Tôhoku Math. Journ.
36 (1984), 99-105.

THE NON-EXISTENCE OF KILLING FIELDS

Dedicated to Professor Shigeo Sasaki on his seventieth birthday

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(Received February 24, 1983, revised April 22, 1983)

1. Introduction. We shall be in the C^∞-category. The manifolds under consideration are connected and orientable. In this note we prove the following theorem:

THEOREM. *Let M be a complete foliated Riemannian manifold with a foliation $\mathscr{F}$, and the Riemannian metric be a bundle-like metric with respect to $\mathscr{F}$. If all leaves of $\mathscr{F}$ are minimal and the Ricci operator ρ_∇ of $\mathscr{F}$ is non-positive everywhere and negative for at least one point of M, then every transverse Killing field of $\mathscr{F}$ with finite global norm is trivial.*

Examples of foliated Riemannian manifolds with bundle-like metrics and minimal leaves are shown in [2] and [6]. We remark that the assumption on the Ricci operator of $\mathscr{F}$ can also be interpreted as the quasi-negativity of the Ricci operator of $\mathscr{F}$ in the sense of Wu [10], [11].

The above theorem seems to be the ultimate generalization of the vanishing theorem of Killing vector fields started by Bochner. So far, we have already obtained the following results:

(i) If M in the above theorem is compact, then every transverse Killing field of $\mathscr{F}$ has finite global norm. The theorem in this case was obtained by Kamber and Tondeur [3].

(ii) If the foliation $\mathscr{F}$ on a complete Riemannian manifold M is the point foliation, then the Ricci operator ρ_∇ of $\mathscr{F}$ is the usual Ricci curvature operator and a transverse Killing field of $\mathscr{F}$ is a usual Killing vector field on M. The theorem in this case was obtained by Yorozu [8], [9].

(iii) The case of the point foliation $\mathscr{F}$ on a compact Riemannian manifold is the well-known theorem of Bochner [1].

Our discussions are essentially based on [3].

The author wishes to express his thanks to Mr. G. Oshikiri for pointing out mistakes. The referee also made several suggestions on the manuscript. The author wishes to thank the referee.

2. Preliminaries. This section is devoted to the review of [3]. Let M be an n-dimensional complete foliated Riemmanian manifold with a foliation $\mathscr{F}$, a Riemannian metric g_M and the Levi-Civita connection ∇^M with respect to g_M. We assume that the foliation $\mathscr{F}$ is of codimension q $(0 \leqq q \leqq n)$ and the metric g_M is a bundle-like metric with respect to $\mathscr{F}$ in the sense of Reinhart [5]. The foliation $\mathscr{F}$ is given by an integrable subbundle E of the tangent bundle TM over M. Let Q denote the normal bundle TM/E. The metric g_M defines a splitting σ of the exact sequence

$$0 \to E \to TM \underset{\pi}{\overset{\sigma}{\leftrightarrows}} Q \to 0$$

with $\sigma(Q) = E^{\perp}$ (the orthogonal complement bundle of E). Thus g_M induces a metric g_Q on Q: $g_Q(\nu, \mu) = g_M(\sigma(\nu), \sigma(\mu))$ for all $\nu, \mu \in \Gamma(Q)$, where $\Gamma(\cdot)$ denotes the space of all sections of a bundle. For any connection D in Q, the torsion T_D of D is given by

$$T_D(X, Y) = D_X\pi(Y) - D_Y\pi(X) - \pi([X, Y])$$

for all $X, Y \in \Gamma(TM)$ and the curvature R_D of D is given by

$$R_D(X, Y)\nu = D_XD_Y\nu - D_YD_X\nu - D_{[X,Y]}\nu$$

for all $X, Y \in \Gamma(TM)$ and all $\nu \in \Gamma(Q)$ (cf. [2], [3]). Now we define a connection ∇ in Q by

$$\nabla_X\nu = \pi([X, Y_\nu])$$

for all $X \in \Gamma(E)$ and all $\nu \in \Gamma(Q)$ with $Y_\nu = \sigma(\nu) \in \Gamma(\sigma(Q))$,

$$\nabla_X\nu = \pi(\nabla^M_X Y_\nu)$$

for all $X \in \Gamma(\sigma(Q))$ and all $\nu \in \Gamma(Q)$ with $Y_\nu = \sigma(\nu) \in \Gamma(\sigma(Q))$.

PROPOSITION 1 (cf. [2]). *The connection ∇ in Q is torsion-free and metrical with respect to g_Q, that is,*

$$T_\nabla = 0 \quad \textit{and} \quad \nabla_X g_Q = 0$$

for all $X \in \Gamma(TM)$.

We remark that $\nabla_X g_Q$ is defined by

$$(\nabla_X g_Q)(\nu, \mu) = X(g_Q(\nu, \mu)) - g_Q(\nabla_X\nu, \mu) - g_Q(\nu, \nabla_X\mu)$$

for all $X \in \Gamma(TM)$ and $\nu, \mu \in \Gamma(Q)$. We have that $i(X)R_\nabla = 0$ for all $X \in \Gamma(E)$, where i denotes the interior product (cf. [2]). We also have the following:

PROPOSITION 2 (cf. [2], [3]). *For all $\nu, \mu \in \Gamma(Q)$, the operator $R_\nabla(\nu, \mu)$:*

$\Gamma(Q) \to \Gamma(Q)$ is a well-defined endomorphism.

We introduce at a point $x \in M$ an orthonormal basis $e_{p+1}, \cdots, e_{p+q} = e_n$ of Q_x with $p = n - q$. Then the Ricci operator $\rho_\nabla: \Gamma(Q) \to \Gamma(Q)$ of $\mathscr{F}$ is defined by

$$(\rho_\nabla \nu)_x = \sum_{\alpha=p+1}^{n} R_\nabla(\nu, e_\alpha)e_\alpha$$

for all $\nu \in \Gamma(Q)$.

DEFINITION. The Ricci operator ρ_∇ of $\mathscr{F}$ is *non-positive* (resp. *negative*) *at a point* $x \in M$ if

$$g_Q(\rho_\nabla \nu, \nu)_x \leqq 0 \quad (\text{resp.} < 0)$$

for all $\nu \in \Gamma(Q)$ satisfying $\nu(x) \neq 0$.

Let $V(\mathscr{F})$ denote the space of all vector fields Y on M satisfying $[Y, Z] \in \Gamma(E)$ for all $Z \in \Gamma(E)$, where [ ] denotes the bracket operator. We define $\theta(Y): \Gamma(Q) \to \Gamma(Q)$ for $Y \in V(\mathscr{F})$ by

$$\theta(Y)\nu = \pi([Y, Y_\nu])$$

for all $\nu \in \Gamma(Q)$ and $Y_\nu \in \Gamma(TM)$ with $\pi(Y_\nu) = \nu$. The right hand side of the above equality is independent of the choice of the representative Y_ν of ν. For $Y \in V(\mathscr{F})$, $\theta(Y)g_Q$ is defined by

$$(\theta(Y)g_Q)(\nu, \mu) = Y(g_Q(\nu, \mu)) - g_Q(\theta(Y)\nu, \mu) - g_Q(\nu, \theta(Y)\mu)$$

for all $\nu, \mu \in \Gamma(Q)$.

DEFINITION (cf. [3]). If $Y \in V(\mathscr{F})$ satisfies $\theta(Y)g_Q = 0$, then $\pi(Y)$ is called a *transverse Killing field of* $\mathscr{F}$.

Let $\Omega^r(M, Q)$ be the space of all Q-valued r-forms on M. We define the exterior differential $d_\nabla: \Omega^r(M, Q) \to \Omega^{r+1}(M, Q)$ $(r \geqq 0)$ by

$$\begin{aligned} d_\nabla \eta(X_1, \cdots, X_{r+1}) = &\sum_{i=1}^{r+1} (-1)^{i+1} \nabla_{X_i} \eta(X_1, \cdots, \hat{X}_i, \cdots, X_{r+1}) \\ &+ \sum_{i<j} (-1)^{i+j} \eta([X_i, X_j], X_1, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_{r+1}) , \end{aligned}$$

and define $d_\nabla^*: \Omega^r(M, Q) \to \Omega^{r-1}(M, Q)$ by

$$d_\nabla^* = (-1)^{rn+n+1} * d_\nabla *$$

where $*$ denotes the Hodge star operator. The Laplacian Δ is defined by $\Delta = d_\nabla d_\nabla^* + d_\nabla^* d_\nabla$. Let $\Omega_o^r(M, Q)$ be the subspace of all Q-valued r-forms on M with compact support. For all $\eta, \tilde{\eta} \in \Omega^r(M, Q)$ with η or $\tilde{\eta}$ in $\Omega_o^r(M, Q)$, we define

$$\langle\langle \eta, \tilde{\eta} \rangle\rangle = \int_M g_Q(\eta \wedge *\tilde{\eta}) \quad (< +\infty)$$

and

$$\| \eta \|^2 = \langle\langle \eta, \eta \rangle\rangle .$$

For example, $g_Q(\eta \wedge *\tilde{\eta}) = g_Q(\nu, \tilde{\nu})\xi \wedge *\tilde{\xi}$ if one of $\eta = \xi \otimes \nu$, $\tilde{\eta} = \tilde{\xi} \otimes \tilde{\nu} \in \Omega^1(M, Q)$ has compact support. If $\eta \in \Omega_o^r(M, Q)$ or $\tilde{\eta} \in \Omega_o^{r+1}(M, Q)$, we have

$$\langle\langle d_\nabla \eta, \tilde{\eta} \rangle\rangle = \langle\langle \eta, d_\Delta^* \tilde{\eta} \rangle\rangle .$$

The space $\Gamma(Q)$ is viewed as the space $\Omega^0(M, Q)$. Let $L_2^0(M, Q)$ be the completion of $\Omega_o^0(M, Q)$ with respect to the inner product $\langle\langle \ , \ \rangle\rangle$.

DEFINITION. If $\nu \in \Gamma(Q)$ belongs to $L_2^0(M, Q) \cap \Omega^0(M, Q)$, then ν is called a *field of $\mathscr{F}$ with finite golbal norm.*

Now, we define $A_\nabla(Y)\colon \Gamma(Q) \to \Gamma(Q)$ for $Y \in V(\mathscr{F})$ by

$$A_\nabla(Y)\nu = \theta(Y)\nu - \nabla_Y \nu$$

for all $\nu \in \Gamma(Q)$ (cf. [3]). By the torsion-freeness of ∇, we have

$$A_\nabla(Y)\nu = -\nabla_{Y_\nu} \pi(Y)$$

where $Y_\nu \in \Gamma(TM)$ with $\pi(Y_\nu) = \nu$. Thus we may define

$$A_\nabla(\nu)\colon \Gamma(Q) \to \Gamma(Q)$$

for all $\nu \in \Gamma(Q)$ by $A_\nabla(\nu) = A_\nabla(Y)$ with $\pi(Y) = \nu$ and $A_\nabla(\nu)f = 0$ for any function f on M.

PROPOSITION 3. *Under the assumption that all leaves of $\mathscr{F}$ are minimal, a transverse Killing field $\nu \in \Gamma(Q)$ of $\mathscr{F}$ satisfies $\Delta\nu = \rho_\nabla \nu$.*

PROOF. Let $X_1, \cdots, X_{p+q}$ be an orthonormal local frame of TM on a neighborhood of $x \in M$ such that $X_1, \cdots, X_p \in \Gamma(E)$, $X_{p+1}, \cdots, X_{p+q} \in \Gamma(\sigma(Q))$, and let $(X_i)_x = e_i$ and $(X_\alpha)_x = e_\alpha$ ($i = 1, \cdots, p$; $\alpha = p+1, \cdots, p+q$). For a transverse Killing field ν of $\mathscr{F}$, we have $\nabla_{X_i}\nu = 0$, and $\nabla_Y \nu = 0$ with $Y = \sum_{i=1}^p (\nabla_{X_i}^M X_i)$, since the minimality of leaves of $\mathscr{F}$ implies that $Y_{\sigma(Q)} = 0$ where $(\ )_{\sigma(Q)}$ denotes the $\sigma(Q)$-component of $(\ )$. Thus we have

$$\begin{aligned}(\Delta\nu)_x &= (d_\nabla^* d_\nabla \nu)_x \\ &= -\sum_{i=1}^p (\nabla_{e_i}\nabla_{X_i}\nu - \nabla_{Y_i}\nu) - \sum_{\alpha=p+1}^{p+q} (\nabla_{e_\alpha}\nabla_{X_\alpha}\nu - \nabla_{U_\alpha}\nu) \\ &= -\sum_{\alpha=p+1}^{p+q} (\nabla_{e_\alpha}\nabla_{X_\alpha}\nu - \nabla_{(U_\alpha)_{\sigma(Q)}}\nu)\end{aligned}$$

with $Y_i = \nabla_{e_i}^M X_i$ and $U_\alpha = \nabla_{e_\alpha}^M X_\alpha$, and

$$(\nabla_{X_\alpha} A_\nabla(\nu))(\pi(X_\beta)) = \nabla_{X_\alpha}(A_\nabla(\nu)\pi(X_\beta)) - A_\nabla(\nu)(\nabla_{X_\alpha}\pi(X_\beta))$$
$$= -\nabla_{X_\alpha}\nabla_{X_\beta}\nu + \nabla_{W_{\sigma(Q)}}\nu$$

with $W = \nabla^M_{X_\alpha} X_\beta$. Hence, in a neighborhood of x, we have

$$\Delta\nu = \sum_{\alpha=p+1}^{p+q} (\nabla_{X_\alpha} A_\nabla(\nu))(\pi(X_\alpha)) .$$

On the other hand, we have

$$(\nabla_{X_\alpha} A_\nabla(\nu))(\pi(X_\beta)) = R_\nabla(\nu, \pi(X_\alpha))\pi(X_\beta)$$

(cf. [3, Proposition 3.17]). In the case of the point foliation, the above equality is well-known (cf. [4, Proposition 2.2]). Therefore we have $\Delta\nu = \rho_\nabla \nu$.

3. Proof of Theorem. Let us pick and fix a point o of M. For $r > 0$, we set

$$B(r) = \{x \in M \mid \rho(x) < r\} ,$$

where $\rho(x)$ denotes the geodesic distance from o to x. There exists a family of Lipschitz continuous functions $\{w_r; r > 0\}$ on M satisfying the following properties:

$$\begin{aligned} &0 \leqq w_r(x) \leqq 1 && \text{for all} \quad x \in M \\ &\operatorname{supp} w_r \subset B(2r) && \\ &w_r(x) = 1 && \text{for all} \quad x \in B(r) \\ &\lim_{r\to\infty} w_r = 1 && \\ &|dw_r| \leqq C/r && \text{almost everywhere on} \quad M , \end{aligned}$$

where C is a positive constant independent of r (cf. [7], [8], [9]). Then we have the following:

LEMMA 1. *For all $\nu \in \Gamma(Q)$, there exists a positive constant A independent of r such that*

$$\| dw_r \otimes \nu \|^2_{B(2r)} \leqq (A/r^2) \| \nu \|^2_{B(2r)}$$

where $\| \nu \|^2_{B(2r)} = \langle\!\langle \nu, \nu \rangle\!\rangle_{B(2r)} = \int_{B(2r)} g_Q(\nu, \nu)^*1.$

We define $d_\nabla(w_r^2\nu)$ by $d_\nabla(w_r^2\nu) = 2w_r dw_r \otimes \nu + w_r^2 d_\nabla \nu$ almost everywhere on M. By the Schwarz inequality and Lemma 1, we have

$$\begin{aligned} \langle\!\langle \Delta\nu, w_r^2\nu \rangle\!\rangle_{B(2r)} &= \langle\!\langle d_\nabla\nu, 2w_r dw_r \otimes \nu + w_r^2 d_\nabla\nu \rangle\!\rangle_{B(2r)} \\ &= \| w_r d_\nabla \nu \|^2_{B(2r)} + 2\langle\!\langle w_r d_\nabla\nu, dw_r \otimes \nu \rangle\!\rangle_{B(2r)} \\ &\geqq \| w_r d_\nabla \nu \|^2_{B(2r)} - 2\| w_r d_\nabla\nu \|_{B(2r)} \| dw_r \otimes \nu \|_{B(2r)} \end{aligned}$$

$$\geqq \| w_r d_\nabla \nu \|^2_{B(2r)} - ((1/2)\| w_r d_\nabla \nu \|^2_{B(2r)} + 2\| dw_r \otimes \nu \|^2_{B(2r)})$$
$$\geqq (1/2)\| w_r d_\nabla \nu \|^2_{B(2r)} - (2A/r^2)\| \nu \|^2_{B(2r)} .$$

Thus, by Proposition 3, we have:

LEMMA 2. *Suppose that all leaves of $\mathscr{F}$ are minimal. For a transverse Killing field ν of $\mathscr{F}$, the following holds*:

$$\langle\langle \rho_\nabla \nu, w_r^2 \nu \rangle\rangle_{B(2r)} \geqq (1/2)\| w_r d_\nabla \nu \|^2_{B(2r)} - (2A/r^2)\| \nu \|^2_{B(2r)} .$$

Since the Ricci operator ρ_∇ of $\mathscr{F}$ is non-positive everywhere, we have that, for a transverse Killing field ν of $\mathscr{F}$ with finite global norm,

$$0 \geqq \limsup_{r\to\infty} \langle\langle \rho_\nabla \nu, w_r^2 \nu \rangle\rangle_{B(2r)}$$
$$\liminf_{r\to\infty} \| w_r d_\nabla \nu \|^2_{B(2r)} \geqq 0$$
$$\lim_{r\to\infty} (2A/r^2)\| \nu \|^2_{B(2r)} = 0 .$$

From these, we have

$$0 \geqq \liminf_{r\to\infty} \langle\langle \rho_\nabla \nu, w_r^2 \nu \rangle\rangle_{B(2r)} \geqq \liminf_{r\to\infty} (1/2)\| w_r d_\nabla \nu \|^2_{B(2r)} \geqq 0 ,$$
$$0 \geqq \limsup_{r\to\infty} \langle\langle \rho_\nabla \nu, w_r^2 \nu \rangle\rangle_{B(2r)} \geqq \limsup_{r\to\infty} (1/2)\| w_r d_\nabla \nu \|^2_{B(2r)} \geqq 0 .$$

Thus $d_\nabla \nu = 0$, i.e., $\nabla_Y \nu = 0$ for all $Y \in \Gamma(TM)$, and $\langle\langle \rho_\nabla \nu, w_r^2 \nu \rangle\rangle_{B(2r)} = 0$ for all $r > 0$. Since the Ricci operator ρ_∇ of $\mathscr{F}$ is negative for at least one point of M, say x_o, we have $\nu(x_o) = 0$. Since $\nabla_Y \nu = 0$, we see that ν vanishes identically. Therefore our theorem is proved.

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