, using the nonlinear function $\Gamma(Z)=(Z-1)^{2} \gamma(Z)^{2}$ in the renormalized formulation (1.19) gives

$$
\begin{equation*}
\left(\epsilon \partial_{t}+v \cdot \nabla_{x}\right)^{b} g_{\epsilon}^{2}=2 \iint q_{\epsilon}{ }^{b} g_{\epsilon} \hat{\gamma}_{\epsilon} b d \sigma_{v, v_{1}}(\omega) M_{1} d v_{1} \tag{8.1}
\end{equation*}
$$

with $\hat{\gamma}$ defined in terms of the truncation $\gamma$ by (4.2). By Lemma 5.4, for each $t^{*}>0$,

$$
\begin{align*}
&\left\|\left(\epsilon \partial_{t}+v \cdot \nabla_{x}\right) g_{\epsilon}\right\|_{L^{1}\left(\left[0, t^{*}\right] ; L^{1}(M d v d x)\right)}  \tag{8.2}\\
& \leq 2\|\hat{\gamma}\|_{L^{\infty}}\left\|q_{\epsilon}{ }^{b} g_{\epsilon}\right\|_{L^{1}\left(\left[0, t^{*}\right] ; L^{1}(d \mu d x)\right)}=O(1)
\end{align*}
$$

On the other hand, the first statement of Proposition 2.7 expresses that, for each sequence $\epsilon_{n} \rightarrow 0$, the extracted sequence $M\left|{ }^{b} \epsilon_{\epsilon_{n}}\right|^{2}$ is locally uniformly integrable in $v$. This and (8.2) imply that this extracted sequence is in fact locally uniformly integrable on $\mathbf{R}_{+} \times \mathbf{R}^{3} \times \mathbf{R}^{3}$ in all variables $t, x$ and $v$, by Lemma 3.6.

Further $(1+|v|)^{s} g_{\epsilon}$ is bounded in $L_{l o c}^{2}\left(d t d x ; L^{2}(M d v)\right)$ for all $s \geq 0$ according to the second statement in Proposition 2.7. This and the local uniform integrability of the sequence $M\left|{ }^{\mathrm{b}} g_{\epsilon_{n}}\right|^{2}$ with respect to the Lebesgue measure on $\mathbf{R}_{+} \times \mathbf{R}^{3} \times \mathbf{R}^{3}$ implies that, for each compact $Q \subset \mathbf{R}_{+} \times \mathbf{R}^{3}$, the sequence $\left.\left.\mathbf{1}_{Q}(t, x)\right|^{b} g_{\epsilon_{n}}(t, x, v)\right|^{2}$ is uniformly integrable on $\mathbf{R}_{+} \times \mathbf{R}^{3} \times \mathbf{R}^{3}$ with respect to the measure $d t d x M d v$, which is precisely the first statement in Proposition 3.8.
8.2. Strong compactness of moments in the $x$-variable. Because $F_{\epsilon}$ is a renormalized solution of (1.13) relative to $M$, using this time the nonlinear function $\Gamma(Z)=(Z-1) \gamma(Z)$ in the renormalized formulation (1.19) gives

$$
\begin{equation*}
\left(\epsilon \partial_{t}+v \cdot \nabla_{x}\right)^{b} g_{\epsilon}=\iint q_{\epsilon} \hat{\gamma}_{\epsilon} b d \sigma_{v, v_{1}}(\omega) M_{1} d v_{1} . \tag{8.3}
\end{equation*}
$$

with $\hat{\gamma}$ defined in terms of $\gamma$ by (4.2). Observe that, denoting $N_{\epsilon}=\frac{2}{3}+\frac{1}{3} G_{\epsilon}$

$$
\begin{equation*}
\left|q_{\epsilon} \hat{\gamma}_{\epsilon}\right|=\left(\frac{2}{3}+\frac{1}{3} G_{\epsilon}\right)\left|\hat{\gamma}\left(G_{\epsilon}\right)\right| \frac{\left|q_{\epsilon}\right|}{N_{\epsilon}} \leq \frac{7}{6}\|\hat{\gamma}\|_{L^{\infty}} \frac{\left|q_{\epsilon}\right|}{N_{\epsilon}} \tag{8.4}
\end{equation*}
$$

since $\hat{\gamma}$ is supported in $\left[\frac{1}{2}, \frac{3}{2}\right]$. As recalled in the last statement in Theorem 11.1 (see Appendix B below), the family $q_{\epsilon} / N_{\epsilon}$ is relatively compact in $w-L_{l o c}^{1}\left(d t d x ; L^{1}\left(\left(1+|v|^{2}\right) d \mu\right)\right)$. This and the inequality (8.4) imply that the family $q_{\epsilon} \hat{\gamma}_{\epsilon}$ also is relatively compact in $w-L_{l o c}^{1}\left(d t d x ; L^{1}(d \mu)\right)$, which implies in turn that the family

$$
\iint q_{\epsilon} \hat{\gamma}_{\epsilon} b d \sigma_{v, v_{1}}(\omega) M_{1} d v_{1} \text { is relatively compact in } w-L_{l o c}^{1}\left(d t d x ; L^{1}(M d v)\right)
$$

By Dunford-Pettis' theorem, this and (8.3) eventually imply that

$$
\begin{aligned}
\left(\epsilon_{n} \partial_{t}+v \cdot \nabla_{x}\right) g_{\epsilon_{n}} & \text { is locally uniformly integrable in }\left[0, t^{*}\right] \times \mathbf{R}^{3} \times \mathbf{R}^{3} \\
& \text { with respect to the measure } d t d x M d v .
\end{aligned}
$$

This and the first statement in Proposition 3.8 entail the second statement in that same proposition by applying Lemma 3.7.

### 8.3. Proof of Corollary 3.9. Consider the operator $|\mathcal{L}|$ defined by ${ }^{8}$

$$
|\mathcal{L}| \phi=\iint\left(\phi+\phi_{1}+\phi^{\prime}+\phi_{1}^{\prime}\right) b\left(v-v_{1}, \omega\right) M_{1} d \sigma_{v, v_{1}}(\omega) d v_{1} .
$$

Using the elementary inequality $a b \leq \frac{1}{2}\left(a^{2}+b^{2}\right)$ repeatedly, one sees that

$$
\begin{equation*}
\left|g_{\epsilon} \mathcal{L}^{\mathrm{b}} g_{\epsilon}\right| \leq 2 b_{\infty}{ }^{\mathrm{b}} g_{\epsilon}^{2}+\frac{1}{2}|\mathcal{L}|\left(\mathrm{g}_{\epsilon}^{2}\right), \quad\left|\mathcal{Q}\left(g_{\epsilon}, g_{\epsilon}\right)\right| \leq \frac{1}{2}|\mathcal{L}|\left(\mathrm{g}_{\epsilon}^{\mathrm{b}}\right) . \tag{8.5}
\end{equation*}
$$

Pick a compact set $Q \subset \mathbf{R}_{+} \times \mathbf{R}^{3}$ and a sequence $\epsilon_{n} \rightarrow 0$. By the first part of Proposition 3.8, the sequence $(t, x, v) \mapsto \mathbf{1}_{Q}(t, x){ }^{\mathrm{b}} \epsilon_{\epsilon_{n}}(t, x, v)^{2}$ (henceforth denoted $\mathbf{1}_{Q} g_{\epsilon_{n}}^{2}$ ) is uniformly integrable in $\mathbf{R}_{+} \times \mathbf{R}^{3} \times \mathbf{R}^{3}$ with respect to the measure $d t d x M d v$. By Dunford-Pettis' theorem it is relatively compact in $w-L^{1}(d t d x M d v)$. On the other hand, the same argument as in the proof of Proposition 1.5 shows that the linear operator $|\mathcal{L}|$ is continuous in $L^{1}(M d v)$. Therefore the sequence $\mathbf{1}_{Q}|\mathcal{L}|\left(g_{\epsilon_{n}}^{2}\right)$ is also relatively compact in $w-L^{1}(d t d x M d v)$. Applying Dunford-Pettis' theorem again implies that this sequence is uniformly integrable in $\mathbf{R}_{+} \times \mathbf{R}^{3} \times \mathbf{R}^{3}$ with respect to the measure $d t d x M d v$. With the inequalities (8.5), this implies the result in Corollary 3.9.

## 9. Conclusions

The proof of the Navier-Stokes limit of the Boltzmann equation presented in this work can be extended in a number of ways.

First, the case of periodic flows (i.e. of the spatial domain $\mathbf{T}^{3}=\mathbf{R}^{3} / \mathbf{Z}^{3}$ as in [7]) may be treated after only slight modifications. Indeed, the compactness results based on dispersion estimates in the present analysis concern local $L^{p}$ spaces.

It is possible to handle a more general class of collision kernels than considered by following the same general strategy with, however, very significant modifications. The case of a hard-sphere gas is of particular importance, since it is so far the only case for which the Boltzmann equation has been rigorously derived from Newtonian mechanics - by O. Lanford [39]. In various applications, it would also be important to extend the convergence results in the present paper to the case of both hard and soft cutoff potentials proposed by H. Grad [32] (see also [18]). In general, neither hard nor soft cutoff potentials satisfy our assumption (H1); this can be remedied at the expense of rather technical modifications of our arguments in Sects. 4-8. As for assumption (H2), its role in the proof of Proposition 5.1 is crucial. As mentioned above, (H2) is thus far known to hold with $p=3$ for cutoff Maxwell molecules only. Whether this assumption (H2) is satisfied by hard-sphere and more general cutoff potentials remains an open

[^6]problem as of now. An alternative to establishing the validity of (H2) for all cutoff potentials would consist of trying to merge the methods leading to Propositions 2.7, 3.4 and 3.8 with those used by C.D. Levermore and N. Masmoudi [41] to treat conservation defects. This however remains an open problem, as the analysis in [41] makes essential use of a variant of the nonlinear control (A2), slightly weaker than (A2) but which we still do not know how to establish by the methods of the present paper, except possibly in the case of soft cutoff potentials

Another natural extension of the present work would be to treat boundary value problems when the spatial domain is a smooth open set in $\mathbf{R}^{3}$. This issue has been rather systematically studied at the formal level by using Hilbert's expansion, for instance by Y. Sone [60]. In view of the importance of this problem for applications to the dynamics of rarefied gases, it would be extremely desirable to confirm these formal asymptotic results by mathematical proofs. This problem can certainly be solved by the methods presented in this paper combined with those developed by N. Masmoudi and L. Saint-Raymond in [52].

Still another natural extension of our results would be to treat the case of two-dimensional flows (more precisely, 2 and 1/2-dimensional flows in the terminology of [44], p. 151). By this, we mean the case where the number density $F_{\epsilon}$ is a function of two space variables only but of all three velocity variables: e.g. $F_{\epsilon} \equiv F\left(t, x_{1}, x_{2}, v_{1}, v_{2}, v_{3}\right)$. This is not a particular case of the theory presented here, since such densities have infinite relative entropy with respect to any uniform Maxwellian state. However, our method carries over to this case by considering as relative entropy the quantity

$$
\frac{1}{\epsilon^{2}} \iint\left[F_{\epsilon} \log \left(\frac{F_{\epsilon}}{M}\right)-F_{\epsilon}+M\right] d v_{1} d v_{2} d v_{3} d x_{1} d x_{2}
$$

instead of (1.15) where the integral bears on all space variables. The NavierStokes limit in this setting leads to velocity fields that have three components but depend on only two space variables, i.e. 2 and $1 / 2$-dimensional flows. What is usually known as a two-dimensional flow is the particular case of a 2 and $1 / 2$-dimensional flow where the third component of the velocity field is identically zero initially, and therefore remains so for all subsequent times.

This case is interesting because global classical solutions to the twodimensional Navier-Stokes equations are known to exist without restrictions on the size of the initial data, at variance with the three-dimensional case: see for instance [44], p. 83. Moreover, weak solutions of the two-dimensional Navier-Stokes equations are known to be uniquely determined by their initial data and satisfy the energy equality - the inequality (1.28) becomes an equality for weak solutions of the two-dimensional Navier-Stokes equations (see for instance Theorem 3.1 in [44], p. 81).

The special properties of two-dimensional flows have interesting consequences on the hydrodynamic limit:

- in the two-dimensional case, the analogue of Theorem 1.6 shows that the family $\frac{1}{\epsilon} \int v F_{\epsilon} d v$ converges in $w-L_{l o c}^{1}\left(\mathbf{R}_{+} \times \mathbf{R}^{2}\right)$ ) to the (unique) weak solution of the two-dimensional Navier-Stokes equations with initial data $u^{i n}$ defined in (1.45); likewise
- in the two-dimensional case, the analogue of Theorem 1.9 holds without regularity assumptions on the limiting velocity field, provided that the initial data $u^{i n}$ defined in (1.45) is smooth - indeed the two-dimensional Navier-Stokes equations propagate the smoothness of the initial data.

In any case, two dimensional flows can be derived in this way from the Boltzmann equation, in contrast with the derivation of the Navier-Stokes equations from the lattice gas considered by J. Quastel and H.-T. Yau [54]. Whether this is a spurious feature of their particle model, or is due to the fact that some fluctuations at the level of the particle dynamics are discarded in the description by the Boltzmann equation is not completely clear.

Finally, a few comments on the role of weak solutions in this work are in order.

It has been repeatedly asserted that both renormalized solutions of the Boltzmann equation and Leray solutions of the Navier-Stokes equations are physically unsatisfying, because to this date, they are not known to be uniquely determined by their initial data and thus have no predictive value. While this might cast doubts on the soundness of the program outlined in [7], we insist that the reason for considering weak solutions in this program is not that they are the only ones known to exist for all time and initial data of arbitrary size. A more crucial reason is that the only a priori estimates known to this date on the scaled Boltzmann equation (1.13) that are uniform as the Knudsen number $\epsilon$ tends to 0 come from the DiPernaLions entropy inequality (1.21). This inequality holds for all renormalized solutions of (1.13) and yields the Leray energy inequality (1.28) in the limit as $\epsilon \rightarrow 0$. Hence, Leray solutions of the Navier-Stokes system and renormalized solutions of the Boltzmann equation should be viewed as the natural objects to which uniform a priori estimates apply rather than a source of spurious technical difficulties caused by our lack of knowledge on the regularity of solutions to the Navier-Stokes or Boltzmann equations in three space dimensions.

To illustrate this, suppose we are given a family $F_{\epsilon}^{i n}$ of initial data leading for each $\epsilon>0$ to a classical solution $F_{\epsilon}$ of the scaled Boltzmann equation (1.13) that satisfies the local conservation laws of momentum and energy. Suppose in addition that $F_{\epsilon}^{i n}$ converges in the strongest possible sense - and at least entropically at rate $\epsilon-$ to $M_{\left(1, u^{i n}, 1\right)}$ with $u^{\text {in }}$ a smooth, divergence-free initial velocity field leading to a classical solution $u$ of the Navier-Stokes equations.

In order to prove for instance that

$$
\frac{1}{\epsilon} \int v F_{\epsilon} d v \rightarrow u
$$

in the weakest possible sense - at least in the sense of distributions - as $\epsilon \rightarrow 0$, it does not seem that one can use the local conservation law of momentum satisfied by $F_{\epsilon}$ for each $\epsilon>0$. Indeed, passing to the limit as $\epsilon \rightarrow 0$ in the momentum flux

$$
\frac{1}{\epsilon^{2}} \int\left(v^{\otimes 2}-\frac{1}{3}|v|^{2} I\right) F_{\epsilon} d v
$$

involves in particular a term of the form

$$
\int\left(v^{\otimes 2}-\frac{1}{3}|v|^{2} I\right) \mathcal{Q}\left(g_{\epsilon}, g_{\epsilon}\right) M d v
$$

(where $g_{\epsilon}$ is the relative fluctuation of $F_{\epsilon}$ about $M$ defined in (1.34)). All that we know about the family $g_{\epsilon}$ from the entropy inequality (1.21) is that $g_{\epsilon}$ is relatively compact in $w-L_{l o c}^{1}\left(d t d x ; w-L^{1}(M d v)\right)$, not in $w-L_{l o c}^{2}\left(d t d x ; w-L^{1}(M d v)\right)$, and this is not enough to guarantee that

$$
\int\left(v^{\otimes 2}-\frac{1}{3}|v|^{2} I\right) \mathcal{Q}\left(g_{\epsilon}, g_{\epsilon}\right) M d v \rightarrow u \otimes u-\frac{1}{3}|u|^{2} I
$$

in the sense of distributions as $\epsilon \rightarrow 0$. Apparently, the only way around this consists of replacing $g_{\epsilon}$ with its $L^{2}$ part $g_{\epsilon}$ in the Flat-Sharp decomposition (2.4). At this point, the benefit of knowing that $F_{\epsilon}$ satisfies the local conservation laws of momentum and energy is lost. Therefore, one has to deal with exactly the same conservation defects as in Sect. 4; the proof of convergence must follow essentially the same steps as in Sect. 5 and Sects. 6 and 7, and so the main technical burden in the present paper cannot be dispensed with. Likewise, although the global weak solutions to the BGK model constructed by B. Perthame [53] satisfy the local conservation law of momentum, the only derivation known to this date of the Navier-Stokes equations from that model in [56] uses a renormalized form of the BGK equation - in other words, renormalization is used in taking the hydrodynamic limit even though it is not needed to define the solution of the kinetic model. This leads to estimating conservation defects in the same manner as in the present paper.

Similar difficulties arise if one tries to use Boltzmann's H Theorem and apply instead Yau's relative entropy method; the proof of convergence would also require all the controls in Propositions 3.4 and 3.8; see the work of F. Golse, C.D. Levermore and L. Saint-Raymond [26].

Thus, to summarize this discussion, it seems doubtful that dealing with classical (instead of weak) solutions would simplify in any significant way the proof of the Navier-Stokes limit of the Boltzmann equation.

## 10. Appendix A. Young's inequality

The functions $h:\left[-1,+\infty\left[\rightarrow \mathbf{R}_{+}\right.\right.$and $\left.r:\right]-1,+\infty\left[\rightarrow \mathbf{R}_{+}\right.$in (2.2), (2.8) are both strictly convex, and satisfy, for all $z>-1$,

$$
\begin{equation*}
h(|z|) \leq h(z), \quad r(|z|) \leq r(z), \quad h(z) \leq r(z) \tag{10.1}
\end{equation*}
$$

The Legendre transform of $h$ is defined for all $p \in \mathbf{R}$ by

$$
\begin{equation*}
h^{*}(p)=\sup _{z>-1}(p z-h(z))=e^{p}-p-1 \tag{10.2}
\end{equation*}
$$

that of $r$ is also defined for all $p \in \mathbf{R}$ by the implicit relation

$$
r^{*}(p)=\sup _{z>-1}(p z-r(z))=\frac{z^{2}}{1+z}, \text { with } \log (1+z)+\frac{z}{1+z}=p
$$

Further, the Legendre transform $h^{*}$ is super-quadratic, i.e.

$$
\begin{equation*}
h^{*}(\eta p) \leq \eta^{2} h^{*}(p), \quad p \in \mathbf{R}_{+}, \quad \eta \in[0,1] \tag{10.3}
\end{equation*}
$$

Finally Young's inequality states that, for all $p \in \mathbf{R}, z>-1$ and $\eta \in[0,1]$,

$$
\begin{equation*}
p|z| \leq \frac{1}{\eta} h(z)+\eta h^{*}(p) \leq \frac{1}{\eta} r(z)+\eta h^{*}(p) . \tag{10.4}
\end{equation*}
$$

## 11. Appendix B. A compendium of results from [7]

Some of the result in [7] were established in the greatest possible generality, and in particular did not use any of the assumptions left unverified there. We have recorded them below without proof; they are used in various places in the present work. While these statements were established in the case of a spatial domain equal to the torus, the proofs from [7] can be adapted to the spatial domain $\mathbf{R}^{3}$.

Theorem 11.1. Under assumptions (H1)-(H2), let $F_{\epsilon}$ be a family of renormalized solutions to (1.13) with initial data $F_{\epsilon}^{\text {in }}$ satisfying (1.14), and define the associated family of fluctuations by

$$
g_{\epsilon}=\frac{F_{\epsilon}-M}{\epsilon M} .
$$

Then

- $g_{\epsilon}$ is relatively compact in $w-L_{l o c}^{1}\left(d t d x ; L^{1}\left(\left(1+|v|^{2}\right) M d v\right)\right)$ and all its limit points as $\epsilon \rightarrow 0$ are local infinitesimal Maxwellians

$$
\begin{equation*}
g(t, x, v)=\rho(t, x)+u(t, x) \cdot v+\theta(t, x) \frac{1}{2}\left(|v|^{2}-3\right) \tag{11.1}
\end{equation*}
$$

where the velocity field $u$ satisfies the incompressibility condition

$$
\begin{equation*}
\nabla_{x} \cdot u=0 \tag{11.2}
\end{equation*}
$$

while the fluctuations of macroscopic density and temperature satisfy the Boussinesq relation

$$
\begin{equation*}
\nabla_{x}(\rho+\theta)=0 \text { which implies that } \rho+\theta=0 \tag{11.3}
\end{equation*}
$$

- the rescaled collision integrands

$$
\begin{equation*}
q_{\epsilon}=\frac{1}{\epsilon^{2}}\left(G_{\epsilon}^{\prime} G_{\epsilon 1}^{\prime}-G_{\epsilon} G_{\epsilon 1}\right) \tag{11.4}
\end{equation*}
$$

are such that the renormalized family $\gamma\left(G_{\epsilon}\right) q_{\epsilon}$ is relatively compact in $w-L_{l o c}^{1}\left(d t d x ; L^{1}\left(\left(1+|v|^{2}\right) d \mu\right)\right)$; further, any of the limit points $q$ of $\gamma\left(G_{\epsilon}\right) q_{\epsilon}$ as $\epsilon \rightarrow 0$ satisfies the $d \mu$-symmetry relations

$$
\begin{equation*}
\langle\langle\phi(v) q\rangle\rangle=\left\langle\left\langle\frac{1}{4}\left(\phi+\phi_{1}-\phi^{\prime}-\phi_{1}^{\prime}\right) q\right\rangle\right\rangle ; \tag{11.5}
\end{equation*}
$$

- for any subsequence $\epsilon_{n} \rightarrow 0$ such that

$$
g_{\epsilon_{n}} \rightarrow g \text { and } \gamma\left(G_{\epsilon_{n}}\right) q_{\epsilon_{n}} \rightarrow q
$$

in $w-L_{l o c}^{1}\left(d t d x ; L^{1}\left(\left(1+|v|^{2}\right) M d v\right)\right)$ and in $w-L_{l o c}^{1}\left(d t d x ; L^{1}((1+\right.$ $\left.\left.|v|^{2}\right) d \mu\right)$ ) respectively, the limits $g$ and $q$ satisfy the limiting Boltzmann equation

$$
\begin{equation*}
v \cdot \nabla_{x} g=\iint q b\left(v-v_{1}, \omega\right) d \sigma_{v, v_{1}}(\omega) M_{1} d v_{1} \tag{11.6}
\end{equation*}
$$

- for any limit point $g$ (of the form (11.1)) of the family of fluctuations $g_{\epsilon}$ as $\epsilon \rightarrow 0$, and for each $t>0$, one has

$$
\begin{align*}
& \frac{1}{2} \int\left(\rho(t, x)^{2}+|u(t, x)|^{2}+\frac{3}{2} \theta(t, x)^{2}\right) d x \\
&+\frac{1}{2} \int_{0}^{t} \int\left(\nu\left|\nabla_{x} u+\left(\nabla_{x} u\right)^{T}\right|^{2}+5 \kappa\left|\nabla_{x} \theta\right|^{2}\right) d x d s  \tag{11.7}\\
& \quad \leq \frac{\lim _{\epsilon \rightarrow 0} \int\left\langle\frac{1}{\epsilon^{2}} h\left(\epsilon g_{\epsilon}^{i n}\right)\right\rangle d x .}{}
\end{align*}
$$

- denoting $N_{\epsilon}=\frac{2}{3}+\frac{1}{3} G_{\epsilon}, g_{\epsilon} / N_{\epsilon}$ is bounded in $L_{t}^{\infty}\left(L^{2}(M d v d x)\right)$ and $q_{\epsilon} / N_{\epsilon}$ is relatively compact in $w-L_{l o c}^{1}\left(d t d x ; L^{1}\left(\left(1+|v|^{2}\right) d \mu\right)\right)$.


## 12. Appendix C. Velocity averaging

Proposition 12.1. Let $\phi_{\epsilon}$ be a bounded family of $L_{l o c}^{2}\left(d t d x ; L^{2}(M d v)\right)$ indexed by $\epsilon \in[0,1]$ such that both families $\left|\phi_{\epsilon}\right|^{2}$ and $\left(\epsilon \partial_{t}+v \cdot \nabla_{x}\right) \phi_{\epsilon}$ are locally uniformly integrable with respect to the measure Mdvdxdt. Then, for each function $\psi \equiv \psi(v)$ in $L^{2}(M d v)$, each $t^{*}>0$ and each compact $Q \subset \mathbf{R}^{3}$, there exists a function $\eta: \mathbf{R}_{+} \rightarrow \mathbf{R}_{+}$such that $\lim _{z \rightarrow 0^{+}} \eta(z)=0$

$$
\left\|\int \phi_{\epsilon}(t, x+y, v) \psi(v) M(v) d v-\int \phi_{\epsilon}(t, x, v) \psi(v) M(v) d v\right\|_{L^{2}\left(\left[0, t^{*}\right] \times Q\right)} \quad \begin{aligned}
& \eta(|y|)
\end{aligned}
$$

for each $y \in \mathbf{R}^{3}$ such that $|y| \leq 1$, uniformly in $\epsilon \in[0,1]$.

Proof. Since $C_{c}\left(\mathbf{R}^{3}\right)$ is dense in $L^{2}(M d v)$, there exists a sequence $\psi_{n} \in$ $C_{c}\left(\mathbf{R}^{3}\right)$ such that $\left\|\psi-\psi_{n}\right\|_{L^{2}(M d v)} \rightarrow 0$ as $n \rightarrow+\infty$. Since the family $\phi_{\epsilon}$ is bounded in $L_{l o c}^{2}\left(d t d x ; L^{2}(M d v)\right)$, for each $t^{*}>0$ and each compact $Q \subset \mathbf{R}^{3}$,

$$
\begin{aligned}
& \| \int \phi_{\epsilon}(t, x)\left(\psi(v)-\psi_{n}(v)\right) M d v \|_{L^{2}\left(\left[0, t^{*}\right] \times Q\right)} \\
& \leq \sup _{\epsilon \in[0,1]}\left\|\phi_{\epsilon}\right\|_{L^{2}\left(\left[0, t^{*}\right] \times Q ; L^{2}(M d v)\right)}\left\|\psi-\psi_{n}\right\|_{L^{2}(M d v)} \rightarrow 0
\end{aligned}
$$

uniformly in $\epsilon \in[0,1]$ as $n \rightarrow+\infty$. Thus we can assume without loss of generality that $\psi \in C_{c}\left(\mathbf{R}^{3}\right)$ and that all the $\phi_{\epsilon}$ are supported in some compact set $K \subset \mathbf{R}_{+}^{*} \times \mathbf{R}^{3} \times \mathbf{R}^{3}$; thus we henceforth consider $\phi_{\epsilon}$ as defined on $\mathbf{R} \times \mathbf{R}^{3} \times \mathbf{R}^{3}$. Let $\lambda>0$; define

$$
\phi_{\epsilon}+\left(\epsilon \partial_{t}+v \cdot \nabla_{x}\right) \phi_{\epsilon}=\Phi_{\epsilon} .
$$

The assumptions made on $\phi_{\epsilon}$ guarantee that the family $\Phi_{\epsilon}$ is uniformly integrable with respect to the measure $M d v d x d t$. Proceeding as in the proof of Theorem 3 of [27] (see pp. 115-116), one shows first that

$$
\begin{equation*}
\left\|\int \phi_{\epsilon}(t, x+y, v) \psi(v) M(v) d v-\int \phi_{\epsilon}(t, x, v) \psi(v) M(v) d v\right\|_{L_{t, x}^{1}} \rightarrow 0 \tag{12.1}
\end{equation*}
$$

uniformly in $\epsilon \in[0,1]$ as $|y| \rightarrow 0$.
It remains to prove that the same convergence holds in $L_{t, x}^{2}$. We therefore split

$$
\begin{align*}
& \iint\left|\int\left[\phi_{\epsilon}(t, x+y, v)-\phi_{\epsilon}(t, x, v)\right] \psi(v) M(v) d v\right|^{2} d x d t  \tag{12.2}\\
\leq & \iint \rho_{\epsilon}^{<}(t, x+y)\left|\int\left[\phi_{\epsilon}(t, x+y, v)-\phi_{\epsilon}(t, x, v)\right] \psi(v) M(v) d v\right| d x d t \\
+ & \iint \rho_{\epsilon}^{<}(t, x)\left|\int\left[\phi_{\epsilon}(t, x+y, v)-\phi_{\epsilon}(t, x, v)\right] \psi(v) M(v) d v\right| d x d t \\
+ & \iint \rho_{\epsilon}^{>}(t, x+y)\left(\int\left[\left|\phi_{\epsilon}(t, x+y, v)\right|+\left|\phi_{\epsilon}(t, x, v)\right|\right] \psi(v) M(v) d v\right) d x d t \\
+ & \iint \rho_{\epsilon}^{>}(t, x)\left(\int\left[\left|\phi_{\epsilon}(t, x+y, v)\right|+\left|\phi_{\epsilon}(t, x, v)\right|\right] \psi(v) M(v) d v\right) d x d t
\end{align*}
$$

with the notations

$$
\begin{aligned}
& \rho_{\epsilon}^{<}(t, x)=\int\left|\phi_{\epsilon} \mathbf{1}_{\left|\phi_{\epsilon}\right| \leq \lambda}\right|(t, x, v)|\psi| M d v \\
& \rho_{\epsilon}^{>}(t, x)=\int\left|\phi_{\epsilon} \mathbf{1}_{\left|\phi_{\epsilon}\right|>\lambda}\right|(t, x, v)|\psi| M d v .
\end{aligned}
$$

The first and second integrals on the right-hand side of the inequality (12.2) are less than

$$
\begin{equation*}
\lambda\langle | \psi\left\rangle\left\|\int \phi_{\epsilon}(t, x+y, v) \psi(v) M(v) d v-\int \phi_{\epsilon}(t, x, v) \psi(v) M(v) d v\right\|_{L_{t, x}^{1}}\right. \tag{12.3}
\end{equation*}
$$

while the third and the fourth are less than

$$
\left.\left.2\langle | \psi\right|^{2}\right\rangle\left\|M^{1 / 2} \phi_{\epsilon} \mathbf{1}_{\left|\phi_{\epsilon}\right|>\lambda}\right\|_{L_{t, x, v}^{2}}\left\|M^{1 / 2} \phi_{\epsilon}\right\|_{L_{t, x, v}^{2}}
$$

This term vanishes as $\lambda \rightarrow+\infty$ uniformly in $\epsilon \in[0,1]$ since the family $\left|\phi_{\epsilon}\right|^{2}$ is uniformly integrable with respect to the measure $M d v d x d t$. On the other hand, for any fixed $\lambda>0$, the first and second integrals in (12.2) vanish as $|y| \rightarrow 0$ uniformly in $\epsilon \in[0,1]$ by (12.3) and (12.1). This concludes the proof.

## 13. Appendix D. Compensated compactness for acoustic waves

We recall below the elegant argument proposed by P.-L. Lions and N. Masmoudi to establish the incompressible limit of the compressible NavierStokes equations [49]. A similar result had been obtained earlier by a somewhat different method: see [33] and [57].

Lemma 13.1. Let $c \neq 0$. Consider two families $\phi_{\epsilon}$ and $\psi_{\epsilon}$ bounded in $L_{l o c}^{\infty}\left(d t ; L_{l o c}^{2}(d x)\right)$ and in $L_{l o c}^{\infty}\left(d t ; H_{l o c}^{1}\left(\mathbf{R}^{3}\right)\right)$ respectively, such that

$$
\begin{aligned}
\partial_{t} \phi_{\epsilon}+\frac{1}{\epsilon} \Delta_{x} \psi_{\epsilon} & =\frac{1}{\epsilon} F_{\epsilon}, \\
\partial_{t} \nabla_{x} \psi_{\epsilon}+\frac{c^{2}}{\epsilon} \nabla_{x} \phi_{\epsilon} & =\frac{1}{\epsilon} G_{\epsilon},
\end{aligned}
$$

where $F_{\epsilon} \rightarrow 0$ and $G_{\epsilon} \rightarrow 0$ in $L_{l o c}^{1}\left(d t ; L_{l o c}^{2}(d x)\right)$. Then

$$
P \nabla_{x} \cdot\left(\left(\nabla_{x} \psi_{\epsilon}\right)^{\otimes 2}\right) \rightarrow 0, \text { and } \nabla_{x} \cdot\left(\phi_{\epsilon} \nabla_{x} \psi_{\epsilon}\right) \rightarrow 0
$$

in the sense of distributions on $\mathbf{R}_{+}^{*} \times \mathbf{R}^{3}$.
Proof. By elementary computations,

$$
\begin{aligned}
\nabla_{x} \cdot\left(\left(\nabla_{x} \psi_{\epsilon}\right)^{\otimes 2}\right) & =\frac{1}{2} \nabla_{x}\left(\left|\nabla_{x} \psi_{\epsilon}\right|^{2}\right)+\nabla_{x} \psi_{\epsilon} \Delta_{x} \psi_{\epsilon} \\
& =\frac{1}{2} \nabla_{x}\left(\left|\nabla_{x} \psi_{\epsilon}\right|^{2}-c^{2}\left|\phi_{\epsilon}\right|^{2}\right)-\partial_{t}\left(\epsilon \phi_{\epsilon} \nabla_{x} \psi_{\epsilon}\right) \\
& +F_{\epsilon} \nabla_{x} \psi_{\epsilon}+\phi_{\epsilon} G_{\epsilon},
\end{aligned}
$$

and, likewise

$$
\begin{aligned}
c^{2} \nabla_{x} \cdot\left(\phi_{\epsilon} \nabla_{x} \psi_{\epsilon}\right) & =c^{2} \phi_{\epsilon} \Delta \psi_{\epsilon}+c^{2} \nabla_{x} \phi_{\epsilon} \cdot \nabla_{x} \psi_{\epsilon} \\
& =-\frac{1}{2} \partial_{t}\left(\epsilon\left(c^{2}\left|\phi_{\epsilon}\right|^{2}+\left|\nabla_{x} \psi_{\epsilon}\right|^{2}\right)\right)+c^{2} \phi_{\epsilon} F_{\epsilon}+G_{\epsilon} \cdot \nabla_{x} \psi_{\epsilon}
\end{aligned}
$$

The bounds assumed on $\phi_{\epsilon}$ and $\psi_{\epsilon}$ ensure that the families $\epsilon \phi_{\epsilon} \nabla_{x} \psi_{\epsilon}$, $\epsilon\left(c^{2}\left|\phi_{\epsilon}\right|^{2}+\left|\nabla_{x} \psi_{\epsilon}\right|^{2}\right), F_{\epsilon} \nabla_{x} \psi_{\epsilon}, G_{\epsilon} \cdot \nabla_{x} \psi_{\epsilon}, \phi_{\epsilon} F_{\epsilon}$ and $\phi_{\epsilon} G_{\epsilon}$ all vanish with $\epsilon$ in $L_{l o c}^{1}(d t d x)$, which, together with the two elementary formulas above, implies the announced convergence.

## 14. Appendix E. Consequences of the Dunford-Pettis theorem

Let $(X, \mathcal{M})$ be a measurable space and $\mu$ be a positive measure on $X$ such that $\mu(X)<+\infty$.

In [24] (Theorem 3.2.1, p. 376), N. Dunford and B.J. Pettis gave a criterion for subsets of $L^{1}(X, \mu)$ to be weakly relatively (sequentially) compact. In the following lemma we apply their result to study the weak $L^{1}$-continuity of certain bilinear expressions.

Lemma 14.1. Let $a_{n}$ and $b_{n}$ be two sequences of real-valued measurable functions defined (a.e.) on $X$, such that $a_{n}$ is bounded in $L^{\infty}(X, \mu)$ and $b_{n} \rightarrow b$ in $w-L^{1}(X, \mu)$ as $n \rightarrow+\infty$.

- Assume that $a_{n} \rightarrow a$ in measure as $n \rightarrow+\infty$; then $a_{n} b_{n} \rightarrow a b$ in $w-L^{1}(X, \mu)$ as $n \rightarrow+\infty$.
- Assume that $a_{n} \rightarrow 0$ in measure as $n \rightarrow+\infty$; then $a_{n} b_{n} \rightarrow 0$ strongly in $L^{1}(X, \mu)$ as $n \rightarrow+\infty$.

Proof. We first prove the second assertion. For each $\epsilon>0$ and each $n \in \mathbf{N}$, let $A(n, \epsilon)=\left\{x \in X| | a_{n}(x) \mid>\epsilon\right\}$. Then

$$
\begin{aligned}
& \int_{X}\left|a_{n}(x) b_{n}(x)\right| d \mu(x) \\
& \quad=\int_{A(n, \epsilon)}\left|a_{n}(x) b_{n}(x)\right| d \mu(x)+\int_{A(n, \epsilon)^{c}}\left|a_{n}(x) b_{n}(x)\right| d \mu(x) \\
& \quad \leq \sup _{k \geq 0}\left\|a_{k}\right\|_{L^{\infty}} \int_{A(n, \epsilon)}\left|b_{n}(x)\right| d \mu(x)+\epsilon \sup _{k \geq 0}\left\|b_{k}\right\|_{L^{1}}
\end{aligned}
$$

Since $a_{n}$ converges to 0 in measure, $\mu(A(n, \epsilon)) \rightarrow 0$ as $n \rightarrow+\infty$. On the other hand, the sequence $b_{n}$ converges weakly in $w-L^{1}(X, \mu)$, and thus is uniformly integrable on $X$ by the Dunford-Pettis theorem. Therefore,

$$
\int_{A(n, \epsilon)}\left|b_{n}(x)\right| d \mu(x) \rightarrow 0
$$

as $n \rightarrow+\infty$. This and the previous inequality imply that

$$
\varlimsup_{n \rightarrow+\infty} \int_{X}\left|a_{n}(x) b_{n}(x)\right| d \mu(x) \leq \epsilon \sup _{k \geq 0}\left\|b_{k}\right\|_{L^{1}}
$$

Since $b_{n}$ is a weakly convergent sequence in $L^{1}(X, \mu)$, it is bounded in $L^{1}(X, \mu)$. The inequality above, which holds for each $\epsilon>0$, shows that $\left\|a_{n} b_{n}\right\|_{L^{1}} \rightarrow 0$ as $n \rightarrow+\infty$.

Next we prove the first assertion. Write $a_{n} b_{n}=\left(a_{n}-a\right) b_{n}+a b_{n}$. Observe that $a \in L^{\infty}(X, \mu)$ : indeed the sequence $a_{n}$ is bounded in $L^{\infty}(X, \mu)$ and there exists a subsequence $a_{n_{k}}$ of $a_{n}$ that converges to $a$ a.e. on $X$. By the second assertion $\left(a_{n}-a\right) b_{n} \rightarrow 0$ in $L^{1}(X, \mu)$; on the other hand, $a b_{n} \rightarrow a b$ in $w-L^{1}(X, \mu)$ as $n \rightarrow+\infty$ since $a \in L^{\infty}(X, \mu)$ and $b_{n} \rightarrow b$ in $w-L^{1}(X, \mu)$. Hence $a_{n} b_{n} \rightarrow a b$ in $w-L^{1}(X, \mu)$ as $n \rightarrow+\infty$.

Lemma 14.1 is a slight amplification of Appendix B in [7] - with the notion of convergence in measure replacing that of a.e. convergence. The first assertion in this Lemma is also a consequence of Proposition 1 of [34], p. 222.

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[^0]:    ${ }^{1}$ R. Esposito, personal communication.

[^1]:    2 Theorem IV. 1 in [43] gives in fact the existence of a weak solution of the Boltzmann equation, a notion defined on pp. 548-549 of [47] that is stronger than the notion of renormalized solution: see [47], p. 551.

[^2]:    3 There is a theory of global existence and uniqueness of classical solutions to the NavierStokes equations (1.25) for all divergence free initial velocity fields that depend on two space variables only and belong to $H^{2}\left(\mathbf{R}^{2}, \mathbf{R}^{3}\right)$ : for a concise presentation of these results, see [44] pp. 83 and 151 . However, these solutions do not have finite kinetic energy (when considered as three-dimensional velocity fields) and thus are not covered by the discussion in the present paper: see Sect. 9 .

[^3]:    ${ }^{4}$ V. Agoshkov [1] quotes without proof a somewhat analogous result obtained independently.

    5 In other words, $f_{n}$ depends on only one space variable, say $x_{1}$, and on all the velocity variables $v_{1}, \ldots, v_{D}$.

[^4]:    ${ }^{6}$ The notation $a_{\epsilon} \lesssim b_{\epsilon}$ as $\epsilon \rightarrow 0$ means that there exists $c_{\epsilon} \sim b_{\epsilon}$ as $\epsilon \rightarrow 0$ such that $a_{\epsilon} \leq c_{\epsilon}$ for all $\epsilon$ small enough in $] 0,1[$.

[^5]:    ${ }^{7}$ For each measurable set $A \subset \mathbf{R}^{d}$, the notation $|A|$ designates the $d$-dimensional Lebesgue measure of $A$.

[^6]:    ${ }^{8}$ This is not to be confused with the common usage for this notation, i.e. the nonnegative self-adjoint operator in the polar decomposition of $\mathcal{L}$ which is of no interest here since $\mathcal{L}$ is itself a nonnegative self-adjoint operator.

