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The Newton and Halley Methods for Complex Roots

Lily Yau and Adi Ben-Israel

1 Introduction

Let $f : \mathbb{C} \to \mathbb{C}$ be analytic. A solution (existence assumed) of

$$f(z) = 0 \tag{1}$$

can be approximated by the *Newton method*

$$z_{k+1} := N_f(z_k) \ , \quad k = 0, 1, \ldots \tag{2}$$

using the iteration

$$N_f(z) := z - \frac{f(z)}{f'(z)} \ , \tag{3}$$

with z_0 sufficiently close to the sought solution. For (local) convergence conditions, see [15, Chapter 7].

Although the complex Newton method has been used at least since 1870, see [18], its geometric interpretation is not well known, if known at all. It is worth studying for several reasons, including:

- The Newton iteration is universal: any iteration

  $$z := g(z) \ , \quad \text{with an arbitrary iteration function } g \ ,$$

  is equivalent to the Newton iteration $z := N_f(z)$ for a function f given by

  $$f(z) := \exp\left\{ \int^z \frac{dt}{t - g(t)} \right\} \ , \quad \text{(see [21, p. 40])} \ ,$$

  in any domain where $(t - g(t))^{-1}$ is integrable, which necessarily excludes the fixed–points of g (where the integrand $(t - g(t))^{-1}$ blows up). The geometry of Newton's method may thus help to study the "far behavior" (i.e., away from fixed–points) of any iteration.

- The Newton iteration, applied to a complex polynomial, is an important model of deterministic chaos. Understanding the geometry of Newton's method may give insights into chaotic behavior; see § 3.

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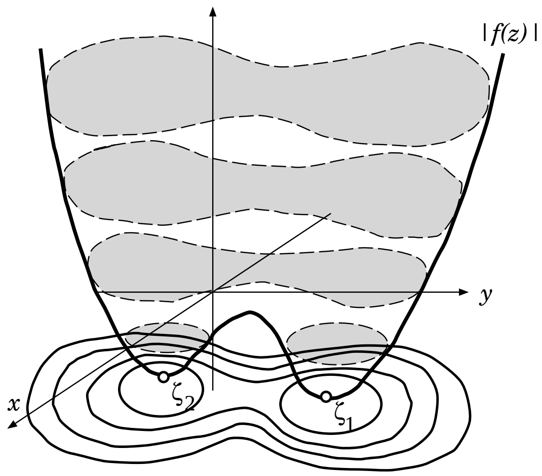
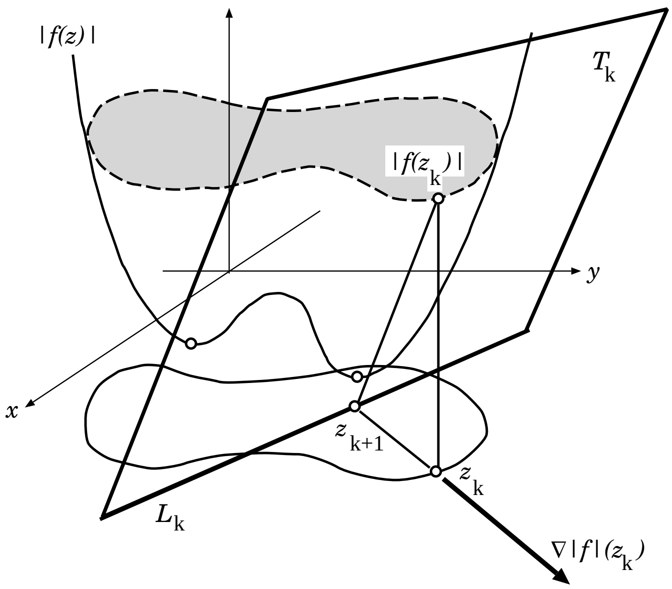


(a) The graph and level sets of $|f(z)|$ showing two zeros ζ_1, ζ_2



(b) The tangent plane T_k and its intersection L_k with the xy–plane

Figure 1: The next Newton iterate z_{k+1} is the point closest to z_k in L_k

The geometric interpretation of the Newton method in the real case is well known: the next iterate x_{k+1} is the zero of the line

$$y = f(x_k) + f'(x_k)(x - x_k)$$

tangent to the graph of f at $(x_k, f(x_k))$. In the complex case one is tempted with analogues of the above interpretation, but what is a "tangent" of a complex function? We show here that the geometry of the complex Newton iteration (3) is indeed analogous to the real case: the next point $z_{k+1} := N_f(z_k)$ is *closest* to z_k in the intersection (line) of the xy–plane and the *tangent plane* T_k of $|f|$ at the point $(z_k, |f(z_k)|)$; see Figure 1.

The direction of the *Newton step* $N_f(z_k) - z_k$ is therefore against the gradient of $|f|$ at z_k; see Figure 1(b). Restricted to the vertical plane V passing through z_k and containing $\nabla|f|(z_k)$, the complex Newton iteration coincides with the real Newton iteration; see Figure 2.

Another well-known method for solving (1) is the *Halley method* $z_{k+1} := H_f(z_k)$, $k = 0, 1, \ldots$ using the iteration

$$H_f(z) := z - \frac{f(z)}{f'(z) - \dfrac{f''(z)}{2f'(z)} f(z)} \ . \tag{4}$$

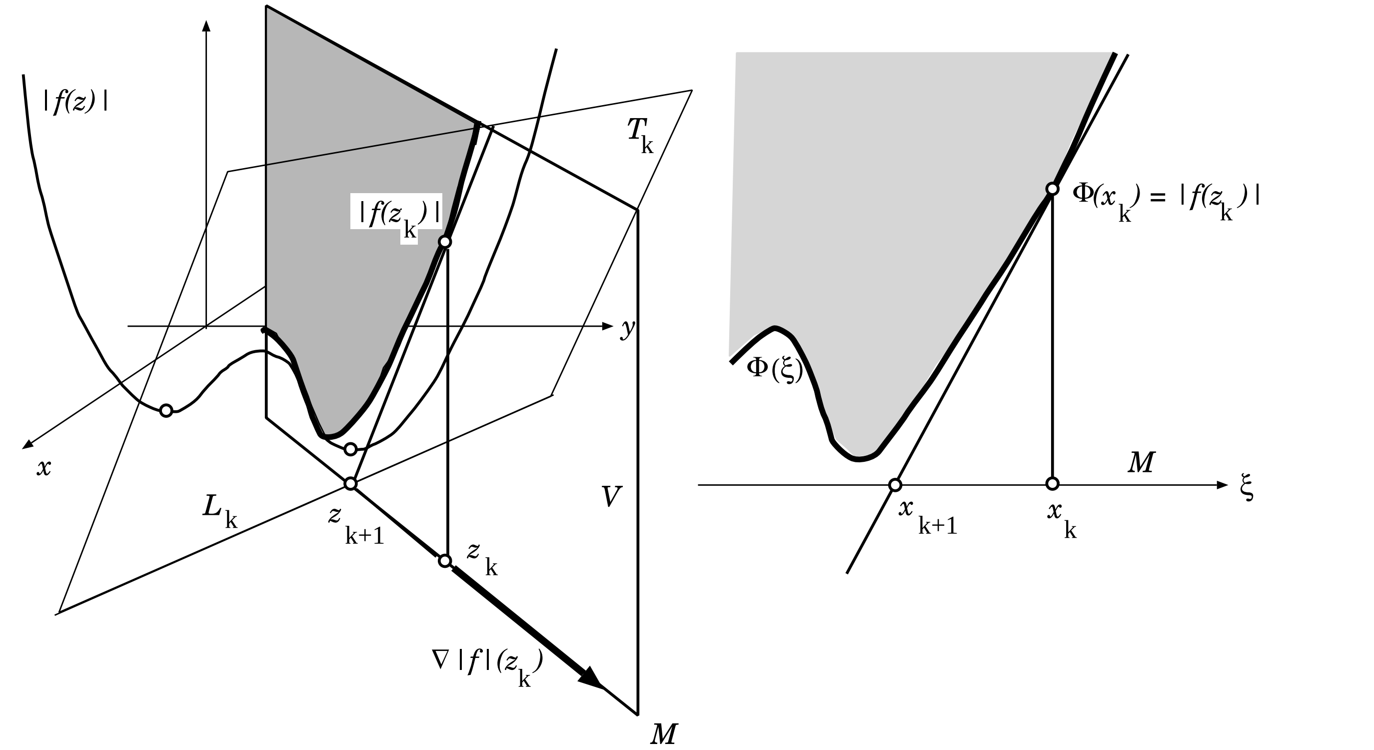
This "most frequently rediscovered" method (quoting [22, p. 91]) has a simple geometric interpretation in the real case: the next iterate $H_f(x_k)$ is the zero of the hyperbola

$$h(x) := b + \frac{a}{x - c} \tag{5}$$

that is *osculating* (or 2nd-*order tangent*) to f at x_k, i.e., $f = h$, $f' = h'$, $f'' = h''$ at $x = x_k$. See also [1], [8], [13], [17], [22] and their references.

To understand the geometry of the complex Halley method, consider the complex "hyperbola"

$$h(z) := b + \frac{a}{z - c} \tag{6}$$



(a) The vertical plane V through z_k, $\nabla|f|(z_k)$ and its intersection with the graph of $|f|$

(b) The next iterate z_{k+1} is obtained by a (real) Newton iteration in the vertical plane V.

Figure 2: A "side view" of the complex Newton iteration

that osculates f at z_k, i.e., the complex numbers a, b, c are determined by

$$h(z_k) = b + \frac{a}{z_k - c} \quad = \quad f(z_k)\ , \tag{7a}$$

$$h'(z_k) = -\frac{a}{(z_k - c)^2} \quad = \quad f'(z_k)\ ,\ \text{and} \tag{7b}$$

$$h''(z_k) = \frac{2a}{(z_k - c)^3} \quad = \quad f''(z_k)\ . \tag{7c}$$

In Theorem 2 we show that the next Halley iterate z_{k+1} is the (unique) zero of h, $z_{k+1} := c - a/b$. In general, the direction of the *Halley step* $H_f(z_k) - z_k$ is different from the direction of the Newton step; see Remark 3.

The complex Halley method has an interesting "top view": the level set of $|h|$ at z_k, $C_k := \{z : |h(z)| = |h(z_k)|\}$, is the *osculating circle* (or *circle of curvature*) of the level set of $|f|$ at z_k, $S_k := \{z : |f(z)| = |f(z_k)|\}$; see Figure 5.

2 The Newton method

We identify complex numbers $z = x + iy \in \mathbb{C}$ with the two–dimensional real vectors $(x, y) \in \mathbb{R}^2$, and denote this correspondence by

$$x + iy \ \longleftrightarrow (x, y)\ .$$

Consider an analytic function $f(z) = u + iv = u(x, y) + iv(x, y)$, and its absolute value $F(x, y) = |f(x + iy)| = \sqrt{u^2(x, y) + v^2(x, y)}$, which is differentiable, as a function of (x, y), except where $f(z) = 0$. The gradient of F is

$$\nabla F(x, y) = \frac{1}{\sqrt{u^2 + v^2}} \begin{pmatrix} uu_x + vv_x \\ uu_y + vv_y \end{pmatrix} \tag{8}$$

where $u_x = \partial u/\partial x$, $u_y = \partial u/\partial y$, etc. Using the Cauchy-Riemann equations: $u_x = v_y$ and $u_y = -v_x$, [14, § II.2], we calculate [10, p. 544]

$$\frac{f(z)}{f'(z)} = \frac{u + iv}{u_x + iv_x} = \frac{(uu_x + vv_x) + i(uu_y + vv_y)}{u_x^2 + v_x^2} \longleftrightarrow \frac{1}{u_x^2 + v_x^2} \begin{pmatrix} uu_x + vv_x \\ uu_y + vv_y \end{pmatrix} . \tag{9}$$

A comparison with (8) shows that the Newton method (3) is a gradient method for the absolute value function $|f|$, i.e.,

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} := \begin{pmatrix} x_k \\ y_k \end{pmatrix} - t \, \nabla F(x_k, y_k) . \tag{10}$$

It remains to determine the step size t in (10). Consider the space $\mathbb{R}^3$, with standard coordinates[1] (x, y, Z). The intersection of a (non-horizontal) plane in $\mathbb{R}^3$ with the xy–plane is called the *trace* of that plane.

A geometric interpretation of the complex Newton method can now be given.

Theorem 1. Let the function f be analytic, and let $z_k = x_k + iy_k$ be a point where $f(z_k)$ and $f'(z_k)$ are nonzero. Let T_k be the tangent plane of $F(x, y) := |f(x + iy)|$ at $(x_k, y_k, F(x_k, y_k))$, and let L_k be the trace of T_k. The next iterate $z_{k+1} = x_{k+1} + iy_{k+1}$ of the Newton method

$$z_{k+1} := z_k - \frac{f(z_k)}{f'(z_k)}$$

corresponds to the point (x_{k+1}, y_{k+1}) on the line L_k that is nearest to (x_k, y_k).

Proof. The plane T_k tangent to the graph of F at $(x_k, y_k, F(x_k, y_k))$ has the equation

$$Z = Z(x, y) = F(x_k, y_k) + \nabla F(x_k, y_k) \cdot \begin{pmatrix} x - x_k \\ y - y_k \end{pmatrix} ,$$

where $\cdot$ denotes inner product. If T_k is non–horizontal (i.e., $\nabla F(x_k, y_k) \neq \mathbf{0}$), its trace is the line L_k given by

$$F(x_k, y_k) + \nabla F(x_k, y_k) \cdot \begin{pmatrix} x - x_k \\ y - y_k \end{pmatrix} = 0 . \tag{11}$$

Let (x^*, y^*) be the point on L_k that is closest to the point (x_k, y_k), i.e., (x^*, y^*) is the orthogonal projection of (x_k, y_k) on L_k. Therefore the difference $(x^*, y^*) - (x_k, y_k)$ is orthogonal to L_k, i.e.,

$$\begin{pmatrix} x^* - x_k \\ y^* - y_k \end{pmatrix} = t \, \nabla F(x_k, y_k) , \tag{12}$$

where the constant t is determined by the condition that (x^*, y^*) lies on L_k. Since (x^*, y^*) satisfies (11), we have

$$F(x_k, y_k) + \nabla F(x_k, y_k) \cdot \begin{pmatrix} x^* - x_k \\ y^* - y_k \end{pmatrix} = F(x_k, y_k) + t \, \|\nabla F(x_k, y_k)\|^2 = 0$$

and

$$t = -\frac{F(x_k, y_k)}{\|\nabla F(x_k, y_k)\|^2} . \tag{13}$$

Using (8) and the Cauchy-Riemann equations, we compute

$$\|\nabla F(x_k, y_k)\|^2 = u_x^2 + v_x^2 . \tag{14}$$

Substituting (13), (8) and (14) into (12) yields

$$\begin{pmatrix} x^* \\ y^* \end{pmatrix} = \begin{pmatrix} x_k \\ y_k \end{pmatrix} - \frac{\sqrt{u^2 + v^2}}{u_x^2 + v_x^2} \, \frac{1}{\sqrt{u^2 + v^2}} \begin{pmatrix} uu_x + vv_x \\ uu_y + vv_y \end{pmatrix} = \begin{pmatrix} x_k \\ y_k \end{pmatrix} - \frac{1}{u_x^2 + v_x^2} \begin{pmatrix} uu_x + vv_x \\ uu_y + vv_y \end{pmatrix} , \tag{15}$$

[1] The 3rd coordinate is denoted Z, to avoid confusion with complex numbers z.

where all functions are evaluated at (x_k, y_k). A comparison with (9) shows that

$$\begin{pmatrix} x^* \\ y^* \end{pmatrix} \longleftrightarrow z_k - \frac{f(z_k)}{f'(z_k)} .$$

$\square$

Remark 1.
(a) The Newton step $z_{k+1} - z_k$ is in the direction of $-\nabla|f|(x_k, y_k)$, and is therefore perpendicular to the level set of $|f|$ at (x_k, y_k).
(b) While $-\nabla|f|$ is the direction of steepest descent for $|f|$, the Newton iteration does not necessarily produce a descent of $|f|$ (if the Newton step is "too long"); see [10, p. 547, Example 2].
(c) We can alternatively interpret $f(z) = 0$ as a system of 2 equations in 2 (real) unknowns

$$\begin{aligned} u(x, y) &= 0 \\ v(x, y) &= 0 \end{aligned}$$

and apply the 2–dimensional Newton method

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} := \begin{pmatrix} x_k \\ y_k \end{pmatrix} - \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}^{-1} \begin{pmatrix} u \\ v \end{pmatrix} , \tag{16}$$

where all functions and derivatives are evaluated at (x_k, y_k). This simplifies (after computing the inverse explicitly, and using the Cauchy-Riemann equations) to

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} := \begin{pmatrix} x_k \\ y_k \end{pmatrix} - \frac{1}{u_x^2 + v_x^2} \begin{pmatrix} uu_x + vv_x \\ uu_y + vv_y \end{pmatrix} , \tag{17}$$

which is identical to the complex Newton method (3), as shown by comparison with (15). The geometric interpretation given in Theorem 1 therefore applies also to the 2–dimensional Newton method (16).

Let V be the vertical plane through the point z_k and containing $\nabla|f|(x_k, y_k)$. Then V contains the next iterate z_{k+1}, by Remark 1(a), but in general V does not contain the sought root of f.

Let Φ be the function whose graph is the intersection of V and the graph of $|f(z)|$. If we perform a (real) Newton iteration for Φ at z_k, then the next iterate is the same as that obtained by a (complex) Newton iteration for f at z_k. The complex Newton iteration is thus equivalent to a real Newton iteration in the vertical plane V; see Figure 2(b). A precise statement is:

Corollary 1. Let the function f be analytic, and let $z_k = x_k + iy_k$ be a point where $f(z_k)$ and $f'(z_k)$ are nonzero. Let V be the vertical plane through the point z_k and containing $\nabla|f|(x_k, y_k)$, and let M be its trace, parametrized by

$$\begin{aligned} x &= \xi \\ y &= m\,\xi + b \end{aligned} \tag{18}$$

Finally let $\Phi(\xi)$ be the function: $M \to \mathbb{R}$ whose graph is the intersection of V and the graph of $|f(z)|$. Then the complex Newton iteration N_f at z_k is equivalent to the real Newton iteration N_Φ at x_k:

$$z_{k+1} := N_f(z_k) = x_{k+1} + iy_{k+1} \longleftrightarrow (x_{k+1}, y_{k+1}) := (N_\Phi(x_k), mN_\Phi(x_k) + b) \tag{19}$$

Proof. We denote $F(x, y) := |f(x + iy)| = \sqrt{u^2(x, y) + v^2(x, y)}$. The slope of the line M is, by (8),

$$m = \frac{uu_y + vv_y}{uu_x + vv_x} , \tag{20}$$

all functions evaluated at (x_k, y_k). Substituting (18) into $F(x, y)$ we get the function

$$\Phi(\xi) = F(\xi, m\,\xi + b) \ , \tag{21}$$

which is the restriction of F to the plane V. In particular,

$$\Phi(x_k) = F(x_k, y_k) = |f(z_k)| \ . \tag{22}$$

The derivative of Φ is

$$\Phi'(\xi) = F_x + mF_y = \frac{uu_x + vv_x}{|f|} + m\,\frac{uu_y + vv_y}{|f|}$$

and at x_k,

$$\begin{aligned}
\Phi'(x_k) &= \frac{1}{|f|}\left((uu_x + vv_x) + \frac{uu_y + vv_y}{uu_x + vv_x}\,(uu_y + vv_y)\right) \ , \quad \text{by (20)} \ , \\
&= \frac{1}{|f|\,(uu_x + vv_x)}\ \left((uu_x + vv_x)^2 + (uu_y + vv_y)^2\right) \\
&= \frac{1}{|f|\,(uu_x + vv_x)}\,(u^2 + v^2)\,(u_x^2 + v_x^2) \ , \quad \text{by the Cauchy–Riemann equations} \ , \\
&= \frac{|f|\,|f'|^2}{uu_x + vv_x} \ .
\end{aligned} \tag{23}$$

The real Newton iteration for Φ at x_k gives the next iterate

$$x_{k+1} := x_k - \frac{\Phi(x_k)}{\Phi'(x_k)} \ ,$$

and substituting (22) and (23) we get

$$x_{k+1} := x_k - \frac{1}{|f'|^2}\,(uu_x + vv_x) \ .$$

The corresponding y–coordinate is

$$y_{k+1} := y_k - m\,\frac{1}{|f'|^2}\,(uu_x + vv_x) = y_k - \frac{1}{|f'|^2}\,(uu_y + vv_y) \ , \quad \text{by (20)} \ .$$

Combining the last two results, we get

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} x_k \\ y_k \end{pmatrix} - \frac{1}{|f'|^2}\begin{pmatrix} uu_x + vv_x \\ uu_y + vv_y \end{pmatrix} \ , \quad \text{all functions evaluated at } (x_k, y_k) \ ,$$

which is the same as (15). □

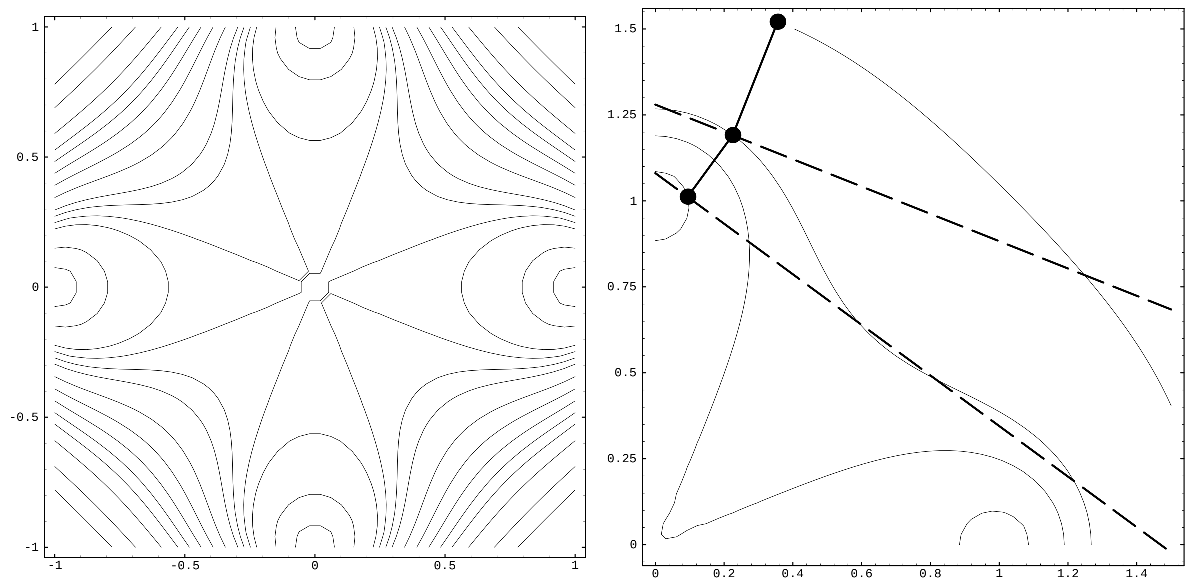
We conclude that the complex Newton method for f is a sequence of real Newton iterations for $|f|$ in vertical planes containing pairs of adjacent iterates (z_k, z_{k+1}).

Remark 2. *Theorem 1 allows tracking the Newton iterates (2) for all initial z_0. The Newton orbit is deflected at the level sets of $|f|$ in analogy with the refraction of light, in geometric optics, by concave or convex lenses.*

The Newton step $z_{k+1} - z_k$ is perpendicular to the level set of $|f|$ at z_k. If the step-lengths are small, as they typically are near a root to which the method converges, the Newton orbits approximate the orthogonal trajectories of the $|f|$ level sets, trajectories that stay close if the level sets are well-behaved. These trajectories are described by the differential equation

$$\dot{z} = -\frac{f(z)}{f'(z)}$$

or the continuous Newton method*; see [5], [11].*



(a) Level sets of $|z^4 - 1|$ (b) Two iterations

Figure 3: Illustration of Newton's method for $f(z) = z^4 - 1$

Exmaple 1. *Consider the polynomial* $f(z) = z^4 - 1$. *The equation* $f(z) = 0$ *has 4 roots:* $\pm 1, \pm i$. *Several level sets of the absolute value* $|f|$ *are plotted in Figure 3(a). Figure 3(b) shows the Newton iterates* $0.356 + i\,1.521,\ 0.225 + i\,1.191,\ 0.094 + i\,1.012$, *and illustrates the fact that each Newton iterate is the orthogonal projection of the previous iterate on the trace of the plane tangent to the graph of* $|f|$. *These traces are shown as dashed lines.*

3 Attraction basins of the Newton method for polynomials

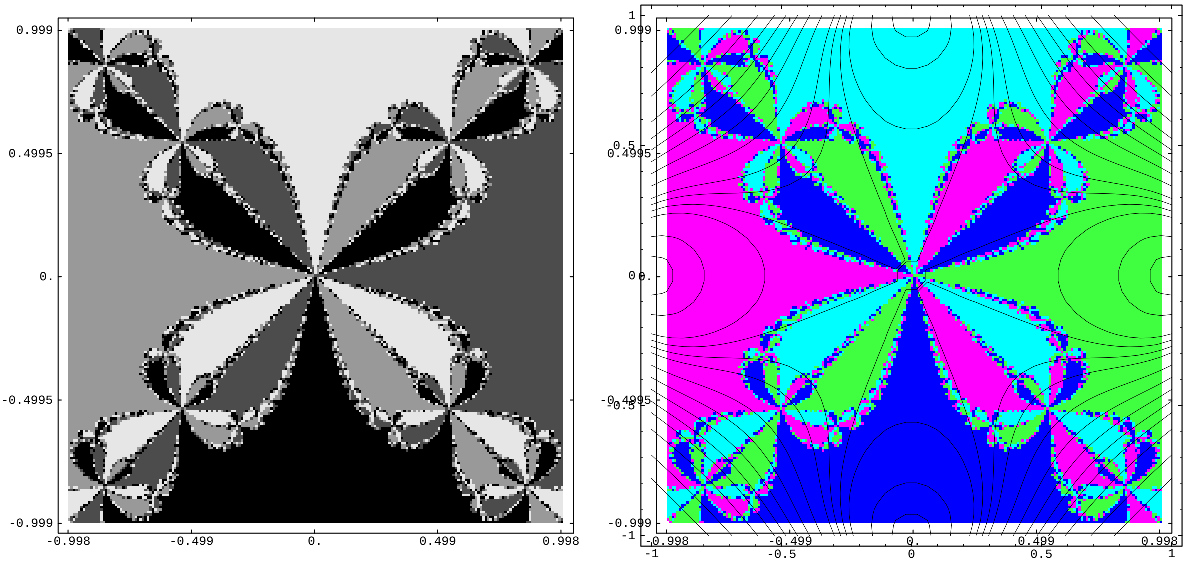
Let p be a polynomial with roots $\zeta_1, \ldots, \zeta_k$. With each root ζ_i we associate a *basin* (or *domain of attraction*), $A(\zeta_i)$, consisting of points from where the Newton method eventually converges to ζ_i.

Exmaple 2. *The polynomial* $p(z) = z^2 - 1$ *has two roots* $\zeta_1 = 1$ *and* $\zeta_2 = -1$. *The Newton iteration is*

$$N_p(z) := \frac{1}{2}\left(z + \frac{1}{z}\right)$$

and its basins are $A(1) = \{z : \mathit{Re}\, z > 0\}$ *(the open right half-plane) and* $A(-1) = \{z : \mathit{Re}\, z < 0\}$ *(the open left half-plane); see [20, p. 20]. Imaginary points are taken by the Newton iteration to imaginary points. The Newton iterations thus take every point to the nearest root, if the nearest root is unique.*

The orderly behavior of Newton's method shown in Example 2 does not hold for polynomials $p(z)$ of degree ≥ 3. Indeed, the Gauss–Lucas Theorem states that the roots of $p'(z)$ lie in the convex hull of the roots of $p(z)$ [10, § 6.5]. Since the Newton iteration $N_p(z) := z - p(z)/p'(z)$ behaves erratically near the zeros of $p'(z)$, there is no guarantee that the Newton iterates converge to the nearest root. In fact, the Newton basins have a common boundary, a *Julia set* [3] consisting of points near which the Newton method leads to any root.



(a) Newton basins

(b) Same, with level sets of $|z^4 - 1|$

Figure 4: Newton basins for $f(z) = z^4 - 1$

The Newton basins give stunning pictures, among the most beautiful in the fractal gallery; see [4, pp. 141–142], [9], [12] and [16]. We illustrate this in Figure 4(a), showing Newton basins of $f(z) = z^4 - 1$.

Since the orbits of the Newton iteration are orthogonal to the level sets of $|f|$, we may gain some insight by superimposing the $|f|$ level sets on the Newton basins. We illustrate this in Figure 4(b) for the function $f(z) = z^4 - 1$. In such a picture, the Julia set includes

• points where the $|f|$ level sets are wild (e.g., along the diagonals of Figure 4(b), near the center), or

• points where the step-lengths $|f(z)|/|f'(z)|$ are big, in which case the Newton orbits may separate, with nearby points bouncing into different basins, and

• points taken by the Newton iteration into the previously mentioned points, e.g., the 4 corners of Figure 4(b).

Historical note: The problem of specifying the Newton basins was posed by Cayley [6] in 1879. Cayley (like Schröder before him) could not go beyond quadratic polynomials because of the chaotic global behavior of the complex Newton method.

The study of iterations with rational functions (such as the Newton iteration (3) for a polynomial) was greatly advanced by Fatou and Julia in 1918–1920; see [3] and the history in [2]. After a hiatus of about 50 years, the subject picked up again in the 1970's, and has been studied intensively ever since; see [3], [4], [9], [19] and references therein.

4 The Halley method

The "complex" Halley method

$$z_{k+1} := z_k - \frac{f(z_k)}{f'(z_k) - \dfrac{f''(z_k)}{2f'(z_k)} f(z_k)} \tag{24}$$

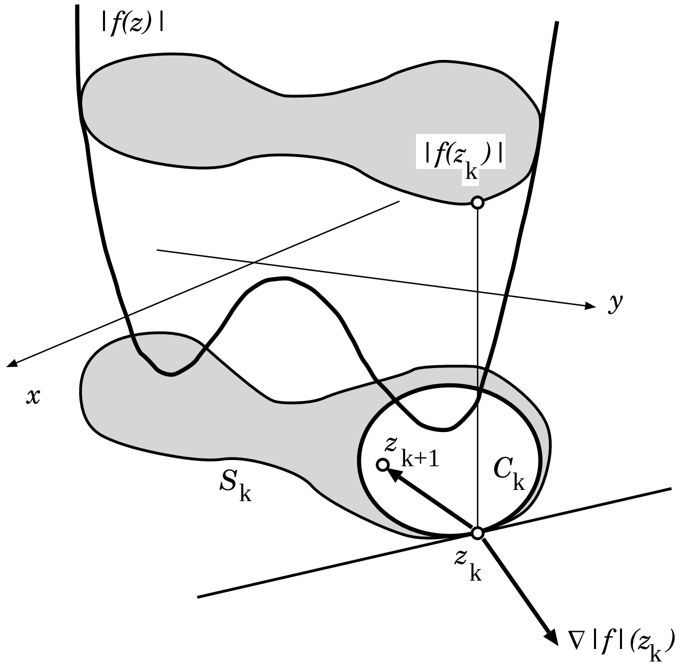


Figure 5: A "top view" of Halley's method: The level set S_k, and its osculating circle C_k at z_k

is interpreted, as in the real case, using the "hyperbola" (6)

$$h(z) := b + \frac{a}{z-c}$$

that is 2nd-order tangent to f at z_k, i.e., the complex numbers a, b, c are determined by the equations (7). We solve these equations to get

$$a = -4\frac{(f_k')^3}{(f_k'')^2}\ , \quad b = f_k - 2\frac{(f_k')^2}{f_k''}\ , \quad c = z_k + 2\frac{f_k'}{f_k''}\ , \tag{25}$$

where f_k, f_k', f_k'' denote their values at z_k. Substituting these values into the zero of h, $\ c - a/b\ $, gives (24). This zero is unique because the "hyperbola" (6) is a *Möbius* (or *linear fractional*) *transformation* [14, § V.3], usually written in the form

$$w = \frac{\alpha z + \beta}{\gamma z + \delta}\ . \tag{26}$$

The Möbius transformation (6) maps c to ∞, z_k to f_k, and z_{k+1} to 0. Therefore, it can be rewritten as

$$w := \frac{(c - z_k)(z - z_{k+1})}{(z_{k+1} - z_k)(z - c)}\, f_k \tag{27}$$

Any line or circle passing through c is mapped by (27) into a line; every other line or circle goes into a circle. In particular, the circle $\{w : |w| = |f_k|\}$ corresponds to

$$\left|\frac{(c - z_k)(z - z_{k+1})}{(z_{k+1} - z_k)(z - c)}\, f_k\right| = |f_k|$$

or

$$\frac{|z - z_{k+1}|}{|z - c|} = \frac{|z_{k+1} - z_k|}{|z_k - c|}\ , \tag{28}$$

which is a line (bisector of the segment $[c, z_{k+1}]$) if the ratio in (28) is 1, and is a circle C_k otherwise. The curvature of the level set

$$S_k = \{(x, y) : F(x, y) = F(x_k, y_k)\} \tag{29}$$

at the point (x_k, y_k) is determined by the values of F, F_x, F_y, F_{xx}, F_{xy}, and F_{yy} at (x_k, y_k). The conditions (7) then imply that S_k and C_k have the same curvature at (x_k, y_k). Since C_k is a circle it must be the *circle of curvature* (or *osculating circle*) of the level set (29) at (x_k, y_k); see Figure 5. We summarize:

Theorem 2. Let the function f be analytic, and let $z_k = x_k + iy_k$ be a point where $f(z_k), f'(z_k)$, and $f''(z_k)$ are nonzero. Let h be the Möbius transformation

(6) $$h(z) := b + \frac{a}{z - c}$$

whose level set $C_k = \{z : |h(z)| = |f(z_k)|\}$ is the osculating circle of $S_k = \{z : |f(z)| = |f(z_k)|\}$ at z_k. The next iterate

$$z_{k+1} := z_k - \frac{f(z_k)}{f'(z_k) - \dfrac{f''(z_k)}{2f'(z_k)} f(z_k)}$$

is the (unique) zero of h. □

Remark 3. *The direction of the Newton step $N_f(z_k) - z_k$ is along $-\nabla|f|(z_k)$, i.e., the Newton step is on the line connecting z_k and the center of the osculating circle of the level set S_k of $|f|$ at z_k.*

To determine the direction of the Halley step we recall that the Halley method for a function $f(z)$ is the same as the Newton method applied to the function

$$g(z) := \frac{f(z)}{\sqrt{f'(z)}} \tag{30}$$

in the sense that $H_f(z) = N_g(z)$ for all z. By Remark 1(a) the Halley step is along $-\nabla|g|(x_k, y_k)$. The gradient of $|g|$ is, by (30),

$$\nabla|g| = \frac{1}{|f'|^{1/2}} \nabla|f| - \frac{|f|}{2|f'|^{3/2}} \nabla|f'| \, ,$$

so the direction of the Halley step is generally different from the direction of the Newton step.

We can show (by analogy with the Newton method, Corollary 1) that the complex Halley method for f is a sequence of real Halley iterations for $|f|$ in vertical planes containing pairs of adjacent iterates (z_k, z_{k+1}).

Exmaple 3. *Figure 6 shows, for $f(z) = z^4 - 1$, the Halley iterates $0.5 + i\,2,\ 0.253 + i\,1.254,\ 0.036 + i$, the associated level sets of $|f|$, and their osculating (dashed) circles.*

Remark 4. *An advantage of writing a Möbius mapping (26) in the form (6) is that the inverse mapping (also Möbius) has the same form, and has the same set of parameters $\{a, b, c\}$ (with b and c transposed):*

$$z := h^{-1}(w) = c + \frac{a}{w - b} \, . \tag{31}$$

A straightforward computation then shows that (31) is a 2nd-order tangent of f^{-1} at $f_k = f(z_k)$, i.e.,

$$\begin{aligned} h^{-1}(f_k) = c + \frac{a}{f_k - b} \quad &= \quad f^{-1}(f_k) = z_k \, , \\ \frac{dh^{-1}}{dw}(f_k) = -\frac{a}{(f_k - b)^2} \quad &= \quad \frac{df^{-1}}{dw}(f_k) \, , \text{ and} \\ \frac{d^2h^{-1}}{dw^2}(f_k) = \frac{2a}{(f_k - b)^3} \quad &= \quad \frac{d^2f^{-1}}{dw^2}(f_k) \, , \end{aligned}$$

so that h^{-1} interpolates f^{-1} in the same way that h interpolates f.

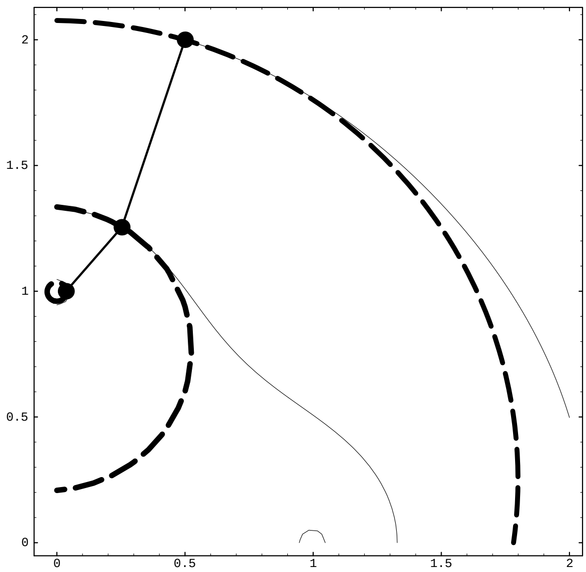


Figure 6: Two Halley iterations for $f(z) = z^4 - 1$, and the osculating circles.

Remark 5. *The global behavior of Halley's method is chaotic, but (as expected) "less chaotic" than Newton's method; compare Figures 4(a) and 7. Figure 7 shows an interesting feature of the Halley method for $f(z) := z^4 - 1$: the Julia set is on the boundaries of circular "half-moons" (comets?). For other polynomials the Julia sets have different shapes, no less pretty than Figure 7.*

For more pictures see our web page at: `http://rutcor.rutgers.edu:80/~bisrael/Halley.html`

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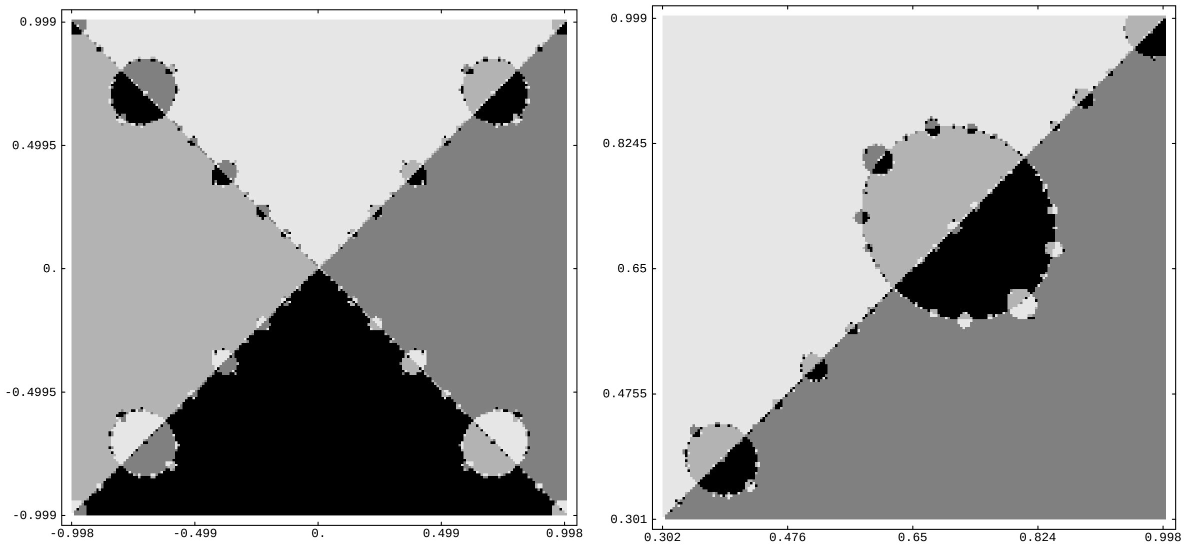

Figure 7: Halley basins for $f(z) = z^4 - 1$

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Operations Research and Computer Science
Columbia University
New York, NY 10027
ty35@columbia.edu

RUTCOR–Rutgers Center for Operations Research
Rutgers University
New Brunswick, NJ 08903-5062
bisrael@rutcor.rutgers.edu