The noisy Hegselmann-Krause model for opinion dynamics

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In the model for continuous opinion dynamics introduced by Hegselmann and Krause, each individual moves to the average opinion of all individuals within an area of confidence. In this work we study the effects of noise in this system. With certain probability, individuals are given the opportunity to change spontaneously their opinion to another one selected randomly inside the opinion space with different rules. If the random jump does not occur, individuals interact through the Hegselmann-Krause's rule. We analyze two cases, one where individuals can carry out opinion random jumps inside the whole opinion space, and other where they are allowed to perform jumps just inside a small interval centered around the current opinion. We found that these opinion random jumps change the model behavior inducing interesting phenomena. Using pattern formation techniques, we obtain approximate analytical results for critical conditions of opinion cluster formation. Finally, we compare the results of this work with the noisy version of the Deffuant et al. model for continuous-opinion dynamics.

PACS numbers:

I. INTRODUCTION

In a social system, the opinion of the individuals determines the character of their mutual interactions. But at the same time, the formation and subsequent evolution of people's opinion are complex phenomena affected by affinities and contracts between the members of the society. This complex behavior is specially observed in situations when a common decision needs to be taken by the individuals. During such a cooperative task it usually happens that either a single position emerges or the population evolves to a state of coexistence of different opinions. It is natural to talk about those processes within the framework of interacting particles, this being one of the reasons why nowadays many physicists address the study of opinion formation in large groups using ideas borrowed from statistical physics and nonlinear science [1, 2]. The introduction of new information-communication technologies and the availability of large data sets have also contributed to develop this interdisciplinary research field.

In recent years, two models where the opinion of an individual can vary continuously have raised the interest of the scientific community [3, 4]. Such continuous models have been introduced independently by Deffuant and collaborators (DW Model) [5] and Hegselmann and Krause (HK model) [6–9]. The two models implement the so-called bounded confidence mechanism by which two individuals only influence each other if their opinions differ less than some given amount [10, 11]. Another common important ingredient of both models is an agreement mechanism, by which individuals that satisfy the bounded confidence condition adjust their opinions towards an average value. The fundamental difference between the models is materialized in the definition of who communicates with whom at once [12]. In the DW model, two randomly chosen individuals meet and a pairwise averaging is implemented, while there is an extra parameter that controls how fast the opinions converge [13, 14]. This model is suitable to describe situations where individuals meet in small groups and exchange information face-to-face. In the HK model, the communication takes place in large groups and individuals move their own opinions to the average opinion of all individuals which lie in the area of confidence.

Although one expects considerable differences between the two models when the number of individuals is large, it has been well established that they always lead to a final state in which either perfect consensus is reached or the population splits into a set of opinion clusters each of them holding exactly the same opinion [3, 15]. However, in real social systems, public opinion does not reach such ideal states of complete consensus. In this regard and with the aim to make models of continuous opinion dynamics more realistic, recent works have introduced additional elements of randomness to the DW model. This new ingredient has been interpreted as a "self-thinking" or "free-will", where individuals change their opinion in a random way [16, 17], as the death of an individual and the birth of a new one [18, 19], or simply, as the replacement of individuals by new ones in systems where the total size is not fixed [20].

Nowadays, with the introduction of new information-communication technologies, an effective global exchange of information in large groups is easily achieved. In this sense, we believe that the HK model deserves more at-

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tention, particularly when a sort of randomness is added to the original rules [21]. Following this motivation, in this paper we generalize the HK dynamical rules to incorporate additional random elements, or "noise". Our aim is to analyze which aspects of the original dynamics are robust against noise and which additional complex collective phenomena can emerge as a result. In our generalization, individuals are allowed to change spontaneously their opinion with certain probability [16, 17]. If this random jump does not occur, individuals can then perform interactions through the HK's rules. We analyze two cases of noise that have been already successfully implemented in the DW model [16, 17]: In the first case, individuals are allowed to perform opinion random jumps to any point in the full opinion space, while in the second case, individuals can perform a random jump in their opinion to a new value located inside a small interval centered around the current opinion. We show that these new ingredients are able to induce novel phenomena in the HK model. In both cases, we have found an order-disorder transition above a critical value of the noise intensity. In the disordered state the opinion distribution tends to be uniform, while for the ordered state, a set of noisy opinion clusters are formed. Using a linear stability analysis we derive approximate conditions for the stability of noisy opinion clusters. Our analytical results are in qualitative agreement with Monte Carlo simulations.

The next section presents the HK model in the presence of noise. Section 3 contains extensive results on the model behavior obtained by Monte Carlo simulations. The order-disorder transition is analyzed through a linear stability analysis in Section 4. Section 5 is devoted to compare the noisy HK model with the noisy DW model. Conclusions are presented in Section 6.

II. THE NOISY HEGSELMANN-KRAUSE MODEL

The original HK model was introduced as a nonlinear extension of previous models of social influence [6, 7, 22]. In this section, we consider a modification of the model in which noise is added to the original HK rules, resulting in a random change of an individual's opinion. To begin the analysis, let us consider a system composed by N individuals ($i = 1, ..., N$). At (discrete) time n each individual i is endowed with a continuous opinion x_i^n, taking values in a continuous one-dimensional interval $x_i^n \in [0, L]$, where L is the range of opinion space. At time-step n a randomly chosen individual i has a probability m of spontaneously changing his opinion to a new random value, and a probability $1-m$ to move to the average opinion of all individuals (including himself) which lie in his interval of confidence of width 2ϵ. The case $m = 0$ corresponds to the standard HK model, in which the opinion of the individual i, at the next step $n+1$, is given by

$$x_i^{n+1} = \frac{\sum\limits_{j:|x_i^n - x_j^n| \le \epsilon} x_j^n}{|\{j : |x_i^n - x_j^n| \le \epsilon\}|} \,, \qquad (1)$$

where the sum is over the individuals j whose opinions differ from x_i^n by at most ϵ, and $|\{j : |x_i^n - x_j^n| \le \epsilon\}|$ is the number of such individuals. The procedure is repeated by selecting at random another individual and so on [8, 23]. The parameter ϵ, which runs from 0 to L, is the confidence parameter. We introduce the time variable $t = n/N$ measuring the number of Monte Carlo steps (MCS), or the number of opinion updates per individual.

As far as the range of the random jumps (the maximum interval in which individuals can change spontaneously their opinions) is concerned, we distinguish two simple scenarios:

(1) Unlimited random jumps to any point inside the interval $[0, L]$, meaning that the new opinion x_i^{n+1} can take any value in the whole opinion space $[0, L]$ [16].

(2) Bounded random jumps inside the interval $[-\gamma, \gamma]$, with $\gamma \le L$. i.e. the new opinion x_i^{n+1} will lie in the interval $(x_i^n - \gamma, x_i^n + \gamma)$ [17].

In both scenarios, the new random value is adopted uniformly within the allowed interval. In the second case, it is possible that opinions leave the bounded opinion space $[0, L]$. To avoid this problem, we will consider *adsorbing boundary conditions* in which opinions that try to go away towards the left or towards the right of the interval $[0, L]$ are set to 0 and L, respectively. The more convenient from the mathematical point of view *periodic boundary conditions*, where the opinion space $[0, L]$ is considered to be wrapped on a circle, will be also considered in particular cases as properly mentioned. For each particular case, the type of final configurations reached by the system will depend on the values of the threshold ϵ, the noise intensity m, and/or the parameter γ. Although we will keep the notation L when referring to the range of opinion space, all the results of this paper are for $L = 1$. Results for other L values can be easily translated from ours by making the rescaling $\epsilon \to \epsilon/L$ and $\gamma \to \gamma/L$.

This noisy HK model can be described in terms of an approximate density-based master equation for the probability density $P(x, t)$ that an individual holds opinion x at time t. This equation can be written as

$$\frac{\partial P(x,t)}{\partial t} = (1-m)\left[\int_L dx_1 P(x_1,t)\left(\delta(x - \langle x\rangle_{x_1}) - \delta(x - x_1)\right)\right] + m\left[G(x,t) - P(x,t)\right], \qquad (2)$$

where $\langle x\rangle_{x_1}$ is the average position of the individuals within distance ϵ of opinion x_1, i.e.

$$\langle x \rangle_{x_1} = \frac{\int_{x_1-\epsilon}^{x_1+\epsilon} uP(u,t)du}{\int_{x_1-\epsilon}^{x_1+\epsilon} P(u,t)du}. \qquad (3)$$

In this average, the denominator is the normalization by the probability mass in the interval $[x_1 - \epsilon, x_1 + \epsilon]$ while the numerator is the first moment in that interval. In Eq. (2) the term proportional to m describes the random jumps, whereas the one proportional to $(1 - m)$ represents the original HK rules. For unlimited random jumps, the function $G(x,t)$ is the homogeneous distribution $P_h(x,t) = 1/L$ [16], whereas for bounded random jumps with adsorbing boundary conditions [17],

$$G(x,t) = \begin{cases} \delta(x) \int_0^{\gamma} dx' \, \frac{\gamma - x'}{2\gamma} P(x',t) \\ + \int_0^{x+\gamma} \frac{dx'}{2\gamma} P(x',t), & \text{if } x \leq \gamma, \\ \\ \int_{x-\gamma}^{x+\gamma} \frac{dx'}{2\gamma} P(x',t), & \text{if } \gamma \leq x \leq L - \gamma, \\ \\ \delta(x-L) \int_{L-\gamma}^{L} dx' \, \frac{-L+\gamma+x'}{2\gamma} P(x',t) \\ + \int_{x-\gamma}^{L} \frac{dx'}{2\gamma} P(x',t), & \text{if } x \geq L - \gamma. \end{cases} \qquad (4)$$

Before we continue with the analysis, let us summarize some of the most relevant features observed in the original noiseless HK model ($m = 0$) [23]. Eq. (2) with $m = 0$ provides a mean-field description (in the sense that correlations between agents' opinions have been neglected) of the process of selecting a random individual and changing his opinion to the average of the individuals in a neighborhood of size 2ϵ. Starting from uniformly distributed random opinions, Monte Carlo simulations show that for $\epsilon > 0$ the system either reaches a final state of complete consensus or splits into a number of opinion clusters separated by a distance larger than ϵ. In the case of $L = 1$ and uniform initial distribution of opinions, $P(x, t = 0) = 1$ for $x \in [0, 1]$ and $P(x, t = 0) = 0$ otherwise, the result given by the master equation is that for $\epsilon \geq 0.19$ only a big cluster emerges and the steady state distribution is $P_\infty(x) = \lim_{t\to\infty} P(x,t) = \delta(x - 1/2)$, whereas for smaller values of ϵ a series of bifurcations and nucleation of clusters occur. In this clustering regime it is found that $P_\infty(x) = \sum_{i=1}^{n_c} m_i \delta(x - x_i)$ with $|x_i - x_j| > \epsilon$ for all $i \neq j$ and $\sum_{i=1}^{n_c} m_i = 1$, where n_c is the number of opinion clusters, x_i is the position of a cluster and m_i its mass. Unlike other bounded confidence models, the noiseless HK model evolving from uniform initial conditions does not exhibit the so-called minor or low-populated clusters at the extreme and between high populated clusters [3, 23]. These minor cluster can appear when starting from more asymmetric initial conditions.

III. MONTE CARLO SIMULATIONS

It is a well-known fact that in continuous opinion dynamics the master equation and the Monte Carlo simulations do not always agree due to finite-size induced fluctuations and to having neglected the correlations between agents. In this section, we present the main phenomena obtained from Monte Carlo simulations with a finite system of N individuals and initial conditions randomly and uniformly distributed in the opinion space interval $[0, L]$.

A. Unlimited random jumps

In this subsection, we will analyze the impact of unlimited opinion random jumps on the original HK model. As it was mentioned above, a randomly chosen individual can change with probability m his opinion to a random opinion inside the full interval $[0, L]$. Otherwise, with probability $1 - m$ the individual interacts with their compatible neighbors following the HK's rule. We will show that the interplay between the confidence parameter ϵ and the noise intensity m induces very interesting phenomena.

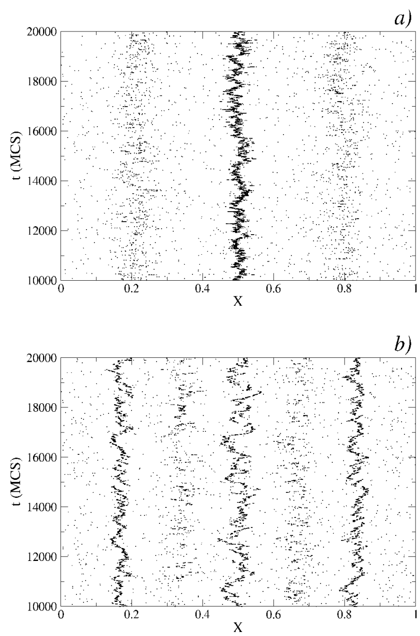

FIG. 1: Time series in opinion space for $m = 0.02$. a) The case $\epsilon = 0.27$, where only one big cluster is formed when $m = 0$. b) The case $\epsilon = 0.127$, where only three big clusters are formed when $m = 0$. The number of individuals is $N = 1000$, but only 100 randomly chosen among them are plotted to avoid saturation of the plot. Note the formation of low-populated opinion clusters at the extremes and between high populated clusters when $m > 0$. The opinion space runs from 0 to $L = 1$. The initial condition at $t = 0$ was uniform in $[0, 1]$ and data starts to be plotted after long enough simulation time.

One of the most distinct features of the noiseless HK model [3, 23] is the lack of low-populated opinion clusters at the extremes and between high populated clusters when the initial condition is uniform in opinion space. The absence of this class of minor clusters, which are typically observed in other models of continuous opinion dynamics, is a consequence of the fully connected and mutual convergence of all the individuals since the very beginning. In other continuous opinion dynamics systems, like the DW model, the interaction is between randomly chosen pairs of individuals and therefore some opinions are not able to interact enough times to enter the basin of attraction of the big clusters. Nevertheless, when noise is introduced, one notices in the HK model the appearance of low-populated clusters for certain values of ϵ. For example, Fig. 1 shows time series of the opinions from Monte Carlo simulations for values of ϵ such that only one (panel a) or three (panel b) clusters are formed when $m = 0$. Figure 1(a) shows that for $m > 0$ a pattern of three opinion clusters is established. The two extreme clusters are low populated and the central one is composed by the vast majority of agents. Figure 1(b) shows a similar case but for a lower value of ϵ. In this case, it is clear that low-populated opinion clusters also appear between clusters with higher populations. Under this type of noise the whole opinion space can be covered and therefore low-populated clusters have more chance to be established out of the range of interaction of highly populated clusters. In fact, they start to increase their population when increasing the noise intensity m.

Similarly to [16], we report a bistable behavior for narrow bands of ϵ near the bifurcation transitions between one stable configuration and the next one. As is typical in bistable situations, we observe that the inherent fluctuations of a finite-size system induce transitions between one state and back. These jumps are, for instance, observed in Monte Carlo simulations for $\epsilon = 0.242$ near the transition for one big cluster to two big ones. Figure 2(b) shows several jumps between both states. Also note that low-populated clusters always exist and play a key role in the transitions [see Figs. 2(b) and 2(c)].

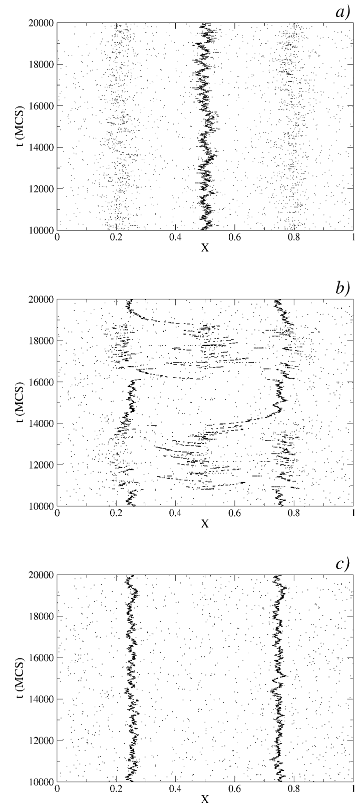

FIG. 2: Time series in opinion space for unlimited jumps at three values of ϵ for $m = 0.02$ and $N = 1000$ (only 100 agents are plotted to avoid saturation of the plot). At $\epsilon = 0.270$ (panel a) a single high populated cluster dominates over two lateral low-populated clusters. At $\epsilon = 0.230$ (panel c) two polarized opinion cluster appear. At $\epsilon = 0.242$ (panel b) the system randomly jumps between these two states. The panels represent values of the confidence parameter ϵ for which the noiseless HK model ($m = 0.0$) is near a transition from one big cluster to two big ones. Note also in (a) the formation of low-populated extreme opinion clusters that play an important role for jumps. The opinion space runs from 0 to $L = 1$ and data starts to be plotted after long enough simulation time.

B. Bounded random jumps

We now allow individuals to perform, with probability m, jumps limited to the interval $[-\gamma, \gamma]$ centered around their current opinion. We found that, when adsorbing boundary conditions are considered, noisy opinion clusters still form for small and moderate noise intensity m. However, for γ small the clusters do not form symmetric patterns around the mean opinion 0.5. Instead, the centers of mass of each one of them perform a random walk along the whole opinion space until eventually they collide to form only one big opinion cluster. Figure 3 shows the successive merging of clusters occurring after collisions

For large values of γ, a stable pattern of opinion clusters with a reduction of their wandering is observed. Under these conditions, one can also find regions of bistability where the inherent fluctuations of our finite system take the system from one state to another. For the case presented in Fig. 4, transitions back were not found even for very long simulation times. The figure just shows an early jump from a state of a big opinion cluster and two smaller ones to a state of two big opinion clusters.

IV. ORDER-DISORDER TRANSITIONS

In many systems, one of the main effects of noise is to induce an order-disorder transition. In this sense, opinion dynamics is not the exception [16–18, 24, 25]. In general we expect that when in our noisy model the intensity m is larger than a critical value m_c, the patterns of opinion would become blurred such that the corresponding maxima of the distributions $P(x, t)$ are not evident, implying the destruction of opinion clusters and the establishment of a highly homogeneous state far from the boundaries.

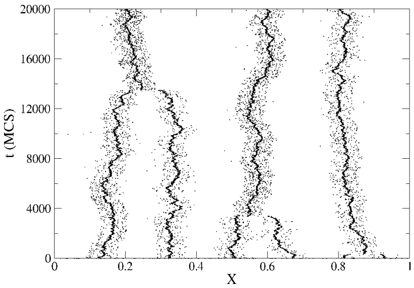

FIG. 3: Time series of the opinion distribution for bounded jumps with $\epsilon = 0.05$, $\gamma = 0.04$, and $m = 0.05$. Opinions form clusters that execute random walks, and successive merging of clusters occurs after collision. At very long time (not shown) only one big cluster of finite width remains. In this simulation adsorbing boundary conditions are considered. The opinion space runs from 0 to $L = 1$ and only 100 opinions are plotted out of $N = 1000$.

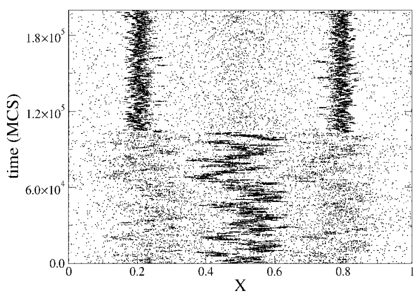

FIG. 4: Time series of the opinion distribution for bounded jumps with $\epsilon = 0.252$, $\gamma = 0.495$, and $m = 0.07$. Note the transition from one big cluster with two sidebands to a state of only two big clusters. We were unable to find transitions back to one single cluster. The opinion space runs from 0 to $L = 1$ and only 100 opinions are plotted out of $N = 1000$.

This effect can be analyzed using Monte Carlo simulations or the corresponding density-based master equation. We now present a linear stability analysis of the master equation in order to obtain analytical conditions for the existence of opinion clusters under noise. In particular, the linear stability analysis of the unstructured solution of Eq. (2) is performed. Then, the obtained expressions are compared with Monte Carlo simulations. If one neglects the influence of the borders or assumes that the opinion space is wrapped on a circle, the steady solution $P_h(x) = 1/L$ is an approximation to the unstructured steady solution of Eq. 2. It allows us to introduce $P(x,t) = 1/L + A_q \exp(iqx + \lambda_q^{\mathrm{HK}} t)$, where λ_q^{HK} represents the growth rate of periodic perturbations, q is the corresponding wavenumber, and A_q the amplitude. After introducing this ansatz in Eq. (2) we find the growth rate of the mode q:

$$\lambda_q^{\mathrm{HK}} = (1-m)\left[\frac{\sin(q\epsilon)}{q\epsilon} - \cos(q\epsilon)\right] + mH(q). \qquad (5)$$

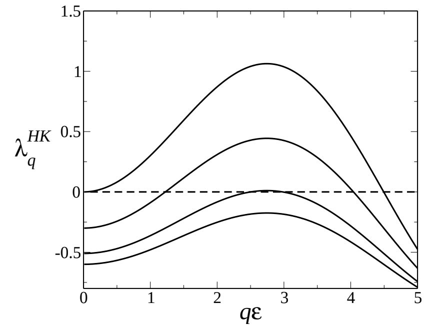

FIG. 5: This figure shows the growth rate, λ_q^{HK}, for the case of unlimited random jumps with noise intensities $m = 0.0$, 0.3, 0.51, 0.6, from top to bottom. Its shows that the growth rate becomes negative for $m > m_c \approx 0.51$.

The function $H(q)$ is equal to -1 for the case of unlimited random jumps inside the whole opinion space. For bounded random jumps inside the interval $[x_i^n - \gamma, x_i^n + \gamma]$, we consider the case of small values of γ because in this case the boundary effects become less important and the linear stability analysis of the homogeneous state $P_h = 1/L$ becomes valid. In this situation, the second case of Eq. (4) applies in the majority of cases and $H(q) = \frac{\sin(q\gamma)}{q\gamma} - 1$. When the growth rate λ_q^{HK} is positive, the homogeneous state is unstable and the subsequent evolution gives rise to cluster formation, a situation identified with order, whereas a negative growth rate implies that the homogeneous state is stable and clusters can not form, a sort of disordered state.

A. Unlimited random jumps

In this case, the opinion jumps are homogeneous around the whole opinion space and therefore $H(q) = -1$. To analyze the impact of noise on the growth rate, Fig. 5 shows λ_q^{HK} versus $q\epsilon$ for several values of noise intensity m. From this figure, one can observe that there is a single wavelength q with the largest growth rate. For large m, the maximum growth rate becomes negative, the homogeneous state is stable and clusters do not develop. This happens for $m > m_c \approx 0.51$, independently of ϵ. This result tells us that well-developed patterns of opinion clusters are possible only for $m < m_c$, and that the unstructured state is unstable in this region. The wavenumber corresponding to the growth rate that dominates and sets the wavelength is $q_{max} \approx 2.8/\epsilon$. This gives us an estimation of the number of opinion clusters by recognizing that the associated periodicity is $2\pi/q_{max}$ and then the number of clusters in the unit interval is $n_{\mathrm{HK}} \approx 0.4/\epsilon$ ($n_{\mathrm{HK}} \approx 0.4L/\epsilon$ in the $[0, L]$ interval). These conclusions can be verified in Fig. 6 which shows time series from Monte Carlo simulations. For strong noise, $m > m_c$, perturbations decay with time and the uniform state is restored. Whereas, for weak noise intensity, $m < m_c$, perturbations are magnified and patterns of opinions are established. This result also means that opinion clusters would be still observed for very small values of ϵ, if $m < m_c$.

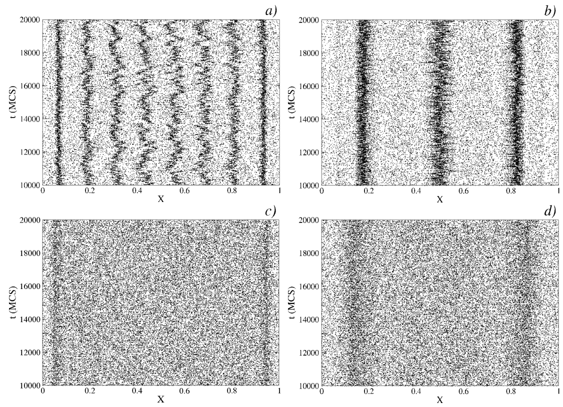

FIG. 6: Group dynamics for unlimited jumps as a function of noise intensity m and confidence parameter ϵ. This figure presents cases for $m = 0.3$ (a-b), and $m = 0.6$ (c-d) at $\epsilon = 0.05$ (Left panels) and 0.125 (Right panels). Its shows that for $m > m_c \approx 0.51$ an unstructured state dominates (except close to the borders, where boundary effects prevail) and opinion clusters do not develop. But, for $m < m_c$, opinion clusters exist even for very small values of ϵ. The opinion space runs from 0 to $L = 1$ and only 100 opinions are plotted out of $N = 1000$. Data is plotted after a long enough simulation time.

B. Bounded random jumps

In this case the confidence mechanism is generalized by allowing individuals to change their opinions randomly inside a small interval $[-\gamma, \gamma]$ centered at the current opinion. As mentioned before, the growth rate in this case involves $H(q) = \dfrac{\sin(q\gamma)}{q\gamma} - 1$ which can also be written as $H(q) = \dfrac{\sin(q\epsilon\gamma/\epsilon)}{q\epsilon\gamma/\epsilon} - 1$ to stress the dependence on ϵ when the growth rate is plotted as a function of $q\epsilon$. Figure 7 shows that the growth rate for a given γ exhibits two regimes as a function of ϵ. For $\gamma = 0.1$ and 0.4, the critical transitions between these two regimes are located at $\epsilon_c \approx 0.068$ and $\epsilon_c \approx 0.28$, respectively. Figure 7(a) shows the shape of $\lambda_q^{\rm HK}$ as a function of $q\epsilon$ for $\gamma = 0.1$ and $\epsilon < \epsilon_c \approx 0.068$. The form of this growth rate allows us to conclude that the perturbation with the largest growth rate dominates. However, the maxima of $\lambda_q^{\rm HK}$ and the appearance of positive values must be obtained numerically. On the other hand, for $\epsilon > \epsilon_c$ [see Fig. 7(b)], the appearance of positive values of $\lambda_q^{\rm HK}$ when varying the noise intensity occurs first at values of q close to zero corresponding, as expected for these large values of ϵ, to a long-wavelength instability. In this limit,

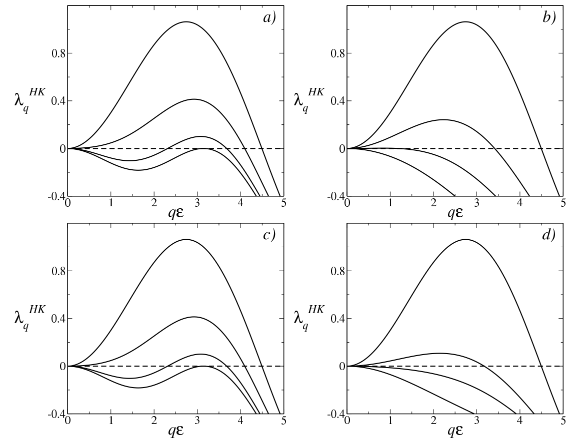

FIG. 7: Growth rate for the case of bounded random jumps. Panels (a and b) show $\gamma = 0.1$. In a) the growth rate for cases $\epsilon < \epsilon_c \approx 0.068$ is presented with $m = 0.0$, 0.3, 0.45, 0.5, from top to bottom. Growth rate becomes positive at a well-defined non-zero q. In b) the growth rate for cases $\epsilon > \epsilon_c \approx 0.068$ is presented with $m = 0.0$, 0.45, 0.65, 0.8, from top to bottom. In this situation, the appearance of positive values occurs first at values of q close to zero and therefore an expansion in powers of q is possible. Panels (c and d) show the same but for $\gamma = 0.4$, where $\epsilon_c \approx 0.28$. In c) $m = 0.0$, 0.3, 0.45, 0.5, from top to bottom. In d) $m = 0.0$, 0.45, 0.55, 0.65, from top to bottom.

approximate analytical expressions can be obtained expanding $\lambda_q^{\rm HK}$ in powers of q:

$$\lambda_q^{\rm HK} = \frac{(1-m)\,(1-\mu)\,\epsilon^2}{3} q^2 - \frac{4(1-m)\epsilon^4}{5!} q^4 + \mathcal{O}(q^6), \quad (6)$$

where $\mu = \frac{m\gamma^2}{(1-m)\epsilon^2}$. Because the q^4 term is always negative, the change of the sign of the q^2 term identifies

$$m_c = \frac{2\epsilon^2}{2\epsilon^2 + \gamma^2} \quad (7)$$

as the value below which opinion clusters appear. Within this approximation and close to the instability threshold the fastest growing mode is:

$$q_{max} \approx \frac{\sqrt{5}}{\epsilon}\,(1-\mu)^{1/2}\,. \quad (8)$$

Figures 7(c) and (d) show that the situation is similar for $\gamma = 0.4$. Figure 8 presents the critical lines for existence of opinion clusters in the parameter space (m, ϵ) for the cases considered in this work. In this case, to identify in a more quantitative way the order-disorder transition from Monte Carlo simulations with adsorbing boundary conditions, we use the so-called cluster coefficient G_M [16, 17]. One divides space $[0, 1]$ in M equal boxes and counts the number of individuals l_i which, at time step n, have their opinion in the box $[(i-1)/M, i/M]$. We choose $M = 100$. Then, one defines an entropy $S_M = -\sum_{i=1}^{M} \frac{l_i}{N} \ln \frac{l_i}{N}$, from

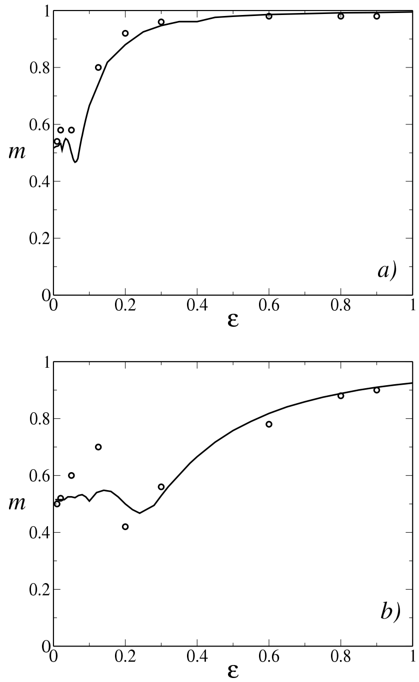

FIG. 8: Phase diagram on the plane (m, ϵ) for the case of bounded jumps obtained from our linear stability analysis (solid lines) and compared with the results coming from the occurrence of the maximum value of the cluster coefficient G_M as a function of m for fixed γ, obtained from Monte Carlo simulations using adsorbing boundary conditions and $N = 10^4$ (open dots). Clusters appear below these lines, whereas the disordered state is stable above. (a) $\gamma = 0.1$. (b) $\gamma = 0.4$. For $\epsilon < \epsilon_c = 0.068$ (case $\gamma = 0.1$) and $\epsilon < \epsilon_c = 0.28$ (case $\gamma = 0.4$), the solid line is obtained numerically from the change of sign of the maximum of the growth rate, Eq. (5), but for $\epsilon > \epsilon_c$ the approximate expression (7) is used, which is virtually identical. In this phase diagram $L = 1$.

which the cluster coefficient is defined as

$$G_M = M^{-1} \left\langle e^{\overline{S}_M} \right\rangle , \tag{9}$$

where the over-bar denotes a temporal average in the long-time asymptotic state and $\langle\cdot\rangle$ indicates an average over different realizations of the dynamics. Note that $1/M \leq G_M \leq 1$. Large values of G_M indicate a situation identified with disorder, while small values of G_M indicate that opinions peak around a finite set of major opinion clusters (a situation identified with order). The adsorption by the borders prevents the fully homogeneous state $G_M = 1$, as two opinion clusters are always formed at the extremes. Therefore, we will consider that the transition from order to disorder is the location m_c of the maximum value of G_M for fixed ϵ and γ (results plotted in Fig. 8).

V. COMPARISON WITH THE NOISY DW MODEL

The bounded confidence mechanism by which two individuals only influence the opinion of each other if their respective opinions differ less than some given amount holds for the DW model and the HK model. The HK model only differs from the DW model in that the interactions take place in groups rather than in pairs. In the noiseless DW model, one starts with a random distribution in opinion space $[0, L]$ and at subsequent time steps two randomly chosen agents may change their opinions to the average of both opinions if their opinions differ less than some given amount ϵ (in the standard particular case in which a convergence parameter in the model is equal 0.5). A detailed analysis of this model shows that the bifurcation of opinion clusters as a function of ϵ differs quantitatively from the noiseless HK model [3]. For instance, they have different critical values of ϵ for the consensus transition and, unlike the HK model, the DW model exhibits low-populated opinion clusters at the extremes and between major clusters even for uniform initial conditions. Nevertheless, they are similar in the fact that the bifurcation and nucleation of clusters observed outside the consensus region seems to repeat itself for decreasing ϵ in such a way that intercluster distances scale approximately with $1/\epsilon$. On the other hand, the DW model has been also studied under opinion random jumps and interesting phenomena arising from this randomness have been reported [16–18]. This section will be devoted to compare the results presented in previous sections with those observed in the noisy DW model.

The first conclusion we arrive is that, under unlimited opinion jumps, both models exhibit low-populated opinion clusters at the extremes and between high populated clusters. We also observed in both cases that the number of individuals belonging to these clusters increases when the noise intensity increases. The remarkable fact of bistability regions, reported first in the noisy DW model [16], is also observed in the noisy HK model. Inside these regions one finds that inherent fluctuations arising from the finite number of individuals take the system from one state to the other and back.

The coarsening process, observed in the noisy DW model [17], is also presented in the HK model when bounded random jumps of opinions are allowed to occur just inside a very small interval centered at the moment opinion. Clusters seem to perform a kind of random walk in opinion space and they merge when they collide. When the interval where jumps occur is larger, we observed that like the DW model the HK model admits a stable pattern of opinion clusters with reduced wandering and with regions of bistability. But it seems that in the HK model it is harder to find multiple jumps between one state to another and back. We just found jumps from one state to another but the new state never comes back to the previous one.

The order-disorder transition as a function of noise in-

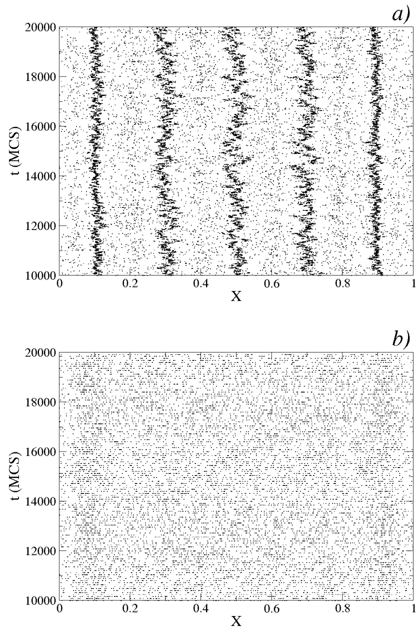

FIG. 9: Opinion dynamics of the HK model (a) and DW model (b) for noise intensity $m = 0.1$ and confidence parameter $\epsilon = 0.08$ in the case of unlimited jumps. In this case $L = 1$ and only 100 opinions are plotted out of $N = 1000$. For the DW model, $\epsilon_c(m = 0.1, L = 1) \approx 0.096$. Our analytical calculations predict that for the HK model the formation of patterns of opinion clusters occurs even for these small values of ϵ, but that this does not occur for DW, as actually seen in the plots. Data starts to be plotted after long enough simulation time

tensity m is also observed in both models [16, 17]. Nevertheless, the linear stability analysis revealed some important differences that we would like to discuss in the rest of this section. The linear stability analysis of the unstructured solution of the DW's density-based master equation under both types of noises and with the opinion space being $[0, 1]$ with periodic boundary conditions gives for the growth rate

$$\lambda_q^{\rm DW} = 4(1-m)\epsilon\left[\frac{4\sin(q\epsilon/2)}{q\epsilon} - \frac{\sin(q\epsilon)}{q\epsilon} - 1\right] + mH(q). \tag{10}$$

$H(q)$ is, for both types of noise, the same function as in the HK case. This result clearly shows that, unlike the HK model, the first term of the growth rate $\lambda_q^{\rm DW}$ carries as a prefactor the confidence parameter ϵ. This difference makes the time scales between the two models to be different, and slows down the DW instability for small ϵ. Since the result for a different value of L is recovered by replacing ϵ by ϵ/L, we also conclude that unlike the HK model a faster instability is expected for the DW model in smaller opinion spaces. More importantly, since the order-disorder transition is determined by a balance between the m and the $1-m$ terms in Eq. (10), the critical noise value m_c below which there is opinion cluster formation is now a function of ϵ for both type of noise, at variance with the HK case.

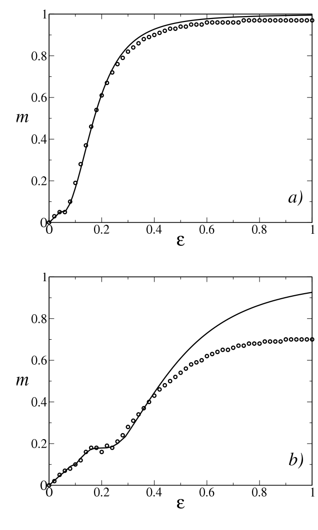

FIG. 10: Phase diagram of the DW model on the plane (m, ϵ) for the case of bounded jumps obtained from our linear stability analysis (solid lines) and compared with the results coming from the occurrence of the maximum value of the cluster coefficient G_M as a function of m for fixed γ, obtained from Monte Carlo simulations using adsorbing boundary conditions and $N = 10^4$ (open dots). Clusters appear below these lines, whereas the disordered state is stable above. (a) $\gamma = 0.1$. (b) $\gamma = 0.4$. For $\epsilon < \epsilon_c = 0.076$ (case $\gamma = 0.1$) and $\epsilon < \epsilon_c = 0.31$ (case $\gamma = 0.4$) the solid line is obtained numerically from the change of sign of the maximum of the growth rate, Eq. (10), but for $\epsilon > \epsilon_c$ the approximate expression (11) is used, which is virtually identical. In this phase diagram $L = 1$. This figure should be compared with Fig. 7 in [17] and Fig. 8 of this work. The discrepancies between lines and dots at large ϵ arise from the influence of the adsorbing boundary conditions of the Monte Carlo case, whereas the analytical calculations assume periodic boundary conditions.

For the case of unlimited random jumps $[H(q) = -1]$ one finds that the maximum value of $\lambda_q^{\rm DW}$ is negative for $m > m_c$ and positive for $m < m_c$, where $m_c \approx \epsilon/(0.8676 + \epsilon)$. Alternatively, for fixed m the maximum growth rate is negative for $\epsilon < \epsilon_c$ and positive for $\epsilon > \epsilon_c$, where $\epsilon_c \approx 0.8676m/(1-m)$ [16]. The absolute maximum of the growth rate occurs at $q_{max} \approx 2.8/\epsilon$, similar to the one of the HK model. It means that the number of clusters predicted as a function of the control parameters is, for both models, $n_{DW} = n_{HK} \approx 0.4L/\epsilon$. Note that while in the HK model one observes the formation of patterns of opinion clusters for small values of ϵ if noise intensities are small ($m < m_c = 0.51$), in the DW model there is a minimal value of $\epsilon_c(m, L)$ below which opinion clusters do not appear. Figure 9 shows Monte Carlo simulations that verify these results for $m = 0.1$ and $L = 1$. With these parameter values we get that the critical condition for cluster formation in the DW model is $\epsilon_c(m = 0.1, L = 1) \approx 0.096$. It means that for $\epsilon = 0.08$ the homogeneous state dominates. But, as predicted above, in the noisy HK model clusters are still possible for these parameter values. In fact, the number of opinion clusters predicted is $n_{HK} \approx 5$, in agreement with the numerical results displayed in this same figure.

For bounded random jumps of opinions, we observe that like in the HK model the growth rate of the DW exhibits two regimes for fixed γ while ϵ varies. In fact, for $\gamma = 0.1$ and 0.4, the critical transitions between regimes are given by $\epsilon_c \approx 0.076$ and 0.31, respectively. Similar to the HK model, for $\epsilon < \epsilon_c$ the critical line must be obtained numerically. For $\epsilon > \epsilon_c$ the appearance of positive values of λ_q^{DW} occurs first at values of q close to zero, identifying again a long-wave instability, and one can find an approximate analytical expression for the critical condition given by

$$m_c^{DW} = \frac{2\epsilon^3}{2\epsilon^3 + \gamma^2} \tag{11}$$

[compare with Eq. (7)]. The phase diagram for the order-disorder transition in the DW model in the parameter space (ϵ, m) is shown in Fig. 10, revealing some differences with the corresponding diagram for HK model (Fig. 8), especially important at small ϵ.

VI. CONCLUSIONS

In this paper, we have analyzed the Hegselmann-Krause model for continuous opinion dynamics under the influence of opinion noise. More precisely, we modify the model by giving each individual the opportunity to change, with a given probability m, his opinion to a randomly selected opinion inside the whole opinion space $[0, L]$ or inside the interval $[\gamma, -\gamma]$, centered around the current opinion. The final behavior, which depends of the confidence parameter ϵ, the noise intensity m and the parameter γ, is compared with the case of zero noise, and with the Deffuant et al. model for continuous opinion dynamics under similar types of noise.

Monte Carlo simulations have shown that, for opinion jumps inside the whole opinion space, the noisy HK model exhibits low-populated clusters at the extremes and between highly populated clusters. We found that the mass of these clusters increases as the noise intensity increases. Similar to the noisy DW model, we also found regions of bistability where the fluctuations present in Monte Carlo simulations are able to induce jumps from one state to another and back. For jumps inside the interval $[\gamma, -\gamma]$, the main dynamics of the system depends strongly on the parameter γ. For small values of γ, wandering of the clusters occurs and a coarsening process develops in which opinion clusters start to collide and merge until a single cluster remains after long time. For large values of γ, the mobility is reduced and the collision of clusters disappears given rise to a stable pattern of opinion clusters with certain regions of bistability.

A density-based master equation is introduced and the order-disorder transition induced by noise is analyzed using a linear stability analysis of the unstructured solution of this equation under periodic boundary conditions. We have derived analytical conditions for opinion pattern formation for both types of noise. We found qualitative, and in some cases even quantitative, agreement between the analytical results and the numerical simulations.

We analyzed in some detail the differences and similarities between the noisy HK model and the noisy DW model. We found that the most striking difference appeared concerning the dependency of the critical conditions for opinion cluster formation with the confidence parameter ϵ.

Finally our work stresses that, although the HK and DW model are similar in nature, their bifurcation behaviors and phenomenology as a function of the control parameters present important differences, also in the present of noise.

Acknowledgments

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