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CONTENT ARTICLES IN ECONOMICS

The Neglect of Monotone Comparative Statics Methods

Carol Horton Tremblay and Victor J. Tremblay

Monotone methods enable comparative static analysis without the restrictive assumptions of the implicit-function theorem. Ease of use and flexibility in solving comparative static and game-theory problems have made monotone methods popular in the economics literature and in graduate courses, but they are still absent from undergraduate mathematical economics courses and textbooks. In this article, the authors illustrate the generality of monotone comparative statics relative to the implicit function approach. For example, to sign the effect of a discrete policy shift on a choice variable, the marginal returns will increase with the policy parameter. They also apply monotone methods in game theory settings. As mathematical economics courses and majors gain popularity, incorporating monotone methods into curriculum and textbooks would provide a modern treatment of comparative static analysis.

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JEL codes *A22, C61, C70*

The use of mathematics has become widely accepted in economics research (Allen 2000). In fact, E. Roy Weintraub (2002) contended that the mathematization of economics is one of the most important developments in the history of the discipline in the last century. Following this trend, more and more economics departments are integrating mathematics into their undergraduate curricula. In 2008, 54 percent of the leading 50 universities and 44 percent of leading 50 liberal arts colleges offered courses in mathematical economics.[1] In addition, undergraduate degrees in mathematical economics are offered by 22 percent of the top 50 universities and 32 percent of the top 50 liberal arts colleges.[2] Given this trend, it is becoming increasingly important for courses and textbooks in mathematical economics to be up to date.

Comparative statics, a cornerstone of economic analysis, enables prediction and understanding of economic effects by comparing equilibria before and after a change in a policy, exogenous

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factor, or parameter. In mathematical economics courses, the classic approach to comparative static analysis, which is discussed in all undergraduate textbooks on mathematical economics, is to apply the implicit-function theorem to equilibrium conditions or the first-order conditions of an optimization problem.[3] To effectively apply the implicit-function theorem, however, derivatives of relevant functions must be continuous, objective functions must be concave, and the stability conditions must be determined. In addition, the implicit-function theorem is only valid for an infinitesimally small change in a policy or exogenous variable.

More recent research in the area of *monotone comparative statics* demonstrates, however, that comparative static analysis can be conducted without many of the restrictions required by the implicit-function theorem (Milgrom and Shannon 1994; Shannon 1995; Edlin and Shannon 1998).[4] One advantage of this new approach is that objective functions need not be continuous or concave. Another is that it works for discrete changes as well as infinitesimally small changes in a policy or exogenous variable. It also has the appeal of ease of use relative to the implicit-function theorem. The drawback of this approach is that monotone comparative statics tell only the direction of change but not the magnitude of change in models with explicit functional forms. This gives no real advantage to the implicit-function approach over monotone methods, however, as we can always use analytic or computational methods to derive comparative static results in models with explicit functions.

The classic method is also difficult to apply in game theoretic settings. With many players and choice variables, the curse of dimensionality is a problem: it becomes increasingly tedious if not impossible to calculate a definitive comparative static result. Multiple equilibria are also common in many games, making it difficult to apply the implicit-function theorem. Monotone methods, however, are tractable in this setting. Recent work shows that unambiguous comparative static results can emerge when a game exhibits increasing monotonic best-reply functions, which occur when all strategic variables are complementary. That is, each player's own choice variables are complementary, and all strategic variables across players are strategic complements. A game with this structure is called *a supermodular game* or a *game with strategic complementarities* (Bulow, Geankoplos, and Klemperer 1985; Milgrom and Roberts 1990; Vives 1999).

Although these modern methods are more than a decade old, they are ignored in undergraduate textbooks in mathematical economics.[5] This is unfortunate because they are relatively easy to apply and are more general than traditional methods. Our goal is to describe these methods and show how they can be applied to problems commonly found in undergraduate courses in economics. In the next section, we provide an example that allows us to compare methods and to show that the implicit-function method is a special case of the monotone method. We then show how monotone methods can be used to do comparative statics in noncooperative games, even when the classic method is intractable.

MONOTONE COMPARATIVE STATICS

The main advantage of monotone comparative statics is that it dispenses with irrelevant assumptions that are required to use the implicit-function theorem. These include differentiability, concavity of the objective function, and convexity of constraint sets.

In the case of differentiability, the function in question must be smooth in the neighborhood around the point being evaluated to use the implicit-function theorem. This is particularly problematic in policy analysis, where a particular policy has a discrete nature. For example, the U.S.

Broadcast Advertising Ban of 1971 made it illegal for cigarette companies to use television and radio advertising to market tobacco products. Likewise, a pollution abatement policy may completely ban the use of a polluting input, and many occupational safety regulations are either in effect or not. These binary types of regulations are likely to cause a discrete jump in a firm's profit function, making it impossible to use the implicit-function theorem. With monotone methods, differentiability and convexity are not required to perform comparative static analysis.

To illustrate the advantages of the modern approach, we investigate the comparative statics of a monopoly firm whose goal is to maximize profit with respect to output. We first investigate the simple case where the firm faces explicit demand and cost functions and show that comparative static analysis can be performed without the use of the implicit-function theorem or monotone methods. Second, we consider a more general specification where the profit function is concave and differentiable, a case where one can use the implicit-function theorem (as well as monotone methods) to perform comparative static analysis. Third, we investigate the case where the profit function is no longer differentiable. Although the implicit-function approach fails, we show how monotone methods can be used to perform comparative static analysis. We also show that the implicit-function theorem is a special case of the strict monotonicity method, present the weak-monotonicity theorem, and discuss the case where the firm has multiple choice variables.

We begin with the most restrictive case where the monopolist faces simple linear demand and cost functions. Let the firm's inverse demand be $p = a - bq$, where q is output, p is price, and parameters a and b are positive. The firm's total cost function (TC) is linear and depends on a regulatory policy (R), such that $TC = cq + R$ and $c > 0$. In this example, the government imposes a per-unit subsidy to encourage monopoly production: $R = -sq$, $s > 0$. Thus, the firm's profit equals $\pi = (a - bq - c + s)q$. To ensure that profits are nonnegative in equilibrium, we assume that $a > c - s$. Our goal is to determine how a change in s will affect the firm's profit-maximizing output.

In this simple model, comparative static analysis can be derived directly from the solution to the monopoly problem.[6] The firm's first- and second-order conditions are

$$\frac{\partial \pi}{\partial q} = a - c + s - 2bq = 0, \quad \frac{d^2\pi}{dq^2} = -2b < 0.$$

From the first-order condition, the firm's profit-maximizing output (q^*) is

$$q^* = \frac{a - c + s}{2b}.$$

Thus, for a marginal change in s, $\partial q^*/\partial s = 1/(2b) > 0$. For a discrete increase in s from s_1 to s_2, the change in q^* is $(s_2 - s_1)/(2b) > 0$. With explicit functions, both the sign and magnitude of change can be obtained, and nothing is gained from using the implicit-function theorem or monotone methods.

This brute-force method of deriving a comparative statics result from a solution to a problem with explicit functions cannot be used with general functional forms, however. For example, assume that the firm faces a general inverse-demand function $p = p(q)$, which is twice continuously differentiable, is strictly decreasing in q, and is not too convex. The firm's total cost function, $TC = C(q) - sq$, is twice continuously differentiable, convex, and strictly increasing in q. In this case, the profit equation, $\pi(q, s) = p(q)q - C(q) + sq$, is strictly concave.[7] The firm's first- and second-order conditions are

$$\frac{\partial \pi}{\partial q} = p + \frac{dp}{dq}q - \frac{dC}{dq} + s = 0, \quad \frac{\partial^2 \pi}{\partial q^2} = 2\frac{dp}{dq} + \frac{d^2 p}{dq^2}q - \frac{d^2 C}{dq^2} < 0.$$

The optimal value of q is embedded in the first-order condition but cannot be derived explicitly when functions are general. In cases in which the brute-force method fails, the implicit-function theorem can be used to do comparative static analysis if we assume an infinitesimally small change in s. From the implicit-function theorem:

$$\frac{dq^*}{ds} = -\frac{\partial^2 \pi / \partial q \partial s}{\partial^2 \pi / \partial q^2}.$$

Because the numerator on the right-hand side of the equality above equals 1 and the denominator is negative, an increase in the subsidy will increase q^*. This shows how differentiability and concavity of the objective function in the neighborhood of q^* are important when one is using the implicit-function theorem to perform comparative static analysis.

An important limitation of the implicit function theorem is that it cannot be used when there is a discrete change in a policy variable.[8] To perform comparative static analysis in this case, we must use Aaron Edlin and Chris Shannon's (1998, 205) strict-monotonicity theorem. We describe the theorem generally, where $f(x, \alpha^*)$ is the objective function, x is the choice variable, and α is the policy parameter that can take on two discrete values α^* and α'.

Strict Monotonicity Theorem: Let f: $\mathbf{R} \to \mathbf{R}$, $S \subset \mathbf{R}$, $\mathbf{x}^* = \text{argmax}_{\mathbf{x} \epsilon s} f(x, a^*)$, and $x' = \text{argmax}_{\mathbf{x} \epsilon s} f(x, a')$. Suppose that x^* is a unique interior solution and that $\partial f / \partial x$ is continuous and has *strictly increasing marginal returns* with respect to the parameter α. Then $x^* > x'$ if $\alpha^* > \alpha'$.[9]

The intuition behind the proof hinges on the assumption of strictly increasing marginal returns, which means that $\partial f / \partial x$ is increasing in α.[10] With strictly increasing marginal returns, an increase in α from α' to α^* causes $f(x, \alpha)$ to increase, implying that $f(x, \alpha^*) > f(x, \alpha')$. Because x' is the unique argmax at α', $\partial f / \partial x\, (x', \alpha') = 0$. Given this and the fact that $f(x, \alpha)$ is increasing in α, $\partial f / \partial x(x', \alpha^*) > 0$. Therefore, the unique argmax at α^* (x^*) must be greater than x'.[11]

Before returning to our monopoly problem, we illustrate the intuition behind the theorem with a simple example. Suppose the objective function is $f(x, \alpha) = g(\alpha)x - x^2$, and the parameter α can take on two discrete values, such that $g(\alpha') = 2$ and $g(\alpha^*) = 3$. These two functions are illustrated in figure 1 and are labeled f' and f^*. Because the function takes a discrete jump when α increases, we cannot use the implicit-function theorem to perform comparative static analysis. The strict monotonicity theorem applies, however, because the function exhibits strictly increasing marginal returns in α: the slope of the tangent to the objective function increases as α increases from α' to α^*. In other words, $\partial f^* / \partial x > \partial f' / \partial x$ as shown in figure 2. Thus, by the strict monotonicity theorem, the argmax of $f(x, \alpha)$ increases as we increase α (i.e., x increases from 1 to 1.5), implying that x and α are complements. This highlights the role of first-order conditions and illustrates how to apply the theorem: essentially all that needs to be checked is whether or not the function exhibits strictly increasing marginal returns with respect to the parameter in question.

We now have the tool needed to determine how a discrete change in s will affect the monopolist's optimal level of output when demand and cost functions are general.[12] Assume that s

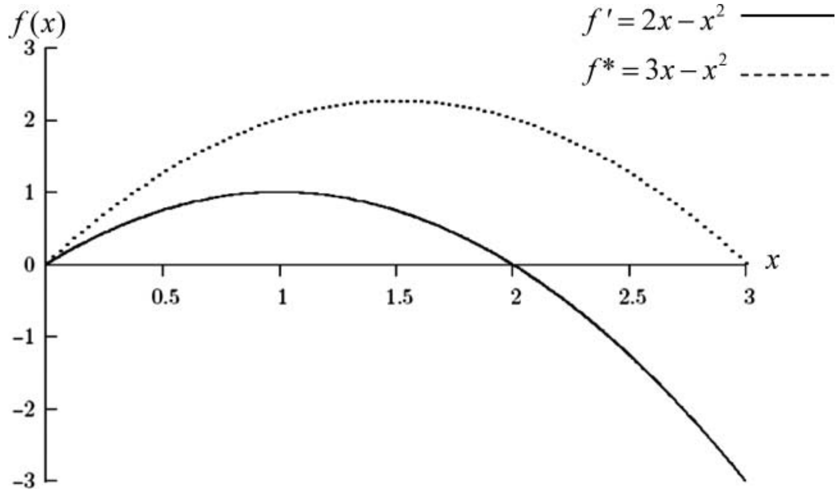

FIGURE 1 A differentiable objective function and a discrete policy change.

increases from s' to s^*, q' is the unique argmax of $\pi(q, s')$, and q^* is the unique argmax of $\pi(q, s^*)$. The firm's total revenue (*TR*) is $p(q)q$, and its total cost (*TC*) is $C(q) - sq$. Under these conditions, the profit equation exhibits increasing marginal returns with respect to q and α because the

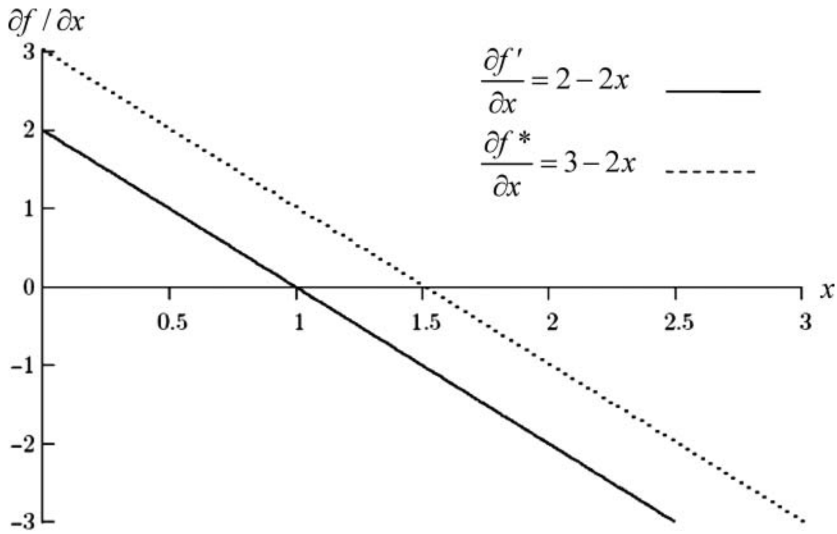

FIGURE 2 Marginal returns that are continuous for a discrete policy change.

following difference in marginal profits is positive.

$$\begin{aligned}\frac{\partial\pi(s^*)}{\partial q}-\frac{\partial\pi(s')}{\partial q}&=\left(\frac{\partial TR}{\partial q}-\frac{\partial TC(s^*)}{\partial q}\right)-\left(\frac{\partial TR}{\partial q}-\frac{\partial TC(s')}{\partial q}\right)\\&=-\frac{\partial TC(s^*)}{\partial q}+\frac{\partial TC(s')}{\partial q}\\&=s^*-s'>0.\end{aligned}$$

This inequality holds because the marginal cost under regime s^* is lower than the marginal cost under regime s' by definition. Thus, by the strict monotonicity theorem, $q^* > q'$. This demonstrates that the monopolist's profit-maximizing level of output will increase with a government policy that reduces marginal cost.

We can use this example to show that the implicit-function theorem is a special case of the strict monotonicity theorem because the strict monotonicity theorem applies to continuous changes as well as discrete changes. Assume that the policy parameter s is a continuous variable. There are strictly increasing marginal returns to s, because $\partial^2\pi/\partial q\partial s =-\partial^2 TC/\partial q\partial s = 1 > 0$. Thus, the strict monotonicity theorem implies that a marginal increase as well as a discrete increase in s will cause an increase in the firm's profit-maximizing output level. Alternatively, by the implicit-function theorem,

$$\frac{\partial q^*}{\partial s}=\frac{\partial^2\pi/\partial q\partial s}{\partial^2\pi/\partial q^2}.$$

To sign this derivative requires additional information, the sign of $\partial^2\pi/\partial q^2$. If the profit function is concave and twice continuously differentiable, then the second-order condition of profit maximization ensures that $\partial^2\pi/\partial q^2 < 0$, which implies that $\partial q^*/\partial a > 0$. This illustrates how easy it is to use the monotone method and shows that the implicit-function approach is a special case of monotone methods. This example also shows that once it is established that there are increasing marginal returns, the other assumptions needed to use the implicit-function theorem are not necessary to sign comparative static expressions.

The monotonicity theorem is weaker, however, when the objective function is not smooth in the choice variable. A classic economics example is where output must be produced in discrete batches of 100 units (i.e., 0, 100, 200, 300, ...), where revenue, cost, and profit are defined as distinct values for each batch of output and are undefined otherwise.[13] In this case, Paul Milgrom and Chris Shannon's (1994) weak-monotonicity theorem applies, where $f(x, \alpha^*)$ is the objective function and x represents one or more choice variables.

Weak-Monotonicity Theorem: Let X be a lattice, $S \subset \mathbf{R}^m$, A be a partially ordered set, and f: $X \times A \to \mathbf{R}$. If $f(x, \alpha)$ has *increasing differences* in x and α and is *supermodular* in x, then $x^* = \text{argmax}_{\mathbf{x}\epsilon s} f(x, \alpha)$ is nondecreasing in α.

We illustrate the important features of the theorem by example.[14] First, we consider the case with only one choice variable, a case where supermodularity does not apply. Figure 3 and figure 4 provide an example in which the objective function is not smooth and not strictly concave but the optimal value of x increases. In this case, $f(x, \alpha)$ exhibits *increasing differences* in α, which represent a weaker discrete version of increasing marginal returns. That is, $f(x, \alpha)$ has increasing differences in x and α when for all $x' > x$, $f(x', \alpha) - f(x, \alpha)$ is nondecreasing in α. We must

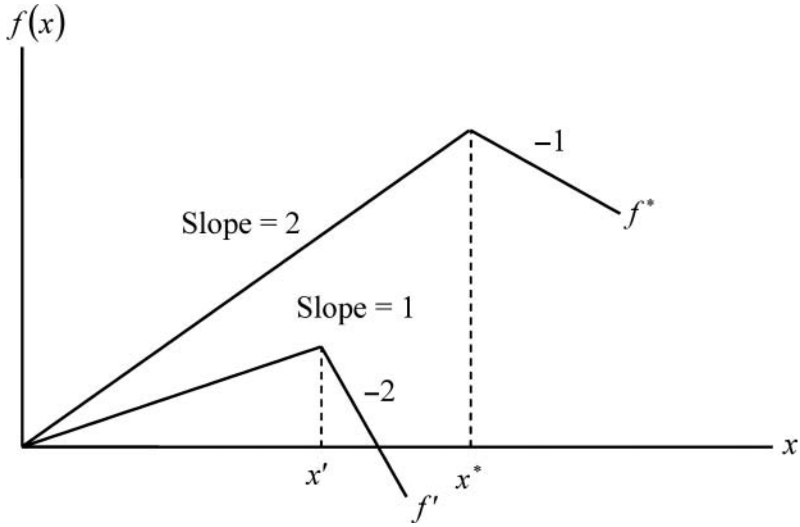

FIGURE 3 A nondifferentiable objective function and a discrete policy change.

now analyze discrete changes because the objective function is no longer differentiable. Figure 5 provides a similar example, this time where the objective function is not concave. Figure 6 illustrates the case in which the equality holds: an increase in α has no effect on the optimal value of x, even though the objective function exhibits increasing differences in α. Given that the objective function is not differentiable at the optimum, this weaker monotonicity theorem implies that an increase in α will have a nonnegative effect on x^*. This illustrates why differentiability

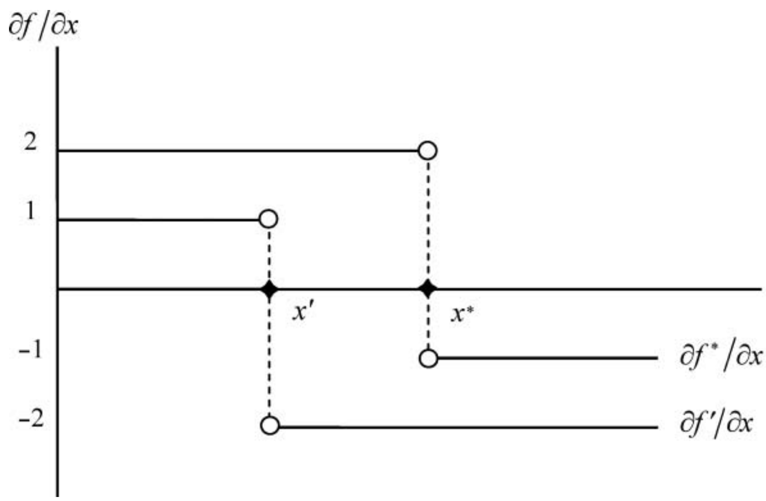

FIGURE 4 Marginal returns that are discontinuous for a discrete policy change.

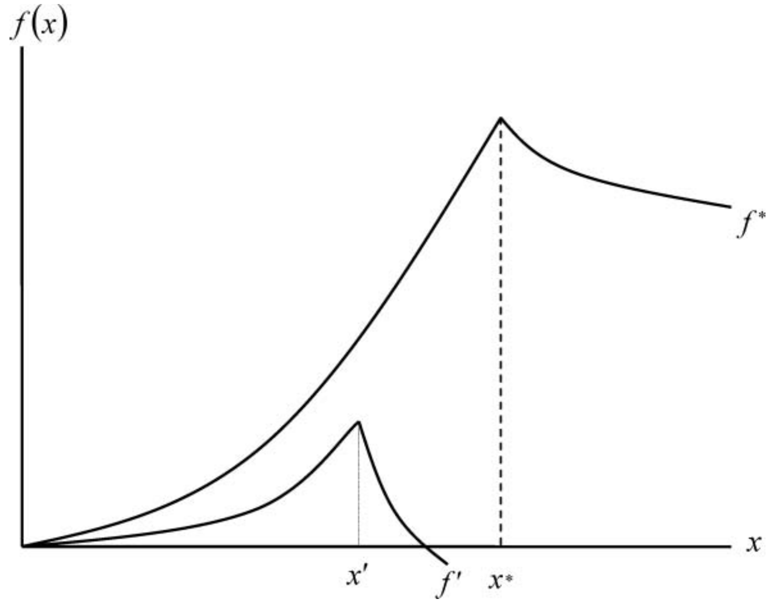

FIGURE 5 A nonconcave objective function and a discrete policy change.

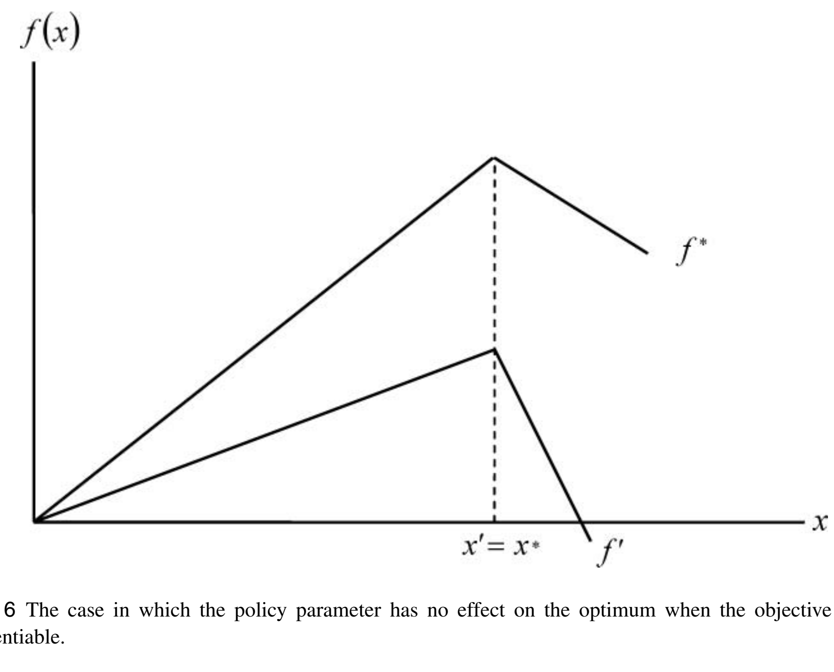

FIGURE 6 The case in which the policy parameter has no effect on the optimum when the objective function is nondifferentiable.

with respect to x is required for the strict-monotonicity theorem to hold when there is just one choice variable. Nevertheless, in this setting the implicit-function theorem cannot be used at all, whereas the weak-monotonicity theorem demonstrates that an increase in a complementary parameter will have a nonnegative effect on the optimal value of the choice variable.

Monotone methods are more complicated when there are multiple choice variables. With a single choice variable, checking for increasing marginal returns or increasing differences is essentially all that is required to do monotone comparative static analysis. With multiple choice variables, the problem is complicated by interaction effects. In this case, the monotonicity theorems also require that all choice variables be complementary.[15] In the continuous case in which the objective function is $f(x_1, x_2, \alpha)$, complementarity of choice variables means that $\partial^2 f/\partial x_1 \partial x_2 \geq 0$. When this condition holds for all choice variables, the objective function is said to be *supermodular*. Thus, the application of monotone comparative static analysis when there are multiple choice variables requires that one check for both supermodularity and increasing marginal returns (or increasing differences).[16] The concept of supermodularity will be especially important in the next section, involving comparative static analysis in game theoretic settings.

MONOTONE METHODS AND GAME THEORY

> This class [of games, called *supermodular games*,] turns out to encompass many of the most important economic applications of noncooperative game theory. (Milgrom and Roberts 1990, 1255)

Game theory has become increasingly important in economics, but comparative static analysis is complicated in game theoretic settings because both optimization and equilibrium concepts are required. Monotone methods simplify this type of analysis, however. The most widely used equilibrium concept in noncooperative game theory is the Nash equilibrium, and it will be our focus here. Augustin Cournot (1838) derived what came to be known as the Nash equilibrium in a duopoly when firms choose output, and Joseph Bertrand (1883) derived the Nash equilibrium when firms choose price.

Because price competition is more common than output competition, we use the Bertrand model to compare and contrast classic methods with monotone methods of comparative statics. We assume that two firms (1 and 2) produce differentiated products in a single market and compete by simultaneously choosing price. In this example, our goal is to analyze how an increase in an excise tax (t) will affect Nash equilibrium prices.

As in the previous section, we demonstrate the advantages of monotone methods by considering progressively more general specifications. We begin by assuming simple linear demand and cost functions. This allows us to illustrate the brute-force method of doing comparative static analysis in a duopoly game. Let firm i's respective demand and total cost functions be $q_i = a - bp_i + dq_j$ and $TC_i = (c + t)\, q_i$, where a, b, d, and c are positive constants. With this notation, subscript i refers to firm 1 or 2 and j refers to the other firm. Throughout this duopoly example, we assume that firm demand is sufficient to assure firm participation (i.e., Nash equilibrium output is positive); this requires that $a > (c - t)(b - d)$ and $b > d/2$. Firm interdependence is revealed in the demand functions as a price increase by one firm raises the demand of the other firm, *ceteris paribus*. The firm's profit equation is $\pi = (p_i - c - t)(a - bp_i + dp_j)$. Information is perfect and complete, meaning that each firm knows all of this information.

Notice that the problem is symmetric, and the respective first and second derivatives of firm i's profit equation are

$$\begin{aligned}
\pi_i &= a - 2bp_i + dp_j + b(c+t),\\
\pi_{ii} &= \pi_{jj} = -2b,\\
\pi_{ij} &= \pi_{ji} = d,\\
\pi_{it} &= b.
\end{aligned}$$

With this notation, $\pi_i \equiv \partial\pi/\partial p_i$, $\pi_{ii} \equiv \partial^2\pi/\partial p_i{}^2$, $\pi_{ij} \equiv \partial^2\pi/\partial p_i \partial p_j$, and $\pi_{it} \equiv \partial^2\pi/\partial p_i \partial t$ for firm i. Solving the first-order condition for p_i gives firm i's best-reply function:

$$p_i^{BR}(p_j) = \frac{a + dp_j + b(c+t)}{2b}.$$

This identifies the optimal p_i for a given value of p_j. The Nash equilibrium satisfies the best-reply functions for both firms simultaneously. In this model, the Nash price for firm i (p_i^*) is:

$$p_i^* = \frac{a + b(c+t)}{2b - d}.$$

For a marginal change, $\partial p_i^*/\partial t = b/(2b - d)$, which is positive given that $b > d/2 > 0$. For a discrete increase in t from t_1 to t_2, the change in p_i^* is $[b(t_2 - t_1)]/(2b - d) > 0$.

Next, we consider a duopoly model that is too general to use the brute-force method but meets the conditions needed to use the implicit-function theorem. In this case, firm i's demand function, $q_i(p_i, p_j)$, is twice continuously differentiable, and has a negative slope ($\partial q_i/\partial p_i < 0$); and, because goods are substitutes, $\partial q_i/\partial p_j > 0$. The total revenue of firm i is $TR_i(p_i, p_j) = p_i\, q_i(p_i, p_j)$. Firm i's total cost is $TC(q_i, t) = C(q_i) + t\, q_i$, with $C(q_i)$ having the same properties as in the monopoly case. Given these definitions, firm i's profit equation is $\pi_i(p_i, p_j, t) = TR_i(p_i, p_j) - TC_i[q_i(p_i, p_j), t]$, which is concave and twice continuously differentiable. The information set is the same as above, and our goal is to determine how a change in t will affect Nash equilibrium prices.

Before performing comparative static analysis, we first summarize the properties of the model. Respective first- and second-order conditions of profit maximization for firm i are:

$$\begin{aligned}
\frac{\partial \pi}{\partial p_i} &= \frac{\partial TR}{\partial p_i} - \frac{\partial TC}{\partial p_i} = 0,\\
\frac{\partial^2 \pi}{\partial p_i^2} &= \frac{\partial^2 TR}{\partial p_i^2} - \frac{\partial^2 TC}{\partial p_i^2} < 0.
\end{aligned}$$

For convenience, subscripts are suppressed on profit, total revenue, and total cost. Because an increase in t increases marginal cost, an increase in t will raise firm i's marginal returns of raising price for a negatively sloped demand function (i.e., $\partial(\partial\pi/\partial p_i)/\partial t \equiv \pi_{it} > 0$).[17]

The best-reply function for firm i is determined by solving the firm's first-order condition for p_i: $p_i^{BR}(p_j) = p_i^*$. We cannot derive this explicitly, but the optimal value of p_i is embedded in firm i's first-order condition, which is identically equal to zero at p_i^*. Thus, we can apply the implicit-function theorem to firm i's first-order condition to determine the slope of its best-reply function.

$$\frac{\partial p_i^*}{\partial p_j} = \frac{-\pi_{ij}}{\pi_{ii}}.$$

Regarding notation, π_{ij} is defined as the second derivative of firm i's profit with respect to p_i and p_j, and π_{ii} is the second derivative of firm i's profit function with respect to p_i. The slope of the best reply for each firm will be positive if prices are strategic complements (i.e., $\pi_{ij} > 0$) and the second-order condition holds (i.e., $\pi_{ii} < 0$).[18] At the Nash equilibrium, both best-reply functions are satisfied simultaneously: $p_1^* = p_1^{BR}(p_2^*)$ and $p_2^* = p_2^{BR}(p_1^*)$. We assume a unique and stable Nash equilibrium. Uniqueness requires that the absolute value of the slope of the best-reply function for each firm is less than one (Kreps and Scheinkman 1983, 328–29), and stability requires that $\pi_{ii}\,\pi_{jj} - \pi_{ij}\,\pi_{ji} > 0$. (The proof is provided in appendix.)

Now we can determine the effect of an increase in parameter t on Nash prices. Substituting the optimal prices into the first-order conditions of each firm and differentiating them with respect to t yields the system of equations:

$$\pi_{11}\frac{\partial p_1}{\partial t} + \pi_{12}\frac{\partial p_2}{\partial t} + \pi_{1t}\frac{\partial t}{\partial t} \equiv 0,$$
$$\pi_{21}\frac{\partial p_2}{\partial t} + \pi_{22}\frac{\partial p_2}{\partial t} + \pi_{2t}\frac{\partial t}{\partial t} \equiv 0.$$

This linear system can be written in matrix form as

$$\begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{pmatrix} \begin{pmatrix} \dfrac{\partial p_1^*}{\partial t} \\ \dfrac{\partial p_2^*}{\partial t} \end{pmatrix} \equiv \begin{pmatrix} -\pi_{1t} \\ -\pi_{2t} \end{pmatrix}.$$

Applying Cramer's rule, in which Π is the 2×2 matrix of second derivatives of profits (i.e., the first matrix in the previous equation),

$$\frac{\partial p_1^*}{\partial t} = \frac{\begin{vmatrix} -\pi_{1t} & \pi_{12} \\ -\pi_{2t} & \pi_{22} \end{vmatrix}}{|\Pi|}.$$

Assuming stability, the determinant of Π is positive (i.e., $\pi_{11}\,\pi_{22} - \pi_{12}\,\pi_{21} > 0$). Given that $\pi_{ii} < 0$, $\pi_{ij} > 0$, and $\pi_{it} > 0$ (i.e., $-\pi_{1t}\,\pi_{22} + \pi_{2t}\,\pi_{12} > 0$), an increase in t will cause p_1^* to increase. Because the problem is symmetric, p_2^* will also increase with t. This example shows how both stability conditions and second-order (concavity) conditions can be important when applying classic comparative static methods to game theoretic problems where both optimization and equilibrium concepts are important.

There are two main weaknesses with the classic approach. First, it suffers from the so-called *curse of dimensionality*, as finding the solution becomes increasingly difficult or impossible as the number of firms and the number of choice variables (e.g., price, advertising, product

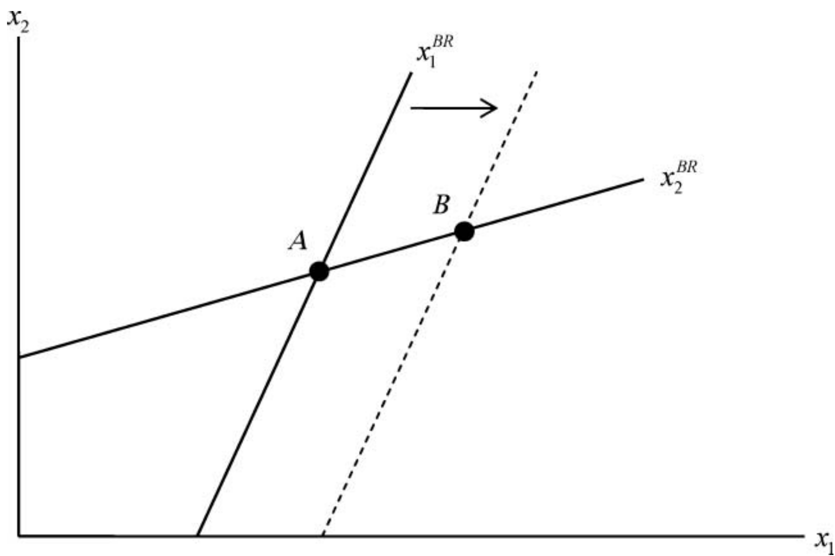

FIGURE 7 Best-reply functions, Nash equilibria, and an excise tax increase on firm 1.

quality) increase. In addition, this technique cannot be used when the relevant functions are not differentiable or when the policy variable takes a discrete change.

Fortunately, comparative static results can still be derived using monotone methods. What is required is that the game be supermodular. Because the main ideas are the same in the differentiable and nondifferentiable cases, and because we have considered the problem of nondifferentiability in the previous section, we will focus on problems associated with the curse of dimensionality, not that of differentiability.[19] These are called smooth supermodular games. To illustrate the power of this approach, we extend the Bertrand model further by assuming that n firms ($1 < n < \infty$) compete in two choice variables: price and marketing expenditures (M). Formal requirements of the game are subsequently described.

SUPERMODULAR GAMES

In a *smooth supermodular game* with n firms, the following assumptions hold for each firm i and each rival j (Milgrom and Roberts 1990, 1264).[20]

1. *Differentiability*: The profit equation is twice continuously differentiable with respect to p_i and M_i.
2. *Complementary strategies* (of each firm's own strategic variables): $\partial^2\pi_i/\partial p_i\, \partial M_i \geq 0$.
3. *Strategic complements* (for strategic variables between firms): $\partial^2\pi_i/\partial p_i \partial p_j \geq 0$, $\partial^2\pi_i/\partial p_i\, \partial M_j \geq 0$, $\partial^2\pi_i/\partial M_i\, \partial p_j \geq 0$, and $\partial^2\pi_i/\partial M_i\, \partial M_j \geq 0$.
4. *Complementary policy parameter*: $\partial^2\pi_i/\partial p_i\, \partial t \geq 0$ and $\partial^2\pi_i/\partial M_i\, \partial t \geq 0$.

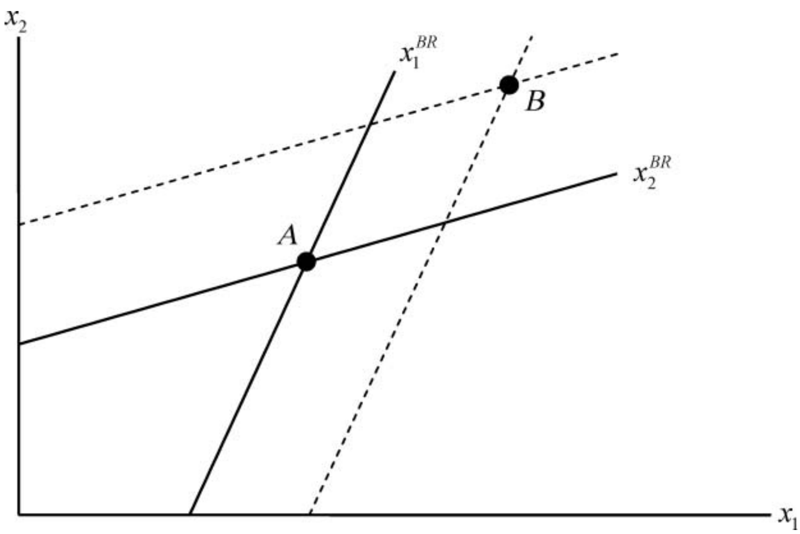

FIGURE 8 Best-reply functions, Nash equilibria, and an excise tax increase on firms 1 and 2.

One can think of this as a game of supercomplementarity, which implies that there is complementarity among all strategies and the policy parameter (Fudenberg and Tirole 1992, 491).[21] That is, all choice variables for each firm are complements (2), all choice variables between different firms are complements (3), and the policy parameter is a complement with all choice variables of each firm (4). Examples of such a policy would include an increase in an excise tax that raises the marginal returns to a price increase or a marketing subsidy that raises the marginal returns to a marketing increase. The important thing to notice is that this specification implies that there are increasing marginal returns between all possible pairs of choice variables and the policy parameter.

Under these conditions and assuming a unique Nash equilibrium,[22] the following comparative static results hold for all firms (Milgrom and Roberts 1990, Theorem 6 and its Corollary):

$$\frac{\partial p_i^*}{\partial t} \geq 0; \quad \frac{\partial M_i^*}{\partial t} \geq 0.$$

That is, when assumptions 1–4 hold, an increase in t will have a nonnegative effect on the Nash prices and marketing expenditures.[23]

Thus, all that is required to apply the theorem is to verify the validity of the assumptions of the Theorem. The critical assumptions are 2 and 3. Differentiability is not required but assumed here for convenience (1). Assumption 4, which implies that there are increasing differences between t and each choice variable, can generally be met by properly defining the policy variable or parameter. So, to apply the theorem, each firm's own strategic variables are complementary (3), and the objective function is supermodular in choice variables: all strategic variables across firms are strategic complements (4).

The proof of this result requires the use of lattice theory, so it will not be presented here. The main idea is intuitive, however. The driving force behind the proof in the case of a strict inequality is the supercomplementarity assumption. That is, any increase in a policy parameter that exhibits increasing marginal returns will cause p_i^* (M_i^*) to increase (assumption 4). This in turn causes M_i^* (p_i^*) to rise because own-choice variables are complements (assumption 2), and it also causes p_j^* and M_j^* to rise for all j because rival choice variables are strategic complements (assumption 3). Finally, this causes a chain of feedback effects that reinforce these increases. That is, the resulting increases in p_j^* and M_j^* cause further increases in p_i^* and M_i^*, and so on. In terms of best-reply functions, this means that the policy change causes one or more of the best-reply functions for each choice variable to shift away from the origin. Examples are given in figure 7 and figure 8, where the parameter change causes a change in equilibrium from A to B, assuming that there are two firms and that the choice variable is $x = p$ or M. Thus, the Nash equilibrium, where the best-reply functions intersect, will support higher optimal values of the strategic variables for an increase in t.

CONCLUSION

A strong foundation in mathematics has become increasingly important to understanding and conducting economics research. In response, minimum math requirements needed to enter graduate programs in economics have risen. To better prepare students for graduate study in economics, and as valuable content in its own right, more and more undergraduate programs encourage students to take courses in mathematical economics and advanced mathematics. Some have added degrees in mathematical economics. For these reasons, it is important that undergraduate textbooks in mathematical economics be updated.

One area where this has not occurred is in the coverage of comparative static analysis. In nonstrategic settings, the strict monotonicity theorem provides a more general method of performing comparative static analysis and demonstrates that many of the assumptions required to use the implicit-function theorem are superfluous. In a game theoretic setting with many players and strategic options, using the implicit-function theorem to do comparative statics suffers from the curse of dimensionality. Fortunately, when the game is supermodular, comparative static analysis can be done with ease by using monotone methods. Although these newer methods have been developed since the early 1990s and are easy to use, they are ignored in the textbooks. We hope that this article will encourage instructors to bring monotone methods into the classroom and will convince textbook writers to include them in future editions of their books on mathematical economics.

NOTES

1. This is based on institutional rankings as defined by *U.S. News and World Report* (2008) and our survey of Web sites. (A table of these data is available upon request from the authors.) In general, those that did not offer courses in mathematical economics required their students to take advanced courses in mathematics and statistics—especially students interested in graduate school. An excellent example of such advice is written by Johnson (2008).
2. Examples of colleges and universities outside the top tiers that offer such degrees include the University of Kentucky, Marquette University, Pacific Lutheran University, Reed College, and Temple University.
3. Most undergraduate textbooks in mathematical economics cover this topic—including those of Simon and Blume (1994), Sydsaeter and Hammond (1995), Hands (2004), Baldani, Bradfield, and Turner (2005), and Chiang and Wainwright (2005). Here, we are referring to the part of the theorem that gives us the derivatives of the function when in implicit form. This is part c of the theorem in Simon and

Blume (1994, 341). Chiang and Wainwright (2005, 196) refer to this portion of the theorem as the implicit-function rule.

4. For a discussion of monotone comparative static methods in constrained optimization problems, see Quah (2007).
5. However, these topics are covered in graduate textbooks, including those of Sundaram (1996) and Carter (2001). For a review of the many uses of these monotone methods, see Amir's (2005) and Vives' (2005a, 2005b) works. For recent applications to the economics of advertising, see Isariyawongse, Kudo, and Tremblay (2007) and Iwasaki et al. (2008).
6. Because the profit function is strictly concave and twice continuously differentiable in q, a unique optimum exists.
7. From the assumptions of the model, $\partial p/\partial q < 0$ and $\partial^2 C/\partial q^2 > 0$. Thus, the second-order condition holds only if the demand function is not too convex (i.e., $\partial^2 p/\partial q^2$ is sufficiently small). This guarantees a strictly concave profit function. For example, this condition holds for a linear demand function, because $\partial^2 p/\partial q^2 = 0$.
8. This normally pertains to small changes in taxes and subsidies, but discrete changes are common. For example, the last increase in the federal excise tax on beer was in 1991, when the rate increased from \$9 to \$18 per (31-gallon) barrel. Alternatively, one could consider s to be a form of government deregulation, where an increase in s leads to a discrete fall in marginal cost.
9. Similarly, $x^* < x'$ if $\alpha^* < \alpha'$. Note that **R** represents a set of real numbers.
10. In the continuous case, strictly increasing marginal returns means that the parameter and the choice variable are complements. That is, $\partial^2 f/\partial x \partial \alpha > 0$.
11. According to Sundaram (1996), when uniqueness is relaxed, the theorem still applies to the greatest element and the least element of the argmax set. We wish to thank an anonymous referee for pointing this out to us.
12. To consider a policy (α) that exhibits decreasing marginal returns, such as an excise tax (t) in our monopoly example, simply redefine the policy variable to equal $-\alpha$. For example, one can analyze an excise tax by defining it as a per-unit subsidy: $s = -t$. Although t exhibits decreasing marginal returns, $-t =$ s exhibits increasing marginal returns.
13. A problem such as this is described in the intermediate microeconomics textbook by Bernheim and Whinston (2008, 297–98). Discrete choice problems are also discussed in Varian's (2006) intermediate microeconomics textbook.
14. Proofs are not discussed here, because they require a knowledge of lattices, which goes beyond the scope of this article. A lattice is a partially ordered binary set that contains its greatest upper bound, infimum, and its greatest lower bound, infimum (Vives 1999, 17–18). For further discussion of lattice theory, see Milgrom and Roberts (1990), Milgrom and Shannon (1994), and Topkis (1998).
15. Again, proofs are not discussed here, because they require the use of lattice theory. For further discussion, see Milgrom and Shannon (1994), Edlin and Shannon (1998), and Vives (1999).
16. Theorem 10.12 in Sundaram (1996) demonstrates that supermodularity and increasing differences are essentially the same. We thank an anonymous referee for pointing this out. The only distinction is that supermodularity applies to complementarity between two different choice variables and increasing differences applies to complementarity between a parameter and a choice variable (Milgrom and Shannon 1994, 164).
17. The reverse is true for output. That is, $\partial(\partial\pi/\partial q_i)/\partial t < 0$. Thus, comparative static results will show that an increase in t will lead to higher prices and lower output.
18. Best-reply functions normally have a negative slope when firms compete in output (Cournot 1838) and a positive slope when they compete in price (Bertrand 1883). Although exceptions are possible, Amir and Grilo (1999) called this the "typical geometry" for Cournot and Bertrand models. For the remainder of our discussion, we assume this typical geometry.
19. This implies that best-reply functions are differentiable. To assume otherwise would require the use of lattice theory. If we were to assume that best replies are complete lattices instead of smooth functions, the main conclusions of the article would remain the same.
20. The model also assumes bounds on the choice variables. In our case, it is natural to assume that prices and marketing expenditures are nonnegative and are less than infinity.

21. For a discussion of monotone comparative static methods when strategies are strategic substitutes instead of complements, see Roy and Sabarwal (2005).
22. The theorem is actually more general than this. The strategy space need not be compact and convex (Fudenberg and Tirole 1992, 489–490). In addition, the theorem also applies to models with multiple equilibria. In that case, the largest and smallest pure-strategy Nash equilibria are nondecreasing in the policy parameter. We thank an anonymous referee for pointing this out.
23. A strict inequality will hold if the best-reply functions have a positive slope, and if an increase in t shifts the best-reply functions away from the origin.

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APPENDIX

STABILITY OF THE BERTRAND-NASH EQUILIBRIUM

Consistent with the duopoly model that we presented, assume smooth best-reply functions and a unique interior Nash equilibrium. For a discussion of stability conditions in the Cournot model, see Fudenberg and Tirole (1992, 23–25). The graph of the best-reply functions assumes that p_2 is on the vertical axis and that p_1 is on the horizontal axis, as in figure 8. Stability requires that for any disequilibrium set of prices in the neighborhood of the Nash equilibrium, the dynamic (myopic) adjustment process causes firms to adjust prices in the direction of Nash prices. This occurs when firm 1's best-reply function is steeper than firm 2's best-reply function. This is easy to verify by starting at a disequilibrium point and showing that prices will converge to the equilibrium in figure 6.

As demonstrated previously, the slopes of the best-reply functions for firm 1 (BR_1') and firm 2 (BR_2') are positive. Recall that these slopes are

$$BR_1' = \frac{-\pi_{12}}{\pi_{11}}; \quad BR_2' = \frac{-\pi_{12}}{\pi_{22}}$$

Because we are interested in solving each best-reply function for p_2 (i.e., p_2 is on the vertical axis), the slope of firm 1's best reply when p_2 is on the vertical axis is $1/BR_1'$. Thus, the Bertrand-Nash equilibrium will be stable if and only if $1/BR_1' > BR_2'$. Thus,

$$\frac{-\pi_{11}}{\pi_{12}} > \frac{-\pi_{21}}{\pi_{22}},$$

$$\frac{\pi_{11}}{\pi_{12}} < \frac{\pi_{21}}{\pi_{22}}.$$

This becomes

$$\pi_{11}\,\pi_{22} > \pi_{12}\,\pi_{21},$$

because $\pi_{12} > 0$ and $\pi_{22} < 0$.