The nemattic phase of a system of long hard rods*

Margherita Disertori

Laboratoire de Mathématiques Raphael Salem,
UMR 6085 CNRS-Université de Rouen,
Av. de l'Université, BP.12, Technopole du Madrillet,
F76801 Saint-Étienne-du-Rouvray

Alessandro Giuliani

Dipartimento di Matematica, Università di Roma Tre,
L.go S. L. Murialdo 1, 00146 Roma - Italy

(Dated: October 9, 2011)

The existence of a nematic phase in a system of long rods with purely hard core interactions is rigorously established.

Dedicated to the 70th birthday of Giovanni Gallavotti

1. INTRODUCTION

In 1949, L. Onsager [21] proposed a statistical theory for a system of elongated molecules interacting via repulsive short-range forces, based on an explicit computation of the first few Mayer's coefficients for the pressure. Onsager's theory predicted the existence at intermediate densities of a nematic liquid crystal phase, that is a phase in which the distribution of orientations of the particles is anisotropic, while the distribution of the particles in space is homogeneous and does not exhibit the periodic variation of densities that characterizes solid crystals (periodicity in all space dimensions) or smectic liquid crystals (periodicity in one dimension).

From a microscopic point of view, the most natural lattice model describing elongated molecules with short-range repulsive forces is a system of rods of length k and thickness 1 at fixed density ρ (here ρ = average number of rods per unit volume), arranged on a cubic lattice (say a large squared box portion of $\mathbb{Z}^2$) and interacting via a purely hard core potential. Even though very natural, this model is not easy to treat and its phase diagram in the plane (ρ, k) is still not understood in many physically relevant parameters' ranges. Of course, for all k's, at very small density there is a unique isotropic Gibbs state, invariant under translations and under discrete rotations of 90^o (this can be proved by standard cluster expansion methods). If $k = 2$, it is known [10] that the state is analytic (and, therefore, there is no phase transition) for all densities but, possibly, at the close packing density (i.e., at the maximal possible density $\rho_{max} = 1/k$). If k is sufficiently large ($k \geq 7$ should be enough [8]) there is numerical evidence [8, 18] for *two* phase transitions as ρ is increased from zero to the maximal density. The first (isotropic to nematic) seems to take place at a $\rho_c^{(1)} \simeq C_1/k^2$, while the second (nematic to isotropic) seems to take place at $\rho_c^{(2)} \simeq \rho_{max} - C_2/k^3$. These findings renovated the interest of the condensed matter community in the phase diagram of long hard rod systems and stimulated more systematic numerical studies of the nature of the critical points at $\rho_c^{(1)}$ and $\rho_c^{(2)}$ [4, 5, 16, 17, 19, 20].

From a mathematical point of view there is no rigorous proof of any of these behaviors yet, with the exception of the "trivial" case of very low densities: namely, there is neither a proof of nematic order at intermediate densities, nor a proof of the absence of orientational order at very high densities, nor a rigorous understanding of the nature of the transitions. In this work we give a rigorous proof to some of the conjectures stated above on the nature of the phase diagram of long hard rods systems. More precisely, we show that well inside the interval $(\rho_c^{(1)}, \rho_c^{(2)})$, the system is in a nematic phase, i.e., in a phase characterized by two distinguished Gibbs states, with different orientational order (horizontal or vertical) but with no translational order.

To the best of our knowledge, this is the first proof of the existence of a nematic phase in a microscopic model with molecules of fixed finite length and finite thickness, interacting via a purely repulsive potential. Important previous results, which our proof builds on, include the following: proof of nematic order in a continuum system of infinitely thin rods with two orientations [2, 23]; proof of an isotropic to nematic transition in an integrable model of polydisperse long rods in $\mathbb{Z}^2$ [13] (a very nice result, obtained by mapping the partition function of the polydisperse hard rods gas into that of the nearest neighbor 2D Ising model); proof of orientational order in (reflection positive) lattice models with both repulsive and attractive interactions [11]. There are several other related models where the existence of orientational order was proved; however, in many cases, orientational comes together with translational order [9, 12, 15], which is not the case in a nematic phase.

Of course, our proof leaves many questions about the phase diagram of long hard rod systems open, the most urgent being, we believe, the question about the nature of the densely packed phase at $\rho \geq \rho_c^{(2)}$: can one prove the absence of orientational order, at least at close packing? Is the densely packed phase characterized by some "hidden" (striped-like) order? Progress on these problems would be important for the understanding of the emergence of hidden order in more complicated systems than elongated molecules with purely hard core interactions, in which short range repulsion competes with attractive forces acting on much longer length scales.

The rest of the paper is organized as follows. In Section 2 we "informally" introduce the model, state the main results and explain the key ideas involved in the proof. In Section 3 we define the model and state the main theorem (Theorem 1 below) in a mathematically precise form. In the following sections we prove Theorem 1: in Section 4 we rewrite the partition function with q boundary conditions in terms of a sum over contours' configurations, where the contours are defined in a way suitable for later application of a Pirogov-Sinai argument. In Section 5 we prove the convergence of the cluster expansion for the pressure, under the assumption that the activity of the contours is small and decays sufficiently fast in the contour's size. In Section 6 we complete the proof of convergence of the cluster expansion for the pressure, by inductively proving the desired bound on the activity of the contours. Finally, in Section 7 we adapt our expansion to the computation of correlation functions and we prove Theorem 1.

2. THE MODEL

We consider a finite square box $\Lambda \subset \mathbb{Z}^2$ of side L, to be eventually sent to infinity. We fix k and the average density $\rho \in (0, 1/k)$. The finite volume Gibbs measure at activity z gives weight z^n to every allowed configuration of n rods: we say that a configuration is allowed if no pair of rods overlaps. Of course, one also needs to specify boundary conditions: we consider, say, periodic boundary conditions, open boundary conditions, horizontal or vertical boundary conditions, the latter meaning that all the rods within a distance $\sim k$ from the boundary of Λ are horizontal or vertical – see below for a more precise definition. The grand canonical partition function is:

$$Z_\Lambda(z) = \sum_{n \geq 0} z^n w_n^\Lambda \ , \tag{2.1}$$

where w_n^Λ is the number of allowed configurations of n rods in the box Λ, in the presence of the prescribed boundary conditions. Note that $w_n^\Lambda = 0$ for all $n \geq |\Lambda|/k$, which shows that $Z_\Lambda(z)$ is a finite (and, therefore, well defined) sum for all finite Λ's. The activity z is fixed in such a way that

$$\lim_{|\Lambda| \to \infty} \frac{\langle n \rangle_\Lambda}{|\Lambda|} = \lim_{|\Lambda| \to \infty} \frac{1}{|\Lambda|} \frac{\sum_{n \geq 0} n z^n w_n^\Lambda}{Z_\Lambda(z)} = \rho \ . \tag{2.2}$$

The goal is to understand the properties of the partition function and of the associated Gibbs state in the limit $|\Lambda| \to \infty$ at fixed ρ. An informal statement of our main result is the following.

Main result. *For k large enough, if $k^{-2} \ll \rho \ll k^{-1}$, the system admits two distinct infinite volume Gibbs states, characterized by long range orientational order (either horizontal or vertical) and no translational order, selected by the boundary conditions.*

Sketch of the proof. The idea is to coarse grain Λ in squares of side $\ell \simeq k/2$. Each square is large, since in average it contains many ($\sim \rho k^2 \gg 1$) rods. On the other hand, its side ℓ is small enough to ensure that only rods of the same orientation are allowed to have centers in the same square. This means that the partition function restricted to a single square contains only sums over vertical or horizontal configurations. Let us consider the case where the rods are all horizontal (vertical is treated in the same way). A typical horizontal configuration consists of many ($\sim \rho k^2$) horizontal rods with centers distributed approximately uniformly (Poisson-like) in the square, since their interaction, once we prescribe their direction, is very weak: they "just" have a hard core repulsion that prevents two rods to occupy the same row, an event that is very rare, since the density of occupied rows ($\sim \rho k^2/k$) is very small, thanks to the condition that $\rho \ll 1/k$. Because of this small density of occupied rows, we are able to quantify via cluster expansion methods how close to Poissonian is the distribution of the centers in the given square (once we condition with respect to a prescribed orientation of the rods).

To control the interaction between different squares we use a Pirogov-Sinai argument. Each square can be of three types: (i) either it is of type $+1$, if it contains only horizontal rods, (ii) or it is of type -1, if it contains only vertical rods, (iii) or it is of type 0, if it is empty. The values $-1, 0, +1$ associated to each square play the role of spin values associated to the coarse grained system. The interaction between the spins is only finite range and squares with vertical ($+1$) and horizontal (-1) spin have a strong repulsive interaction, due to the hard core constraint. On the other hand, the vacuum configurations (the spins equal to 0) are very unlikely, since the probability of having a large deviation event such that a square of side ℓ is empty is expected to be exponentially small $\sim \exp\{-c\rho k^2\}$, for a suitable constant c.

Therefore the typical spin configurations consist of big connected clusters of "uniformly magnetized spins", either of type $+1$ or of type -1 separated by boundary layers (the contours), which contain zeros or pairs of neighboring opposite spins. These contours can be shown to satisfy a Peierls' condition. The contour theory is not symmetric under spin flip and, therefore, we are forced to study it by the (non-trivial although standard) methods first introduced by Pirogov and Sinai [22].

Before we move to discuss the details of our proof, let us state our main results in a mathematically more sound form.

3. MAIN RESULTS

Definitions. For any region $X \subseteq \mathbb{Z}^2$ we call Ω_X the set of rod configurations $R = (r_1, \ldots, r_n)$ where all the rods belong to the region X. A rod r "*belongs to*" a region X if the center of the rod is inside the region, in which case we write $r \in X$. Here each rod is identified with a sequence of k adjacent sites of $\mathbb{Z}^2$ in the horizontal or vertical direction. If k is odd, the center of the rod belongs to the lattice $\mathbb{Z}^2$ itself and, therefore, the notion of "rod belonging to X" is unambiguously defined. On the contrary, if k is even, the geometrical center of the rod does not belong to the original lattice $\mathbb{Z}^2$; however, for what follows, it is convenient to pick one of the sites belonging to r and elect it to the role of "center of the rod": if r is horizontal (vertical), we decide that the "center of r" is the site of r that is closest to its geometrical center from the left (bottom). We shall also say that: a rod r "*touches*" a region X, if $r \cap X \neq \emptyset$; a rod r "*is contained in*" a region X, if $r \cap X^c = \emptyset$, in which case we write $r \subset X$.

The rod configurations in Ω_X can contain overlapping and even coinciding rods; we denote by $R(r)$ the multiplicity of r in $R \in \Omega_X$ and by $\mathrm{supp}(R)$ the support of R, i.e., the set of rods that are in R, each counted without taking multiplicity into account. The grand canonical partition function in X with open boundary conditions is

$$Z_0(X) = \sum_{R \in \Omega_X} z^{|R|} \varphi(R) \tag{3.1}$$

where $|R| := \sum_{r \in \mathrm{supp}(R)} R(r)$ and $\varphi(R)$ implements the hard core interaction:

$$\varphi(R) = \prod_{r, r' \in R} \varphi(r, r'), \qquad \varphi(r, r') = \begin{cases} 1 & \text{if } r \cap r' = \emptyset \\ 0 & \text{if } r \cap r' \neq \emptyset. \end{cases} \tag{3.2}$$

Let $\ell := \lceil k/2 \rceil$ and assume that $\Lambda \subseteq \mathbb{Z}^2$ is a square box of side divisible by 4ℓ. We pave Λ by squares of side ℓ, called "tiles", and by squares of side 4ℓ, called "smoothing squares". The lattice of the tiles' centers is a coarse grained lattice of mesh ℓ, called Λ'; similarly, the lattice of the smoothing squares' centers is a coarse grained lattice of mesh 4ℓ, called Λ''. Given $\xi \in \Lambda'$, the tile centered at ξ is denoted by Δ_ξ; given $a \in \Lambda''$, the smoothing square centered at a is denoted by $\mathcal{S}_a$. Given two sets $X, Y \subseteq \Lambda$, we indicate their (euclidean) distance by $\mathrm{dist}(X, Y) = \min_{x \in X, y \in Y} |x - y|$. If X and Y are union of tiles, we shall also indicate by $X', Y' \subset \Lambda'$ the coarse versions of X and Y, i.e., the sets of sites in Λ' such that $X = \cup_{\xi \in X'} \Delta_\xi$ and $Y = \cup_{\xi \in Y'} \Delta_\xi$. The distance between X' and Y' is denoted by $\mathrm{dist}(X', Y')$ and their rescaled distance by $\mathrm{dist}'(X', Y') := \ell^{-1}\mathrm{dist}(X', Y')$; with these conventions, if ξ and η are nearest neighbor sites on Λ', then $\mathrm{dist}(\xi, \eta) = |\xi - \eta| = \ell$ and $\mathrm{dist}'(\xi, \eta) = 1$. The complement of Λ is denoted by $\Lambda_c := \mathbb{Z}^2 \setminus \Lambda$ and its coarse version by Λ'_c, with obvious meaning.

The size of the tiles is small enough to ensure that if one vertical (horizontal) rod belongs to a given tile, then all other rods belonging to the same tile and respecting the hard core repulsion condition must be vertical (horizontal). If a tile is empty, i.e., no rod belongs to it, then we assign it an extra (fictitious) label, which can take three possible values, either 0 or + or −. A rod configuration $R \in \Omega_\Lambda$ (combined with an assignment of these extra fictitious labels) induces a spin configuration $\sigma = \{\sigma_\xi\}_{\xi \in \Lambda'}$ on Λ', $\sigma_\xi \in \{-1, 0, +1\}$, via the following rules:

- $\sigma_\xi = +1$, if all rods belonging to Δ_ξ are horizontal or if the tile is empty with the extra label equal to +,

- $\sigma_\xi = -1$, if all rods belonging to Δ_ξ are vertical or if the tile is empty with the extra label equal to −,

- $\sigma_\xi = 0$, if Δ_ξ is empty with the extra label equal to 0.

The corresponding set of rod configurations in the tile Δ_ξ is denoted by $\Omega_{\Delta_\xi}^{\sigma_\xi}$: $\Omega_{\Delta_\xi}^{+}$ ($\Omega_{\Delta_\xi}^{-}$) is the set of rod configurations in Δ_ξ consisting either of horizontal (vertical) rods or of the empty configuration; similarly, $\Omega_{\Delta_\xi}^{0}$ consists only of the empty configuration.

Note that the grand canonical partition function in Λ with open boundary conditions can be rewritten as

$$Z_0(\Lambda) = \sum_{\sigma \in \Theta_{\Lambda'}} \sum_{R \in \Omega_\Lambda(\sigma)} \bar{\varphi}(R) \, , \tag{3.3}$$

where $\Theta_{\Lambda'} := \{-1, 0, +1\}^{\Lambda'}$ and $\Omega_\Lambda(\sigma) := \cup_{\xi \in \Lambda'} \Omega_{\Delta_\xi}^{\sigma_\xi}$. Moreover,

$$\bar{\varphi}(R) := \Big[\prod_{\xi \in \Lambda'} \zeta(\xi) \Big] \varphi(R) \, , \tag{3.4}$$

where the activity of a tile is defined as

$$\zeta(\xi) = \begin{cases} z^{|R_\xi|} & \text{if } \sigma_\xi = \pm 1 \\ -1 & \text{if } \sigma_\xi = 0. \end{cases} \tag{3.5}$$

The sign −1 is necessary to avoid over-counting of the empty configurations. Note that $\bar{\varphi}(R)$ depends both on σ and on R; however, in order not to overwhelm the notation, we shall drop the label σ.

The partition function with q boundary conditions, $q = \pm$, denoted by $Z(\Lambda|q)$, can be defined in a similar fashion:

$$Z(\Lambda|q) = \sum_{\sigma \in \Theta^q_{\Lambda'}} \sum_{R \in \Omega_\Lambda(\sigma)} \bar{\varphi}(R) \tag{3.6}$$

where $\Theta^q_{\Lambda'} \subset \Theta_{\Lambda'}$ is the set of spin configurations such that $\mathrm{dist}'(\xi, \Lambda'_c) \leq 5 \Rightarrow \sigma_\xi = q$. Correspondingly, the ensemble $\langle \cdot \rangle^q_\Lambda$ with q boundary conditions is defined by

$$\langle A_X \rangle^q_\Lambda = \frac{1}{Z(\Lambda|q)} \sum_{\sigma \in \Theta^q_{\Lambda'}} \sum_{R \in \Omega_\Lambda(\sigma)} \bar{\varphi}(R)\, A_X(R) \ , \tag{3.7}$$

where A_X is a local observable, depending only on the restriction R_X of the rod configuration R to a given finite subset $X \subset \Lambda$. The infinite volume states $\langle \cdot \rangle^q$ with q boundary conditions are defined by

$$\langle A_X \rangle^q = \lim_{|\Lambda| \to \infty} \langle A_X \rangle^q_\Lambda \ , \tag{3.8}$$

if the limit exists for all local observables A_X, $X \subset \mathbb{Z}^2$. Our main results can be stated as follows.

Theorem 1 *If zk and $(zk^2)^{-1}$ are small enough, then the two infinite volume states $\langle \cdot \rangle^q$, $q = \pm$, exist. They are translationally invariant and are different among each other. In particular, if $\chi^\sigma_{\xi_0}$ is the projection onto the rod configurations such that $R_{\xi_0} \in \Omega^\sigma_{\Delta_{\xi_0}}$, then*

$$\langle \chi^{-q}_{\xi_0} \rangle^q \leq e^{-czk^2} \ , \tag{3.9}$$

for a suitable constant c. Moreover, if n_{x_0} is the indicator function equal to 1 if a rod touches $x_0 \in \mathbb{Z}^2$ and 0 otherwise, then

$$\rho = \langle n_{x_0} \rangle^+ = \langle n_{x_0} \rangle^- = z(1 + O(zk, e^{-czk^2})) \tag{3.10}$$

and

$$\rho(x-y) = \langle n_x n_y \rangle^+ = \langle n_x n_y \rangle^- = \rho^2 \Big(1 + O(e^{-c|x-y|/k})\Big) \ , \tag{3.11}$$

for a suitable $c > 0$.

Eq.(3.9) proves the existence of orientational order in the system. Eqs.(3.10)-(3.11) prove the absence of translational symmetry breaking. These two behavior together prove that the system is in a nematic liquid crystal phase, as announced in the introduction. The rest of the paper is devoted to the proof of Theorem 1, which is based on a two-scales cluster expansion. As it will be clear from the discussion in the next sections, our construction proves much more than what is explicitly stated in Theorem 1, namely it allows us to compute the averages of all the local observables in terms of an explicit exponentially convergent series.

4. THE CONTOUR THEORY.

The proof of the Theorem 1 will be split in several steps. We start by developing a representation of the partition function $Z(\Lambda|q)$ with q boundary conditions in terms of a set of interacting contours. Later, we will adapt the contour expansion to the computation of the correlations. The contour theory can be studied by an adaptation of Pirogov-Sinai's method to the present context. See [22] for the original version of this method and [1, 14, 25, 26] for several alternative simplified versions of it. In the following we will try to be as self-consistent as possible and to keep things simple, by avoiding as much as we can general and abstract settings. We first need some more definitions.

Definition 1: sampling squares. Given a spin configuration $\sigma \in \Theta^q_{\Lambda'}$, this induces a partition of Λ' into regions where the spins are "uniformly magnetized up or down" (i.e., regions where the spins are constantly equal to $+1$ or to -1) and boundary regions separating the "uniformly magnetized regions" among each other, which can possibly contain spins equal to zero. To make this more precise we introduce the notion of "sampling squares", defined as follows: given $\xi \in \Lambda'$, the sampling square associated to ξ is defined as $S_\xi = \cup_{\eta\in\Lambda':\ 0\le\eta_i-\xi_i\le\ell}\,\Delta_\eta$, where η_i and η_i, $i = 1, 2$, are the coordinates of $\xi, \eta \in \Lambda'$. Note that if $\mathrm{dist}'(\xi, \Lambda'_c) > 1$, then S_ξ contains exactly 4 tiles. We say that a sampling square is

- **good** if the spins inside S_ξ are all equal either to $+1$ or to -1. Each good sampling square comes with a *magnetization* $m = \pm 1$.
- **bad** otherwise; note that each bad sampling square is such that either it contains at least one spin equal to zero, or it contains at least one pair of neighboring spins with opposite values, $+1$ and -1.

Definition 2: connectedness, good and bad regions. Given a configuration $\sigma \in \Theta_{\Lambda'}$, we call

$$B(\sigma) = \cup_{\substack{\xi\in\Lambda':\\ S_\xi \text{ is bad}}} S_\xi \tag{4.1}$$

the union of all bad sampling squares. The "smoothening" of $B(\sigma)$ on scale 4ℓ is defined as:

$$\overline{B}(\sigma) = \cup_{\substack{a\in\Lambda'':\\ \mathcal{S}_a\cap B(\sigma)\neq\emptyset}} \mathcal{S}_a\ , \tag{4.2}$$

where the lattice Λ'' and the smoothing squares $\mathcal{S}_a$ were defined after Eq.(3.2).

Let $X \subset \Lambda$ be a union of tiles: we say that X is connected if, given any pair of points $x, y \in X$, there exists a sequence $(x_0 = x, x_1, \dots, x_{n-1}, x_n = y)$ such that $x_i \in X$ and $|x_i - x_{i-1}| = 1$, for all $i = 1, \dots, n$. We also say that X is D-connected (with the prefix "D" meaning "diagonal") if, given any pair of points $x, y \in X$, there exists a sequence $(x_0 = x, x_1, \dots, x_{n-1}, x_n = y)$ such that $x_i \in X$

and $|x_i - x_{i-1}| \leq \sqrt{2}$, for all $i = 1, \ldots, n$ (here $|x - y|$ is the euclidean distance between x and y).

The maximal D-connected components of $\overline{B}(\sigma)$ are denoted by Γ_j and are the geometric supports of the contours that we will introduce below. The complement of the bad region,

$$G(\sigma) := \Lambda \setminus \overline{B}(\sigma) \,, \tag{4.3}$$

can be split into uniformly magnetized disconnected regions, each of which is a union of tiles; these are denoted by Y_j and m_j are the corresponding magnetizations.

Remarks.

1. Note that distinct D-disconnected regions, $\Gamma_j(\sigma), \Gamma_{j'}(\sigma)$ with $j \neq j'$, do not interact directly; i.e., $\varphi(R_\xi, R_\eta) = 1$ for all $\xi \in \Gamma_j$, $\eta \in \Gamma_{j'}$. This is because $\Gamma_j(\sigma)$ and $\Gamma_{j'}(\sigma)$ are separated by at least one smoothing square. Similarly, distinct uniformly magnetized disconnected regions, $Y_j(\sigma), Y_{j'}(\sigma)$ with $j \neq j'$ and magnetizations m_j, $m_{j'}$, do not interact directly; i.e., $\varphi(R_\xi, R_\eta) = 1$ for all $\xi \in Y_j$, $\eta \in Y_{j'}$ and all $R_\xi \in \Omega_\xi^{m_j}$, $R_\eta \in \Omega_\eta^{m_{j'}}$. In fact, given $\xi \in Y_j$, $\eta \in Y_{j'}$, then: either $|\xi_i - \eta_i| \geq 2\ell$ for some $i \in \{1, 2\}$, in which case the rods in R_ξ are so far from those in R_η that they certainly do not interact, whatever is their orientation; or $|\xi_1 - \eta_1| = |\xi_2 - \eta_2| = 1$, in which case necessarily $m_j = m_{j'}$, so that the rods in R_ξ have the same orientation as those in R_η, while their centers belong to different rows and columns and, therefore, do not interact.

2. In terms of the definitions above, the set $\Theta^q_{\Lambda'} \subset \Theta_{\Lambda'}$ of spin configurations with q boundary conditions can be thought as the set of spin configurations such that all the contours' supports $\Gamma_j \subset \overline{B}(\sigma)$ are D-disconnected from Λ^c and separated from it by at least one smoothing square.

Definition 3: contours. Given a spin configuration with q boundary conditions $\sigma \in \Theta^q_{\Lambda'}$ and a rod configuration $R \in \Omega_\Lambda$ compatible with it, let Γ be one of the maximal connected components of $\overline{B}(\sigma)$. By construction, the complement of Γ, $\Lambda \setminus \Gamma$, consists of one or more connected components: one of these components touches Λ^c and is naturally identified as the exterior of Γ; it is denoted by $\mathrm{Ext}\,\Gamma$. If Γ is simply connected this is the only connected component of $\Lambda \setminus \Gamma$; if not, i.e., if Γ has $h_\Gamma \geq 1$ holes, then there are other connected components of $\Lambda \setminus \Gamma$, to be called the interiors of Γ and denoted by $\mathrm{Int}_j\Gamma$, $j = 1, \ldots, h_\Gamma$. The interior of Γ is then $\mathrm{Int}\Gamma = \cup_j \mathrm{Int}_j\Gamma$. For what follows, it is also convenient to introduce the 1-tile-thick peel of Γ:

$$P_\Gamma = \cup_{\substack{\xi \in \Lambda': \\ \mathrm{dist}'(\xi, \Gamma') = 1}} \Delta_\xi \,. \tag{4.4}$$

Note that, since distinct D-disconnected regions are separated by at least one smoothing square, then also the peels associated to distinct Γ's are mutually D-disconnected. The *contour* γ associated to the support $\Gamma = \text{supp}(\gamma)$ is defined as the collection:

$$\gamma = (\Gamma, \sigma_\gamma, R_\gamma, m_{ext}, \underline{m}_{int}) \tag{4.5}$$

where

- σ_γ is the restriction of the spin configuration σ to Γ;
- R_γ is the restriction of the rod configuration R to Γ;
- m_{ext} is the magnetization of $P_\Gamma^{ext} := \text{Ext}\,\Gamma \cap P_\Gamma$;
- $\underline{m}_{int} = \{m_{int}^1, \ldots, m_{int}^{h_\Gamma}\}$, with m_{int}^j the magnetization of $\text{Int}_j\Gamma \cap P_\Gamma$; if $h_\Gamma = 0$, then $\underline{m}_{int}$ is the empty set. In the following we shall also denote by $P_\Gamma^{int} := \text{Int}\Gamma \cap P_\Gamma$ the internal peel of Γ.

If $m_{ext} = q$, then we say that γ is a q-contour.

Remark. The set γ must satisfy a number of constraints. In particular, given m_{ext} and $\underline{m}_{int}$, σ_γ must be compatible with the conditions that: (i) all the sampling squares touching P_Γ are good; (ii) each smoothing square contained in Γ has non zero interesection with at least one bad sampling square. Moreover, R_γ must be compatible with σ_γ itself.

In the following we want to write an expression for $Z(\Lambda|q)$ purely in terms of contours. Roughly speaking, given a contour configuration contributing to the r.h.s. of Eq.(3.6), we first want to freeze the rods inside the supports of the contours, next sum over all the rod configurations in the good regions and show that the resulting effective theory is a contour theory treatable by the Pirogov-Sinai method. The resummation of the configurations within the good regions can be performed by standard cluster expansion methods, as explained in the following digression.

Partition function restricted to a good region. Given a set $X \subseteq \Lambda$ consisting of a union of tiles, let $\Omega_X^q = \cup_{\xi \in X'} \Omega_{\Delta_\xi}^q$, $q = \pm$. The restricted theory of the "uniformly q-magnetized" region X (with open boundary conditions) is associated to the partition function:

$$Z^q(X) = \sum_{R \in \Omega_X^q} z^{|R|} \varphi(R) \;, \tag{4.6}$$

which can be easily computed by standard cluster expansion methods. In particular,

$$\log Z^q(X) = \sum_{R \in \Omega_X^q} z^{|R|} \varphi^T(R) = z|X|(1 + O(zk)) \tag{4.7}$$

where φ^T are the *Mayer's coefficients*, which admit the following explicit representation. Given the rod configuration $R = (r_1, \ldots, r_n)$, consider the graph $\mathcal{G}$ with n nodes, labelled by $1, \ldots, n$, with edges connecting all pairs i, j such that r_i, r_j overlap ($\mathcal{G}$ is sometimes called the connectivity graph of R). Then one has $\varphi^T(\emptyset) = 0$, $\varphi^T(r) = 1$ and, for $|R| > 1$:

$$\varphi^T(R) = \frac{1}{R!} \sum_{C \subseteq \mathcal{G}}^{*} (-1)^{\text{number of edges in } C}, \tag{4.8}$$

where $R! = \prod_{r \in \mathrm{supp}(R)} R(r)!$ and the sum runs over all the connected subgraphs C of G *that visit all the n points* $1, \ldots, n$. In particular, if $|R| > 1$, then $\varphi^T(R) = 0$ unless R is connected.

The sum in the r.h.s. of Eq.(4.7) is exponentially convergent for $zk \ll 1$; in particular, if $x_0 \in \Lambda$, then

$$\sum_{\substack{R \in \Omega^q_\Lambda: \\ R \ni x_0,\ \mathrm{diam}(R) \geq d}} |z|^{|R|} |\varphi^T(R)| \leq (\mathrm{const.})(zk)^{d/(\ell-1)} , \tag{4.9}$$

uniformly in Λ, where $R \ni x_0$ means that R contains at least one rod with center in x_0 and $\mathrm{diam}(R)$ is the diameter of the union of the rods $r \in R$ (thought of as a subset of Λ). Moreover, the sum $\sum_{R \in \Omega^+_\Lambda :\, R \ni x_0} z^{|R|} \varphi^T(R)$ is analytic in zk, uniformly in Λ, for zk small enough and its limit as $\Lambda \nearrow \mathbb{Z}^2$ is analytic, too. Similarly, all the correlation functions can be computed in terms of convergent series, as long as zk is small enough. This result is classical, see [24] or, e.g., [3, 6, 7]. The restricted theory is applied to the computation of the sums over the rod configurations in the good regions, as described in the following.

Contour representation of the partition function. Given a contour γ, let $Z_\gamma(\mathrm{Int}_j \Gamma | m^j_{int})$ be the partition function on the j-th interior of Γ with the boundary conditions created by the presence of the "frozen" rods R_γ. Moreover, if $\xi \in P'_\Gamma$, let

$$A_\gamma(\Delta_\xi) = \cup_{\eta \in a_\gamma(\xi)} \Delta_\eta , \qquad C_\gamma(\Delta_\xi) = \cup_{\substack{\eta \in \Gamma': \\ \mathrm{dist}'(\eta,\xi) \leq 2}} \Delta_\eta , \tag{4.10}$$

where

$$a_\gamma(\xi) := \{\xi\} \cup \{\eta \in \Lambda' : \mathrm{dist}'(\eta, \xi) = 1, \mathrm{dist}'_1(\eta, \Gamma') = 2, \eta_{j(-q)} = \xi_{j(-q)}\} , \tag{4.11}$$

with $\text{dist}_1'(\cdot,\cdot)$ the rescaled ("coarse") L_1 distance on Λ' and $j(+)=1$, $j(-)=2$. Finally, given $\Delta \subseteq P_\Gamma$, let f_Δ and g_Δ be the following characteristic functions:

$$f_\Delta(R) = \begin{cases} 1 & \text{if } R \text{ has at least one rod belonging to } A_\gamma(\Delta) \\ & \text{and one belonging to } C_\gamma(\Delta)\ , \\ 0 & \text{otherwise}\ , \end{cases} \tag{4.12}$$

$$g_\Delta(R) = \begin{cases} 1 & \text{if } R \cap R_\gamma \neq \emptyset,\ R \text{ has at least one rod belonging to } A_\gamma(\Delta) \\ & \text{and } R_\gamma \text{ has at least one rod belonging } C_\gamma(\Delta)\ , \\ 0 & \text{otherwise}\ . \end{cases} \tag{4.13}$$

Pictorially speaking, f_Δ is the characteristic function of the event "R crosses the boundary of Γ at Δ", while g_Δ is the characteristic function of the event "R intersects R_γ across Δ". Note that, by construction, given two distinct tiles $\Delta_1 \subseteq P_{\Gamma_1}$ and $\Delta_2 \subseteq P_{\Gamma_2}$, then $A_{\gamma_1}(\Delta_1) \cap A_{\gamma_2}(\Delta_2) = \emptyset$.

In terms of these definitions, the following contours' representation for $Z(\Lambda|q)$ is valid.

Lemma 1 *The conditioned partition function $Z(\Lambda|q)$, $q=\pm 1$, can be written as*

$$Z(\Lambda|q) = Z^q(\Lambda) \sum_{\partial \in \mathcal{C}(\Lambda,q)} \Big[\prod_{\gamma\in\partial} \zeta_q(\gamma)\Big]\ e^{-W(\partial)}\ , \tag{4.14}$$

where:

- *$\mathcal{C}(\Lambda,q)$ is the set of all the D-disconnected q-contour configurations in Λ;*
- *$\zeta_q(\gamma)$ is the activity of γ:*

$$\zeta_q(\gamma) = \zeta_q^0(\gamma)\ \exp\Big\{ - \sum_{R\in\Omega_\Lambda^q} \varphi^T(R) z^{|R|} \sum_{\Delta\subseteq P_\Gamma} F_\Delta(R)\Big\}\ , \tag{4.15}$$

  where

$$\zeta_q^0(\gamma) = \frac{\bar\varphi(R_\gamma)}{Z^q(\Gamma)} \prod_{j=1}^{h_\Gamma} \frac{Z_\gamma(\text{Int}_j\Gamma|m^j_{int})}{Z(\text{Int}_j\Gamma|q)} \tag{4.16}$$

  and $F_\Delta = f_\Delta$ if $\Delta \subseteq P_\Gamma^{int}$ while $F_\Delta = f_\Delta + g_\Delta(1-f_\Delta)$ if $\Delta \subseteq P_\Gamma^{ext}$.

- *$W(\partial)$ is the interaction between the contours in ∂:*

$$W(\partial) = \sum_{R\in\Omega_\Lambda^q} \varphi^T(R) z^{|R|} \sum_{n\geq 2} (-1)^{n+1} \sum^{*}_{\Delta_1<\cdots<\Delta_n} F_{\Delta_1}(R)\cdots F_{\Delta_n}(R)\ , \tag{4.17}$$

  where the $*$ on the sum indicates the constraint that $\Delta_1,\ldots,\Delta_n$ are all contained in the peel of some contour of ∂ and their centers $\xi_1,\ldots\xi_n$ all *belong to the same row (if $q = +$) or column (if $q = -$) of Λ', namely $\xi_{1,j(-q)} = \cdots = \xi_{n,j(-q)}$. Moreover, by writing $\Delta_1 < \cdots < \Delta_n$, we mean that $\xi_{1,j(q)} < \cdots < \xi_{n,j(q)}$. Finally, $F_\Delta = f_\Delta$ if Δ is contained in the internal peel of some contour in ∂ or $F_\Delta = f_\Delta + g_\Delta(1 - f_\Delta)$ if Δ is contained in the external peel of some contour in ∂.*

Remarks.

1. The contour configurations $\{\gamma_1, \ldots, \gamma_n\} \in \mathcal{C}(\Lambda, q)$ consist of n-ples of q-contours whose geometric supports $\Gamma_1, \ldots, \Gamma_n$ are D-disconnected. Note, however, that their external and internal magnetizations are not necessarily compatible among each other: for instance, Γ_1 may have one hole surrounding Γ_2, and the internal magnetization of Γ_1 may be different from the external magnetization of Γ_2 (which is q). It is actually an important point of the representation Eq.(4.14) that we can forget about the compatibility conditions among the internal and external magnetizations of different contours. There exist different (and even more straightforward) contour representation of $Z(\Lambda|q)$ where the internal and external contours' magnetizations satisfy natural but non-trivial constraints (e.g., in the example above, the natural constraint is that the internal magnetization of Γ_1 is the same as the external magnetization of Γ_2). However, the magnetization constraints are not suitable to apply cluster expansion methods to the resulting contour theory. Therefore, it is convenient to eliminate such constraints, at the price of adding the extra factors $Z_\gamma(\mathrm{Int}_j\Gamma|m^j_{int})/Z(\mathrm{Int}_j\Gamma|q)$ in the definition of the contours' activities, see Eq.(4.16).

2. The interest of the representation Eq.(4.14) is that the contour activities and the multi-contour interaction satisfy suitable bounds, allowing us to study the r.h.s. Eq.(4.14) by cluster expansion methods. In particular, $\sup^*_{\sigma_\gamma} \sum_{R_\gamma \in \Omega_\Gamma(\sigma_\gamma)} |\zeta_q(\gamma)| \le \exp\{-(\mathrm{const.}) z k^2 |\Gamma'|\}$, where the $*$ on the sup reminds the constraint that all the smoothing squares in Γ must have a non-zero intersection with at least one bad sampling square. Moreover, $W(\partial)$ is a quasi-one-dimensional potential, exponentially decaying to zero in the mutual distance between the supports of the contours in ∂. The proofs of these claims will be postponed to the next sections.

Proof. Given $\sigma \in \Theta^q_{\Lambda'}$ a spin configuration with q boundary conditions consider the corresponding set of contours $\{\gamma_1, \ldots, \gamma_n\}$. Some of them are *external*, in the sense that they are not surrounded by any other contour in $\{\gamma_1, \ldots, \gamma_n\}$. By construction, these external contours are all q-contours. We denote by $\mathcal{C}_{ext}(\Lambda, q)$ the set of external q-contour configurations. Given $\partial \in \mathcal{C}_{ext}(\Lambda, q)$, there is a common external region to all the contours in ∂, which we denote by $\mathrm{Ext}(\partial)$. Besides this, there are several internal regions within each contour $\gamma \in \partial$. For each external contour $\gamma \in \partial$, we freeze the corresponding rod configuration R_γ and sum over the rod configurations inside all the internal regions $\mathrm{Int}_j\Gamma$, $j = 1, \ldots, h_\Gamma$. In this way, for each such interior, we reconstruct the partition function $Z_\gamma(\mathrm{Int}_j\Gamma|m^j_{int})$. On the other hand, by construction all rods inside $\mathrm{Ext}(\partial)$ are either horizontal or vertical, according to the value of q. Therefore, if we sum over all the allowed rod configurations inside this region we get the restricted partition function $Z^q_\partial(\mathrm{Ext}(\partial))$, where the subscript ∂ reminds the fact that the rods R_∂ create an excluded volume for the rods in $\mathrm{Ext}(\partial)$. Using these definitions, we can rewrite

$$Z(\Lambda|q) = \sum_{\partial \in \mathcal{C}_{ext}(\Lambda,q)} Z^q_\partial(\mathrm{Ext}(\partial)) \prod_{\gamma\in\partial} \left[\bar{\varphi}(R_\gamma) \prod_{j=1}^{h_\Gamma} Z_\gamma(\mathrm{Int}_j\Gamma|m^j_{int})\right] . \tag{4.18}$$

Note that here we used the fact that the exterior and the interior(s) of ∂ do not interact directly (i.e., they only interact through R_γ). Using the definition of $\zeta^0_q(\gamma)$, Eq.(4.16), we can rewrite $Z(\Lambda|q)$ as

$$\frac{Z(\Lambda|q)}{Z^q(\Lambda)} = \sum_{\partial \in \mathcal{C}_{ext}(\Lambda,q)} \prod_{\gamma\in\partial} \left[\zeta^0_q(\gamma) \prod_{j=1}^{h_\Gamma} \frac{Z(\mathrm{Int}_j\Gamma|q)}{Z^q(\mathrm{Int}_j\Gamma)}\right] e^{-W^{ext}_0(\partial)} , \tag{4.19}$$

where

$$e^{-W^{ext}_0(\partial)} = \frac{Z^q_\partial(\mathrm{Ext}(\partial)) \prod_{\gamma\in\partial} \left[Z^q(\Gamma) \prod_{j=1}^{h_\Gamma} Z^q(\mathrm{Int}_j\Gamma)\right]}{Z^q(\Lambda)}. \tag{4.20}$$

The factors $\frac{Z(\mathrm{Int}_j\Gamma|q)}{Z^q(\mathrm{Int}_j\Gamma)}$ have the same form as the l.h.s. of Eq.(4.19) itself, with Λ replaced by $\mathrm{Int}_j\Gamma$: therefore, the equation can be iterated until the interior of all the contours is so small that it cannot contain other contours. The result of the iteration is

$$\frac{Z(\Lambda|q)}{Z^q(\Lambda)} = \sum_{\partial \in \mathcal{C}(\Lambda,q)} \left[\prod_{\gamma\in\partial} \zeta^0_q(\gamma)\right] e^{-W_0(\partial)} , \tag{4.21}$$

where

$$e^{-W_0(\partial)} = \frac{Z^q_\partial(\Lambda(\partial)) \prod_{\gamma\in\partial} Z^q(\Gamma)}{Z^q(\Lambda)}, \tag{4.22}$$

$\Lambda(\partial) = \Lambda \setminus \cup_{\gamma\in\partial} \Gamma$ is the complement of the contours' supports and $Z^q_\partial(\Lambda(\partial))$ is the restricted partition function with magnetization q in the volume $\Lambda(\partial)$ and in the presence of the hard rod constraint generated by the frozen rods R_γ in the region $\cup_{\gamma\in\partial} \cup_{\Delta\subseteq P^{ext}_\Gamma} A_\gamma(\Delta)$.

We now use Eq.(4.7) and the analogous expression for $Z^q_\partial(\Lambda(\partial))$, i.e.,

$$\log Z^q_\partial(\Lambda(\partial)) = \sum_{\substack{R \overset{ext}{\cap} R_\partial = \emptyset \\ R\in\Omega^q_{\Lambda(\partial)}}} z^{|R|} \varphi^T(R) , \tag{4.23}$$

where $R\overset{ext}{\cap}R_\partial=\emptyset$ means that R does not intersect R_∂ from the outside, namely:

$$R\overset{ext}{\cap}R_\partial=\emptyset \quad \overset{def}{\Leftrightarrow} \quad \prod_{\gamma\in\partial}\prod_{\Delta\subseteq P_\Gamma^{ext}}(1-g_\Delta(R))=1\;, \tag{4.24}$$

where g_Δ was defined in Eq.(4.13). Then we can rewrite:

$$\begin{aligned} e^{-W_0(\partial)} =& \frac{Z^q(\Lambda(\partial))\prod_{\gamma\in\partial}Z^q(\Gamma)}{Z^q(\Lambda)}\cdot\frac{Z^q_\partial(\Lambda(\partial))}{Z^q(\Lambda(\partial))}\;, \qquad (4.25)\\ =& \exp\Big\{-\sum_{R\in\Omega^q_\Lambda}^{R\overset{\partial}{\rightsquigarrow}2} z^{|R|}\varphi^T(R)\Big\}\cdot\exp\Big\{-\sum_{R\in\Omega^q_{\Lambda(\partial)}}^{R\cap R_\partial^{ext}\neq\emptyset} z^{|R|}\varphi^T(R)\Big\}\;, \end{aligned}$$

where $R\overset{\partial}{\rightsquigarrow}2$ means that R must touch at least two distinct elements of the partition $\mathcal{P}(\partial)$ of Λ induced by the contours in ∂; i.e., R must touch either two disconnected components of $\Lambda(\partial)$, or two different contours' supports, or one contour's support and one of the components of $\Lambda(\partial)$. Using the definitions of the characteristic functions f_Δ and g_Δ defined in Eqs.(4.12)-(4.13), the two exponential in the r.h.s. of Eq.(4.25) can be written as

$$\sum_{R\in\Omega^q(\Lambda)}^{R\overset{\partial}{\rightsquigarrow}2} z^{|R|}\varphi^T(R)=\sum_{R\in\Omega^q_\Lambda} z^{|R|}\varphi^T(R)\Big[1-\prod_{\gamma\in\partial}\prod_{\Delta\subseteq P_\Gamma}(1-f_\Delta(R))\Big]\;, \tag{4.26}$$

$$\begin{aligned}\sum_{R\in\Omega^q_{\Lambda(\partial)}}^{R\cap R_\partial^{ext}\neq\emptyset} z^{|R|}\varphi^T(R)=\sum_{R\in\Omega^q_\Lambda} z^{|R|}\varphi^T(R)\Big[1-\prod_{\gamma\in\partial}\prod_{\Delta\subseteq P_\Gamma^{ext}}(1-g_\Delta(R))\Big]\cdot \qquad (4.27)\\ \cdot\Big[\prod_{\gamma\in\partial}\prod_{\Delta\subseteq P_\Gamma}(1-f_\Delta(R))\Big]\;.\end{aligned}$$

Using the representations Eqs.(4.25), (4.26), (4.27) into Eq.(4.21), we find

$$\begin{aligned}\frac{Z(\Lambda|q)}{Z^q(\Lambda)}=\sum_{\partial\in\mathcal{C}(\Lambda,q)}\Big[\prod_{\gamma\in\partial}\zeta^0_q(\gamma)\Big]\exp\Big\{-\sum_{R\in\Omega^q_\Lambda} z^{|R|}\varphi^T(R)\cdot \qquad (4.28)\\ \cdot\Big[1-\Big(\prod_{\gamma\in\partial}\prod_{\Delta\subseteq P_\Gamma^{ext}}(1-g_\Delta(R))\Big)\cdot\Big(\prod_{\gamma\in\partial}\prod_{\Delta\subseteq P_\Gamma}(1-f_\Delta(R))\Big)\Big]\Big\}\;.\end{aligned}$$

Note that the expression in square brackets in the second line can be conveniently rewritten as

$$\begin{aligned}&1-\prod_{\gamma\in\partial}\Big(\prod_{\Delta\subseteq P_\Gamma^{ext}}(1-g_\Delta(R))(1-f_\Delta(R))\Big)\cdot\Big(\prod_{\Delta\subseteq P_\Gamma^{int}}(1-f_\Delta(R))\Big)\equiv\\ &\equiv 1-\prod_{\gamma\in\partial}\prod_{\Delta\subseteq P_\Gamma}(1-F_\Delta)\;, \qquad (4.29)\end{aligned}$$

where F_Δ was defined in the statement of Lemma 1. Plugging Eq.(4.29) into Eq.(4.28) gives

$$\frac{Z(\Lambda|q)}{Z^q(\Lambda)} = \sum_{\partial\in\mathcal{C}(\Lambda,q)} \Big[\prod_{\gamma\in\partial}\zeta_q^0(\gamma)\Big] \exp\Big\{ -\sum_{R\in\Omega_\Lambda^q} z^{|R|}\varphi^T(R)\cdot \tag{4.30}$$
$$\cdot\Big[\sum_{\Delta\subseteq P_\partial} F_\Delta(R) + \sum_{n\geq 2}(-1)^{n+1}\sum_{\{\Delta_1,\ldots,\Delta_n\}} F_{\Delta_1}(R)\cdots F_{\Delta_n}(R)\Big] ,$$

where the sum $\sum_{\{\Delta_1,\ldots,\Delta_n\}}$ runs over collections of distinct tiles $\Delta_i \subseteq P_\partial$, with $P_\partial := \cup_{\gamma\in\partial} P_\Gamma$. Finally, using the fact that $\varphi^T(R)$ forces R to be connected and, therefore, to live on a single row or column, depending on whether q is $+$ or $-$, we find that the only non-vanishing contributions in the latter sum come from n-ples of tiles all living on the same row or column. This proves the desired result. ∎

5. CONVERGENCE OF THE CONTOURS' EXPANSION

In this and in the next section we prove the convergence of the cluster expansion for the logarithm of the partition function with q boundary conditions, starting from Eq.(4.14). The proof will be split in two main steps: first, in this section, we prove convergence under the assumption that the activities $\zeta_q(\gamma)$ satisfy suitable decay bounds in the size of $|\Gamma|$. Then, in the next section, we prove the validity of such a decay bound via an induction in the size of $|\Gamma|$. From now on, $C, C', \ldots$ and $c, c', \ldots$, indicate universal positive constants (to be thought of as "big" and "small", respectively), whose specific values may change from line to line.

Lemma 2 *Let zk and $(zk^2)^{-1}$ be small enough and assume that for a suitable constant $c > 0$*

$$\sup_{\sigma_\gamma}{}^{*} \sum_{R_\gamma\in\Omega_\Gamma(\sigma_\gamma)} |\zeta_q(\gamma)| \leq e^{-c\,zk^2|\Gamma'|} , \tag{5.1}$$

where the $*$ on the sup reminds the constraint that all the smoothing squares in Γ must have a non-zero intersection with at least one bad sampling square. Then the logarithm of the partition function admits a convergent cluster expansion

$$\log Z(\Lambda|q) = \sum_{R\in\Omega_\Lambda^q} z^{|R|}\varphi^T(R) + \sum_{\mathcal{X}\subseteq\Lambda}\Big[\prod_{X\in\mathcal{X}} K_q^{(\Lambda)}(X)\Big]\varphi^T(\mathcal{X}) , \tag{5.2}$$

where $\mathcal{X} = \{X_1, \ldots, X_n\}$ is a polymers' configuration (possibly, some of the X_i's can coincide), each polymer X being a connected subset of Λ consisting of a union of tiles. The polymers' activities $K_q^{(\Lambda)}(X)$ satisfy $|K_q^{(\Lambda)}(X)| \leq Ce^{-c'zk^2|X'|}(zk)^{c'\delta'(X')}$, where $\delta'(X')$ is the rescaled tree length of the (coarse) set $X' \subset \Lambda'$, i.e., it is the number of nearest neighbor edges of the smallest tree on Λ' that covers X'.

Remark. The function $\varphi^T(\mathcal{X})$ in Eq.(5.2) is the Mayer's coefficient of $\mathcal{X}$, defined as in Eq.(4.8), with R replaced by $\mathcal{X}$, r_i by X_i, and the notion of "r_i overlaps with r_j" replaced by "X_i is D-connected to X_j".

Proof. Given ∂, let Y be a collection of $n \geq 2$ distinct tiles, all contained in the peel of some contour $\gamma \in \partial$ and all belonging to the same row or column, depending on whether $q = +$ or $q = -$; we shall write $Y = \{\Delta_1, \ldots, \Delta_n\}$, with $\Delta_1 < \cdots < \Delta_n$, and $\mathrm{supp}(Y) = \cup_{i=1}^n \Delta_i$. Moreover, we shall indicate by $\gamma(\Delta_i)$ the contour, which Δ_i belongs to the peel of. Eqs.(4.14)-(4.17) can be equivalently rewritten as

$$\frac{Z(\Lambda|q)}{Z^q(\Lambda)} = \sum_{\partial \in \mathcal{C}(\Lambda,q)} \Big[\prod_{\gamma\in\partial} \zeta_q(\gamma)\Big] \prod_Y e^{\mathcal{F}(Y)} \;, \tag{5.3}$$

where, if $Y = \{\Delta_1, \ldots, \Delta_n\}$,

$$\mathcal{F}(Y) := (-1)^n \sum_{R\in\Omega^q_\Lambda} z^{|R|} \varphi^T(R) F_{\Delta_1}(R) \cdots F_{\Delta_n}(R) \;. \tag{5.4}$$

Note that Eq.(5.4) is at least of order n (with $n \geq 2$) in z, by the very definition of the characteristic function F_Δ. In fact, $F_\Delta(R)$ is either equal to f_Δ or to $f_\Delta + g_\Delta(1 - f_\Delta)$; therefore, using the definitions of f_Δ and g_Δ, Eqs.(4.12)-(4.13), we see that $F_\Delta(R)$ is different from zero only if R contains a rod belonging to $A_{\gamma(\Delta)}(\Delta)$. Now recall that, as already observed after Eq.(4.13), distinct tiles $\Delta_1 \neq \Delta_2$ correspond to distinct sets $A_{\gamma(\Delta_1)}(\Delta_1)$ and $A_{\gamma(\Delta_2)}(\Delta_2)$, such that $A_{\gamma(\Delta_1)}(\Delta_1) \cap A_{\gamma(\Delta_2)}(\Delta_2) = \emptyset$; therefore, the r.h.s. of Eq.(5.4) is non zero only if R contains at least n distinct rods.

Using Eq.(4.9), we find that

$$|\mathcal{F}(Y)| \leq C' z k^2 (zk)^{\max\{|Y|-1,\, c'\cdot \mathrm{diam}'(Y)\}} \;, \tag{5.5}$$

where, if $Y = \{\Delta_{\xi_1}, \ldots, \Delta_{\xi_n}\}$ with $\Delta_{\xi_1} < \cdots < \Delta_{\xi_n}$, then $\mathrm{diam}'(Y) = (\xi_n - \xi_1)/\ell$ is the rescaled diameter of the set $\cup_{\Delta\in Y}\Delta$.

Let us now add and subtract 1 to each of the factors $e^{\mathcal{F}(Y)}$ in Eq.(5.4): in this way we turn each factor into a binomial $1 + (e^{\mathcal{F}(Y)} - 1)$. If $Y = \{\Delta_1, \ldots, \Delta_m\}$ with $\Delta_1 < \cdots < \Delta_m$, we associate the quantity $(e^{\mathcal{F}(Y)} - 1)$ with the region $\overline{Y}$ consisting of the union of all the tiles between Δ_1 and Δ_m. Similarly, we associate the activities $\zeta(\gamma)$ with the region Γ. Then we develop the binomials $1+(e^{\mathcal{F}(Y)}-1)$ and collect together the contribution corresponding to the maximally D-connected regions, obtained as unions of Γ_i's and $\overline{Y}_j$'s. The result is:

$$\frac{Z(\Lambda|q)}{Z^q(\Lambda)} = 1 + \sum_{m\geq1} \sum_{\{X_1,\ldots,X_m\}} K_q^{(\Lambda)}(X_1)\cdots K_q^{(\Lambda)}(X_m)\varphi(X_1,\ldots,X_m) \;, \tag{5.6}$$

where φ implements the hard core interaction, i.e., $\varphi(X_1,\ldots,X_m)$ is equal to 1 if the X_i's are all mutually D-disconnected and 0 otherwise, and

$$K_q^{(\Lambda)}(X)=\sum_{n\geq 1,\,p\geq 0}\ \sum^{*}_{\substack{\{\gamma_1,\ldots,\gamma_n\}\\ \{Y_1,\ldots,Y_p\}\\ \{\cup_i\Gamma_i\}\cup\{\cup_j\overline{Y}_j\}=X}}\zeta_q(\gamma_1)\cdots\zeta_q(\gamma_n)(e^{\mathcal{F}(Y_1)}-1)\cdots(e^{\mathcal{F}(Y_p)}-1)\ , \tag{5.7}$$

where the $*$ on the sum reminds the constraint that the contours γ_i must all be well-separated among each other and from the boundary of Λ; i.e., they must be separated among each other and from Λ^c by at least one smoothing square. As we will prove below, the assumption Eq.(5.1) on the contours' activities, together with Eq.(5.5), implies that

$$|K_q^{(\Lambda)}(X)|\leq\varepsilon_1\varepsilon^{|X'|}\ ,\qquad \varepsilon_1:=e^{-c''zk^2}\ ,\qquad \varepsilon:=\max\{\varepsilon_1,(zk)^{c''}\}\ , \tag{5.8}$$

for some $c''>0$. This decay bound on $K(Y)$ is sufficient to apply the standard cluster expansion strategy for computing the logarithm of Eq.(5.6) and put it in the form of the exponentially convergent sum Eq.(5.2); see, e.g., [7, Proposition 7.1.1]. The result is

$$\log\frac{Z(\Lambda|q)}{Z^q(\Lambda)}=\sum_{\mathcal{X}\subseteq\Lambda}\Big[\prod_{X\in\mathcal{X}}K_q^{(\Lambda)}(X)\Big]\varphi^T(\mathcal{X})\ . \tag{5.9}$$

Combing this equation with Eq.(4.7) gives Eq.(5.2).

We are left with proving Eq.(5.8). Using the bound Eq.(5.1 on the contours activities, the bound Eq.(5.5) on $\mathcal{F}(Y)$ and the fact that $|e^x-1|\leq|x|e^{|x|}$, we find:

$$|K_q^{(\Lambda)}(X)|\leq\sum_{n\geq 1}C^n\sum^{*}_{\substack{\{\Gamma_1,\ldots,\Gamma_n\}:\\ \cup_i\Gamma_i=X_0\subseteq X}}6^{|X_0'|}e^{-c\cdot zk^2|X_0'|}\sum_{\substack{X_1\subseteq X:\\ X_0\cup X_1=X}}\ \sum_{Q\subseteq X_1}\sum_{p\geq 0}\ \sum_{\substack{\{Y_1,\ldots,Y_p\}\\ \cup_i\mathrm{supp}(Y_i)=Q\\ \cup_i\overline{Y}_i=X_1}}$$
$$\prod_{j=1}^{p}\Big[C'zk^2(zk)^{c'\cdot\mathrm{diam}'(Y_j)}\exp\big\{C'zk^2(zk)^{c'\cdot\mathrm{diam}'(Y_j)}\big\}\Big]\ , \tag{5.10}$$

where, again, the $*$ on the sum over the Γ_i's indicates the constraint that the contours' supports must be separated among each other and from Λ^c by at least one smoothing square. Moreover, the factor $6^{|X_0'|}$ bounds the sums over σ_γ and $\underline{m}_{int}$ at Γ fixed.

Now, note that: (i) $\sum_{j=1}^p\mathrm{diam}'(Y_j)\geq|X_1'|$; (ii) $|X_0'|\geq|Q'|$, because every element of $\cup_jY_j$ belongs to the peel of some contour;

$$\text{(iii)}\quad \sum_{j=1}^{p}(zk)^{c'\cdot\mathrm{diam}'(Y_j)}\leq\sum_{\xi\in Q'}\sum_{\overline{Y}'\ni\xi}(zk)^{c'|\overline{Y}'|}\leq C''|Q'|(zk)^{c'}\ .$$

Plugging these estimates into Eq.(5.10) we find

$$|K_q^{(\Lambda)}(X)| \le \sum_{\emptyset \neq X_0 \subseteq X}^{*} (6Ce^{-\frac{c}{2}\cdot zk^2})^{|X_0'|} \sum_{\substack{X_1 \subset X: \\ X_0 \cup X_1 = X}} (zk)^{\frac{c'}{2}|X_1'|} \cdot \tag{5.11}$$

$$\cdot \sum_{Q \subseteq X_1} e^{-zk^2|Q'|(\frac{c}{2} - C'C''(zk)^{c'})} \sum_{p \ge 0} \sum_{\substack{\{Y_1,\ldots,Y_p\} \\ \cup_i \mathrm{supp}(Y_i) = Q \\ \cup_i \overline{Y}_i = X_1}} \prod_{j=1}^{p} \Big[C' zk^2 (zk)^{\frac{c'}{2}\mathrm{diam}'(Y_j)} \Big] ,$$

which can be further bounded by:

$$|K_q^{(\Lambda)}(X)| \le \sum_{\emptyset \neq X_0 \subseteq X}^{*} e^{-\frac{c}{3}zk^2|X_0'|} \sum_{\substack{X_1 \subset X: \\ X_0 \cup X_1 = X}} (zk)^{\frac{c'}{2}|X_1'|} \cdot \tag{5.12}$$

$$\cdot \sum_{Q \subseteq X_1} e^{-\frac{c}{3}zk^2|Q'|} \sum_{p \ge 0} \frac{1}{p!} \Big[C' zk^2 \sum_{A \cap Q \neq \emptyset} (zk)^{\frac{c'}{2}\delta'(A')} \Big]^p ,$$

where in the last sum A is a generic subset of Λ consisting of a union of tiles, and $\delta'(A')$ is its rescaled tree length. The expression in square brackets in the second line is bounded above by $C''zk^2|Q'|(zk)^{\frac{c'}{2}}$, so that

$$\begin{aligned} |K_q^{(\Lambda)}(X)| &\le \sum_{\emptyset \neq X_0 \subseteq X}^{*} e^{-\frac{c}{3}zk^2|X_0'|} \sum_{\substack{X_1 \subset X: \\ X_0 \cup X_1 = X}} (zk)^{\frac{c'}{2}|X_1'|} \sum_{Q \subseteq X_1} e^{-zk^2|Q'|(\frac{c}{3} - C''(zk)^{\frac{c'}{2}})} \\ &\le \sum_{\emptyset \neq X_0 \subseteq X}^{*} e^{-\frac{c}{3}zk^2|X_0'|} \sum_{\substack{X_1 \subset X: \\ X_0 \cup X_1 = X}} (zk)^{\frac{c'}{2}|X_1'|} \sum_{Q \subseteq X_1} e^{-\frac{c}{4}zk^2|Q'|} . \end{aligned} \tag{5.13}$$

The last sum can be bounded as $\sum_{Q \subseteq X_1} e^{-\frac{c}{4}zk^2|Q'|} \le (1 + e^{-\frac{c}{4}zk^2})^{|X_1'|}$, so that, defining $\tilde{\varepsilon}_1 := e^{-\frac{c}{3}zk^2}$, $\tilde{\varepsilon}_2 := (zk)^{\frac{c'}{2}}$ and $\tilde{\varepsilon} := \max\{\tilde{\varepsilon}_1, \tilde{\varepsilon}_2\}$:

$$\begin{aligned} |K_q^{(\Lambda)}(X)| &\le \sum_{\emptyset \neq X_0 \subseteq X}^{*} \tilde{\varepsilon}_1^{|X_0'|} \sum_{\substack{X_1 \subset X: \\ X_0 \cup X_1 = X}} \big((1 + e^{-\frac{c}{4}zk^2}) \tilde{\varepsilon}_2 \big)^{|X_1'|} \\ &\le \sqrt{\tilde{\varepsilon}_1} \Big[2\sqrt{\tilde{\varepsilon}} (1 + e^{-\frac{c}{4}zk^2}) \Big]^{|X'|} , \end{aligned} \tag{5.14}$$

which is the desired estimate on $K(X)$. This concludes the proof of the lemma. ■

Remark. The dependence of the activities $K_q^{(\Lambda)}(X)$ on Λ is inherited from the constraint that X must be separated from Λ^c by at least one smoothing square, and by the fact that the quantities $\zeta(\gamma)$ and $\mathcal{F}(Y)$ themselves are Λ-dependent, simply because their definitions involve sums over rods collections in Ω_Λ^q. However, this dependence is very weak: in fact, if $K_q(X)$ is the infinite volume limit of $K_q^{(\Lambda)}(X)$, we have:

$$\big| K_q^{(\Lambda)}(X) - K_q(X) \big| \le \sqrt{\varepsilon_1 \varepsilon^{|X'|}}\, \varepsilon^{c \cdot \mathrm{dist}'(X, \Lambda^c)} , \tag{5.15}$$

for some $c > 0$. The proof of Eq.(5.15) proceeds along the same lines used to prove Eq.(5.8) and, therefore, we will not belabor the details of this computation.

6. THE ACTIVITY OF THE CONTOURS

In this section we prove the assumption Eq.(5.1) used in the proof of Lemma 2. Let us first remind, for the reader's convenience, the definition of $\zeta_q(\gamma)$:

$$\zeta_q(\gamma) = \zeta_q^0(\gamma) \exp\Big\{ - \sum_{R\in\Omega_\Lambda^q} \varphi^T(R) z^{|R|} \sum_{\Delta\subseteq P_\Gamma} F_\Delta(R)\Big\} , \tag{6.1}$$

where

$$\zeta_q^0(\gamma) = \frac{\bar\varphi(R_\gamma)}{Z^q(\Gamma)} \prod_{j=1}^{h_\Gamma} \frac{Z_\gamma(\mathrm{Int}_j\Gamma|m^j_{int})}{Z(\mathrm{Int}_j\Gamma|q)} . \tag{6.2}$$

By using the same considerations used to get the bound Eq.(5.5), we see that the expression in braces in the r.h.s. of Eq.(6.1) is equal to a contribution of order one in z plus a rest, which is bounded in absolute value by $Czk^2|\Gamma'|(zk)^c$. On the other hand, the contribution of order one in z is equal to $-z\sum_{R\in\Omega^q_\Lambda:\ |R|=1}\sum_{\Delta\subseteq P_\Gamma} F_\Delta(R)$, which is *negative*, simply because $F_\Delta \geq 0$. Therefore,

$$|\zeta_q(\gamma)| \leq |\zeta_q^0(\gamma)| e^{Czk^2|\Gamma'|(zk)^c} , \tag{6.3}$$

which makes apparent that, in order to prove Eq.(5.1), we need to prove an analogous bound for $\zeta_q^0(\gamma)$. By definition, $Z_\gamma(X|m) \leq Z(X|m)$, so that

$$|\zeta_q^0(\gamma)| \leq |\overline{\zeta}_q^0(\gamma)| \prod_{j=1}^{h_\Gamma} \max\Big\{1, \frac{Z(\mathrm{Int}_j\Gamma|-q)}{Z(\mathrm{Int}_j\Gamma|q)}\Big\} , \qquad \overline{\zeta}_q^0(\gamma) := \frac{\bar\varphi(R_\gamma)}{Z^q(\Gamma)} . \tag{6.4}$$

The estimate that we need on the quantities $\overline{\zeta}_q^0(\gamma)$ and $\frac{Z(\mathrm{Int}_j\Gamma|-q)}{Z(\mathrm{Int}_j\Gamma|q)}$ is summarized in the following two lemmas.

Lemma 3 *Let zk and $(zk^2)^{-1}$ be small enough. Then, for a suitable constant $\alpha > 0$,*

$$\sup_{\sigma_\gamma}{}^* \sum_{R_\gamma\in\Omega_\Gamma(\sigma_\gamma)} |\overline{\zeta}_q^0(\gamma)| \leq e^{-\alpha zk^2|\Gamma'|} , \tag{6.5}$$

where the $*$ on the sup reminds the constraint that all the smoothing squares in Γ must have a non-zero intersection with at least one bad sampling square.

Lemma 4 *Let zk and $(zk^2)^{-1}$ be small enough. Then there exist two positive constants $C, c > 0$ such that, for any simply connected region $X \subset \mathbb{Z}^2$ consisting of a union of smoothing squares,*

$$e^{-|P'_X|(Czk^2(zk)+\varepsilon^c)} \leq \frac{Z(X|+)}{Z(X|-)} \leq e^{|P'_X|(Czk^2(zk)+\varepsilon^c)} , \tag{6.6}$$

where ε was defined in Eq.(5.8) and P_X is the 1-tile-thick peel of X.

These two estimates combined with Eq.(6.3) prove Eq.(5.1) under the only assumptions that zk and $(zk^2)^{-1}$ are small enough. Therefore, these two lemmas imply the convergence of the cluster expansion Eq.(5.2), which completes the computation of the partition function of our hard rod system with q boundary conditions. A computation of the correlation functions based on a similar expansion will be discussed in the next section. The rest of this section is devoted to the proofs of Lemma 3 and 4.

Proof of Lemma 3. Let σ_γ be a spin configuration compatible with the fact that γ is a contour. In particular, let us recall that every smoothing square contained in Γ has a non zero intersection with at least one bad sampling square; moreover, by its very definition, each such bad square must contain either one tile with magnetization equal to 0, or one pair of neighboring tiles with magnetizations $+$ and $-$, respectively. Therefore, given σ_γ, it is possible to exhibit a partition $\mathcal{P}$ of Γ such that: (i) all the elements of the partition consist either of a single tile or of a pair of neighboring tiles with opposite magnetizations $+$ and $-$ (we shall call such pairs "domino tiles"); (ii) if $\mathcal{N}_0$ is the number of tiles in $\mathcal{P}$ with magnetization equal to 0 and $\mathcal{N}_d$ is the number of domino tiles in $\mathcal{P}$, then $\mathcal{N}_0+\mathcal{N}_d \ge |\Gamma'|/64$. The factor 64 comes from the consideration that in Γ, by definition, we have at least one bad square every four smoothing squares, and by the fact that four smoothing squares contain 64 tiles.

By the definition of $\bar\varphi(R_\gamma)$, we have: $\bar\varphi(R_\gamma)\le\prod_{P\in\mathcal{P}}\bar\varphi(R_P)$. Moreover, using the standard cluster expansion described after Eq.(4.6), we find that $Z^q(\Gamma)\ge\prod_{P\in\mathcal{P}}Z^q(P)e^{-Czk^2(zk)|\Gamma'|}$. By combining these two bounds we get

$$\sum_{R_\gamma\in\Omega_\Gamma(\sigma_\gamma)}|\bar\zeta^0_q(\gamma)|\le e^{Czk^2(zk)|\Gamma'|}\prod_{P\in\mathcal{P}}\Big|\sum_{R_P}\frac{\bar\varphi(R_P)}{Z^q(P)}\Big|\,, \tag{6.7}$$

where the sum over R_P runs over rods configurations in $\Omega_P(\cup_{\xi\in P'}\sigma_\xi)$. Now, if P is a single tile with magnetization either $+$ or $-$, then $\sum_{R_P}\frac{\bar\varphi(R_P)}{Z^q(P)}=1$. Moreover, if P is a single tile with magnetization equal to 0, then $\sum_{R_P}\frac{\bar\varphi(R_P)}{Z^q(P)}=-\frac{1}{Z^q(P)}=-e^{-z\ell^2(1+O(zk))}$.

Finally, let us consider the case that P is a domino tile. We assume without loss of generality that $P=\{\Delta_{\xi_1},\Delta_{\xi_2}\}$, with $\xi_2-\xi_1=(\ell,0)$, and $\sigma_{\xi_1}=-\sigma_{\xi_2}=+$. Since the rods interact via a hard core, $\bar\varphi(R_{\xi_1},R_{\xi_2})$ is different from zero only if at least one of the two rod configurations R_{ξ_1} and R_{ξ_2} is *untypical*: here we say that R_{ξ_1} is untypical if it does not contain any rod in the right half of Δ_{ξ_1} and, similarly, that R_{ξ_2} is untypical if it does not contain any rod in the left half of

Δ_{ξ_2}. Therefore,

$$\sum_{R_P} \frac{\bar{\varphi}(R_P)}{Z^q(P)} \le e^{Czk^2(zk)} \Big[\sum_{\substack{R_{\xi_1} \in \Omega^+_{\Delta_{\xi_1}}: \\ R_{\xi_1} \text{ untypical}}} \frac{\bar{\varphi}(R_{\xi_1})}{Z^+(\Delta_{\xi_1})} + \sum_{\substack{R_{\xi_2} \in \Omega^-_{\Delta_{\xi_2}}: \\ R_{\xi_2} \text{ untypical}}} \frac{\bar{\varphi}(R_{\xi_2})}{Z^-(\Delta_{\xi_2})} \Big] , \tag{6.8}$$

where we used that $\sum_{R \in \Omega^q_\Delta} \bar{\varphi}(R) = Z^q(\Delta)$. Eq.(6.8) can be rewritten and estimated (defining $\Delta^L_{\xi_1}$ to be the left half of Δ_{ξ_1}) as

$$\sum_{R_P} \frac{\bar{\varphi}(R_P)}{Z^q(P)} \le 2e^{Czk^2(zk)} \frac{Z^+(\Delta^L_{\xi_1})}{Z^+(\Delta_{\xi_1})} \le 2e^{C'zk^2(zk)} e^{-z\ell^2/2} . \tag{6.9}$$

Plugging the bounds on $\sum_{R_P} \frac{\bar{\varphi}(R_P)}{Z^q(P)}$ into Eq.(6.7) gives:

$$\begin{aligned} \sum_{R_\gamma \in \Omega_\Gamma(\sigma_\gamma)} |\bar{\zeta}^0_q(\gamma)| &\le e^{Czk^2(zk)|\Gamma'|} e^{-z\ell^2(1-Czk)(\mathcal{N}_0 + \frac{1}{2}\mathcal{N}_d)} \\ &\le e^{-z\ell^2(1-C'zk)|\Gamma'|/128} , \end{aligned} \tag{6.10}$$

where in the last line we used the bound $\mathcal{N}_0 + \mathcal{N}_d \ge |\Gamma'|/64$. Eq.(6.10) is the desired bound, so the proof of the lemma is complete. ■

Proof of Lemma 4. We proceed by induction on the size of X. If X is so small that it cannot contain contours D-disconnected from X^c, then

$$\frac{Z(X|+)}{Z(X|-)} = \frac{Z^+(X)}{Z^-(X)} = \exp\Big\{ \sum_{R \in \Omega^+_X} \varphi^T(R) z^{|R|} - \sum_{R \in \Omega^-_X} \varphi^T(R) z^{|R|} \Big\} . \tag{6.11}$$

Let $V(R)$ be the union of the centers of the rods in R. Then

$$\begin{aligned} \sum_{R \in \Omega^q_X} \varphi^T(R) z^{|R|} &= \sum_{x \in X} \sum_{\substack{R \in \Omega^q_X \\ V(R) \ni x}} \frac{\varphi^T(R) z^{|R|}}{|V(R)|} \\ &= \sum_{x \in X} \sum_{\substack{R \in \Omega^q_{\mathbb{Z}^2} \\ V(R) \ni x}} \frac{\varphi^T(R) z^{|R|}}{|V(R)|} + \sum_{x \in X} \sum_{\substack{R \in \Omega^q_{\mathbb{Z}^2} \setminus \Omega^q_X \\ V(R) \ni x}} \frac{\varphi^T(R) z^{|R|}}{|V(R)|} . \end{aligned} \tag{6.12}$$

The first sum in the second line is equal to

$$\sum_{x \in X} \sum_{\substack{R \in \Omega^q_{\mathbb{Z}^2} \\ V(R) \ni x}} \frac{\varphi^T(R) z^{|R|}}{|V(R)|} = |X| s(z) , \tag{6.13}$$

where

$$s(z) := \sum_{\substack{R \in \Omega^q_{\mathbb{Z}^2} \\ V(R) \ni x}} \frac{\varphi^T(R) z^{|R|}}{|V(R)|} \tag{6.14}$$

is an analytic function of z, of the form $s(z) = z(1 + O(zk))$, independent of q and x. The second sum in the second line of Eq.(6.12) involves rod configurations containing at least one rod belonging to X and one belonging to X^c. Therefore, it is of order at least 2 in z and scales like the boundary of X:

$$\Big|\sum_{x\in X}\sum_{\substack{R\in\Omega^q_{\mathbb{Z}^2}\setminus\Omega^q_X\\ V(R)\ni x}} \frac{\varphi^T(R)z^{|R|}}{|V(R)|}\Big| \le C_1 zk^2(zk)|P'_X| \,, \tag{6.15}$$

for a suitable constant $C_1 > 0$, independent of q. Plugging Eqs.(6.12)–(6.15) into Eq.(6.11) gives:

$$\frac{Z(X|+)}{Z(X|-)} = \exp\Big\{\sum_{x\in X}\sum_{\substack{R\in\Omega^+_{\mathbb{Z}^2}\setminus\Omega^+_X\\ V(R)\ni x}} \frac{\varphi^T(R)z^{|R|}}{|V(R)|} - \sum_{x\in X}\sum_{\substack{R\in\Omega^-_{\mathbb{Z}^2}\setminus\Omega^-_X\\ V(R)\ni x}} \frac{\varphi^T(R)z^{|R|}}{|V(R)|}\Big\} \,, \tag{6.16}$$

which is bounded from above and below by $e^{2C_1zk^2(zk)|P'_X|}$ and $e^{-2C_1zk^2(zk)|P'_X|}$, respectively. This proves the inductive hypothesis Eq.(6.6) at the first step, i.e., for regions X small enough, provided that $C \ge 2C_1$.

Let us now assume the validity of Eq.(6.6) for all the regions of size strictly smaller than Λ_0, and let us prove it for Λ_0. As explained in Section 5, $Z(\Lambda_0|q)$ admits the cluster expansion Eq.(5.2) involving polymers X that are D-disconnected from Λ_0^c, whose activities are defined in Eq.(5.7). In particular, the cluster expansion is convergent provided that $\zeta_q(\gamma)$ is bounded as in Eq.(5.1). Now, note that the interiors of the contours γ_i involved in the cluster expansion for $Z(X_0|q)$ via Eqs.(5.2) and (5.7) have all sizes strictly smaller than Λ_0. Therefore, using the inductive hypothesis, the product $\max\Big\{1, \frac{Z(\mathrm{Int}_j\Gamma|-q)}{Z(\mathrm{Int}_j\Gamma|q)}\Big\}$ in Eq.(6.4) can be bounded from above by $e^{|\Gamma'|(Czk^2(zk)+\varepsilon^c)}$ that, if combined with Eqs.(6.3), (6.5), implies Eq.(5.1) with $c = \alpha/2$, for all the the contours γ_i involved in the cluster expansion for $Z(X_0|q)$. We can then write:

$$\frac{Z(\Lambda_0|+)}{Z(\Lambda_0|-)} = \frac{Z^+(\Lambda_0)}{Z^-(\Lambda_0)} \exp\Big\{\sum_{\mathcal{X}\subseteq\Lambda_0}\Big[K_+^{(\Lambda_0)}(\mathcal{X}) - K_-^{(\Lambda_0)}(\mathcal{X})\Big]\varphi^T(\mathcal{X})\Big\} \,, \tag{6.17}$$

where $K_q^{(\Lambda_0)}(\mathcal{X}) = \prod_{X\in\mathcal{X}} K_q^{(\Lambda_0)}(X)$ and $K_q^{(\Lambda_0)}(X)$ admits the bound Eq.(5.8). The first factor in the r.h.s. of Eq.(6.17) is rewritten as in Eq.(6.16) and is bounded from above and below by $e^{2C_1zk^2(zk)|P'_{\Lambda_0}|}$ and $e^{-2C_1zk^2(zk)|P'_{\Lambda_0}|}$, respectively, exactly in the same way as Eq.(6.17) itself.

The second factor in the r.h.s. of Eq.(6.17) can be bounded as follows. We rewrite

$$\exp\Big\{\sum_{\mathcal{X}\subseteq\Lambda_0}\Big[K_+^{(\Lambda_0)}(\mathcal{X})-K_-^{(\Lambda_0)}(\mathcal{X})\Big]\varphi^T(\mathcal{X})\Big\}= \tag{6.18}$$
$$=\exp\Big\{\sum_{\substack{\mathcal{X}\subseteq\Lambda_0\\ q=\pm}}qK_q(\mathcal{X})\varphi^T(\mathcal{X})\Big\}\cdot\exp\Big\{\sum_{\substack{\mathcal{X}\subseteq\Lambda_0\\ q=\pm}}q\Big[K_q^{(\Lambda_0)}(\mathcal{X})-K_q(\mathcal{X})\Big]\varphi^T(\mathcal{X})\Big\}\;,$$

where $K_q(\mathcal{X})=\prod_{X\in\mathcal{X}}K_q(X)$ and, using Eq.(5.15),

$$|K_q^{(\Lambda_0)}(\mathcal{X})-K_q(\mathcal{X})|\le\varepsilon^{c_1\mathrm{dist}'(\mathcal{X},\Lambda_0^c)}\prod_{X\in\mathcal{X}}\varepsilon^{c_1|X'|}\;, \tag{6.19}$$

for a suitable constant $c_1>0$. Therefore, the second factor in the second line of Eq.(6.18) can be bounded from above and below by $e^{\varepsilon^{c_2}|P'_{\Lambda_0}|}$ and $e^{-\varepsilon^{c_2}|P'_{\Lambda_0}|}$, respectively. We are left with the first factor in the second line of Eq.(6.18), which involves the partition sum

$$\sum_{\mathcal{X}\subseteq\Lambda_0}K_q(\mathcal{X})\varphi^T(\mathcal{X})=\sum_{\xi\in\Lambda_0'}\sum_{\substack{X\supseteq\Delta_\xi\\ X\subseteq\Lambda_0}}\frac{K_q(\mathcal{X})\varphi^T(\mathcal{X})}{|\mathcal{X}'|}\;, \tag{6.20}$$

where $|\mathcal{X}'|$ is number of tiles in $\cup_{X\in\mathcal{X}}X$. Eq.(6.20) can be further rewritten as

$$\sum_{\mathcal{X}\subseteq\Lambda_0}K_q(\mathcal{X})\varphi^T(\mathcal{X})=|\Lambda_0'|\mathcal{S}+\sum_{\xi\in\Lambda_0'}\sum_{\substack{X\supseteq\Delta_\xi\\ X\cap\Lambda_0^c\neq\emptyset}}\frac{K_q(\mathcal{X})\varphi^T(\mathcal{X})}{|\mathcal{X}'|}\;, \tag{6.21}$$

where

$$\mathcal{S}:=\sum_{\substack{X\supseteq\Delta_\xi\\ X\subseteq\mathbb{Z}^2}}\frac{K_q(\mathcal{X})\varphi^T(\mathcal{X})}{|\mathcal{X}'|}\;, \tag{6.22}$$

is independent of q and ξ. The second term in the r.h.s. of Eq.(6.21) is bounded in absolute value from above by $|P'_{\Lambda_0}|\varepsilon^{c_3}$ for a suitable $c_3>0$; therefore,

$$\exp\Big\{\sum_{\substack{\mathcal{X}\subseteq\Lambda_0\\ q=\pm}}qK_q(\mathcal{X})\varphi^T(\mathcal{X})\Big\}=\exp\Big\{\sum_{\substack{\xi\in\Lambda_0'\\ q=\pm}}\sum_{\substack{X\supseteq\Delta_\xi\\ X\cap\Lambda_0^c\neq\emptyset}}q\frac{K_q(\mathcal{X})\varphi^T(\mathcal{X})}{|\mathcal{X}'|}\Big\}\le e^{2|P_{\Lambda_0'}|\varepsilon^{c_3}} \tag{6.23}$$

and is bounded from below by $e^{-2|P_{\Lambda_0'}|\varepsilon^{c_3}}$. This completes the inductive proof of Eq.(6.6), provided that $\varepsilon^c\ge\varepsilon^{c_2}+2\varepsilon^{c_3}$. ∎

7. EXISTENCE OF NEMATIC ORDER

In this section we prove Theorem 1. We start by proving Eq.(3.9). The probability that the tile centered at ξ_0 has magnetization $-q$ in the presence of boundary conditions q can be written as

$$\langle \chi_{\xi_0}^{-q} \rangle_\Lambda^q = \frac{\partial}{\partial z_0} \log Z_{z_0}(\Lambda|q)\Big|_{z_0=1} , \tag{7.1}$$

where $Z_{z_0}(\Lambda|q)$ is defined in a way completely analogous to Eqs.(3.4)-(3.6), with the only difference that the activity $\zeta(\xi)$ in Eq.(3.4) is replaced by $\tilde\zeta(\xi)$, where $\tilde\zeta(\xi) = \zeta(\xi)$ if $\xi \neq \xi_0$, while

$$\tilde\zeta(\xi_0) = \begin{cases} z^{|R_\xi|} & \text{if } \sigma_\xi = q \\ z_0 z^{|R_\xi|} & \text{if } \sigma_\xi = -q \\ -1 & \text{if } \sigma_\xi = 0. \end{cases} \tag{7.2}$$

The change of $\zeta(\xi)$ into $\tilde\zeta(\xi)$ induces a corresponding change of $\zeta_q(\gamma)$ and $K_q^{(\Lambda)}(X)$ into $\tilde\zeta_q(\gamma)$ and $\tilde K_q^{(\Lambda)}(X)$, respectively. The activity $\tilde K_q^{(\Lambda)}(X)$ admits the same bound Eq.(5.8) (possibly with a slightly different constant c''), uniformly in z_0 for z_0 close to 1, and it depends explicitly on z_0 only if $X \supseteq \Delta_{\xi_0}$. In such a case, the derivative of $\tilde K_q^{(\Lambda)}(X)$ with respect to z_0 is bounded by $\sqrt{\varepsilon_1 \varepsilon^{|X'|}}$, uniformly in z_0 for z_0 close to 1.

The logarithm of the modified partition function admits a convergent cluster expansion analogous to Eq.(5.2):

$$\log \frac{Z_{z_0}(\Lambda|q)}{Z^q(\Lambda)} = \sum_{\mathcal{X} \subseteq \Lambda} \tilde K_q^{(\Lambda)}(\mathcal{X}) \varphi^T(\mathcal{X}) , \tag{7.3}$$

so that

$$\langle \chi_{\xi_0}^{-q} \rangle_\Lambda^q = \sum_{\mathcal{X} \subseteq \Lambda} \partial_{z_0} \tilde K_q^{(\Lambda)}(\mathcal{X}) \varphi^T(\mathcal{X})\Big|_{z_0=1} . \tag{7.4}$$

The sum in the r.h.s. of Eq.(7.4) is exponentially convergent for ε small enough, and it only involves polymer configurations containing Δ_{ξ_0}, simply because $\tilde K_q^{(\Lambda)}(X)$ is independent of z_0 whenever $\Delta_{\xi_0} \cap X = \emptyset$. Therefore,

$$\begin{aligned} \langle \chi_{\xi_0}^{-q} \rangle_\Lambda^q \leq \sum_{\mathcal{X} \subseteq \Lambda} |\varphi^T(\mathcal{X})| \cdot |\partial_{z_0} \tilde K_q^{(\Lambda)}(\mathcal{X})|_{z_0=1} &\leq \sqrt{\varepsilon_1} \sum_{\mathcal{X} \supseteq \Delta_{\xi_0}} |\varphi^T(\mathcal{X})| \prod_{X \in \mathcal{X}} \varepsilon^{\frac{1}{2}|X'|} \\ &\leq (\text{const.}) \sqrt{\varepsilon_1 \varepsilon} , \end{aligned} \tag{7.5}$$

which proves Eq.(3.9).

In order to compute the density-density correlation functions we proceed in a similar fashion. We replace the activity z of a rod r centered at x by z_x and we define $\tilde Z_{\mathbf{z}}(\Lambda|q)$ to be the modified partition function with boundary conditions q and variable rod activities $\mathbf{z} = \{z_x\}_{x \in \Lambda}$. Correspondingly, we rewrite:

$$\begin{aligned} &\langle n_x \rangle_\Lambda^q = z \partial_{z_x} \log \tilde Z_{\mathbf{z}}(\Lambda|q)\Big|_{\mathbf{z}=z} , \\ &\langle n_x n_y \rangle_\Lambda^q - \langle n_x \rangle_\Lambda^q \langle n_y \rangle_\Lambda^q = z^2 \partial_{z_x} \partial_{z_y} \log \tilde Z_{\mathbf{z}}(\Lambda|q)\Big|_{\mathbf{z}=z} , \end{aligned} \tag{7.6}$$

where $\mathbf{z} = z$ means that $z_x = z$, $\forall x \in \Lambda$; the higher order density correlation functions have a similar representation. Once again, $\log \tilde{Z}_{\mathbf{z}}(\Lambda|q)$ admits a cluster expansion completely analogous to $\log Z(\Lambda|q)$:

$$\log \tilde{Z}_{\mathbf{z}}(\Lambda|q) = \sum_{R \in \Omega_\Lambda^q} \big[\prod_{r \in R} z_{x(r)}\big] \varphi^T(R) + \sum_{\mathcal{X} \subseteq \Lambda} \tilde{K}_{q,\mathbf{z}}^{(\Lambda)}(\mathcal{X}) \varphi^T(\mathcal{X}) \Big|_{\mathbf{z}=z} , \tag{7.7}$$

where $x(r)$ is the center of r. Moreover, $\tilde{K}_{q,\mathbf{z}}^{(\Lambda)}(X)$, together with its derivatives with respect to z_x and/or z_y, admit the same bound Eq.(5.8), possibly with a different constant c''; the derivative of $\tilde{K}_{q,\mathbf{z}}^{(\Lambda)}(X)$ with respect to z_x and/or z_y is different from zero only if $X \ni x$ and/or $X \ni y$. Therefore,

$$\begin{aligned} \langle n_x \rangle_\Lambda^q &= \sum_{R \in \Omega_\Lambda^q} z^{|R|} R(r(x)) \varphi^T(R) + \sum_{\mathcal{X} \subseteq \Lambda} \partial_{z_x} \tilde{K}_{q,\mathbf{z}}^{(\Lambda)}(\mathcal{X}) \varphi^T(\mathcal{X}) \Big|_{\mathbf{z}=z} , \\ \langle n_x n_y \rangle_\Lambda^q &- \langle n_x \rangle_\Lambda^q \langle n_y \rangle_\Lambda^q = \\ &= \sum_{R \in \Omega_\Lambda^q} z^{|R|} R(r(x)) R(r(y)) \varphi^T(R) + \sum_{\mathcal{X} \subseteq \Lambda} \partial^2_{z_x\, z_y} \tilde{K}_{q,\mathbf{z}}^{(\Lambda)}(\mathcal{X}) \varphi^T(\mathcal{X}) \Big|_{\mathbf{z}=z} , \end{aligned} \tag{7.8}$$

where $R(r)$ is the multiplicity of r in R. The sums in the first line involve connected rod or polymer configurations containing at least one rod centered at x; similarly, the sums in the second line involve connected rod or polymer configurations containing at least one rod centered at x and one rod centered at y. All the sums are exponentially convergent and their evaluation finally leads to the finite volume analogues of Eqs.(3.10)-(3.11). The infinite volume counterparts are obtained simply by replacing all the finite volume activities with their infinite volume counterparts and by dropping the constraints that the polymers should be contained in Λ. The infinite volume limit is reached exponentially fast and all the observables share the same invariance properties as the infinite volume activities themselves. In particular, the infinite volume Gibbs measures $\langle \cdot \rangle^q$ are translation invariant, and the averages $\langle \chi_{\xi_0}^{-q} \rangle^q$ and $\langle \prod_j n_{x_j} \rangle^q$ are all independent of q. We will not belabor the proofs of these claims, since they are all straightforward consequences of the cluster expansion described in the previous sections, in the same sense as the representations for $\langle \chi_{\xi_0}^{-q} \rangle^q$, $\langle n_x \rangle^q$ and $\langle n_x n_y \rangle^q$ and the proof of their convergence, discussed in this section, are a consequence of the bounds of sections 4 and 5. This concludes the proof of the main theorem. ∎

Acknowledgements. We gratefully acknowledge financial support from the ERC Starting Grant CoMBoS-239694. We warmly thank Emanuele Caglioti, for several key ideas and illuminating suggestions, which stimulated us to start this project and allowed us to complete it successfully. We thank G. Gallavotti, J.

Imbrie, J. Lebowitz and E. Lieb for many useful discussions, and H. Tasaki, for making us aware of this problem.

---

[1] C. Borgs and J. Z. Imbrie: *A Unified Approach to Phase Diagrams in Field Theory and Statistical Mechanics*, Commun. Math. Phys. **123**, 305-328 (1989).

[2] J. Bricmont, K. Kuroda and J. L. Lebowitz: *The structure of Gibbs states and phase coexistence for nonsymmetric continuum Widom-Rowlinson models*, Z. Wahrsch. Verw. Gebiete **67**, 121-138 (1984).

[3] D. C. Brydges: *A short course on cluster expansions*, in K. Osterwalder & R. Stora eds., "Critical Phenomena, Random Systems, Gauge Theories", Les Houches Summer School, North Holland, Amsterdam, New York, pp. 131183 (1984).

[4] D. Dhar, R. Rajesh and J. F. Stilck: *Hard rigid rods on a Bethe-like lattice*, Phys. Rev. E **84**, 011140 (2011).

[5] T. Fischer and R. L. C. Vink: *Restricted orientation "liquid crystal" in two dimensions: Isotropic-nematic transition or liquid-gas one (?)*, Europhysics Letters **85**, 56003 (2009).

[6] G. Gallavotti, *Statistical mechanics: a short treatise*, Springer, 1999.

[7] G. Gallavotti, F. Bonetto, G. Gentile, *Aspects of ergodic, qualitative, and statistical theory of motion*, Springer, 2004.

[8] A. Ghosh and D. Dhar: *On the orientational ordering of long rods on a lattice*, Europhysics Letters **78**, 20003 (2007).

[9] O. J. Heilmann: *Existence of phase transition in certain lattice gases with repulsive potential*, Lett. Nuovo Cim. **3**, 95 (1972).

[10] O. J. Heilmann and E. H. Lieb: *Monomers and Dimers*, Phys. Rev. Lett. **24**, 1412 (1970); and: *Theory of monomer-dimer systems*, Communications in Mathematical Physics **25**, 190-232 (1972).

[11] O. J. Heilmann and E. H. Lieb: *Lattice Models for Liquid Crystals*, J. Stat. Phys. **20**, 679-693 (1979).

[12] D. A. Huckaby: *Phase transitions in lattice gases of hard-core molecules having two orientations*, J. Statist. Phys. **17**, 371-375 (1977).

[13] D. Ioffe, Y. Velenik and M. Zahradnik: *Entropy-Driven Phase Transition in a Polydisperse Hard-Rods Lattice System*, Journal of Statistical Physics **122**, 761-786 (2006).

[14] R. Kotecky: *Pirogov-Sinai Theory*, in J.-P. Francoise, G. L. Naber and T. S. Tsun eds, Encyclopedia of Mathematical Physics, Elsiever, pp.60-65 (2006).

[15] J. L. Lebowitz and G. Gallavotti: *Phase transitions in binary lattice gases*, J. Math. Phys. **12**, 1129-1133 (1971).

[16] L. G. Lopez, D. H. Linares, and A. J. Ramirez-Pastor: *Critical exponents and universality for the isotropic-nematic phase transition in a system of self-assembled rigid rods on a lattice*, Phys. Rev. E **80**, 040105(R) (2009).

[17] L. G. Lopez, D. H. Linares, A. J. Ramirez-Pastor and S. A. Cannas: *Phase diagram of self-assembled rigid rods on two-dimensional lattices: Theory and Monte Carlo simulations*, J. Chem. Phys. **133**, 134706 (2010).

[18] D. A. Matoz-Fernandez, D. H. Linares, and A. J. Ramirez-Pastor: *Critical behavior of long straight rigid rods on two-dimensional lattices: Theory and Monte Carlo simulations*, J. Chem. Phys. **128**, 214902 (2008).

[19] D. A. Matoz-Fernandez, D. H. Linares and A. J. Ramirez-Pastor: *Determination of the critical exponents for the isotropic-nematic phase transition in a system of long rods on two-dimensional lattices: Universality of the transition*, Europhysics Letters **82**, 50007 (2008).

[20] D. A. Matoz-Fernandez, D. H. Linares and A. J. Ramirez-Pastor: *Critical behavior of long linear k-mers on honeycomb lattices*, Phys. A **387**, 6513-6525 (2008).

[21] L. Onsager: *The effects of shape on the interaction of colloidal particles*, Ann. N. Y. Acad. Sci. **51**, 627-659 (1949).

[22] S. Pirogov and Ya. Sinai: *Phase diagrams of classical lattice systems*, Theor. Math. Phys. **25**, 1185-1192 (1975) and **26**, 39-49 (1976).

[23] D. Ruelle: *Existence of a Phase Transition in a Continuous Classical System*, Phys. Rev. Lett. **27**, 1040-1041 (1971).

[24] D. Ruelle, *Statistical mechanics: rigorous results*, World Scientific, 1999.

[25] M. Zahradnik: *An alternative version of Pirogov-Sinai theory*, Commun. Math. Phys. **93**, 559-581 (1984).

[26] M. Zahradnik: *A short course on the Pirogov-Sinai theory*, Rendiconti Math. Serie VII **18**, 411-486 (1998).