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THE NISHIMORI DECOMPOSITIONS OF CODIMENSION-ONE FOLIATIONS AND THE GODBILLON-VEY CLASSES

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Introduction. It seems to be an interesting problem to study the relations between the geometric behaviour of leaves in codimension-one foliations and their characteristic classes. The first important result in this direction was that of Herman [Her]. He proved that the Godbillon-Vey invariants of foliations by planes vanish. This result was generalized by Morita and Tsuboi to the case of foliations without holonomy ([Mo-T], see also [Mi-T]).

Recently, two remarkable results in this line were obtained. First, Nishimori looked at a certain class of codimension-one foliations (of finite depth and with abelian holonomy in his terminology [Ni 2, 3]), and saw that such foliations admit nice decompositions—so-called SRH-decompositions. By using the decomposition, he proved that the Godbillon-Vey numbers of such foliations are zero if the dimension of the foliated manifold is three [Ni 4].

Secondly, there has been another class of foliations whose qualitative natures were fairly well-known. Recall that a codimension-one foliation is called almost without holonomy if the holonomy group of each non-compact leaf is trivial. For such foliations there is a structure theorem due mainly to Hector [Hec 1] and Imanishi [Im 2]. Let U be a connected component of the complement of the union of compact leaves in a foliation almost without holonomy. There is a homomorphism, called the Novikov transformation, from the fundamental group of U to the group of diffeomorphisms of the real line which describes the qualitative behaviour of each leaf in U ([No], [Im 1, 2, 3], [T 2]). Mizutani, Morita and Tsuboi defined the notion of foliated J-bundles in order to relate this homomorphism to the holonomy groups of compact leaves in the boundary of U and thus to treat functorially foliations almost without holonomy. They proved that the Godbillon-Vey classes of such foliations are all trivial [M-M-T].

Our goal in this paper is to enlarge the above list of foliations with trivial charactristic classes. We consider the class of codimension-one foliations which satisfy the following conditions (P) and (A):

(P) Each leaf has polynomial growth.

(A) The holonomy group of each leaf is abelian.

We say that a foliation is a PA-foliation if it satisfies the above conditions (P) and (A). It is known that the foliations studied by Nishimori [Ni 4] and foliations almost without holonomy are PA [T 2]. Thus the purpose of this paper is two-fold. On the one hand we study the qualitative property of PA-foliations, and on the other hand we prove the vanishing of the Godbillon-Vey classes by using the qualitative theory.

First of all, we prove that there is a finite upper bound for the degrees of growth of leaves of a PA-foliation (Theorem 1). Using the boundedness property, we prove that a PA-foliation admits a certain decomposition similar to Nishimori's SRH-decomposition (Theorem 2). This theorem follows rather easily from known results about polynomial leaves (e.g., [C-C 1, 2, 3], [T 1, 2, 4]), and the method of Nishimori [Ni 4]. So we take the liberty of calling it the (regular) Nishimori decomposition. Then we prove the nullity of the Godbillon-Vey invariants of PA-foliations by using the Nishimori decomposition (Theorem 3). As an application, we see that a transversely analytic foliation has an exponential leaf if its Godbillon-Vey class is non-trivial (Corollary 4). This is our partial answer to Problem 17.3 of [S].

The paper is organized as follows. In §1, we collect some known facts about the qualitative behaviour of non-exponential leaves from the papers of Cantwell and Conlon, Hector and myself. In §2, we prove Theorem 1. First we prove the theorem in the special case where the foliated manifold has a structure of a foliated interval bundle. Then the general case follows by an argument using Dippolito's filtration theorem [D]. In §3, we prove the decomposition theorem (Theorem 2). The proof is parallel to that of Nishimori's SRH-decomposition theorem. In §4, we prove Theorem 3. We observe that a Godbillon-Vey form is restricted to relative cocycles of the units of a decomposition. And we show that the class vanishes in each unit of a Nishimori decomposition. The proof of the latter fact depends heavily on the results of Wallett [W] and Mizutani-Morita-Tsuboi [M-M-T]. In §5, we see how Corollary 4 follows from Theorem 3. In the Appendix, we review and generalize the notion, given in [M-M-T], of foliated J-bundles associated to foliated manifolds whose interior leaves are without holonomy.

In this paper, all manifolds and foliations are smooth (C^∞), although most of the results are valid in the C^2-case. For simplicity we assume all manifolds are oriented and all foliations are transversely oriented.

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1. Preliminaries. In this section we review some terminologies and facts about leaves with polynomial growth from [C-C 1, 2, 3, 4], [Hec 2, 3, 4] and [T 1, 2, 3, 4]. We refer the reader to [C-C 3] for a comprehensive and beautiful exposition.

(1.1) *Growth.* The *growth function* $g_F: \boldsymbol{Z}^+ \to \boldsymbol{R}^+$ of a leaf F is defined as in [P 2], via chains of plaques from a suitable regular cover of the foliated manifold. The (strong) growth type $\mathrm{gr}(F) = \mathrm{gr}(g_F)$ is defined as in [Hec 2] via the equivalence relation of mutual dominance and is independent of all choices.

We say that $\mathrm{gr}(F)$ is *polynomial* if g_F is dominated by a polynomial. It is *exactly polynomial of degree* k if $\mathrm{gr}(g_F) = \mathrm{gr}(m^k)$. It is *non-exponential* if $\liminf_{m\to\infty}(1/m)\log(g_F(m)) = 0$. Otherwise, $\mathrm{gr}(F)$ is *exponential*.

(1.2) *Saturated open sets.* Let M be a closed n-manifold with a foliation $\mathscr{F}$ and fix a one dimensional foliation $\mathscr{L}$ transverse to $\mathscr{F}$. Let $U \subset M$ be an open, connected, $\mathscr{F}$-saturated set. As in [D], let $\hat{U}$ denote the completion of U in any Riemannian metric inherited from M. The set $\hat{U}$ is a manifold whose boundary $\partial\hat{U}$ has finitely many components. The natural immersion $\iota: \hat{U} \to M$ is one-to-one on U and on each connected component of $\partial\hat{U}$, carrying such a component onto a leaf of $\bar{U} - U$. We let $\delta U = \iota(\partial\hat{U})$ (the union of border leaves to U). Foliations $\hat{\mathscr{F}}$ and $\hat{\mathscr{L}}$ are induced by $\mathscr{F}$ and $\mathscr{L}$. By [D, Theorem 1], $\hat{U}$ is the union of a compact *nucleus* K (which is a manifold with corner) and finitely many complete, connected, non-compact *arms* $\hat{U}_i \simeq B_i \times [-1, 1]$, $B_i \subset \partial\hat{U}$ (for which the factors $\{x\} \times [-1, 1]$ are leaves of $\hat{\mathscr{L}}$).

DEFINITION (1.2.1) ([C-C 2, (6.8)], [Im 3, (4.7)]). If the nucleus $K \subset \hat{U}$ can be chosen so that, in each arms $\hat{U}_i \simeq B_i \times [-1, 1]$, $\hat{\mathscr{F}}$ restricts to the product foliation by leaves $B_i \times \{t\}$, then U is said to be *trivial at infinity*. Such a nucleus will be called a *trivializing nucleus*.

The following proposition follows from the proof of [T 2; (6.1)] .

PROPOSITION (1.2.2). *Let U be an open $\mathscr{F}$-saturated set. Suppose $\bar{U} - U$ consists of finitely many leaves with abelian holonomy groups. Then U is trivial at infinity.*

(1.3) *Levels and totally proper leaves.* The theory of levels of

Cantwell-Conlon [C-C 3] and Hector provides us with a systematic treatment of the limit set of an arbitrary leaf. Here we only extract some notions and facts that are used later.

DEFINITION (1.3.1). Let $X \subset M$ be a non-empty $\mathscr{F}$-saturated set. If $\bar{X} - X$ is compact and if each leaf in X is dense in $\bar{X}$, then X is called a *local minimal set* of $\mathscr{F}$.

REMARK. A subset X of M is a local minimal set if and only if there is an open $\mathscr{F}$-saturated set U and $X \subset U$ is a minimal set of the foliation $\mathscr{F}|_U$.

There are three types of local minimal sets;

(a) every proper leaf is a local minimal set;

(b) a connected, open, $\mathscr{F}$-saturated set $U \subset M$, in which each leaf of $\mathscr{F}|_U$ is dense in U, is a local minimal set of *locally dense type*;

(c) a local minimal set of neither type (a) nor type (b) is said to be of *exceptional type*.

DEFINITION (1.3.2). A minimal set of $\mathscr{F}$, and each of its leaves, are said to be at *level* 0. A local minimal set X, and each of its leaves, is at level $k > 0$, if $\bar{X} - X$ consists entirely of leaves at levels at most $k-1$, at least one of which is at level $k-1$.

A leaf belonging to no local minimal set is said to be at infinite level.

THEOREM (1.3.3) ([C-C 3; Theorem (4.0)]). *Each local minimal set is at some finite level.*

DEFINITION (1.3.4). The *substructure* $S(F)$ of a leaf F is the union of all leaves F' such that $F' \subset \bar{F}$ and $\bar{F}' \neq \bar{F}$.

A leaf F is *totally proper* if each leaf in $\bar{F}$ is proper. In [T 4], we used the notion of *depth* $d(F)$ of a leaf F. By (1.3.3), $d(F) = k < \infty$ if and only if F is a totally proper leaf of level k.

A totally proper leaf "spirals in" on leaves at lower levels very finely ([C-C 3; §6], [T 4; Theorem 1]), and as a consequence we get the following.

THEOREM (1.3.5) ([C-C 3; Theorem (6.0)], [T 4; Theorem 2]). *If a totally proper leaf F is at level k (equivalently, $d(F) = k$), then F has exactly polynomial growth of degree k.*

(1.4) *Non-exponential leaves.* The results of Cantwell-Conlon on leaves of non-exponential growth [C-C 1, 2, 3] can be assembled in the following theorem.

THEOREM (1.4.1) ([C-C 3; Theorem (7.0)]). *If F is a leaf with non-exponential growth, then F has totally proper substructure, that is, $S(F)$ is a union of totally proper leaves.*

If, in addition, F is at finite level, then either

(a) *F is a totally proper leaf;*

or

(b) *the set $U = \bar{F} - S(F)$ is an open local minimal set, $\mathscr{F}|_U$ has trivial holonomy, the leaves of $\mathscr{F}|_U$ are mutually diffeomorphic, and these leaves have the same growth type as F.*

Let U be as in (1.4.1, (b)). Then $\bar{U} - U$ consists of finitely many totally proper leaves. As in [T 2, §5], one can define a homomorphism $q: \pi_1(U) \to \mathrm{Diff}(\boldsymbol{R})$ which is called the *Novikov transformation* (see also [Im 1, 2, 3]). Its image $\mathrm{Im}(q)$ acts freely on $\boldsymbol{R}$ and is abelian. The image of q is called the *group of periods* of U [C-C 2]. If the group of periods is not finitely generated, then a leaf in U need not have polynomial growth. It may have exponential growth, or neither exponential nor polynomial growth [C-C 2]. It may have "fractional" or "oscillating" polynomial growth [T 3], [C-C 2, 4].

If the group of periods of U is finitely generated, we define the *rank* of U as the rank of the group of periods.

Suppose U is trivial at infinity. Then the group of periods of U is finitely generated [T 2; (6.1)]. In this case we get the following.

THEOREM (1.4.2) ([T 2; Theorem 2], [C-C 2; (6.10)]). *Let U be an open local minimal set at level k. Assume that $\mathscr{F}|_U$ is without holonomy and U is trivial at infinity. Then each leaf in U has exactly polynomial growth of degree $k + r - 1$, where r is the rank of U.*

PROPOSITION (1.4.3). *A leaf F at infinite level has non-polynomial growth. If a leaf F is at finite level k, then the growth function g_F of F dominates the polynomial m^k.*

PROOF. The first assertion follows from [C-C 3; Proposition 2]. We prove the second assertion. If F is totally proper, then $\mathrm{gr}(g_F) = \mathrm{gr}(m^k)$ from (1.3.5). If the closure of F contains a local exceptional minimal set, then F has exponential growth from (1.4.1). Otherwise, F is contained in an open local minimal set and the substructure $S(F)$ of F is totally proper. In this case, the growth function g_F dominates m^{k+1} from [T 1; Theorem 2].

COROLLARY (1.4.4). *Assume $\mathscr{F}$ is* PA; *that is, each leaf has polynomial growth and the holonomy group of each leaf is abelian. Then* *each leaf of $\mathscr{F}$ has exactly polynomial growth.*

2. Boundedness of growth of leaves. In this section we prove the following theorem.

THEOREM 1. *Let M be a compact manifold and $\mathscr{F}$ a codimension one foliation of M (tangent to the boundary). Assume that $\mathscr{F}$ is* PA, *that is, each leaf of $\mathscr{F}$ has polynomial growth and the holonomy group of each leaf is abelian. Then there is a finite upper bound for the degrees of growth of leaves of $\mathscr{F}$.*

In (2.1), we prove the theorem in the case of foliated interval bundles. In (2.2), we prove it in the general case.

(2.1) *Foliated I-bundle case.* Let I denote the unit interval $[-1, 1]$. Assume that M has a structure of a differentiable I-bundle over a connected manifold B, and $\mathscr{F}$ is transverse to the fibres. Such an object $(M, B, \mathscr{F})$ is called a *foliated I-bundle.* In (2.1), we prove the following.

PROPOSITION (2.1.1). *Let $(M, B, \mathscr{F})$ be a foliated I-bundle over a compact connected manifold B. Assume that $\mathscr{F}$ is* PA. *Then there is a finite upper bound for the degrees of growth of leaves of $\mathscr{F}$.*

Choose a base point b in B, and fix an identification of the fibre over b with I. Then one can define the total holonomy homomorphism $q\colon \pi_1(B, b) \to \mathrm{Diff}(I)$, where $\mathrm{Diff}(I)$ denotes the group of all orientation preserving diffeomorphisms of I. It is well known that the foliated I-bundle $(M, B, \mathscr{F})$ is completely described by this total holonomy homomorphism [Haef]. The image of q is called the *total holonomy group* and is denoted by G. We denote by $\mathrm{Fix}(G)$ the set of fixed points of G. For a point x of I, we denote by F_x the leaf through x, and by Gx the orbit of x by G.

For each non-negative integer k, let M_k denote the union of all leaves at levels at most k. Then M_k is a closed subset of M [C-C 3]. Let U be a connected component of $M - M_k$, J a connected component of $U \cap I$ and F_0 a border leaf to U. Then $\hat{U}$ has the structure of a foliated interval $(=\bar{J})$ bundle over F_0 with the total holonomy group $G_J = \{g|_{\bar{J}}; g \in G, g(J) = J\}$. By (1.2.2), G_J is finitely generated. We say that U is a *type* (A)*-component* if the level of each leaf in U is $k+1$, and that U is a *type* (B)*-component* otherwise.

LEMMA (2.1.2). *If U is a type* (A)*-component, then $\mathscr{F}|_U$ is without holonomy, G_J is abelian and the rank of U is equal to the rank of G_J. If the rank of U is* 1, *then each leaf in U is proper. Otherwise each* *leaf in U is dense in U.*

PROOF. If there is a leaf in U with non-trivial holonomy, then there is a leaf F in U with a contracting holonomy f. Then F is totally proper by (1.4.1), and each leaf which intersects the domain of f is at level $\geqq k+2$, a contradiction. So $\mathscr{F}|_U$ is without holonomy. The remaining assertions are easy to prove.

LEMMA (2.1.3). *Assume U is a type* (B)*-component. Then we have the following.*

(1) *There is $x \in J$ such that F_x is a totally proper leaf of level $k+1$.*

(2) *The action of G_J on Gx is cyclic and is generated by a contraction, that is, there is $f \in G_J$ such that $f|_J < \mathrm{id}_J$ and $Gx = \{f^n(x); n \in \boldsymbol{Z}\}$.*

(3) *Let $G_J^1 = \{g|_{\bar{J}}; g \in G, g(x) = x\}$. Then G_J is a semi-direct product of G_J^1 and the infinite cyclic subgroup of G generated by f.*

(4) *For each $y \in J$ such that F_y is a totally proper leaf of level $k+1$, and for each $g \in G_J^1$, we have $g(y) = y$.*

(5) *The group G_J^1 is finitely generated and non-trivial.*

(6) *The group G_J is non-abelian.*

PROOF. Let F be a leaf of level $> k+1$ in U. Then the substructure $S(F)$ of F is totally proper from (1.4.1), and hence $S(F)$ contains a totally proper leaf F_x at level $k+1$. Since $G_J x$ has no accumulation points other than those in $\partial \bar{J}$, we can write $G_J x = \{x_i; i \in \boldsymbol{Z}\}$ so that $x_i < x_{i+1}$ for each $i \in \boldsymbol{Z}$. Define a map $\alpha: G_J \to \boldsymbol{Z}$ by $g(x_0) = x_{\alpha(g)}$. Then it is easy to see that $g^n(x_i) = x_{i+n\alpha(g)}$ for each i and n, and α is a surjective homomorphism (see e.g., [Ni 1]). Choose $f \in G_J$ satisfying $\alpha(f) = -1$. Then we have $G_J x = \{f^n(x); n \in \boldsymbol{Z}\}$ and $f|_J < \mathrm{id}_J$. Obviously, the homomorphism α defines a split short exact sequence $\{1\} \to G_J^1 \to G_J \xrightarrow{\alpha} \boldsymbol{Z} \to \{0\}$. We have proved the first three statements of the lemma.

The proof of the fourth statement is easy and is omitted.

The group G_J^1 is finitely generated since the open saturated set $U - F_x$ is trivial at infinity from (1.2.2). The group G_J^1 is non-trivial since there is a leaf at level $k+1$. So G_J is non-abelian by Kopell's lemma [K]. Thus we are done.

LEMMA (2.1.4). *For each k, the number of type* (B)*-components of $M - M_k$ is finite.*

PROOF. We prove the lemma by induction on k. Suppose that there are infinitely many type (B)-components U_i $(i = 1, 2, \cdots)$ of $M - M_0$.

For each i, let $J_i = (x_i, y_i)$ denote the open interval $U_i \cap I$. Taking a subsequence of $\{J_i\}$ and reversing the orientation of I if necessary, we may assume that the sequenc $\{J_i\}$ converges to a point x of I from the positive side. Then for large i, the natural homomorphism from $G_{(x,y_i)} = \{g|_{[x,y_i]};\ g \in G\}$ to $\mathrm{hol}_+(F_x)$ is an isomorphism, where $\mathrm{hol}_+(F_x)$ is the one-side holonomy group of the compact leaf F_x. From (2.1.3) the group G_{J_i} is non-abelian for each i. It follows that $\mathrm{hol}_+(F_x)$ is non-abelian. This contradicts our assumption.

Suppose that the number of type (B)-components of $M - M_{k-1}$ is finite and that there are infinitely many type (B)-components of $M - M_k$ ($k > 0$). Then there are a type (B)-component U of $M - M_{k-1}$ and type (B)-components U_i ($i = 1, 2, \cdots$) of $M - M_k$ such that $U_i \subset U$ for each i. Let J be a connected component of $U \cap I$, and $J_i = (x_i, y_i)$ be a connected component of $U_i \cap J$. As before, we may suppose that the sequence $\{J_i\}$ converges to a point x of $\bar{J}$ from the positive side. The level of the leaf F_x is at most k since M_k is a closed subset of M. Let $G_{(x,y_i)} = \{g|_{[x,y_i]};\ g \in G,\ g(x) = x$ and $g(y_i) = y_i\}$. Then from (2.1.3), $G_{(x,y_i)} = \{g|_{[x,y_i]};\ g \in G_J^1\}$ and $G_{(x,y_i)}$ is finitely generated. So, for large i, the natural homomorphism $G_{(x,y_i)} \to \mathrm{hol}_+(F_x)$ is injective. Since G_{J_i} is non-abelian for every i, it follows that $\mathrm{hol}_+(F_x)$ is non-abelian. This contradicts our assumption and we are done.

LEMMA (2.1.5). *The number of type* (B)*-components is finite. In particular, there is a finite upper bound for the levels of all leaves of* $\mathscr{F}$.

PROOF. Assume the contrary. Then, by (2.1.4), there is an infinite sequence $U_1 \supset U_2 \supset \cdots \supset U_i \supset U_{i+1} \supset \cdots$ of type (B)-components. Let F be a leaf contained in $\bigcap_{i=1}^{\infty} \bar{U}_i$. Then F is at infinite level, and hence has non-polynomial growth by (1.4.3). This contradicts our assumption, and the lemma is proved.

LEMMA (2.1.6). *For each k, there is a finite upper bound for the ranks of type* (A)*-components of* $M - M_k$.

PROOF. Let B_k denote the union of type (B)-components of $M - M_k$. And let U_i ($i = 1, \cdots, p$) be the connected components of $M - \bar{B}_k$ (they are finite in number by (2.1.4)). For each i, let J_i be a connected component of $I \cap U_i$. Then $G_{J_i} = \{g|_{\bar{J}_i};\ g \in G,\ g(J_i) = J_i\}$ is finitely generated from (1.2.2). Let J be a connected component of $J_i - \mathrm{Fix}(G_{J_i})$. Then the saturation $\mathrm{Sat}(J)$ of J is a type (A)-component of $M - M_k$. So by (2.1.2), $G_{J_i}|_J$ is abelian. Since this is true for each connected component of $J_i - \mathrm{Fix}(G_{J_i})$, the group G_{J_i} is abelian. Let r be the maximum of the ranks of the finitely generated abelian groups G_{J_i} $(i = 1, \cdots, p)$. Then the rank of each type (A)-component of $M - M_k$ is not greater than r. We have proved the lemma.

PROOF OF PROPOSITION (2.1.1). By (2.1.5) and (2.1.6), there are finite upper bounds k for the levels of all leaves of $\mathscr{F}$ and r for the ranks of all open local minimal sets of $\mathscr{F}$. By (1.3.5) and (1.4.2), each leaf has exactly polynomial growth of degree $\leqq k + r$. Thus we have proved the proposition.

(2.2) *Proof of Theorem* 1. To deduce Theorem 1 from (2.1.1), we use the following theorem of Dippolito.

DIPPOLITO'S FILTRATION THEOREM (2.2.1) [D]. *Let $\mathscr{F}$ be a codimension one foliation of a compact manifold M (tangent to the boundary). Then there exists a finite filtration*

$$M = U_0 \supset U_1 \supset \cdots \supset U_{2m} \supset U_{2m+1} = \varnothing$$

by $\mathscr{F}$-saturated open sets such that for each $i = 0, 1, \cdots, m$,

(a) *$U_{2i} - U_{2i+1}$ is a local minimal set in U_{2i},*

and

(b) *the foliation on $\widehat{U_{2i+1} - U_{2i+2}}$ induced by $\mathscr{F}$ has a structure of a foliated I-bundle.*

PROOF OF THEOREM 1. Let $M = U_0 \supset U_1 \supset \cdots \supset U_{2m} \supset U_{2m+1} = \varnothing$ be the Dippolito's filtration. By (1.4.1), all leaves contained in a local minimal set have the same growth type. So, for each i, all leaves in $U_{2i} - U_{2i+1}$ have the same growth type. Since there is no local exceptional minimal set in M, the number of connected components of $U_{2i+1} - U_{2i+2}$ is finite for each i. Let U be such a component. Obviously, the boundary $\partial\hat{U}$ consists of finitely many totally proper leaves. Let $d = \max\{d(F); F \subset \partial\hat{U}\} < \infty$. By (1.2.2), U is trivial at infinity. Let $K \subset \hat{U}$ be a trivializing nucleus of $\hat{U}$. Then $\hat{\mathscr{F}}|_K$ is a foliated I-bundle and $\hat{\mathscr{F}}|_K$ is PA. By (2.1.1), there are a finite upper bound k for the level of each leaf of $\hat{\mathscr{F}}|_K$ and a finite upper bound r for the rank of each open local minimal set in $\hat{\mathscr{F}}|_K$. Then by (1.3.5) and (1.4.2), $d + k + r$ is an upper bound for the degrees of growth of leaves in $\bar{U}$. Since these components were finite in number, we have proved the theorem.

3. Nishimori decompositions of PA-foliations. In this section, we prove that a PA-foliation admits a Nishimori decomposition (Theorem 2).

(3.1) *Nishimori decompositions.* Let M be a compact connected manifold possibly with corner and $\mathscr{F}$ a codimension one foliation of M.

We assume that the boundary ∂M of M is divided by the corner into two parts, the tangent boundary $\partial_{\tan}M$ and the transverse boundary $\partial_{\mathrm{tr}}M$, and the foliation $\mathscr{F}$ is tangent (resp. transverse) to $\partial_{\tan}M$ (resp. $\partial_{\mathrm{tr}}M$). We call the foliated manifold $(M, \mathscr{F})$ a *unit* if the foliation $\mathscr{F}$ is trivial near $\partial_{\mathrm{tr}}M$.

Let C be a connected component of the corner of M. We say C is *convex* (resp. *concave*) if each point of C has a neighbourhood diffeomorphic to a neighborhood of the origin in $\boldsymbol{R}^n_{++} = \{(x_1, \cdots, x_n);\ x_{n-1} \geqq 0$ and $x_n \geqq 0\}$ (resp. $\boldsymbol{R}^n_{--} = \{(x_1, \cdots, x_n);\ x_{n-1} \leqq 0$ or $x_n \leqq 0\}$, where n is the dimension of M [In]. By definition, each connected component of the corner is either convex or concave.

DEFINITION (3.1.1). Let M be a closed manifold of dimension n, and $\mathscr{F}$ a codimension one foliation of M. A pair (Δ, ϕ), where $\Delta = \{(M_i, \mathscr{F}_i);\ i = 1, \cdots, m\}$ is a finite family of n-dimensional units and ϕ is a foliation preserving immersion from the disjoint union $\bigsqcup_{i=1}^m (M_i, \mathscr{F}_i)$ to $(M, \mathscr{F})$, is called a *decomposition* of $(M, \mathscr{F})$ if the following conditions are satisfied;

(1) for each i, $\phi|_{M_i - \partial_{\tan} M_i}$ is an imbedding,

(2) if $i \neq i'$, then $\phi(\mathrm{Int}\,(M_i)) \cap \phi(\mathrm{Int}\,(M_{i'})) = \emptyset$,

and

(3) $\bigcup_{i=1}^m \phi(M_i) = M$.

Following Nishimori [Ni 4], we define three types of units which are constituents of the Nishimori decomposition.

Let K be a compact manifold with or without boundary, and let N be a codimension one transversely oriented closed submanifold of K which does not separate K. Let $C(K, N)$ denote the manifold with boundary which is obtained from $K - N$ by attaching two copies N_1 and N_2 of N as boundary. Let $f\colon [0, \delta_1] \to [0, \delta_2]$ be a diffeomorphism such that $f(\mathrm{t}) < t$ for any $t \in (0, \delta_1]$. We denote by $X(K, N, f)$ the manifold with corner which is the quotient space of $C(K, N) \times [0, \delta_1]$ by the equivalence relation $\sim$ which is defined by $(x_1, t) \sim (x_2, f(t))$, where $x_1 \in N_1$ and $x_2 \in N_2$ are the same point in N. Let $\mathscr{F}(K, N, f)$ denote the foliation of $X(K, N, f)$ induced by the product foliation $\{C(K, N) \times \{t\}\}$, $t \in [0, \delta_1]$, of $C(K, N) \times [0, \delta_1]$.

DEFINITION (3.1.2). A unit $(M, \mathscr{F})$ is called a *regular staircase* if there are K, N, f as above, and a foliation preserving diffeomorphism h from $(X(K, N, f), \mathscr{F}(K, N, f))$ to $(M, \mathscr{F})$. We call $C(M) = h(C(K, N) \times \{\delta_1\})$, $F(M) = h(C(K, N) \times \{0\})$, $W(M) = h(N_2 \times [\delta_2, \delta_1])$ and $D(M) = h(\partial K \times [0, \delta_1])$, the *ceiling*, the *floor*, the *wall* and the *door* of

$(M, \mathscr{F})$, respectively.

Note that $\partial_{\tan} M = C(M) \cup F(M)$ and $\partial_{\mathrm{tr}} M = W(M) \cup D(M)$. Each connected component of $h(N \times \{\delta_2\})$ is a concave corner, and all other corners are convex.

DEFINITION (3.1.3). A unit $(M, \mathscr{F})$ is called an *abelian room* if it admits a structure of a foliated I-bundle of abelian total holonomy. In this case $D(M) = \partial_{\mathrm{tr}} M$ is called the *door* of $(M, \mathscr{F})$.

Note that each connected component of the corner of an abelian room is convex by definition.

DEFINITION (3.1.4). A unit $(M, \mathscr{F})$ is called a *hall* if the following conditions are satisfied;

(1) each connected component of the corner of M is convex,

(2) each connected component D of $\partial_{\mathrm{tr}} M$ is diffeomorphic to $C \times I$, where C is a connected component of ∂D, and

(3) each leaf except the boundary leaves has trivial holonomy group.

Again we call $D(M) = \partial_{\mathrm{tr}} M$ the *door* of $(M, \mathscr{F})$.

REMARK. Our definition of halls is different from Nishimori's.

DEFINITION (3.1.5). Let $(M, \mathscr{F})$ be a closed foliated manifold. A decomposition $(\Delta = \{(M_i, \mathscr{F}_i); i = 1, \cdots, m\}, \phi)$ is called a *quasi-Nishimori decomposition* if the following conditions are satisfied;

(1) each unit $(M_i, \mathscr{F}_i)$ is either a regular staircase, abelian room or a hall,

(2) for each i, and for each connected component D of the door of $(M_i, \mathscr{F}_i)$, there is a regular staicase $(M_j, \mathscr{F}_j)$ of Δ such that $\phi(D)$ is contained in $\phi(W(M_j))$, and

(3) if $(M_i, \mathscr{F}_i)$, and $(M_j, \mathscr{F}_j)$ are two distinct regular staircases of Δ, then the images of the walls $\phi(W(M_i))$ and $\phi(W(M_j))$ are disjoint.

Let $(\Delta = \{(M_i, \mathscr{F}_i)\}, \phi)$ be a quasi-Nishimori decomposition. We denote by $\mathscr{S}(\Delta)$ the set of regular staircases of Δ. We define a relation $<$ in the set $\mathscr{S}(\Delta)$ as follows. We write $M_i < M_j$ if there is a sequence $M_i = M_{i_0}, M_{i_1}, \cdots, M_{i_a} = M_j$ of elements of $\mathscr{S}(\Delta)$ such that $\phi(W(M_{i_k})) \cap \phi(D(M_{i_{k+1}})) \neq \emptyset$, for $k = 0, \cdots, a-1$.

DEFINITION (3.1.6). A quasi-Nishimori decomposition (Δ, ϕ) is called a (regular) *Nishimori decomposition* if the following conditions are satisfied;

(1) $\mathscr{S}(\Delta)$ has no cycle with respect to the relation $<$, and

(2) for each regular staircase $(M_i, \mathscr{F}_i) \in \mathscr{S}(\Delta)$, the image $\phi(C(M_i))$ of the ceiling of M_i has the trivial holonomy group in $(M, \mathscr{F})$.

Now we can state the main therem of this section.

THEOREM 2. *Let $\mathscr{F}$ be a codimension one foliation of a closed manifold M. Then $\mathscr{F}$ is* PA, *that is, each leaf of $\mathscr{F}$ has polynomial growth and the holonomy group of each leaf is abelian, if and only if $(M, \mathscr{F})$ admits a Nishimori decomposition.*

In (3.2), we prove the "if" part, and in (3.3) we prove the "only if" part. Since the proof goes parallel with the proof of Nishimori's SRH-decomposition theorem [Ni 4], we only sketch it.

(3.2) *Proof of the "if" part of Theorem* 2. Let $(M, \mathscr{F})$ be a closed foliated manifold, and $(\Delta = \{(M_i, \mathscr{F}_i)\}, \phi)$ be a Nishimori decomposition of $(M, \mathscr{F})$. As before, we denote by $\mathscr{S}(\Delta)$ the subset of Δ consisting of regular staircases of Δ.

LEMMA (3.2.1). *For each leaf F of $\mathscr{F}$, the holonomy group* $\mathrm{hol}(F)$ *of F is abelian.*

For a proof, see [Ni 4; Proposition 1].

For an element $(M_i, \mathscr{F}_i)$ of $\Delta - \mathscr{S}(\Delta)$, we denote by U_i the $\mathscr{F}$-saturation of $\phi(M_i - \partial_{\tan} M_i)$, that is, $U_i = \mathrm{Sat}(\phi(M_i - \partial_{\tan} M_i))$.

LEMMA (3.2.2). (1) *The boundary $\partial \hat{U}_i$ consists of finitely many totally proper leaves.*

(2) *The set $\phi(M_i)$ is a trivializing nucleus of $\hat{U}_i$. That is, $\hat{U}_i - \iota^{-1} \circ \phi(M_i)$ is a trivial foliated I-bundle, where $\iota: \hat{U}_i \to M$ is the natural immersion.*

(3) *If $(M_i, \mathscr{F}_i)$ and $(M_j, \mathscr{F}_j)$ are two distinct elements of $\Delta - \mathscr{S}(\Delta)$, then $U_i \cap U_j = \varnothing$.*

(4) $M = \bigcup \{\bar{U}_i; (M_i, \mathscr{F}_i) \in \Delta - \mathscr{S}(\Delta)\}$.

The proof is omitted (see [Ni 4; §2] [T 4; §2]).

COROLLARY (3.2.3). *Each leaf F of $\mathscr{F}$ has polynomial growth.*

PROOF. Assume that F is contained in $U_i = \mathrm{Sat}(\phi(M_i - \partial_{\tan} M_i))$, where $(M_i, \mathscr{F}_i)$ is an element of $\Delta - \mathscr{S}(\Delta)$, and F is non-proper. Then by (3.2.2), F is contained in an open local minimal set without holonomy which is trivial at infinity. So it has polynomial growth from (1.4.2). Otherwise, F is totally proper and hence F has polynomial growth.

By (3.2.1) and (3.2.3), we have proved the "if" part of Theorem 2.

(3.3) *The Nishimori decompositions of* PA-*foliations.*

LEMMA (3.3.1). *Let* $(M, \mathscr{F})$ *be a closed foliated manifold and let* (Δ, ϕ) *be a quasi-Nishimori decomposition of* $(M, \mathscr{F})$. *Suppose there are* $M_i, M_j \in \mathscr{S}(\Delta)$ *such that* $M_i < M_j < M_i$. *Then the leaf through the image* $\phi(F(M_i))$ *of the floor of* M_i *has non-polynomial growth.*

PROOF. It is easy to see that the leaf through $\phi(F(M_i))$ is a non-proper leaf with non-trivial holonomy [T 4; Lemma 2.5]. So it has non-polynomial growth from (1.4.1) and (1.4.4).

REMARK. The leaf in (3.3.1) is, in fact, a resilient leaf (see [C-C 2], [Hec 3]) and hence has exponential growth.

We are going to prove Theorem 2. In the following three lemmas, $(M, \mathscr{F})$ denotes a unit which is immersed in a PA-foliation of a closed manifold.

LEMMA (3.3.2). *Let* $(M, \mathscr{F})$ *be as above. Then there are finitely many abelian rooms* $(M_i, \mathscr{F}_i)$ *and foliation preserving immersions* $\phi_i: (M_i, \mathscr{F}_i) \to (M, \mathscr{F})$ $(k=1, \cdots, m)$ *which satisfy the following conditions.*

(1) *For each* i, $\phi_i|_{M_i - \partial_{\tan} M_i}$ *is an imbedding.*

(2) $\phi_i(M_i - \partial_{\tan} M_i)$*'s are disjoint.*

(3) *For each* i, $\phi_i(\partial_{\mathrm{tr}} M_i)$ *is contained in* $\partial_{\mathrm{tr}} M$.

(4) *Let* M' *be the manifold obtained from* $M - \bigcup \phi_i(M_i)$ *by attaching the boundary, and* $\mathscr{F}'$ *be the induced foliation. Then each compact leaf of* $\mathscr{F}'$ *is isolated.*

For a proof, see [Ni 3; Lemma 9], [Ni 4; p. 18].

LEMMA (3.3.3). *Let* $(M, \mathscr{F})$ *be as above and assume that each leaf which intersects the interior* $\mathrm{Int}(M)$ *of* M *is non-compact. Then there are finitely many regular staircases* $(M_i, \mathscr{F}_i)$ *and foliation preserving imbeddings* $\phi_i: (M_i, \mathscr{F}_i) \to (M, \mathscr{F})$ *which satisfy the following conditions.*

(1) $\phi_i(M_i)$*'s are disjoint.*

(2) *For each* i, *the image of the floor* $F(M_i)$ *is contained in* $\partial_{\tan} M$, *the image of the door* $D(M_i)$ *is contained in* $\partial_{\mathrm{tr}} M$, *the image of the wall* $W(M_i)$ *is contained in the interior of* M *and the image of the ceiling* $C(M_i)$ *is contained in a leaf of finite depth of* $\mathscr{F}$.

(3) *Let* $M' = M - \bigcup \phi_i(M_i - C(M_i) \cup W(M_i))$. *Let* F *be a leaf of* $\mathscr{F}$. *Then* $F \cap M'$ *is connected. If* F *has depth* k *with* $1 \leqq k < \infty$, *then* $F \cap M'$ *is a leaf of depth* $k-1$ *of* $\mathscr{F}|_{M'}$.

PROOF. Let K be a connected component of $\partial_{\tan} M$ such that there is a leaf F of finite depth which contains K in its limit set. Since the holonomy group $\mathrm{hol}(K)$ of K is abelian, and the depth of F is finite, $\mathrm{hol}(K)$ is generated by a contraction and there is a regular staircase whose floor is K and whose ceiling is contained in F [Ni 2].

Let $(M_i, \mathscr{F}_i)$, $\phi_i: (M_i, \mathscr{F}_i) \to (M, \mathscr{F})$ $(i = 1, \cdots, m)$ be all regular staircases obtained in this way. We may suppose $\phi_i(M_i)$'s are disjoint by "thinning" them if necessary (see [Ni 4]). It is easy to see that they satisfy the conditions (2) and (3).

LEMMA (3.3.4). *Let $(M, \mathscr{F})$ be as above. Suppose that each connected component of the corner of M is convex and all interior leaves are non-proper. Then each interior leaf has trivial holonomy group, in other words, $(M, \mathscr{F})$ is a hall.*

This follows directly from (1.4.1).

Now we prove Theorem 2. Let $(M, \mathscr{F})$ be a PA-foliation of a closed manifold. Put $k = \sup\{d(F); F \text{ is a leaf of finite depth of } \mathscr{F}\}$. Then k is finite from Theorem 1. By (3.3.2), (3.3.3) and (3.3.4), one can construct a filtration $M = A_0 \supset B_0 \supset A_1 \supset B_1 \supset \cdots \supset A_k \supset B_k$ of M by compact submanifolds which satisfies the following conditions (here, by abuse of language, we identify a unit and its immersed image in $(M, \mathscr{F})$).

(1) For each i, the closure $\mathrm{Cl}(A_i - B_i)$ of $A_i - B_i$ is a finite union of abelian rooms and halls. For each abelian room or a hall R in $\mathrm{Cl}(A_i - B_i)$, the door $D(R)$ is contained in $\partial_{\mathrm{tr}} A_i$, and $\partial_{\tan} R$ is contained in a leaf of finite depth.

(2) For each i, $\mathrm{Cl}(B_i - A_{i+1})$ is a finite union of regular staircases. For each regular staircase S in $\mathrm{Cl}(B_i - A_{i+1})$, the floor $F(S)$ is contained in a compact leaf of $\mathscr{F}|_{B_i}$, the door $D(S)$ is contained in $\partial_{\mathrm{tr}} B_i$, the wall $W(S)$ is contained in the interior of B_i and the ceiling $C(S)$ is contained in a leaf of finite depth of $\mathscr{F}|_{B_i}$.

(3) If F is a leaf of depth d of $\mathscr{F}$, $1 \leqq d < \infty$, and $F \cap A_i$ is non-empty, then $i \leqq d$ and $F \cap A_i$ is a leaf of depth $d - i$ of $\mathscr{F}|_{A_i}$.

It is easy to see that the collection of abelian rooms, halls and regular staircases in the above filtration constitutes a quasi-Nishimori decomposition of $(M, \mathscr{F})$. As was observed in (3.3.1), the decomposition has no cycle, since all leaves of $\mathscr{F}$ have polynomial growth. By taking 1-thinning of the decomposition (see [Ni 4]) if necessary, we may suppose that the holonomy group of the ceiling of each regular staircase is trivial. We have thus proved Theorem 2.

4. Nishimori decompositions and the Godbillon-Vey classes. In this section, we prove the following theorem.

THEOREM 3. *Let M be a compact manifold and $\mathscr{F}$ a codimension one foliation of M (tangent to the boundary). Assume that $\mathscr{F}$ is* PA, *that is, each leaf of $\mathscr{F}$ has polynomial growth and the holonomy group of each leaf is abelian. Then the Godbillon-Vey class* $\mathrm{gv}(\mathscr{F})$ *of $\mathscr{F}$ is trivial.*

We consider in (4.1) a localization of the Godbillon-Vey class of a codimension one foliation when a nice decomposition is given. And we prove in (4.2) that localized classes vanish in case of a Nishimori decomposition.

(4.1) OGV-*decompositions*. Let $(M, \mathscr{F})$ be a unit. Let ω be a non-singular 1-form on M which defines $\mathscr{F}$ and η a 1-form which satisfies the Frobenius condition $d\omega = \omega \wedge \eta$. Following [F-G-G], we call such a pair (ω, η) a $\mathscr{C}(\mathscr{F})$*-structure*. To a $\mathscr{C}(\mathscr{F})$-structure (ω, η) is associated the *Godbillon-Vey form* $\Omega = \eta \wedge d\eta$. It is easy to see that Ω is closed, and if M is without boundary, the de Rham cohomology class $[\Omega]$ of Ω is independent of the choice of a $\mathscr{C}(\mathscr{F})$-structure [G-V]. The class $[\Omega]$ is called the Godbillon-Vey class of $(M, \mathscr{F})$ and is denoted by $\mathrm{gv}(\mathscr{F})$. If the boundary ∂M of M is non-empty, $\mathrm{gv}(\mathscr{F})$ is defined in the relative cohomology group $H^3(M, \partial M; \boldsymbol{R})$ [M-M-T].

We say that a $\mathscr{C}(\mathscr{F})$-structure (ω, η) is *special* if $d\omega = 0$ and $\eta = 0$ near the transverse boundary $\partial_{\mathrm{tr}}M$. Since we assumed $\mathscr{F}$ is trivial near $\partial_{\mathrm{tr}}M$, there is always a special $\mathscr{C}(\mathscr{F})$-structure.

Let $A^q(M, \mathrm{rel})$ be the set of all smooth q-forms ϕ on M such that $\phi|_{\partial_{\mathrm{tan}}M} = 0$ and $\phi = 0$ near $\partial_{\mathrm{tr}}M$. There is a differential $d\colon A^q(M, \mathrm{rel}) \to A^{q+1}(M, \mathrm{rel})$.

LEMMA (4.1.1). *Let $(M, \mathscr{F})$ be a unit and (ω, η) be a special $\mathscr{C}(\mathscr{F})$-structure. Then the associated Godbillon-Vey form $\eta \wedge d\eta$ is an element of $A^3(M, \mathrm{rel})$. If (ω', η') is another special $\mathscr{C}(\mathscr{F})$-structure, then the difference of the associated Godbillon-Vey forms is exact in the relative complex. That is, there is a 2-form $\xi \in A^2(M, \mathrm{rel})$ such that $d\xi = \eta' \wedge d\eta' - \eta \wedge d\eta$.*

PROOF. The first assertion is obvious. We prove the second assertion. Since the 1-forms ω and ω' define the same foliation $\mathscr{F}$, there is a function $f\colon M \to \boldsymbol{R}^+$ such that $\omega' = f \cdot \omega$. Then $d\omega' = (df + f \cdot \eta) \wedge \omega = (d \log(f) + \eta) \wedge \omega'$. So there is a function g on M for which $\eta' - \eta - d\log(f) = g \cdot \omega$ holds. Put $\xi = \log(f) \cdot d\eta - d\log(f) \wedge g \cdot \omega + g \cdot d\omega$. Then $\xi \in A^2(M, \mathrm{rel})$, and an easy calculation shows $d\xi = \eta' \wedge d\eta' - \eta \wedge d\eta$. Thus we have proved the lemma.

DEFINITION (4.1.2). A unit $(M, \mathscr{F})$ is called an OGV-*unit* if for some special $\mathscr{C}(\mathscr{F})$-structure (ω, η), there is $\xi \in A^2(M, \mathrm{rel})$ such that $d\xi = \eta \wedge d\eta$.

REMARK. If $(M, \mathscr{F})$ is an OGV-unit and (ω', η') is a special $\mathscr{C}(\mathscr{F})$-structure, then there is $\xi' \in A^2(M, \mathrm{rel})$ such that $d\xi' = \eta' \wedge d\eta'$ by (4.1.1).

DEFINITION (4.1.3). Let $(M, \mathscr{F})$ be a closed foliated manifold. A decomposition (Δ, ϕ) is called an OGV-*decomposition* if each unit $(M_i, \mathscr{F}_i) \in \Delta$ is an OGV-unit, and if there is a $\mathscr{C}(\mathscr{F})$-structure (ω, η) such that $d\omega = 0$ near the image of the transverse boundaries $\bigcup \{\phi(\partial_{\mathrm{tr}} M_i); (M_i, \mathscr{F}_i) \in \Delta\}$.

PROPOSITION (4.1.4). *Let $(M, \mathscr{F})$ be a closed foliated manifold. Assume that $(M, \mathscr{F})$ admits an* OGV-*decomposition $(\Delta = \{(M_i, \mathscr{F}_i)\}, \phi)$. Then the Godbillon-Vey class* $\mathrm{gv}(\mathscr{F})$ *of $\mathscr{F}$ is zero.*

PROOF. From the assumption, there is a $\mathscr{C}(\mathscr{F})$-structure (ω, η) such that $d\omega = 0$ and $\eta = 0$ near $\bigcup \{\phi(\partial_{\mathrm{tr}} M_i); (M_i, \mathscr{F}_i) \in \Delta\}$. Let $\omega_i = \phi^*\omega$ and $\eta_i = \phi^*\eta$. Then (ω_i, η_i) is a special $\mathscr{C}(\mathscr{F}_i)$-structure. Since $(M_i, \mathscr{F}_i)$ is OGV, there is a 2-form $\xi_i \in A^2(M_i, \mathrm{rel})$ such that $d\xi_i = \eta_i \wedge d\eta_i$. To prove $\eta \wedge d\eta$ is exact, take a 3-cycle σ in M. Since we are considering real cohomology classes, we may assume that σ is represented by a closed oriented 3-dimensional submanifold N of M. Furthermore we may assume that N is in general position with respect to the decomposition. Then

$$\langle \mathrm{gv}(\mathscr{F}), [N] \rangle = \sum_i \int_{N \cap M_i} \eta_i \wedge d\eta_i = \sum_i \int_{N \cap M_i} d\xi_i = \sum_i \int_{N \cap \partial_{\tan} M_i} \xi_i = 0 ,$$

since $\xi_i \in A^2(M_i, \mathrm{rel})$. This completes the proof.

(4.2) *Nishimori decompositions are* OGV.

LEMMA (4.2.1). *A regular staircase is an* OGV-*unit.*

PROOF. We use the notations of (3.1.2). Let $(X(K, N, f), \mathscr{F}(K, N, f))$ be a regular staircase. Consider the 2-dimensional regular staircase $(S, \mathscr{F}_S) = (X(S^1, \{0\}, f), \mathscr{F}(S^1, \{0\}, f))$ which is defined by the same contraction f as $X(K, N, f)$. From the definition of a regular staircase, there is a C^∞-map $\Phi: X(K, N, f) \to S$ such that (1) Φ is transverse to $\mathscr{F}_S$ and $\Phi^*(\mathscr{F}_S) = \mathscr{F}(K, N, f)$, and (2) near $N \times [f(\delta_1), \delta_1]$, Φ is the projection to the second factor.

Now let (ω_0, η_0) be a special $\mathscr{C}(\mathscr{F}_S)$-structure. Then $\eta_0 \wedge d\eta_0 = 0$, since S is 2-dimensional. Then $(\omega, \eta) = (\Phi^*\omega_0, \Phi^*\eta_0)$ is a special $\mathscr{C}(\mathscr{F}(K, N, f))$-structure from the condition (2) above, and $\eta \wedge d\eta = 0$. The lemma is proved.

LEMMA (4.2.2). *An abelian room is an* OGV-*unit.*

PROOF. Let $(M, \mathscr{F})$ be an abelian room. Then, by definition, M has a structure of a differentiable I-bundle over a compact manifold B and $\mathscr{F}$ is transverse to the fibres. Let $q: \pi_1(B) \to \mathrm{Diff}(I)$ be the total holonomy map. By assumption, the image of q is abelian. Let k be the rank of this abelian group.

ASSERTION (4.2.3). *There are a foliated I-bundle $(E, T^k, \mathscr{F}_1)$ over the k-dimensional torus T^k and smooth maps $f: B \to T^k$ and $\tilde{f}: M \to E$ with the following conditions*:

(1) $\tilde{f}$ *is a foliation preserving bundle map over* f;

(2) f *collapses a neighbourhood of* ∂B *to the base point of* T^k.

PROOF. Consider the following commutative diagram

$$\begin{array}{ccc} \pi_1(B) & \xrightarrow{\quad q \quad} & \mathrm{Diff}(I) \\ & {\scriptstyle q_0}\searrow \quad \nearrow{\scriptstyle q_1} & \\ & \pi_1(T^k) \simeq Z^k & \end{array}$$

where q_1 is the inclusion map $Z^k \simeq \mathrm{Image}\,(q) \subset \mathrm{Diff}\,(I)$. Let $(E, T^k, \mathscr{F}_1)$ be the foliated I-bundle over T^k which is defined by the total holonomy map q_1. Since T^k is a $K(Z^k, 1)$, there is a smooth map $f_0: B \to T^k$ which induces q_0 in the fundamental group. Let L be a connected component of ∂B and $\iota: L \to B$ be the inclusion map. Then the composed map $q_0 \circ \iota_\#: \pi_1(L) \to \pi_1(B) \to \pi_1(T^k)$ is the zero map because the restriction of $\mathscr{F}$ to $\pi^{-1}(L)$ is trivial. So $f_0(L)$ is contractible in T^k. Since this is true for each connected component of ∂B, the map f_0 is homotopic to a smooth map which collapses a neighbourhood of ∂B to the base point of T^k. From the construction, it is easily seen that f is covered by a foliation preserving bundle map $\tilde{f}$. We have proved the assertion.

To prove (4.2.2), we use the following theorem of Wallett and Mizutani-Morita-Tsuboi.

THEOREM (4.2.4) ([W], [M-M-T; Theorem 1]). *Let $(E, T^k, \mathscr{F}_1)$ be a foliated I-bundle over the k-dimensional torus T^k. Then the Godbillon-Vey class* $\mathrm{gv}\,(\mathscr{F}_1)$ *of* $\mathscr{F}_1$ *is zero.*

We return to the proof of (4.2.2). By (4.2.3), we have a foliated I-bundle $(E, T^k, \mathscr{F}_1)$ and a foliation preserving bundle map $\tilde{f}: M \to E$ over f. Choose a $\mathscr{C}(\mathscr{F}_1)$-structure (ω_1, η_1) for $(E, \mathscr{F}_1)$ such that $d\omega_1 = 0$ and $\eta_1 = 0$ near the fibre over the base point of T^k. Then $(\tilde{f}^*\omega_1, \tilde{f}^*\eta_1)$

is a special $\mathscr{C}(\mathscr{F})$-structure, since f collapses a neighbourhood of ∂B to the base point of T^k. By (4.2.4), there is a 2-form $\xi_1 \in A^2(E, \text{rel})$ such that $\eta_1 \wedge d\eta_1 = d\xi_1$. Then $\tilde{f}^*\xi_1 \in A^2(M, \text{rel})$, and $d\tilde{f}^*\xi_1 = \tilde{f}^*\eta_1 \wedge d\tilde{f}^*\eta_1$. This proves (4.2.2).

LEMMA (4.2.5). *A hall is an* OGV-*unit.*

PROOF. Let $(M, \mathscr{F})$ be a hall. To $(M, \mathscr{F})$ one can associate a *foliated J-bundle* $(\boldsymbol{E}, M, \boldsymbol{F})$ over $(M; L^+, L^-)$ where L^+ and L^- are union of connected components of $\partial_{\tan}M$ (see Appendix). The total space $\boldsymbol{E}$ is a manifold with "cubic" corner. There is a neighbourhood Σ of $\partial_{\text{tr}}M$ such that $\boldsymbol{F}$ restricted to $\pi^{-1}(\Sigma)$ is a trivial foliated I-bundle, where $\pi: \boldsymbol{E} \to M$ is the natural projection. There is a section $s: M \to \boldsymbol{E}$ such that $s^*(\boldsymbol{F}) = \mathscr{F}$. The principal total holonomy group of $\boldsymbol{E}$ is fixed point free and abelian. Let k be the rank of the principal total holonomy group.

ASSERTION (4.2.6). *There are open submanifolds N^+ and N^- of the k-dimensional torus T^k which have the homotopy type of union of various subtori, a foliated J-bundle $(\boldsymbol{E}_1, T^k\boldsymbol{F}_1)$ over the triad (T^k, N^+, N^-) and a morphism of foliated J-bundles $\tilde{f}: \boldsymbol{E} \to \boldsymbol{E}_1$ which satisfy the following conditions.*

(1) *The principal total holonomy group of $\boldsymbol{E}_1$ is fixed point free.*

(2) *The restriction of $\tilde{f}$ to $\pi^{-1}(\Sigma)$, $\tilde{f}|_{\pi^{-1}(\Sigma)}: \pi^{-1}(\Sigma) \simeq \Sigma \times I \to \pi^{-1}(b) \simeq I \subset \boldsymbol{E}_1$, is the projection to the second factor, where b is a base point of T^k.*

The proof is similar to that of (4.2.3), and is omitted.

We return to the proof of (4.2.5). Let $(\boldsymbol{E}_1, T^k, \boldsymbol{F}_1)$ and $\tilde{f}$ be as in (4.2.6). Let (ω_1, η_1) be a special $\mathscr{C}(\boldsymbol{F}_1)$-structure for $\boldsymbol{F}_1$ such that $d\omega_1 = 0$ and $\eta_1 = 0$ near $\pi^{-1}(b)$. Then $(\tilde{f}^*\omega_1, \tilde{f}^*\eta_1)$ is a $\mathscr{C}(\boldsymbol{F})$-structure for $\boldsymbol{F}$ and $(\omega, \eta) = (s^* \circ \tilde{f}^*\omega_1, s^* \circ \tilde{f}^*\eta_1)$ is a special $\mathscr{C}(\mathscr{F})$-structure for $(M, \mathscr{F})$, from the condition (2) of (4.2.6). By a theorem of Mizutani-Morita-Tsuboi (see Appendix (A.2)), there is $\xi_1 \in A^2(\boldsymbol{E}_1, \text{rel})$ such that $d\xi_1 = \eta_1 \wedge d\eta_1$. Let $\xi = s^* \circ \tilde{f}^*\xi_1$. Then $\xi \in A^2(M, \text{rel})$ and $d\xi = \eta \wedge d\eta$. Thus we have proved (4.2.5).

PROPOSITION (4.2.7). *A Nishimori decomposition is an* OGV-*decomposition.*

PROOF. Let $(\Delta = \{(M_i, \mathscr{F}_i)\}, \phi)$ be a Nishimori decomposition of a compact foliated manifold. We have just seen that each unit is an OGV-unit. By definition the restricted foliation $\mathscr{F}|_{\cup_i \phi(\partial_{\text{tr}}M_i)}$ is trivial. Note that $\bigcup_i \phi(\partial_{\text{tr}}M_i) = \bigcup \{\phi(W(M_i)); (M_i, \mathscr{F}_i) \in \mathscr{S}(\Delta)\}$, and the latter is the disjoint union. Since the image of the ceiling $C(M_i)$ of each regular staircase $(M_i, F_i) \in \mathscr{S}(\Delta)$ has the trivial holonomy group, the holonomy groups of $\phi(\partial W(M_i)) = \phi(W(M_i) \cap C(M_i))$ are trivial. So there is a 1-form ω which defines $\mathscr{F}$ and which is closed near $\bigcup_i \phi(\partial_{tr} M_i)$. We have thus proved the proposition.

PROOF OF THEOREM 3. Theorem 3 now follows directly from Theorem 2, (4.1.4) and (4.2.7).

5. Remarks on the tranversely analytic case. If a foliation $\mathscr{F}$ is transversely analytic, then the leaves of $\mathscr{F}$ do not exhibit pathological growth types.

THEOREM (5.1). *Let $\mathscr{F}$ be a transversely analytic codimension one foliation of a closed manifold. Then there are integers d and r with the following property. Let F be a leaf of non-exponential growth of $\mathscr{F}$. Then either F is totally proper of level $\leqq d$ or F is contained in an open local minimal set without holonomy which is trivial at infinity and whose rank is $\leqq r$. In particular, each non-exponential leaf of $\mathscr{F}$ has exactly polynomial growth of degree $\leqq d + r$.*

This theorem was conjectured by Hector [Hec 2], and is announced by Cantwell-Conlon [C-C 2; (6.13)]. A proof of (5.1) and related results will appear elsewhere.

From a theorem of Hector [Hec 4], one gets the following (see [T 2; (9.1)]).

PROPOSITION (5.2). *Let $\mathscr{F}$ be a transversely analytic codimension one foliation of a closed manifold. Let F be a leaf of $\mathscr{F}$ such that the holonomy group* hol (F) *of F is non-abelian. Let F' be a leaf which contains F in its limit set. Then F' has exponential growth.*

COROLLARY (5.3). *Let $\mathscr{F}$ be a transversely analytic codimension one foliation of a closed manifold. Assume that each leaf of $\mathscr{F}$ has non-exponential growth. Then $\mathscr{F}$ is* PA, *that is, each leaf has polynomial growth and the holonomy group of each leaf is abelian.*

From Theorem 3 and (5.3), we get the following corollary.

COROLLARY 4. *Let $\mathscr{F}$ be a transversely analytic codimension one foliation of a closed manifold. Assume that the Godbillon-Vey class* gv $(\mathscr{F})$ *of $\mathscr{F}$ is non-trivial. Then $\mathscr{F}$ contains a leaf of exponential growth.*

COROLLARY (5.4). *Let M be a closed manifold. Assume that M admits a transversely analytic codimension one foliation with the non-*

trivial Godbillon-Vey class. Then the fundamental group of M has exponential growth.

This follows from Corollary 4 and a theorem of Plante [P 1].

Appendix. *Foliated J-bundles associated to halls.* In this appendix, we recall from [M-M-T], the notion of foliated J-bundles. Let Diff $(\mathring{I})$ denote the group of all orientation preserving diffeomorphisms of the open interval $\mathring{I}=(-1, 1)$, and let Diff (I_+) (resp. Diff (I_-)) be the group of all orientation preserving diffeomorphisms of the half open interval $I_+=(-1, 1]$ (resp. $I_-=[-1, 1)$). Suppose that we are given a connected CW-complex K and its subcomplexes L^+ and L^-.

DEFINITION (A.1). A family of foliated bundles and foliated bundle isomorphisms $(E; E^+, E^-; b^+, b^-)$ is said to be a *foliated J-bundle* over the triad $(K; L^+, L^-)$ if the following conditions are satisfied;

(1) E is a locally trivial foliated $\mathring{I}$-bundle, E^+ is a locally trivial foliated I_+-bundle and E_- is a locally trivial foliated I_--bundle over K, L^+ and L^-, respectively, and

(2) b^+ and b^- are isomorphisms of foliated $\mathring{I}$-bundles

$$b^+\colon \mathring{E}^+ \to E|_{L^+}\,, \qquad b^-\colon \mathring{E}^- \to E|_{L^-}$$

where $\mathring{E}^+$ (resp. $\mathring{E}^-$) is the associated foliated $\mathring{I}$-bundle to E^+ (resp. E^-).

Let $\boldsymbol{E}$ be the space obtained from the disjoint union of E, E^+ and E^- by identifying $\mathring{E}^+$ (resp. $\mathring{E}^-$) with $E|_{L^+}$ (resp. $E|_{L^-}$) by the isomorphism b^+ (resp. b^-). We have a natural projection $\pi\colon \boldsymbol{E}\to K$ and we call $\boldsymbol{E}$ the total space of the foliated J-bundle $(E; E^+, E^-; b^+, b^-)$. The space $\boldsymbol{E}$ has a "foliation" $\boldsymbol{F}$ which is transverse to the fibres of π, and sometimes we call the triple $(\boldsymbol{E}, K, \boldsymbol{F})$ a foliated J-bundle. We can consider that E, E^+ and E^- are subspaces of $\boldsymbol{E}$. Hereafter, $\boldsymbol{E}-E$ is denoted by $\hat{\partial}\boldsymbol{E}$.

Morphisms of foliated J-bundles and cross sections of foliated J-bundles are defined naturally.

If we replace $(K; L^+, L^-)$ by a homotopy equivalent manifold triple $(M; N^+, N^-)$, where $\partial M=\varnothing$ and N^+ and N^- are open submanifolds of M, then the corresponding total space $\boldsymbol{E}$ has a structure of a manifold with boundary $\hat{\partial}\boldsymbol{E}=\boldsymbol{E}-E$. The foliated J-bundle structure on $\boldsymbol{E}$ induces a codimension one foliation $\boldsymbol{F}$ on $\boldsymbol{E}$, tangent to the boundary $\hat{\partial}\boldsymbol{E}$. Thus we can speak of its Godbillon-Vey class gv$(\boldsymbol{F})$, which is an element of $H^3_{DR}(\boldsymbol{E}, \hat{\partial}\boldsymbol{E}; \boldsymbol{R})\simeq H^3(K\times I, L^+\times\{1\}\cup L^-\times\{-1\}; \boldsymbol{R})$. The Godbillon-Vey class is natural with respect to the morphisms of foliated J-bundles.

The foliation of E is defined by the total holonomy homomorphism $q: \pi_1(K, b) \to \mathrm{Diff}(\mathring{I})$, where b is a base point of K. We call q the *principal total holonomy homomorphism*, and the image of q the *principal total holonomy group* of the foliated J-bundle $(\boldsymbol{E}, K, \boldsymbol{F})$.

Now a theorem of Mizutani-Morita-Tsuboi can be stated as follows.

THEOREM (A.2) [M-M-T; Lemma 2 and Lemma 5]. *Assume that the principal total holonomy group is fixed point free. Then the Godbillon-Vey class* $\mathrm{gv}(\boldsymbol{F})$ *of the foliated J-bundle* $(\boldsymbol{E}, K, \boldsymbol{F})$ *is zero.*

Let $(M, \mathscr{F})$ be a hall. Choose a non-singular flow ϕ which is transverse to $\mathscr{F}$ and tangent to $\partial_{\mathrm{tr}}M$. To $(M, \mathscr{F}, \phi)$, one can canonically associate a foliated J-bundle.

THEOREM (A.3) (cf. [M-M-T; Theorem 5]). *Let* $(M, \mathscr{F}, \phi)$ *be as above. Then there exist a foliated J-bundle* $(\boldsymbol{E}, M, \boldsymbol{F})$ *and a cross section s which satisfy the following conditions.*

(1) *The cross section s is transverse to* $\boldsymbol{F}$ *and* $s^*(\boldsymbol{F}) = \mathscr{F}$.

(2) *The total space* $\boldsymbol{E}$ *has a structure of a "manifold with cubic corner", that is, each point of* $\boldsymbol{E}$ *has a neighbourhood which is diffeomorphic to an open subset of* $\boldsymbol{R}^{n+1}_{+++} = \{(x_1, \cdots, x_{n+1}) \in \boldsymbol{R}^{n+1}; x_{n-1} \geqq 0,\ x_n \geqq 0$ *and* $x_{n+1} \geqq 0\}$, *where n is the dimension of M.*

(3) *The principal total holonomy group of* $(\boldsymbol{E}, M, \boldsymbol{F})$ *is fixed point free.*

(4) *Let* $\pi: \boldsymbol{E} \to M$ *be the projection. There is a collar neighbourhood* Σ *of* $\partial_{\mathrm{tr}}M$ *such that* $\boldsymbol{F}|_{\pi^{-1}(\Sigma)}$ *is trivial, that is,* $\pi^{-1}(\Sigma)$ *is diffeomorphic to* $\Sigma \times I$ *and the foliation* $\boldsymbol{F}|_{\pi^{-1}(\Sigma)}$ *is the product foliation by leaves* $\Sigma \times \{t\}$, $t \in I$.

Indication of proof (see [M-M-T; §3]). Define $\boldsymbol{E}'$ to be the domain of ϕ; $\boldsymbol{E}' = \{(t, x) \in \boldsymbol{R} \times M;\ \phi(t, x) \text{ is defined}\}$. It is easy to see that $\boldsymbol{E}'$ is an $(n+1)$-manifold with cubic corner. The manifold $\boldsymbol{E}'$ has a foliation $\boldsymbol{F} = \phi^*(\mathscr{F})$. Let $\pi: \boldsymbol{E}' \to M$ denote the natural projection. We have a natural section $s: M \to \boldsymbol{E}'$ which is defined by $s(x) = (0, x) \in \boldsymbol{E}' \subset \boldsymbol{R} \times M$, $x \in M$. This section is transverse to $\boldsymbol{F}$ and $s^*(\boldsymbol{F}) = \mathscr{F}$.

To obtain $\boldsymbol{E}$, we write $\partial_{\tan}M$ as a union of leaves

$$\partial_{\tan}M = L_1^+ \cup \cdots \cup L_k^+ \cup L_1^- \cup \cdots \cup L_l^-$$

where ϕ is directed outwards on $L^+ = \bigcup L_i^+$ and inwards on $L^- = \bigcup L_j^-$ (Note that this decomposition is possible since each connected component of the corner of M is convex). Let $\hat{\partial}\boldsymbol{E}' = \{(t, x) \in \boldsymbol{E}';\ \phi(t, x) \in \partial_{\tan}M\}$. Put $\boldsymbol{E} = \boldsymbol{E}' - \hat{\partial}\boldsymbol{E}'$, $E^+ = E|_{L^+} \cup s(L^+)$ and $E^- = E|_{L^-} \cup s(L^-)$. Then one can show that $(E; E^+, E^-)$ is a foliated J-bundle over (M, L^+, L^-) and the principal total holonomy group of E is fixed point free, by using a result of Imanishi [Im 1].

Since the foliation $\boldsymbol{F}$ is trivial near $\partial_{tr}M$, there is a collar neighbourhood Σ of $\partial_{tr}M$ which satisfies the condition (4) of the theorem.

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