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THE NEUMANN PROBLEM FOR THE EQUATION $\Delta u - k^2 u = 0$ IN THE EXTERIOR OF NON-CLOSED LIPSCHITZ SURFACES

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Abstract. We study the Neumann problem for the equation $\Delta u - k^2 u = 0$ in the exterior of non-closed Lipschitz surfaces in R^3. Theorems on existence and uniqueness of a weak solution of the problem are proved. The integral representation for a solution is obtained in the form of a double layer potential. The density in the potential is defined as a solution of the operator (integral) equation, which is uniquely solvable.

Weak solvability of elliptic boundary value problems with Dirichlet, Neumann and mixed Dirichlet–Neumann boundary conditions in Lipschitz domains has been studied in [1], [2], [3], [4]. It is pointed out in the book [1, p. 91] that domains with cracks (cuts) are not Lipschitz domains. So, solvability of elliptic boundary value problems in domains with cracks does not follow from general results on solvability of elliptic boundary value problems in Lipschitz domains. In the present paper, the weak solvability of the Neumann problem for the equation $\Delta u - k^2 u = 0$ in the exterior of non-closed Lipschitz surfaces (cracks) in R^3 is studied. Theorems on existence and uniqueness of a weak solution are proved, integral representation for a solution in the form of a double layer potential is obtained, and the problem is reduced to the uniquely solvable operator equation.

The weak solvability of the Neumann problem for the Laplace equation in the exterior of several smooth non-closed surfaces in R^3 has been studied in [5]. Boundary value problems for the Helmholtz equation in the exterior of smooth non-closed screens in R^3 have been studied in [6], [7].

In Cartesian coordinates $x = (x_1, x_2, x_3)$ in R^3, we consider a bounded Lipschitz domain G with the boundary S; i.e. S is a closed Lipschitz surface. Note that a normal vector exists on the Lipschitz surface almost everywhere [1, p. 96]. Let γ be a nonempty subset of the boundary S and $\gamma \neq S$. Assume that γ is a non-closed Lipschitz surface with Lipschitz boundary $\partial\gamma$ in the space R^3, and assume that γ includes its limiting points, or, alternatively, assume that γ is a union of a finite number of such non-closed surfaces, which do not have common points; in particular, they do not have common boundary points. In the latter case, γ is not a connected set. Notice that γ is a closed

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set. Let us introduce Sobolev spaces on γ:

$$H^{-1/2}(\gamma) = \{v: \ v = V|_{\gamma\setminus\partial\gamma}, \ V \in H^{-1/2}(S)\},$$

$$\tilde{H}^{1/2}(\gamma) = \{v: \ v \in H^{1/2}(S), \ \ \mathrm{supp}\, v \subset \gamma\}.$$

Spaces $H^{-1/2}(\gamma)$ and $\tilde{H}^{1/2}(\gamma)$ are dual spaces in the sense of a scalar product in $L_2(\gamma)$ [1, p. 91-92]. Furthermore, one can set $\|v\|_{\tilde{H}^{1/2}(\gamma)} = \|v\|_{H^{1/2}(S)}$ for $v \in \tilde{H}^{1/2}(\gamma)$ (see [1, p. 79]), and $\|v\|_{H^{-1/2}(\gamma)} = \min\limits_{V|_{\gamma\setminus\partial\gamma}=v, \ V\in H^{-1/2}(S)} \|V\|_{H^{-1/2}(S)}$ (see [1, p. 77, p. 99]). Spaces $H^{1/2}(S)$ and $H^{-1/2}(S)$ on a closed Lipschitz surface S and their norms are defined, for example, in [1, p. 98].

Let Δ be a Laplacian in R^3; then for the equation

$$\Delta u(x) - k^2 u(x) = 0, \quad k = \mathrm{const} > 0, \tag{1}$$

consider the double layer potential

$$W[h](x) = \frac{1}{4\pi}\int_S h(y)\frac{\partial}{\partial n_y}\frac{\exp(-k|x-y|)}{|x-y|}ds_y, \tag{2}$$

with the density $h \in H^{1/2}(S)$. By n denote the outward unit normal vector on S where it exists. The function (2) is defined for $x \in R^3 \setminus S$. According to Theorem 6.11 in [1], the potential $W[h](x)$ belongs to $H^1_{loc}(R^3\setminus\bar{G})\cap H^1(G)$, and its normal derivative $\dfrac{\partial W[h]}{\partial n}$ does not have jump on S; when approaching S from G and from $R^3 \setminus \bar{G}$ it has the same trace $\left.\dfrac{\partial W[h]}{\partial n}\right|_S \in H^{-1/2}(S)$. The overline means closure. Moreover, potential $W[h](x)$ belongs to $C^\infty(R^3 \setminus S)$ (see [1, p. 202]) and obeys equation (1) in $R^3 \setminus S$ and conditions at infinity:

$$u = o(|x|^{-1}), \quad |\nabla u| = o(|x|^{-1}), \qquad |x| \to \infty. \tag{3}$$

Lemma. Let $h \in H^{1/2}(S)$, $k > 0$, and let S be a boundary of an open bounded Lipschitz domain G. Then there is such a constant $c > 0$, that inequality

$$\left(-\left.\frac{\partial W[h]}{\partial n}\right|_S, h\right)_{L_2(S)} \geq c\|h\|^2_{H^{1/2}(S)}$$

holds.

Proof. Note that a normal vector exists on the Lipschitz surface almost everywhere [1, p. 96]. Let B_r be an open ball of radius r with the center in the origin and $\bar{G} \subset B_r$. By n denote the outward (with respect to G) unit normal vector on S where it exists as well as the outward unit normal vector on ∂B_r. Writing down Green's formula [1, p. 118] for the function $W[h](x)$ in $B_r \setminus \bar{G}$ and in G, we obtain

$$\begin{aligned} &\|\nabla W[h]\|^2_{L_2(B_r\setminus\bar{G})} + k^2\|W[h]\|^2_{L_2(B_r\setminus\bar{G})} \\ &= -\left((W[h])^+, \left.\frac{\partial W[h]}{\partial n}\right|_S\right)_{L_2(S)} + \left(W[h], \frac{\partial W[h]}{\partial n}\right)_{L_2(\partial B_r)}, \end{aligned} \tag{4}$$

$$\|\nabla W[h]\|^2_{L_2(G)} + k^2\|W[h]\|^2_{L_2(G)} = \left((W[h])^-, \left.\frac{\partial W[h]}{\partial n}\right|_S\right)_{L_2(S)}. \tag{5}$$

By $(W[h])^-$ and $(W[h])^+$, we mean traces of the function $W[h](x)$ on S when approaching S from G and from $R^3 \setminus \bar{G}$ respectively. According to Theorem 6.11 in [1], traces $(W[h])^-$ and $(W[h])^+$ exist and belong to $H^{1/2}(S)$. We remind the reader that, under conditions of the lemma, the normal derivative of the function $W[h](x)$ has the same trace $\left.\dfrac{\partial W[h]}{\partial n}\right|_S \in H^{-1/2}(S)$ when approaching S both from G and from $R^3 \setminus \bar{G}$. Since spaces $H^{-1/2}(S)$ and $H^{1/2}(S)$ are dual, the scalar products are defined in $L_2(S)$ on the right-hand sides of (4) and (5). Tending $r \to \infty$ in (4) and taking into account that the potential $W[h](x)$ satisfies conditions (3), we obtain

$$\|\nabla W[h]\|^2_{L_2(R^3\setminus\bar{G})} + k^2\|W[h]\|^2_{L_2(R^3\setminus\bar{G})} = -\left((W[h])^+, \left.\frac{\partial W[h]}{\partial n}\right|_S\right)_{L_2(S)}. \tag{6}$$

By Theorem 6.11 in [1], the jump of the potential $W[h]$ on S is defined by the formula

$$(W[h])^- - (W[h])^+ = -h.$$

Adding (5) and (6), we obtain

$$\|\nabla W[h]\|^2_{L_2(R^3\setminus S)} + k^2\|W[h]\|^2_{L_2(R^3\setminus S)} = \left(-h, \left.\frac{\partial W[h]}{\partial n}\right|_S\right)_{L_2(S)}.$$

Taking into account the theorem on equivalence of Sobolev spaces [1, Theorems 3.16, 3.30], we observe that there is such a constant $c_1 > 0$ for which inequality holds:

$$\begin{gathered} c_1\|W[h]\|^2_{H^1(R^3\setminus S)} \\ \le \min\{k^2,1\}(\|\nabla W[h]\|^2_{L_2(R^3\setminus S)} + \|W[h]\|^2_{L_2(R^3\setminus S)}) \le \left(-h, \left.\frac{\partial W[h]}{\partial n}\right|_S\right)_{L_2(S)}. \end{gathered} \tag{7}$$

Using the fact that the trace operator of a double layer potential $W[h] \in H^1(G) \cap H^1_{loc}(R^3 \setminus \bar{G})$ is bounded on S (see [1, Theorem 3.37]) when approaching S both from G and from $R^3 \setminus \bar{G}$ for some constant $c_2 > 0$, we obtain (see [1, p. 203-204])

$$\begin{aligned} \|h\|^2_{H^{1/2}(S)} &= \left\|(W[h])^+ - (W[h])^-\right\|^2_{H^{1/2}(S)} \le \|(W[h])^+\|^2_{H^{1/2}(S)} + \|(W[h])^-\|^2_{H^{1/2}(S)} \\ &\le c_2\|W_0[h]\|^2_{H^1(R^3\setminus S)}. \end{aligned}$$

Here $W_0[h](x) = \delta(x)W[h](x)$, where $\delta(x) \in C^\infty(R^3)$ is a cutoff function, such that $\delta(x) \le 1$ for all $x \in R^3$, $\delta(x) \equiv 1$ in an open bounded domain containing $\bar{G}$, and $\delta(x) \equiv 0$ in the exterior of some ball with the center in the origin. Clearly,

$$\|W_0[h]\|^2_{H^1(R^3\setminus S)} \le c_3\|W[h]\|^2_{H^1(R^3\setminus S)},$$

for some constant $c_3 > 0$, so

$$\|h\|^2_{H^{1/2}(S)} \le c_2c_3\|W[h]\|^2_{H^1(R^3\setminus S)}.$$

Using (7), we obtain

$$c\|h\|^2_{H^{1/2}(S)} \le \left(-\left.\frac{\partial W[h]}{\partial n}\right|_S, h\right)_{L_2(S)}, \quad c = c_1/(c_2c_3).$$

The lemma is proved. $\square$

Let us formulate the Neumann problem for the equation (1) in the exterior of non-closed Lipshitz surfaces γ.

PROBLEM $\mathcal{N}$. Find a function $u(x) \in H^1_{loc}(R^3 \setminus \bar{G}) \cap H^1(G) \cap C^2(R^3 \setminus \gamma)$ that satisfies equation (1) in $R^2 \setminus \gamma$ the boundary condition

$$\left.\frac{\partial u}{\partial n}\right|_{\gamma} = f \in H^{-1/2}(\gamma) \tag{8}$$

and conditions at infinity (3).

Boundary condition (8) implies that the normal derivative of the function $u(x)$ has the same trace $\left.\frac{\partial u}{\partial n}\right|_{\gamma}$ when approaching γ from G and from $R^3 \setminus \bar{G}$, and this trace has to satisfy condition (8).

Let us construct the solution of the problem. We look for a solution in the form of a double layer potential

$$u(x) = W[g](x) = \frac{1}{4\pi} \int_{\gamma} g(y) \frac{\partial}{\partial n_y} \frac{\exp(-k|x-y|)}{|x-y|} ds_y \tag{9}$$

$$= \frac{1}{4\pi} \int_{S} g(y) \frac{\partial}{\partial n_y} \frac{\exp(-k|x-y|)}{|x-y|} ds_y$$

with the density $g \in \tilde{H}^{1/2}(\gamma) \subset H^{1/2}(S)$. The function (9) is defined if $x \in R^3 \setminus \gamma$.

It follows from the aforementioned properties of a double layer potential (2) that the potential $W[g](x)$ belongs to $H^1_{loc}(R^3 \setminus \bar{G}) \cap H^1(G)$, its normal derivative has a trace on S: $\left.\frac{\partial W[g]}{\partial n}\right|_{S} \in H^{-1/2}(S)$, and it has a trace on γ: $\left.\frac{\partial W[g]}{\partial n}\right|_{\gamma} \in H^{-1/2}(\gamma)$. Furthermore, the potential $W[g](x)$ belongs to $C^{\infty}(R^3 \setminus \gamma)$ (see [1, p. 202]) and satisfies equation (1) in $R^3 \setminus \gamma$ as well as conditions at infinity (3). Therefore, for any function g from the space $\tilde{H}^{1/2}(\gamma)$, the potential $W[g](x)$ satisfies all conditions of the problem $\mathcal{N}$, except for the boundary condition (8). We have to find the function $g \in \tilde{H}^{1/2}(\gamma)$ to satisfy the boundary condition (8). Substituting (9) into the boundary condition (8), we arrive at the operator equation

$$Mg|_{\gamma} = f \in H^{-1/2}(\gamma), \qquad Mg = \frac{\partial W[g]}{\partial n}. \tag{10}$$

To prove the solvability of equation (10), we have to study properties of the operator M on the left-hand side of the equation.

Operator M is bounded when acting from $H^{1/2}(S)$ into $H^{-1/2}(S)$ by Theorem 6.11 in [1], so when acting from $\tilde{H}^{1/2}(\gamma) \subset H^{1/2}(S)$ into $H^{-1/2}(S)$ it is bounded as well. If a set of functions is bounded (in norm) in $H^{-1/2}(S)$ by a constant, then a set of narrowing of these functions to γ is bounded (in norm) in $H^{-1/2}(\gamma)$ also and by the same constant. Therefore, the operator M is bounded when acting from $\tilde{H}^{1/2}(\gamma)$ into $H^{-1/2}(\gamma)$. Since $g \in \tilde{H}^{1/2}(\gamma) \subset H^{1/2}(S)$, by applying the lemma to the operator M, we have

$$\left(- \left.\frac{\partial W[g]}{\partial n}\right|_{S}, g \right)_{L_2(S)} = \left(- \left.\frac{\partial W[g]}{\partial n}\right|_{\gamma}, g \right)_{L_2(\gamma)} \geq c\|g\|^2_{H^{1/2}(S)} = c\|g\|^2_{\tilde{H}^{1/2}(\gamma)}.$$

Therefore, for some constant $c > 0$, we have

$$(-Mg|_\gamma, g)_{L_2(\gamma)} \geq c\|g\|^2_{\tilde{H}^{1/2}(\gamma)}. \tag{11}$$

Here by $Mg|_\gamma = \left.\dfrac{\partial W[g]}{\partial n}\right|_\gamma$ we mean the trace of the normal derivative of the function (9) on γ; this trace belongs to $H^{-1/2}(\gamma)$. Note that the operator M acts from $\tilde{H}^{1/2}(\gamma)$ into $H^{-1/2}(\gamma)$ and bounded, while spaces $\tilde{H}^{1/2}(\gamma)$, $H^{-1/2}(\gamma)$ are dual in the sense of a scalar product in $L_2(\gamma)$. Inequality (11) implies that the operator $(-M)$ is positive and bounded below. Consequently, from Lemma 2.32 in [1, p. 43] it follows that the operator $(-M)$ is invertible (it has bounded inverse operator). Therefore, equation (10) has a unique solution $g \in \tilde{H}^{1/2}(\gamma)$ for any function $f \in H^{-1/2}(\gamma)$. The potential (9) constructed on this solution satisfies all conditions of the problem $\mathcal{N}$. From the above considerations follows

THEOREM 1. The solution of the problem $\mathcal{N}$ exists and is given by formula (9), where $g \in \tilde{H}^{1/2}(\gamma)$ is a solution of equation (10), which is uniquely solvable in $\tilde{H}^{1/2}(\gamma)$.

Let us prove the uniqueness of a solution to the problem $\mathcal{N}$.

THEOREM 2. The problem $\mathcal{N}$ has at most one solution.

Proof. Let $u(x)$ be a solution of the homogeneous problem $\mathcal{N}$. Consider the ball B_r of large enough radius r with the center in the origin. Suppose that $\bar{G} \subset B_r$ and $\bar{G} \cap \partial B_r = \emptyset$. The overline means closure, while ∂B_r is a sphere, which is the boundary of the ball B_r. Since $u(x) \in H^1_{loc}(R^3 \setminus \bar{G}) \cap H^1(G)$, the Green's formulae [1, Theorem 4.4, p. 118],

$$\|\nabla u\|^2_{L_2(G)} + k^2\|u\|^2_{L_2(G)} = \left(u^-, \left.\frac{\partial u}{\partial n}\right|_S\right)_{L_2(S)}, \tag{12}$$

$$\|\nabla u\|^2_{L_2(B_r\setminus\bar{G})} + k^2\|u\|^2_{L_2(B_r\setminus\bar{G})} = -\left(u^+, \left.\frac{\partial u}{\partial n}\right|_S\right)_{L_2(S)} + \left(u, \frac{\partial u}{\partial n}\right)_{L_2(\partial B_r)}, \tag{13}$$

hold for the function u. By n on ∂B_r, the outward (regarding B_r) unit normal vector is understood, while by n on S, the outward (regarding G) unit normal vector is understood (where it exists). Since the function $u(x)$ belongs to $H^1_{loc}(R^3 \setminus \bar{G}) \cap H^1(G)$, its traces on S exist when approaching S both from G and from $R^3 \setminus \bar{G}$; these traces are denoted by u^- and u^+ respectively and belong to $H^{1/2}(S)$ (see [1, Theorems 3.37, 3.38, p. 102]). Since, in addition, the function $u(x)$ obeys equation (1) outside S, the traces of the normal derivative of the function u exist on S by Lemma 4.3 in [1] when approaching S both from G and from $R^3 \setminus \bar{G}$. Moreover, it follows from the formulation of the problem $\mathcal{N}$ that these traces of the normal derivative of the function u on S from G and from $R^3 \setminus \bar{G}$ are the same; they are denoted by $\left.\dfrac{\partial u}{\partial n}\right|_S$ and belong to $H^{-1/2}(S)$ by Lemma 4.3 in [1]. Since spaces $H^{-1/2}(S)$ and $H^{1/2}(S)$ are dual, the scalar product in $L_2(S)$ on the right-hand sides of (12) and (13) is defined. Note that $\left.\dfrac{\partial u}{\partial n}\right|_\gamma = 0 \in H^{-1/2}(\gamma)$, since u is a solution of the homogeneous problem $\mathcal{N}$. Moreover, $u^+ = u^-$ on $S \setminus \gamma$, since $u \in C^2(R^3 \setminus \gamma)$. Adding (12) and (13) we obtain

$$\|\nabla u\|^2_{L_2(B_r \setminus S)} + k^2 \|u\|^2_{L_2(B_r \setminus S)} = \left(u, \frac{\partial u}{\partial n}\right)_{L_2(\partial B_r)} = \int_{\partial B_r} u \frac{\partial u}{\partial n} ds. \tag{14}$$

Using conditions (3) at infinity, we obtain from (14) as $r \to \infty$

$$\lim_{r\to\infty} (\|\nabla u\|^2_{L_2(B_r \setminus S)} + k^2 \|u\|^2_{L_2(B_r \setminus S)}) = \|\nabla u\|^2_{L_2(R^3 \setminus S)} + k^2 \|u\|^2_{L_2(R^3 \setminus S)} = 0.$$

Therefore, $u \equiv 0$ in $R^3 \setminus S$ because $k > 0$. Since $u \in C^2(R^3 \setminus \gamma)$, we observe that $u \equiv 0$ in $R^3 \setminus \gamma$. Thus, the homogeneous problem $\mathcal{N}$ has only the trivial solution. In view of the linearity of the problem $\mathcal{N}$, the inhomogeneous problem $\mathcal{N}$ has at most one solution. The theorem is proved. □

In conclusion, we note that papers [8], [9] treat the Neumann problem for the Laplace equation in planar domains with cracks. Interior domains are studied in [8], while exterior domains are studied in [9]. The well-posed classical formulation of the problems is given. Existence of classical solutions is proved under certain conditions, and theorems on a number of solutions are given. The integral representation for solutions is obtained in the form of potentials, densities in which satisfy the uniquely solvable integral equations. Note that the Neumann problem for the 2D Laplace equation is not uniquely solvable in both interior and exterior domains, but it can be reduced to the uniquely solvable integral equation by a special technique. Results obtained in [8], [9] for single-sided cracks are extended to the case of double-sided cracks in [10] in the case of interior domains, and in [11] in the case of exterior domains.

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