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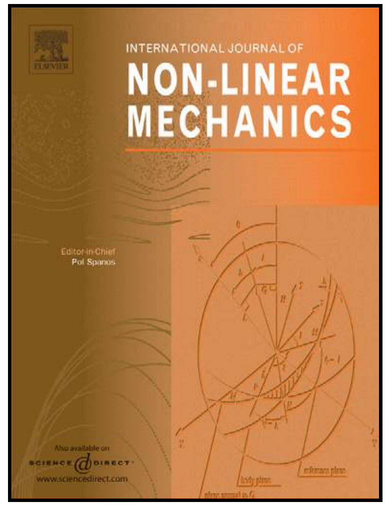

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THE NON-LINEAR DYNAMIC RESPONSE OF THE EULER-BERNOULLI BEAM WITH AN ARBITRARY NUMBER OF SWITCHING CRACKS

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ABSTRACT

In this study the nonlinear dynamic response of the Euler-Bernoulli beam in presence of multiple concentrated switching cracks (i.e. cracks that are either fully open or fully closed) is addressed. The overall behaviour of such a beam is non-linear due to the opening and closing of the cracks during the dynamic response, however it can be regarded as a sequence of linear phases each of them characterised by different number and positions of the cracks in open state. In the paper the nonlinear response of the beam with switching cracks is evaluated by determining the exact modal properties of the beam in each linear phase and evaluating the corresponding time history linear response through modal superposition analysis. Appropriate initial conditions at the instant of transition between two successive linear phase have been considered and an energy control has been enforced with the aim of establishing the minimum number of linear modes that must be taken into account in order to obtain accurate results. Some numerical applications are presented in order to illustrate the efficiency of the proposed approach for the evaluation of the nonlinear dynamic response of beams with multiple switching cracks. In particular, the behaviour under different boundary conditions both for harmonic loading and free vibrations has been investigated.

Keywords: Euler-Bernoulli beam; Non-linear dynamic response; Damaged beams; Switching cracks; Breathing cracks; Closing cracks; Modal analysis.

1. INTRODUCTION

In the last decades several authors devoted considerable interest in developing suitable models able to describe the influence of damage along the span of beam-like structures.. This increased interest has led to the improvement of the existing methods as well as to the development of new procedures for the analysis of the dynamic response of damaged structures . Particular attention has been devoted to the occurrence of cracks, i.e. damages concentrated at cross-sections of the beam. There are different approaches for crack modelling in beam structures reported in the literature; a great part of the considered approaches can be attributed to one of the following categories: spring models or elastic hinges [1-3], local stiffness reduction [4], and finite element models [5-7]. Friswell and Penny in [8] compare some different approaches for crack modelling and demonstrate that simple models of crack flexibility based on beam elements are adequate. A great diffusion has been reached by those models belonging to the category relying on the spring models. According to the latter approach, following the examples provided in [9-11], a crack can be macroscopically represented as an elastic link connecting the two adjacent beam segments. In particular, a model in which an internal hinge endowed with a rotational spring, whose stiffness is dependent on the extent of the damage, has been proved to be accurate [2,3,12-15].

Most of the procedures proposed in the literature are based on the strong assumption that the damaged structure behaves linearly since the crack is supposed to remain always open during the dynamic response. However theoretical and experimental studies have demonstrated that in many cases a state of damage in a structure can cause a nonlinear behaviour in its dynamic response [16] due to the so called '*closing crack*' phenomenon , i.e. a crack which opens and closes during the dynamic response. Within the context of the closing crack phenomenon, two different models can be distinguished [17]: *i)* the '*switching crack*' model in which the crack is either fully open or fully closed showing a bilinear behaviour; *ii)* the more realistic '*breathing crack*' model, showing a response-dependent behaviour, in which a smooth transition phase between open and closed crack occurs.

The presence of damage in a beam structure causes a decrease of the natural frequencies with respect to the undamaged beam; both experimental and theoretical investigations show that the frequencies of a beam with switching and breathing cracks are intermediate between the natural frequency of the undamaged beam and the model of the beam with open cracks. As a consequence, attention should be paid in the adoption of the open crack model when the crack closure occurs. Furthermore several studies highlighted that, in presence of switching cracks, there is a significant change of the response spectrum that is characterised by the presence of sub-harmonics typical of non-linear systems.

Although the study of the dynamic behaviour of cracked beams has been investigated by several authors, the great part of the surveys are relative to open cracks, while recently very few studies have been devoted to the non linear behaviour of beams due to the presence of switching and breathing cracks. In the studies investigating the non linear behaviour the presence of a single crack is usually considered [5,7,17-25], while multiple cracks have been analysed under an harmonic excitation only [26] and by means of a finite element approach. A short comprehensive review of these studies can be found in references [19,26].

The bi-linear behaviour of a beam with a single switching crack was recognized by Zastrau in [5] with reference to a simple supported beam whose response has been evaluated by means of the finite element method. Also Chu and Shen [20] highlighted the bi-linear behaviour of a single cracked beam, they used a Galerkin procedure for obtaining a bi-linear equation for each vibration mode. Shen and Chu [21] obtained a closed-form solution for a bi-linear oscillator subjected to low frequency excitations. Qian *et al*. [22] observed that the amplitude of the forced vibration response of cracked beams with switching cracks is over-estimated if a model with open cracks is considered. Ibrahim et al. [1] investigated the effects of crack closure on the frequency changes of cracked cantilever beams, both simulations and experimental results lead to the conclusion that, relying on the drop in the natural frequency alone may lead to serious underestimation of the crack severity.

In reference [23] Friswell and Penny analysed the non-linear behaviour of a beam with a switching crack under harmonic excitation in a range of frequency near to the first natural frequency of the beam, such that it can be considered as a single-degree-of-freedom system with bilinear stiffness.

The frequency response functions, obtained through numerical integration, highlighted the presence of peaks in the response spectrum at integer multiples of the excitation frequency, a common property for non-linear systems. Similar results were obtained by Ruotolo et al. [7] by performing numerical integration of the equation of motion and using a finite element model of the beam.

The non-linear behaviour of beams with a single switching crack has been also highlighted by Crespo et al. [24] and by Pugno et al. [25] where the concept of higher order frequency response functions has been applied to characterise the non-linearity due to the closing crack phenomenon.

In reference [27] Ostachowicz and Krawczuk used the harmonic balance method to determine the response of a cantilever beam with a single switching crack under harmonic excitation showing the reduced calculation time with respect to numerical integration.

Later, Pugno et al. [26] under the assumption of periodic response and adopting the harmonic balance method treated the case of several cracks by introducing a smooth crack closure according to [28].

An appealing approach for the modal analysis of nonlinear systems, even with strong nonlinearities, has been proposed in the literature by means of the definition of the so called '*non-linear normal modes*' [29].

In this study the problem of the evaluation of the nonlinear dynamic response of the Euler-Bernoulli beam under a generic excitation in presence of multiple concentrated switching cracks is addressed. Although the breathing crack model can be considered more realistic, the authors adopted the switching crack model since exact closed form expressions of the mode shapes of cracked beams can be obtained only for fully open or fully closed cracks, to be adopted also for the case of non-periodic response, and allow a complete treatment of the case of multiple switching cracks. Although the overall behaviour of the beam is highly nonlinear the problem can be treated as a multi-linear system each corresponding to a particular state of the cracks in the beam.

In previous papers the authors [30,31] treated both the stability and the dynamic behaviour of the Euler-Bernoulli beam with an arbitrary number of open cracks and presented, in closed form, the correspondent linear eigen-properties.

As far as the dynamic behaviour is concerned, the overall response of a beam with several switching cracks can be regarded as a sequence of linear phases, each of them characterised by different number and positions of the cracks in open state. Therefore, in this paper the response of the beam with switching cracks, is evaluated by determining the modal properties of the beam in each linear phase and calculating the time history responses through modal superposition analysis. Appropriate initial conditions at the instant of transition between two successive linear phases have been considered and an energy control criteria has been enforced in order to establish the minimum number of modes that must be taken into account in order to obtain accurate results. Some numerical applications are presented with the aim to illustrate efficiency of the proposed approach for beams with multiple switching cracks under different boundary and loading conditions. In order to compare the proposed approach with other accurate results reported in the literature, the harmonic responses of two-cracked cantilever steel beams reported by Pugno *et al.* in [26] have been considered. Furthermore, the presented procedure has been applied to evaluate several free vibration responses.

2. FORMULATION OF THE PROBLEM

The considered model is represented by an Euler-Bernoulli vibrating beam, of length L and uniform mass per unit length m, in presence of multiple concentrated closing cracks, with general boundary restraint conditions subjected to a generic load function p. A reference system is chosen with the origin at the first end of the beam, therefore each cross-section can be identified by the value of the normalised abscissa:

$$\xi = \frac{x}{L}, \qquad \text{with} \qquad 0 \le \xi \le 1 \tag{1}$$

The basic concept adopted in this study is that the concentrated cracks may be open or closed; when the generic crack is open, it affects locally the flexural stiffness of the beam and its influence can be modelled by means of generalised functions. The adoption of generalised functions to treat singularities in the flexural stiffness both in the context of static, stability and dynamic analyses has been previously considered by the authors in [30÷34]. According to the latter model, if a finite number of open cracks N_c are considered along the span of the beam at abscissas $\xi_{o,i}$, i=1, 2, ... , N_c, punctual reductions of the stiffness are introduced, so that the following expression of uniform flexural stiffness with Dirac's delta singularities is adopted to treat the concentrated open cracks:

$$EI(\xi) = E_o I_o \left[1 - \sum_{i=1}^{N_c} \gamma_i \cdot \delta\left(\xi - \xi_{o,i}\right) \right] \tag{2}$$

where $E_o I_o$ is the flexural stiffness of the undamaged beam, γ_i is a dimensionless damage intensity parameter and $\delta(\xi)$ is the Dirac's delta function.

The exact explicit expressions of the vibration modes and the corresponding frequencies of a multi-cracked beam with open cracks has been presented in [31]. Here, the above mentioned solution is employed for analysing, through modal analysis, the nonlinear dynamic response of beams with switching cracks. In this context, a switching crack is intended as a crack that is fully open for a given sign of the curvature of the beam in the current position and is fully closed otherwise. In such a system the variability of the stiffness of each crack, associated to its state (closed or open), can be conveniently described by the following flexural stiffness model

$$EI(\xi, \mathbf{b}) = E_o I_o \left[1 - \sum_{i=1}^{N_c} b_i \cdot \gamma_i \cdot \delta\left(\xi - \xi_{o,i}\right) \right] \tag{3}$$

where the *i*-th component b_i of the state vector **b** is assumed equal to 1, if the *i*-th crack is open, or zero, if the integrity of the cross-section is assumed as the fracture surfaces are in a full contact state.

According to the flexural stiffness model represented by Eq. (3), the dynamic differential equation of the Euler-Bernoulli beam with an arbitrary number of switching cracks subjected to a general transversal load distribution $p(\xi, t)$ can be written as

$$\left\{\left[1-\sum_{i=1}^{N_c} b_i\,\gamma_i\,\delta\left(\xi-\xi_{o,i}\right)\right]u''\left(\xi,t\right)\right\}'' + \frac{mL^4}{E_o I_o}\ddot{u}\left(\xi,t\right) = \frac{L^4}{E_o I_o}p\left(\xi,t\right) \tag{4}$$

where the apex indicates differentiation with respect to the normalised abscissa ξ, and the dot indicates differentiation with respect to time t.

In order to obtain the time-history response of equation (4) through the modal analysis, the eigen-properties of the beam in a generic state, identified by the Boolean vector **b**, must be evaluated.

2.1 EIGEN-PROPERTIES OF THE BEAM IN A GENERIC CRACK CONFIGURATION

The classical mode shapes and the corresponding frequencies of the beam subjected to a generic cracked configuration must be evaluated by considering the following dynamic differential equation that governs the free vibration of the beam

$$\left\{\left[1-\sum_{i=1}^{N_c} b_i\,\gamma_i\,\delta\left(\xi-\xi_{o,i}\right)\right]u''\left(\xi,t\right)\right\}'' + \frac{mL^4}{E_o I_o}\ddot{u}\left(\xi,t\right) = 0 \tag{5}$$

The solution of equation (5) with the use of separation of variables can be written as:

$$u(\xi,t) = y(t)\,\phi(\xi) \tag{6}$$

Substitution of equation (6) into equation (5) yields to the following differential equation for the modal displacements that, after some simple algebraic manipulation, can be written in the form:

$$\left\{\left[1-\sum_{i=1}^{N_c} b_i\cdot\gamma_i\cdot\delta\left(\xi-\xi_{o,i}\right)\right]\phi''\left(\xi\right)\right\}'' - \alpha^4\,\phi\left(\xi\right) = 0 \tag{7}$$

where the frequency parameter $\alpha^4 = \omega^2 mL^4/(E_o I_o)$ has been introduced.

Equation (7), by performing double differentiation with respect to ξ of the first term containing the Dirac's delta distribution, and after simple algebra, may be given the following form:

$$\phi^{IV}\left(\xi\right) - \alpha^4\phi\left(\xi\right) = B\left(\xi\right) \tag{8}$$

where the function $B\left(\xi\right)$ collects all the terms with the Dirac's deltas and their derivatives as follows:

$$B(\xi)=\left[\sum_{i=1}^{N_c} b_i\,\gamma_i\phi^{\prime v}(\xi)\delta(\xi-\xi_{oi})+2\sum_{i=1}^{N_c} b_i\,\gamma_i\phi'''(\xi)\delta'(\xi-\xi_{oi})+\sum_{i=1}^{N_c} b_i\,\gamma_i\phi''(\xi)\delta''(\xi-\xi_{oi})\right] \tag{9}$$

The general solution of equation (8) has been presented in [31] by making use of the theory of the generalised functions and may be written, for the case under study, in closed form as follows:

$$\begin{aligned}\phi(\mathbf{b},\xi)=&\,C_1\left\{\frac{1}{2\alpha}\sum_{i=1}^{N_c} b_i\lambda_i\mu_i\left[\sin\alpha(\xi-\xi_{oi})+\sinh\alpha(\xi-\xi_{oi})\right]U(\xi-\xi_{oi})+\sin\alpha\xi\right\}+\\&+C_2\left\{\frac{1}{2\alpha}\sum_{i=1}^{N_c} b_i\lambda_i\upsilon_i\left[\sin\alpha(\xi-\xi_{oi})+\sinh\alpha(\xi-\xi_{oi})\right]U(\xi-\xi_{oi})+\cos\alpha\xi\right\}+\\&+C_3\left\{\frac{1}{2\alpha}\sum_{i=1}^{N_c} b_i\lambda_i\zeta_i\left[\sin\alpha(\xi-\xi_{oi})+\sinh\alpha(\xi-\xi_{oi})\right]U(\xi-\xi_{oi})+\sinh\alpha\xi\right\}+\\&+C_4\left\{\frac{1}{2\alpha}\sum_{i=1}^{N_c} b_i\lambda_i\eta_i\left[\sin\alpha(\xi-\xi_{oi})+\sinh\alpha(\xi-\xi_{oi})\right]U(\xi-\xi_{oi})+\cosh\alpha\xi\right\}\end{aligned} \tag{10}$$

where $U(\xi-\xi_{oi})$ is the unit step (Heaviside) function, which is the distributional derivative of the Dirac's delta distribution, and the terms $\mu_i,\nu_i,\zeta_i,\eta_i$ are given by the following expressions.

$$\begin{aligned}\mu_j&=\frac{\alpha}{2}\sum_{i=1}^{j-1} b_i\lambda_i\mu_i\left[-\sin\alpha(\xi_{oj}-\xi_{oi})+\sinh\alpha(\xi_{oj}-\xi_{oi})\right]-\alpha^2\sin\alpha\xi_{oj}\\ \upsilon_j&=\frac{\alpha}{2}\sum_{i=1}^{j-1} b_i\lambda_i\upsilon_i\left[-\sin\alpha(\xi_{oj}-\xi_{oi})+\sinh\alpha(\xi_{oj}-\xi_{oi})\right]-\alpha^2\cos\alpha\xi_{oj}\\ \zeta_j&=\frac{\alpha}{2}\sum_{i=1}^{j-1} b_i\lambda_i\zeta_i\left[-\sin\alpha(\xi_{oj}-\xi_{oi})+\sinh\alpha(\xi_{oj}-\xi_{oi})\right]+\alpha^2\sinh\alpha\xi_{oj}\\ \eta_j&=\frac{\alpha}{2}\sum_{i=1}^{j-1} b_i\lambda_i\eta_i\left[-\sin\alpha(\xi_{oj}-\xi_{oi})+\sinh\alpha(\xi_{oj}-\xi_{oi})\right]+\alpha^2\cosh\alpha\xi_{oj}\end{aligned} \tag{11}$$

The dimensionless parameters λ_i appearing in equations (10),(11) are related to γ_i as follows:

$$\lambda_i=\frac{\gamma_i}{(1-A\gamma_i)},\quad i=1,\ldots,N_c \tag{12}$$

and will be considered in the sequel as "*damage parameters"* and will be adopted in the applications in order to represent the intensity of concentrated damages. In equation (12) A is a constant which does not influence the damage modelling since the damage intensity can be correlated directly to the damage parameters λ_i as discussed in reference [30].

The integration constants C_1,C_2,C_3,C_4, appearing in equation (10), can be easily evaluated in explicit form by imposing the boundary conditions of the eigen-mode and its derivatives.

2.2 EVALUATION OF THE TIME-HISTORY RESPONSE THROUGH MODAL ANALYSIS IN A GENERIC CRACK CONFIGURATION

The adopted model for the closing crack, in which the generic crack may be either fully open or fully closed, implies that the nonlinear response of the system can be considered as a sequence of linear states each of them can be evaluated through a classical modal analysis. Therefore, by considering a time interval in which the system maintains the same crack configuration, i.e. the state vector **b** does not change, the corresponding displacement time-history response can be expressed exactly by an infinite series through modal superposition as follows:

$$u(t,\xi)=\sum_{n=1}^{\infty}\phi_n(\mathbf{b},\xi)\cdot y_n(t) \tag{13}$$

where $\phi_n(\mathbf{b},\xi)$ is the n-th mode shape corresponding to the state vector **b** and $y_n(t)$ is the n-th normal coordinate. By substituting expression (13) in the governing equation of motion (4) the following expression is obtained:

$$\sum_{n=1}^{\infty}\left[\left(1-\sum_{i=1}^{N_c}b_i\gamma_i\delta\left(\xi-\xi_{o,i}\right)\right)\phi_n\left(\mathbf{b},\xi\right)\right]'' y_n(t)+\frac{mL^4}{E_oI_o}\sum_{n=1}^{\infty}\phi_n\left(\mathbf{b},\xi\right)\ddot{y}_n\left(t\right)=\frac{L^4}{E_oI_o}p\left(\xi,t\right) \tag{14}$$

Multiplying each of equation (14) term by $\phi_m\left(\mathbf{b},\xi\right)$ and integrating along the length of the beam yield to:

$$\begin{aligned}&\sum_{n=1}^{\infty}y_n(t)\int_0^1\phi_m\left(\mathbf{b},\xi\right)\left[\left(1-\sum_{i=1}^{N_c}b_i\gamma_i\delta\left(\xi-\xi_{o,i}\right)\right)\phi_n''(\mathbf{b},\xi)\right]''d\xi+\frac{mL^4}{E_oI_o}\sum_{n=1}^{\infty}\ddot{y}_n\left(t\right)\int_0^1\phi_m\left(\mathbf{b},\xi\right)\phi_n\left(\mathbf{b},\xi\right)d\xi=\\&=\frac{L^4}{E_oI_o}\int_0^1\phi_m\left(\mathbf{b},\xi\right)p\left(\xi,t\right)d\xi\end{aligned} \tag{15}$$

It has to be remarked that the following orthogonality properties of the mode shapes with respect to the mass and to the stiffness hold:

$$m\int_0^1\phi_m\left(\mathbf{b},\xi\right)\phi_n\left(\mathbf{b},\xi\right)d\xi=\left\langle\begin{array}{ll}0 & \text{for } n\neq m\\ M_n & \text{for } n=m\end{array}\right. \tag{16}$$

$$E_o I_o \int_0^1 \phi_m(\mathbf{b},\xi)\left[\left(1-\sum_{i=1}^{N_c} b_i \gamma_i \delta\left(\xi-\xi_{o,i}\right)\right)\phi_n''(\mathbf{b},\xi)\right]'' d\xi = \left\langle \begin{array}{ll} 0 & \text{for } n \neq m \\ K_n & \text{for } n = m \end{array} \right. \tag{17}$$

where M_n and K_n are the generalised modal mass and stiffness of the *n*-th mode, respectively.

By accounting for the relationship $K_n = \omega_n^2 M_n$, involving the generalised modal mass and stiffness, the following modal equation is obtained:

$$\ddot{y}_n(t) + \alpha_n{}^4 y_n(t) = \frac{P_n(t)}{M_n} \tag{18}$$

where

$$P_n(t) = \int_0^1 \phi_n(\mathbf{b},\xi)\, p(\xi,t) d\xi \tag{19}$$

If classical modal damping [35] is assumed, the damping can be considered by simply assigning a modal damping ratio ζ_n to each considered mode. In this case the modal equation (18) can be rewritten as follows:

$$\ddot{y}_n(t) + 2\zeta_n \alpha_n{}^2 \dot{y}_n(t) + \alpha_n{}^4 y_n(t) = \frac{P_n(t)}{M_n} \tag{20}$$

Equation (20) coincides formally with the equation of motion of a single-degree-of-freedom system and, if solved, yields to the normalised coordinate $y_n(t)$ relative to the *n*-th mode. In each linear phase, the response of the beam in terms of displacement $u(\xi,t)$ can therefore be estimated by choosing a finite number of modes *N*, by solving the *N* independent equations of the normalised coordinates (18) and superposing the modal responses by means of equation (13), where *N* terms are accounted for in the summation, as follows:

$$u(t,\xi) \cong \sum_{n=1}^{N} \phi_n(\mathbf{b},\xi) y_n(t) \tag{21}$$

Once the modal truncation has been performed, the correspondent residual displacement function $\Delta u(\xi,t)$ is evaluated as follows:

$$\Delta u(\xi,t) = u(\xi,t) - \sum_{n=1}^{N} \phi_n(\mathbf{b},\xi) y_n(t) \tag{22}$$

It is important to consider a sufficient number of vibration modes in order to maintain the error due to the modal truncation within an acceptable tolerance. In the following paragraph this aspect will be further discussed.

The response in terms of velocity can be easily obtained by means of differentiation with respect to time: as follows:

$$v(t,\xi) \cong \sum_{n=1}^{N} \phi_n(\mathbf{b},\xi)\dot{y}_n(t) \qquad (23)$$

3. THE NON-LINEAR DYNAMIC RESPONSE OF THE BEAM WITH SWITCHING CRACKS

Let us now consider a beam with N_c switching cracks. The initial configuration, open or closed, of each crack is known and therefore the initial state vector **b** can be assigned. For this initial condition the first N natural frequency parameters and the corresponding modes of vibration can be derived by means of equation (10), and its derivatives, by enforcing the boundary condition, once the zeros of the corresponding frequency equations have been evaluated. The response of the system during each linear phase, i.e. a phase characterised by the same state vector **b**, is obtained by using modal superposition as outlined in the previous paragraph. When one or more cracks open or close the system is subjected to a state change. In this case, the definition of the phase transition conditions is necessary in order to characterise the initial conditions for the new linear phase to be solved with a new set of modal coordinates.

3.1 PHASE TRANSITION CONDITIONS

Without loss of generality it is assumed that a generic closed crack opens when the curvature at the crack position $\xi_{o,i}$ reaches the positive sign (upward concavity), while an open crack closes when the curvature at the crack position $\xi_{o,j}$ attains the negative sign (downward concavity). A generic configuration of a simply supported beam with some cracks closed and others open is qualitatively represented in Fig.1. The incipient opening condition for a closed crack is characterised by the transition of the curvature from a negative toward a positive value, therefore this condition can be written as follows:

$$u''(t,\xi) = 0\,;\ \frac{du''}{dt}(t,\xi) > 0 \tag{24}$$

Viceversa, the incipient closing conditions for an open crack may be expressed as follows:

$$u''(t,\xi) = 0\,;\ \frac{du''}{dt}(t,\xi) < 0 \tag{25}$$

Let us suppose that the opening/closing conditions (24)/(25) are satisfied for the *i*-th crack at the time instant t_o. At t_o a change in the *i*-th component of the state vector **b** occurs. Therefore a new set of N natural frequency parameters $\alpha_1^+, \alpha_2^+, ..., \alpha_N^+$ and the corresponding mode shapes $\phi_1(\mathbf{b}^+,\xi), \phi_2(\mathbf{b}^+,\xi), \ldots \phi_N(\mathbf{b}^+,\xi)$ must be evaluated for this new linear phase that, for simplicity, can be identified by the vector $\mathbf{b}^+$. Beyond the time instant t_o, the response of the beam must be evaluated in terms of the normalised coordinates $y_n^+(t)$ relative to the new mode shapes. Obviously this new solution is valid until a new event, associated to opening or closing of one or more cracks, occurs.

In the simplifying hypothesis that no dissipation energy is associated to the opening or closing of a crack, the initial conditions that must be enforced at the beginning of each linear phase must be determined by imposing the continuity of the displacement and velocity flexural response at time t_o. These conditions can be expressed as follows:

$$u(t_o^+,\xi) = u(t_o^-,\xi) \quad \Rightarrow \quad \sum_{n=1}^{N} \phi_n(\mathbf{b}^+,\xi)\, y_n(t_o^+) = \sum_{n=1}^{N} \phi_n(\mathbf{b}^-,\xi)\, y_n(t_o^-) \tag{26}$$

$$\dot{u}(t_o^+,\xi) = \dot{u}(t_o^-,\xi) \quad \Rightarrow \quad \sum_{n=1}^{N} \phi_n(\mathbf{b}^+,\xi)\, \dot{y}_n(t_o^+) = \sum_{n=1}^{N} \phi_n(\mathbf{b}^-,\xi)\, \dot{y}_n(t_o^-) \tag{27}$$

where $y_n(t_o^+)$ and $\dot{y}_n(t_o^+)$ are unknowns to be evaluated by taking advantage of the orthogonality condition (15), pre-multiplying both members of expressions (26) and (27) by $\phi_m(\mathbf{b}^+,\xi)$ and integrating along the span of the beam, as follows:

$$M_m^+ y_m(t_o^+) = \int_0^1 u(t_o^-,\xi)\,\phi_m(\mathbf{b}^+,\xi)\cdot d\xi \qquad \Rightarrow \qquad y_m(t_o^+) = \frac{\int_0^1 u(t_o^-,\xi)\,\phi_m(\mathbf{b}^+,\xi)\cdot d\xi}{M_m^+} \tag{28}$$

$$M_m^+ \dot{y}_m(t_o^+) = \int_0^1 \dot{u}(t_o^-,\xi)\,\phi_m(\mathbf{b}^+,\xi)\cdot d\xi \qquad \Rightarrow \qquad \dot{y}_m(t_o^+) = \frac{\int_0^1 \dot{u}(t_o^-,\xi)\,\phi_m(\mathbf{b}^+,\xi)\cdot d\xi}{M_m^+} \tag{29}$$

It must be noticed that, in view of the adopted modal truncation (a finite number of modes considered), the transition from one finite set of mode shapes, at the $\mathbf{b}^-$ state, to another, at the $\mathbf{b}^+$ state, to represent the same displacement and velocity configuration of the beam, introduces an additional error associated to the change of the mode shapes basis. As a consequence, the continuity conditions reported in equations (26) and (27), strictly speaking, are satisfied in an approximated way. An energy balance at the transition instant t_o can be adopted, as shown in what follows, to provide an estimate of the error due to the change of mode shape basis. To this purpose it can be observed that the strain energy $E_S(t)$

$$E_S(t) = \frac{1}{2}\frac{E_o I_o}{L^3}\int_0^1 \left[1 - \sum_{i=1}^{N_c} b_i \cdot \gamma_i \cdot \delta\left(\xi - \xi_{o,i}\right)\right]\left[u''\left(\xi,t\right)\right]^2 \cdot d\xi \tag{30}$$

and the kinetic energy $E_k(t)$

$$E_k(t) = \frac{1}{2} mL \int_0^1 \left[\dot{u}\left(\xi,t\right)\right]^2 \cdot d\xi \tag{31}$$

at the transition instant t_o must be conserved since no loss of energy has been associated to the opening/closing of the switching crack.

The above mentioned error due to the change of the mode shape basis, occurring at every transition phase, can be estimated by expressing both the strain and the kinetic energy at the instants t_o^- and t_o^+, considering the different sets of modal coordinate corresponding to the states $\mathbf{b}^-$ and $\mathbf{b}^+$, and verifying that the energy loss associated to the variation of the modal basis is within a certain tolerance ε.

The latter condition can be expressed as follows:

$$\frac{\left[E_S(t_o^-)+E_k(t_o^-)\right]-\left[E_S(t_o^+)+E_k(t_o^+)\right]}{\left[E_S(t_o^-)+E_k(t_o^-)\right]}<\varepsilon \tag{32}$$

4. NUMERICAL APPLICATIONS

The numerical applications presented in this section are relative to beams with multiple closing cracks subject to different boundary and loading conditions. The frequency equation for a multi-cracked beam, in a fixed configuration, can be derived by means of the closed form solution reported in equation (10) by simply enforcing the standard boundary conditions, including the general case of rotational and translational spring supports. In particular, in this section the closed form solution given by equation (10) for a general state of open/closed cracks is adopted to treat the cases of cantilever, and simply supported. Euler Bernoulli beams. The damage parameter λ has been chosen as representative of the damage intensities, and the correspondent crack depths can be easily inferred through existing damage models as reported in [30]. The response of the considered beams to harmonic loadings is first analysed and the results are reported in terms of frequency response functions and compared with other results provided in the literature. Subsequently, it is shown how the proposed modal analysis can also be efficaciously applied to determine the free vibration response of beams with multiple switching cracks.

4.1 HARMONIC LOADING

In order to compare the proposed approach with other accurate results reported in the literature, the first application considered herein is relative to a prismatic cantilever steel beam, in presence of two closing cracks and subjected to a harmonic concentrated load at the free end, considered by Pugno et al. [26]. The beam has length L=0.70 m, square cross-section with height h=20 mm, Young modulus $E=2.06\cdot10^8\ KN/m^2$ and mass density $\rho=8\ KN\cdot s^2/m^4$. According to the referenced paper [26] three different configurations of the crack depths and positions, as reported in Table 1, have been considered. In the analyses considered by Pugno et al. [26] the structure has been discretised by using Euler-type finite elements with two nodes and two degrees of freedom per node, furthermore, they assumed that the transition from closed to open crack, and *vice versa*, is smooth rather than instantaneous. Assuming that the dynamic response is periodic, they employed the harmonic balance method to solve the equations of motion, furthermore, in order to demonstrate the efficiency of their procedure, in the same paper a comparative analysis with results, previously obtained by the same authors through direct numerical integration according to a different procedure presented in [25], has been also reported.

In Figure 2 the results obtained by Pugno et al. [26] in terms of maximum displacement u of the free end, normalised with respect to the maximum load value P, are compared with those obtained by the proposed approach. In order to perform the comparison, the values of damage intensity parameter λ, adopted in this work, corresponding to the relative crack depths is defined as follows [30]:

$$\lambda = \frac{h}{L} C(\beta) \tag{33}$$

where $C(\beta)$ is the local compliance due to the concentrated crack, which is here adopted according to the model proposed in [15] as follows:

$$C(\beta) = \frac{\beta(2-\beta)}{0.9(\beta-1)^2} \tag{34}$$

The values of the damage intensity parameters corresponding to the cases considered by Pugno et al. [26] are reported in Table 1.

The cantilever beam, is characterised by the first fundamental frequency in the undamaged configuration equal to $f = 33.7Hz$ and has been subjected to a concentrated harmonic force $P\sin(\omega t)$ at the free end section in order to obtain the corresponding frequency response. The results have been reported in term of the maximum displacement response of the beam, at the same section, discarding the transient part of the response. Nine mode shapes have been considered for each linear phases of the analyses and a modal damping ratio equal to 2% has been set for all the needed vibration modes. From the comparisons reported in Figure 2, it can be observed a good agreement with the results obtained by Pugno et al. [26]. However, since the model adopted in this work considers instantaneous crack closure, the results obtained with the proposed procedure are closer to those obtained according to the method reported in [25] that considers the same model of 'switching' crack for representing the opening and closing of the crack. The small differences, that can be observed by the comparison, can also be associated to the different beam models adopted by each approach, to the different damage models adopted for the crack depth, and, finally, to the unavoidable numerical errors.

The results plotted in Figure 3a are relative to the beam, in presence of two cracks, corresponding to configuration 3 of Table 1 with a modal damping ratio equal to 2% and are compared with the response of both the undamaged beam and the beam with open cracks. The analyses have been extended to a frequency range wider than that reported in [26] in order to show an additional fundamental frequency (the second fundamental frequency in the undamaged configuration is $f = 211.0\ Hz$). As highlighted by many authors in previous studies [7,8,21,27], it can be observed that the fundamental frequency of the beam with switching cracks is collocated between those of the undamaged beam and the beam with open cracks. The analysis has been also conducted for reduced values of the damping ratio such as 1% and 0.5%, and the results have been reported in Figs 3b and 3c. It must be noted that, for lower levels of damping, additional extra peaks can be observed.

The results reported in Figure 4 are relative to a simply supported beam with different number of equally spaced cracks subjected to a harmonic uniform load $p(\xi,t) = p_0 \sin(\omega t)$. The latter results are reported in dimensionless form, by making use of the frequency parameter $\alpha^4 = \omega^2 m L^4 /(E_o I_o)$ introduced in equation (7), and are expressed in terms of maximum displacement u of the middle span cross-section normalised with respect to the value of the static displacement u_S due to the amplitude of the distributed load for the undamaged beam. The simply supported beam is characterised by the value of the frequency parameter $\alpha = 3.15$ corresponding to the first natural frequency in the undamaged configuration.

In the adopted ‘switching crack model’ the nonlinear behaviour of the beam is associated to a succession of linear phases. According to this model the state of each crack depends on the sign of the curvature only, as a consequence the normalised frequency responses, reported in what follows, do not depend on the amplitude of the load but on the load condition only.

Figure 4 is relative to 1, 2, 4 and 8 cracks characterised by equal damage intensity parameters $\lambda = 0.05$, and reports a comparison of the results provided by the proposed approach, for beams with switching cracks, with the response of the corresponding undamaged beam and the beam with open cracks. Once again, the fundamental natural frequencies of the beam with switching cracks are collocated between those corresponding to the always-open and to the always-closed (undamaged) beams. In Figure 4, for the four analysed beams with switching cracks, it can be also observed the reduction of the fundamental frequency with the increase of the number of equally spaced and equally damaged cracks.

A further significant difference in the frequency response functions is associated to the presence of peaks, for the case of switching cracks, at lower and higher frequencies with respect to the fundamental one, indicating that the structures behave non-linearly.

In Figure 5 the time histories correspondent to each peak of the frequency response function of the simply supported beam with 4 cracks are reported. The crack state histories, indicating whether each crack is open or closed at each time instant, are also reported. The latter figure highlights the roughly bi-linear behaviour of the beam with multiple closing cracks for each considered peak since all the cracks open and close nearly at the same time.

In Figure 6 the snap-shots relative to the time-history correspondent to the frequency parameter $\alpha = 3.05$ are represented. The circles present, along the beam span, in some snap-shots of Figure 6 coincide with the crack positions and their appearance indicate that the corresponding cracks are in open state. It has to be noted that, since at $\alpha = 3.05$ a forced resonance occurs, the sequence of snapshots proposed in Fig.6 might be interpreted as an approximated nonlinear normal mode since in [36,37] it is stated that "forced resonances in nonlinear systems occur in neighborhood of nonlinear normal modes".

A further insight into the appearance of peaks in the frequency response spectrum is pursued by analysing the case of a simply supported beam with two cracks subjected to an anti-symmetric uniform load $p(\xi,t) = p_0\left[1-2\,U(\xi-0.5)\right]\sin(\omega t)$ such that the cracks are at anti-phase positions, i.e. the two cracks open and close alternatively.

The relevant frequency response spectrum is reported in Figure 7 showing that the beam with closing cracks exhibits extra peaks if compared to the undamaged beam and the beam with open cracks. At this stage, it is not clear how the number of extra peaks is related to the number of cracks.

In Figure 8 the time histories corresponding to the peaks at $\alpha = 6.06$ and $\alpha = 6.78$ of the frequency response function are reported together with the crack state histories, indicating whether each crack is open or closed at each time instant. In Figure 9 the snap-shots relative to the time-histories correspondent to the frequency parameters $\alpha = 6.06$ and $\alpha = 6.78$ are represented. It can be noted that the time history relative to Fig.8a (frequency parameter $\alpha = 6.06$) is relative to a forced resonance in fact the correspondent snapshots reported in Fig.9 are representative of a synchronous periodic oscillation and, according to [36], may be interpreted as an approximation of the nonlinear normal mode.

4.2 FREE VIBRATIONS

The results reported in Figures 10 and 11 are relative to the undamped free vibration responses of a simply supported beam with three closing crack, collocated at the positions ξ= 0.1, 0.2 and 0.3, all characterised by the damage intensity parameter λ=0.15, correspondent to a significant damage.

The normalised formulation, by making use of the frequency parameter α, whose value corresponding to the first natural frequency in the undamaged configuration is $\alpha = 3.15$, has been once again adopted. The beam is subjected to an initial displacement shape equivalent to the first mode of the undamaged beam. In Figure 10 the time-history displacement response of the cross-section at $\xi = 0.25$, normalised with respect to the maximum initial value u_0 is displayed together with the modal contributions of the first three modes, the overall contribution of the other considered six modes. It can be observed as the mode contributions are variable during the time history response, and, although the beam has been subjected to its first mode, after the first linear phase all the other modes provide a significant contribution to the free vibration response.

The crack state histories, indicating whether each crack is open or closed at each time instant, are also reported at the bottom of Figure 10. The crack state histories highlight the roughly bi-linear behaviour of the beam with multiple closing cracks since the three cracks open and close nearly at the same time.

In order to obtain a discrete time history representation of the deformed shape of the beam, a series of snap-shots in a time interval equal to a complete cycle of response, are reported in Figure 11. Once again, the circles appearing in some snap-shots of Figure 11 indicate that the corresponding cracks are in open state. The latter figure shows clearly the variability of the deformed shape during the nonlinear response due to the open/closing crack phenomenon.

Moreover, the results reported in Figures 12 and 13 are relative to the above considered beam, however subjected to the second mode of the undamaged beam. In this case the contribution of the other modes are more pronounced with respect to the previous case. The corresponding time discrete snap-shots representation of the nonlinear response is reported in Figure 13, where the modification of the initial undamaged mode shape during a complete cycle of response can be observed.

Finally, the results regarding the free vibrations of the simply supported beam under study, with three cracks and in presence of a damping $\zeta = 2\%$, subjected to initial conditions obtained by a combination of the first and second vibration modes of the undamaged beam, have been presented in Figures 14 and 15. In this case all cracks do not open and close at the same time and in addition the sequence of the crack state is not subjected to a fixed pattern, as shown by the crack state histories, reporting the open state of the cracks, at the bottom of Figure 14. The chosen initial conditions, together with the presence of damping, generate a nonlinear behaviour, with multiple linear sub-regions, which is easily handled by the proposed procedure.

CONCLUSIONS

In this work the non-linear dynamic behaviour of beams with multiple concentrated cracks has been analysed. The non-linearity, due to the crack opening and closing phenomenon, has been studied by means of the switching crack model. No discretisation of the beam has been adopted. The cracks have been modelled by means of Dirac's deltas which allowed the closed form evaluation of the beam mode shapes for a generic crack configuration.

An integration procedure has been proposed to compute the time history through modal analysis by considering the sequence of crack opening/closing phenomena together with the relative phase transition conditions.

Numerical analyses regarding beams with different boundary conditions have been presented, first, for the case of harmonic loading in order to compare the results with others available in the literature, then, different cases of free vibrations have been investigated.

The presented procedure can be extensively applied to a wide range of analyses and proved to be efficacious to treat the case of a nonlinear behaviour with multiple linear sub-regions.

Furthermore, the results obtained by means of the proposed procedure could be helpful in order to validate alternative appealing methods such as the '*Proper Orthogonal Decomposition*' [39], requiring a collection of data of the original response, obtained by a spatial discretisation of the continuous system, and could be the object of future work.

Finally, the application of the present work towards vibration based damage identification procedures involving non-linear behaviour is currently under investigation.

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TABLE CAPTION

Table 1. Cantilever beam configurations in precence of two cracks.

FIGURE CAPTIONS

Figure 1. The considered beam model with open/closed cracks.

Figure 2. Frequency response functions *u/P* (*u*=maximum displacement of the free end, *P*=harmonic load amplitude) for the cantilever beam with two cracks of Table 1: a) configuration 1; b) configuration 2; c) configuration 3; Harmonic Balance Method (HBM) [12] (continuous line); Time Numerical Integration (TNI) [18] (broken line); proposed approach (bold line).

Figure 3. Frequency response function *u/P* (*u*=maximum displacement of the free end, *P*=harmonic load amplitude) with the proposed approach for the cantilever beam [undamaged (continuous line); open cracks (broken line); closing cracks (bold line)] with two cracks in configuration 3 of Table 1: a) damping ratio 2%; b) damping ratio 1%; c) damping ratio 0.5%.

Figure 4. Frequency response functions u/u_S (u=maximum displacement of the middle span cross-section, u_S=static displacement of the undamaged beam) for the simply-supported beam with equally spaced cracks with intensity λ=0.05 subjected to an uniform harmonic load: a) 1 crack; b) 2 cracks; c) 4 cracks; d) 8 cracks.

Figure 5. Time response function u/u_S (u=maximum displacement of the middle span cross-section, u_S=static displacement of the undamaged beam) and crack state history [closed crack (empty plot), open crack (bold line)] for the simply-supported beam with 4 cracks with intensity λ=0.05 subjected to an uniform harmonic load at the peak frequencies $\alpha = 1.53$ (a and b), $\alpha = 1.75$ (c and d), $\alpha = 2.15$ (e and f); $\alpha = 3.05$ (g and h).

Figure 6. Sequence of snap-shots at different time instants for the simply-supported beam with 4 cracks subjected to an uniform harmonic load correspondent to the peak frequency $\alpha = 3.05$.

Figure 7. Frequency response functions u/u_S (u=maximum displacement of the cross-section at $\xi = 0.25$, u_S=static displacement of the undamaged beam due to the anti-symmetric uniform harmonic load) for the simply-supported beam with two cracks with intensity λ=0.1.

Figure 8. Time response function u/u_S (u=maximum displacement of the cross-section at $\xi = 0.25$, u_S=static displacement of the undamaged beam due to the anti-symmetric uniform harmonic load) for the simply-supported beam with two cracks with intensity λ=0.1 subjected to an uniform harmonic load at the peak frequencies.

Figure 9. Sequence of snap-shots at different time instants for the simply-supported beam with two cracks subjected to an anti-symmetric uniform harmonic load correspondent to the peak frequencies $\alpha = 6.06$ and $\alpha = 6.78$.

Figure 10. Time history of the free vibration response u/u_0 (u=displacement of the cross-section at ξ=0.25, u_0=maximum initial displacement value) of the simply supported beam with 3 cracks for the undamaged first mode initial condition: (a) total response; (b,c,d,e) modal contributions; (f) crack state history [closed crack (empty plot), open crack (bold line)].

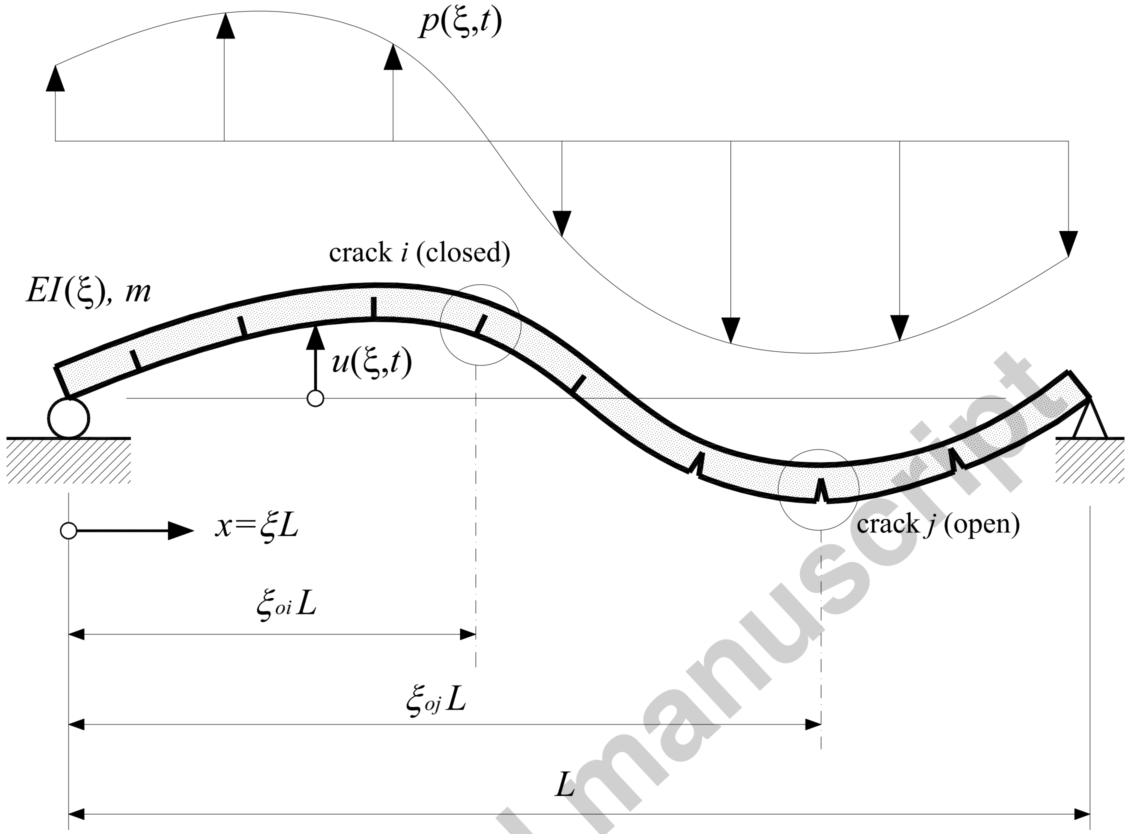
Figure 11. Sequence of snap-shots of the deformed shape at different time instants for the simply-supported beam with 3 cracks subjected to the undamaged first mode initial condition.

Figure 12. Time history of the free vibration response u/u_0 (u=displacement of the cross-section at ξ=0.25, u_0=maximum initial displacement value) of the simply supported beam with 3 cracks for the undamaged second mode initial condition: (a) total response; (b,c,d,e) modal contributions; (f) crack state history [closed crack (empty plot), open crack (bold line)].

Figure 13. Sequence of snap-shots of the deformed shape at different time instants for the simply-supported beam with 3 cracks subjected to the undamaged second mode initial condition.

Figure 14. Time history of the free vibration response u/u_0 (u=displacement of the cross-section at ξ=0.25, u_0=maximum initial displacement value) of the simply supported beam with 3 cracks for mixed undamaged first/second mode initial condition: (a) total response; (b,c,d,e) modal contributions; (f) crack state history [closed crack (empty plot), open crack (bold line)].

Figure 15. Sequence of snap-shots of the deformed shape at different time instants for the simply-supported beam with 3 cracks subjected to the mixed undamaged first/second mode initial condition.



$p(\xi,t)$
crack i (closed)
$EI(\xi)$, m
$u(\xi,t)$
$x=\xi L$
crack j (open)
$\xi_{oi} L$
$\xi_{oj} L$
L

Figure 1

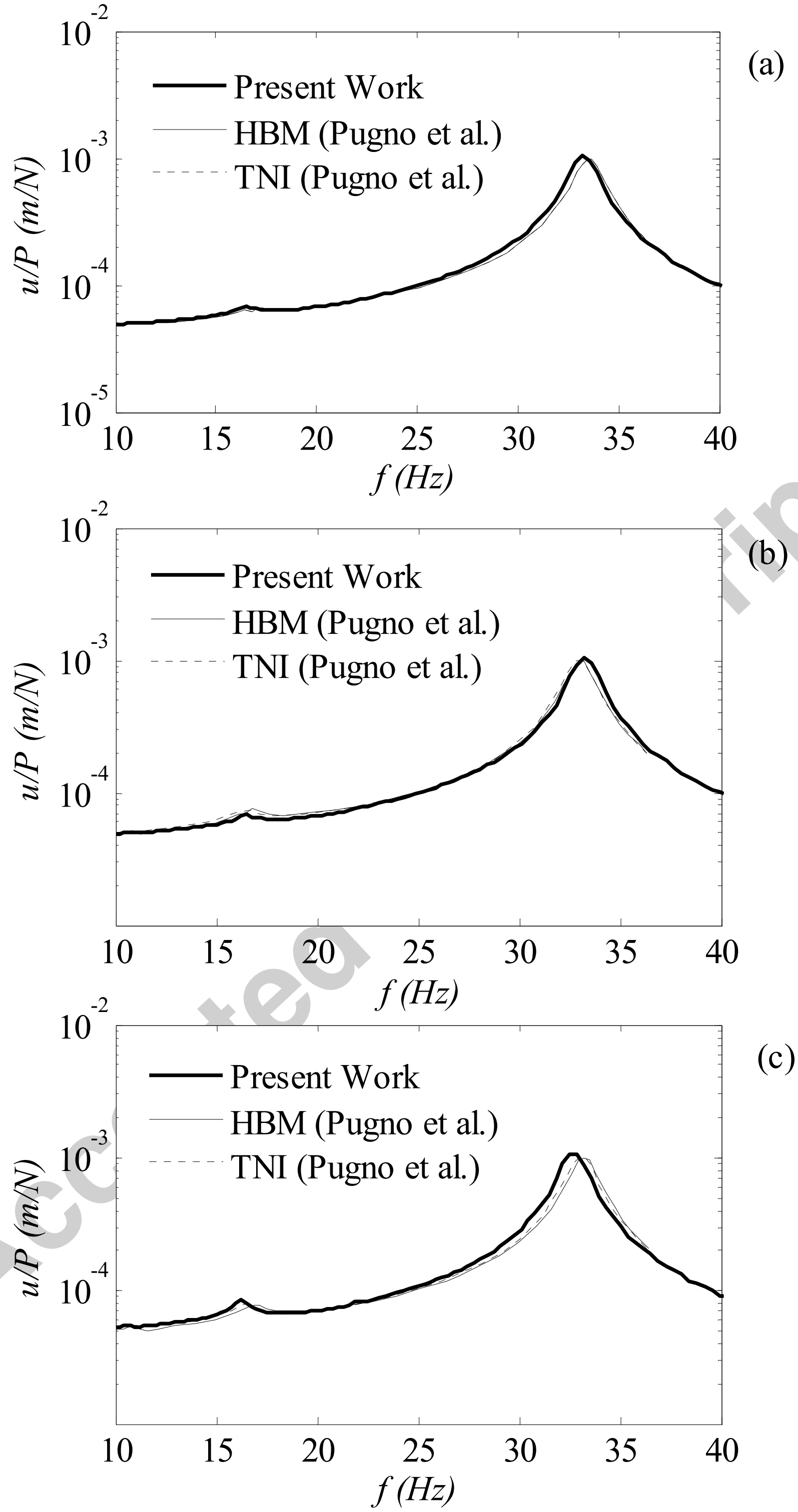

10^{-2}
10^{-3}
10^{-4}
10^{-5}
(a)
Present Work
HBM (Pugno et al.)
TNI (Pugno et al.)
u/P (m/N)
10 15 20 25 30 35 40
f (Hz)
(b)
Present Work
HBM (Pugno et al.)
TNI (Pugno et al.)
u/P (m/N)
10 15 20 25 30 35 40
f (Hz)
(c)
Present Work
HBM (Pugno et al.)
TNI (Pugno et al.)
u/P (m/N)
10 15 20 25 30 35 40
f (Hz)

Figure 2

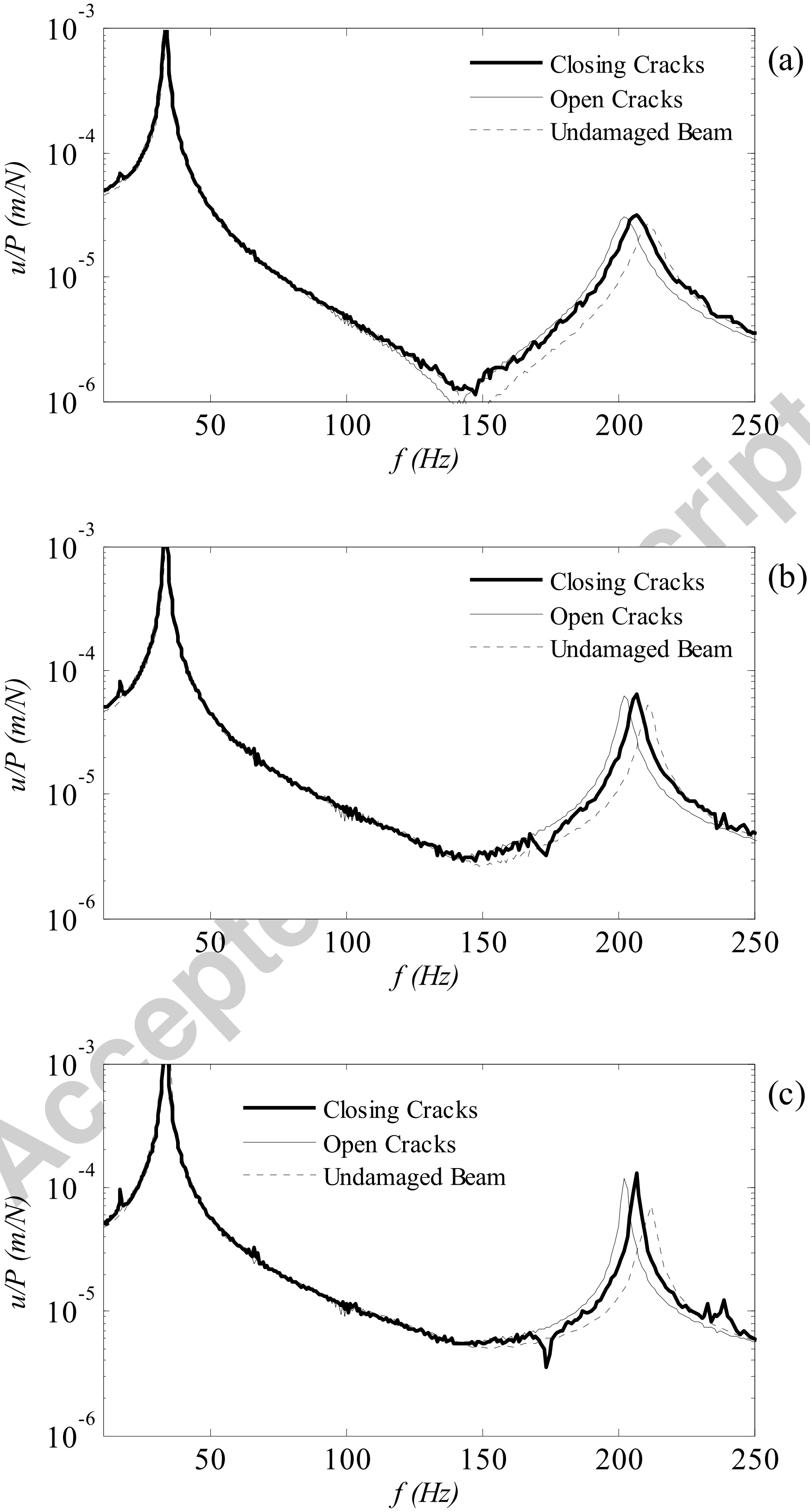
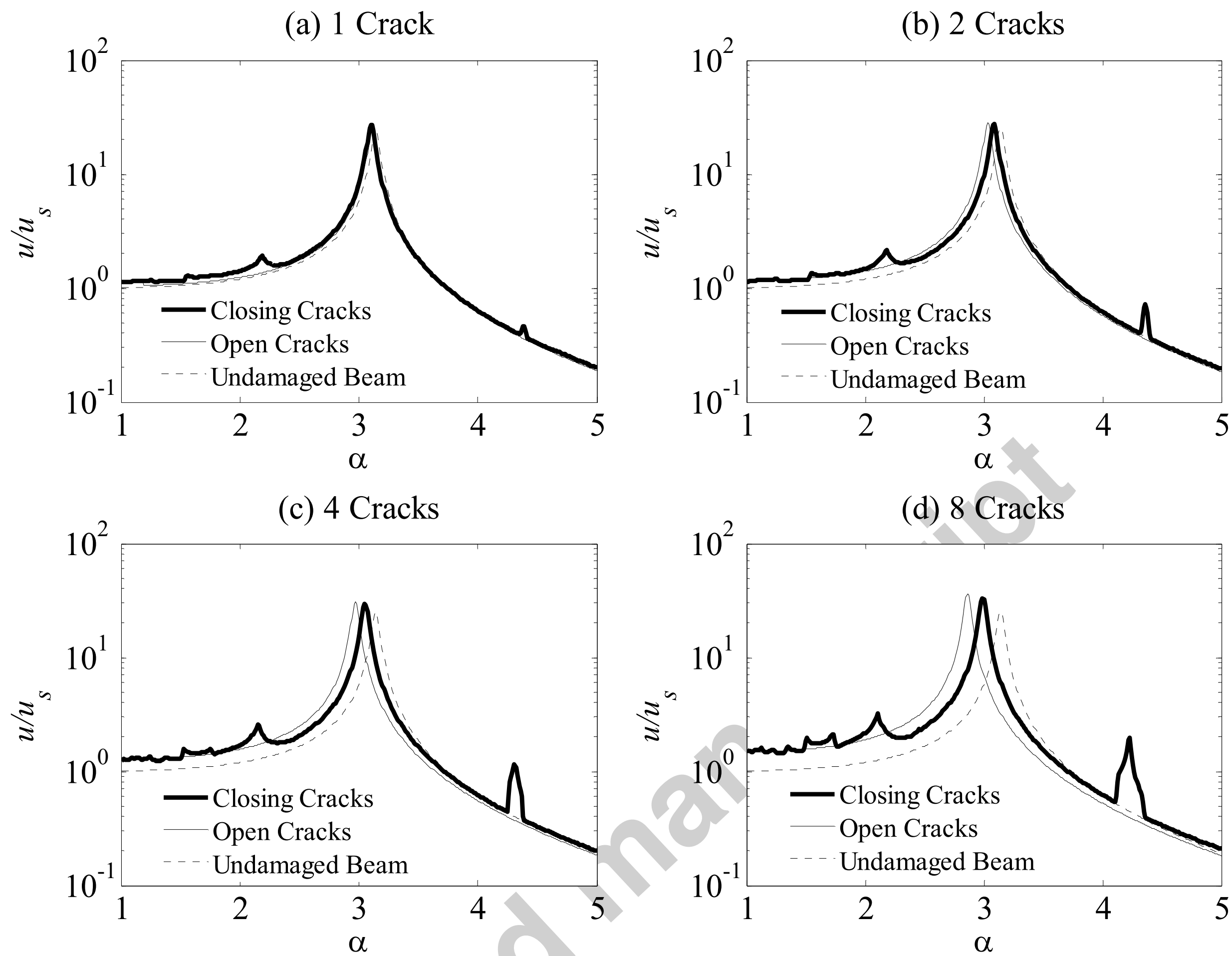

(a)
Closing Cracks
Open Cracks
Undamaged Beam
u/P (m/N)
f (Hz)
(b)
Closing Cracks
Open Cracks
Undamaged Beam
u/P (m/N)
f (Hz)
(c)
Closing Cracks
Open Cracks
Undamaged Beam
u/P (m/N)
f (Hz)

Figure 3



(a) 1 Crack
(b) 2 Cracks
(c) 4 Cracks
(d) 8 Cracks
10^2
10^1
10^0
10^{-1}
u/u_s
α
1
2
3
4
5
Closing Cracks
Open Cracks
Undamaged Beam

Figure 4

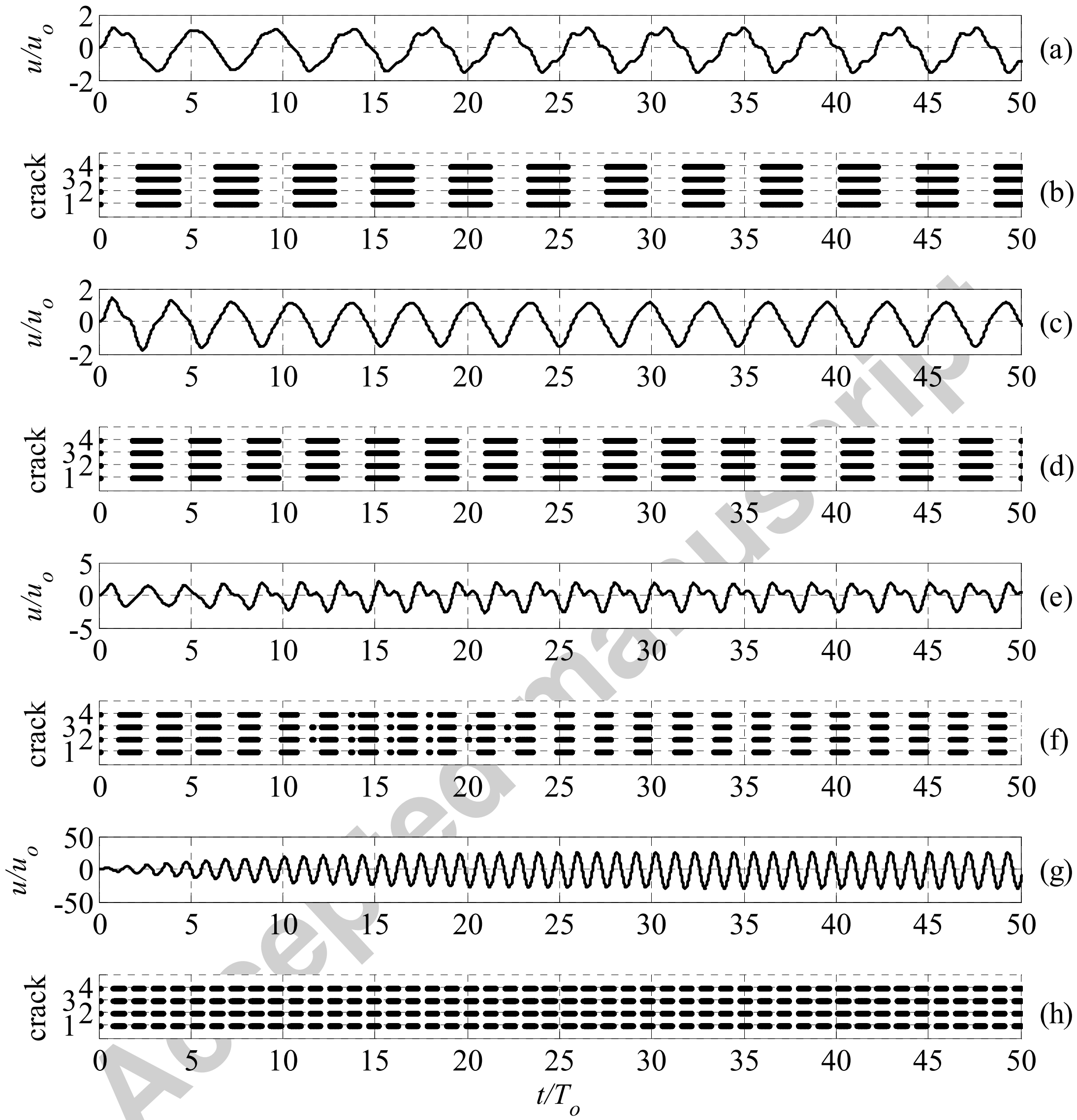


Figure 5

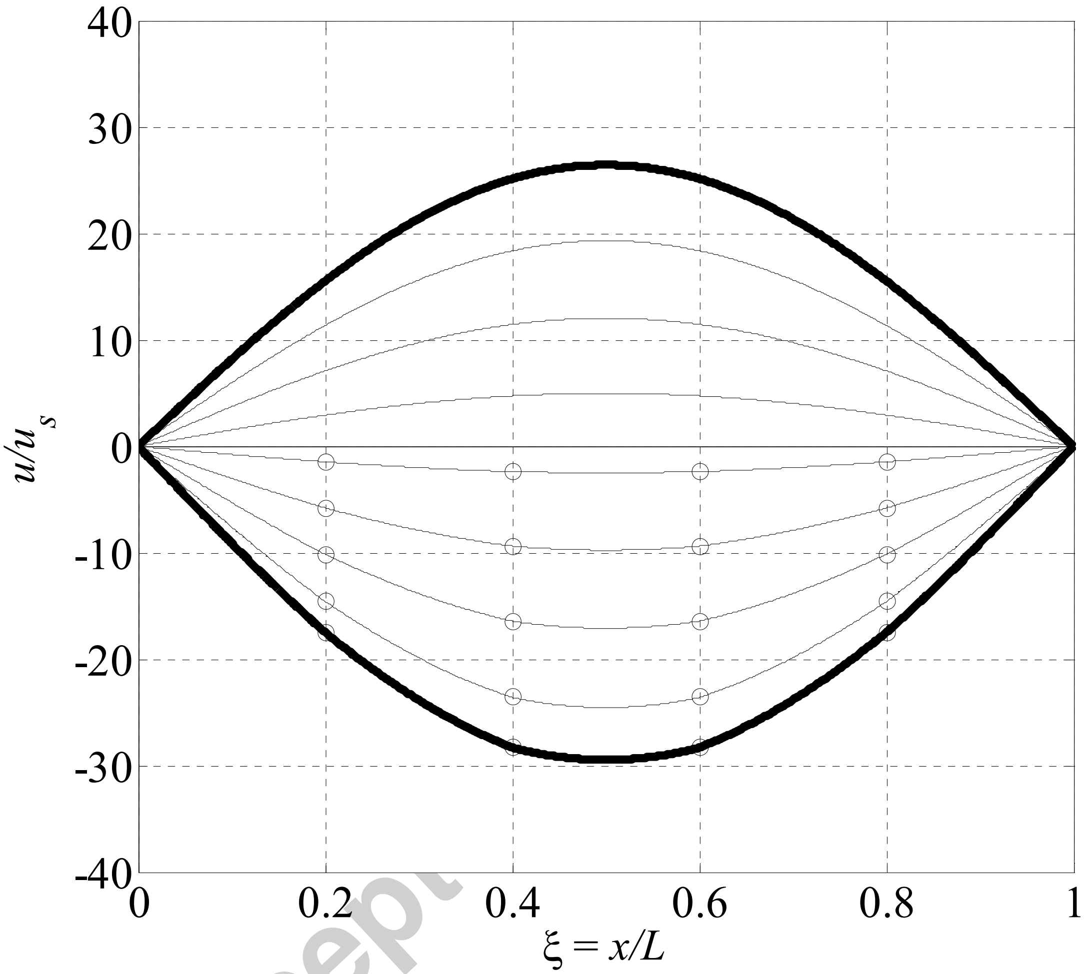


Figure 6

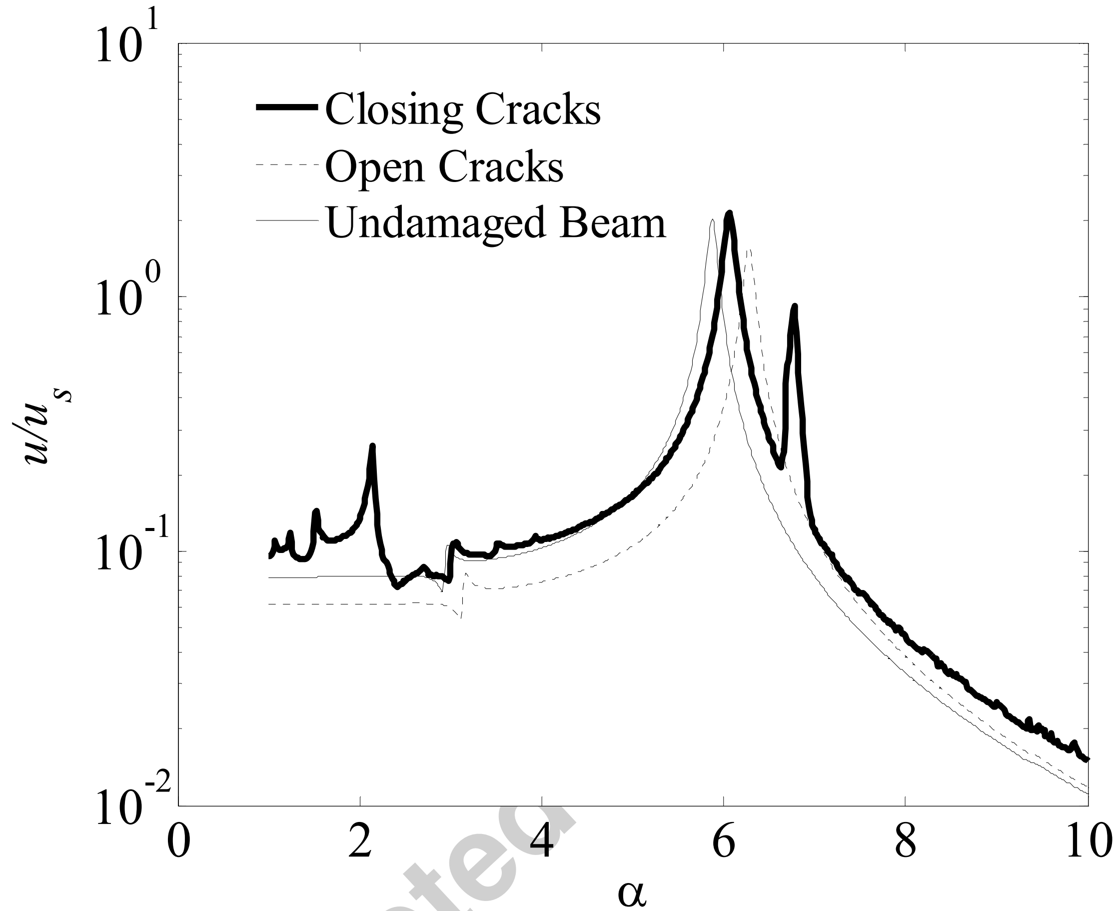

10^1
10^0
10^-1
10^-2
0
2
4
6
8
10
u/u_s
α
Closing Cracks
Open Cracks
Undamaged Beam

Figure 7

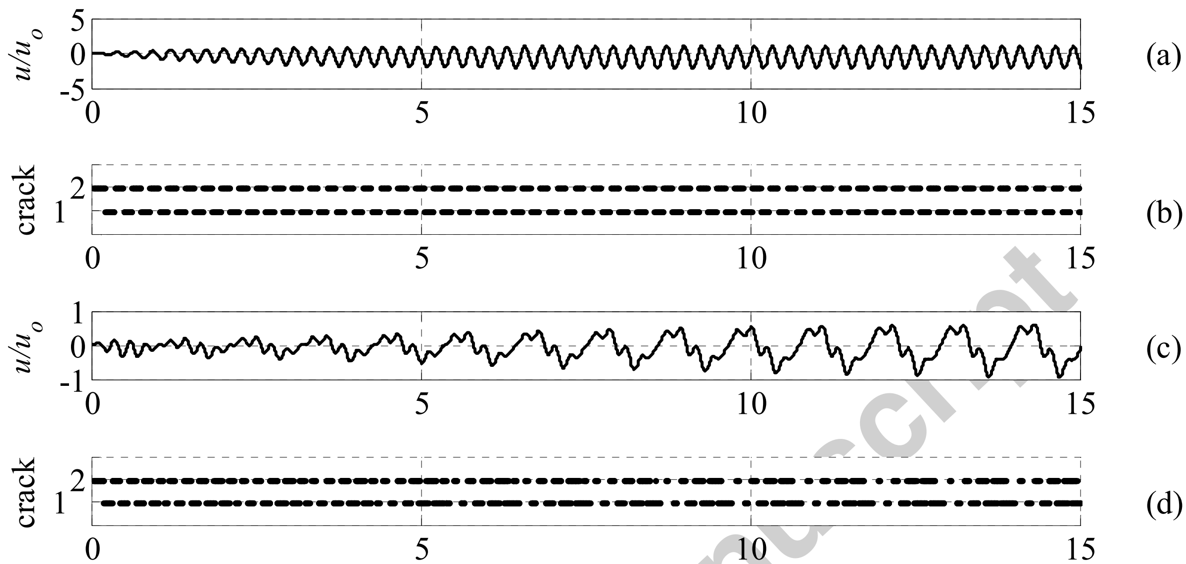


Figure 8

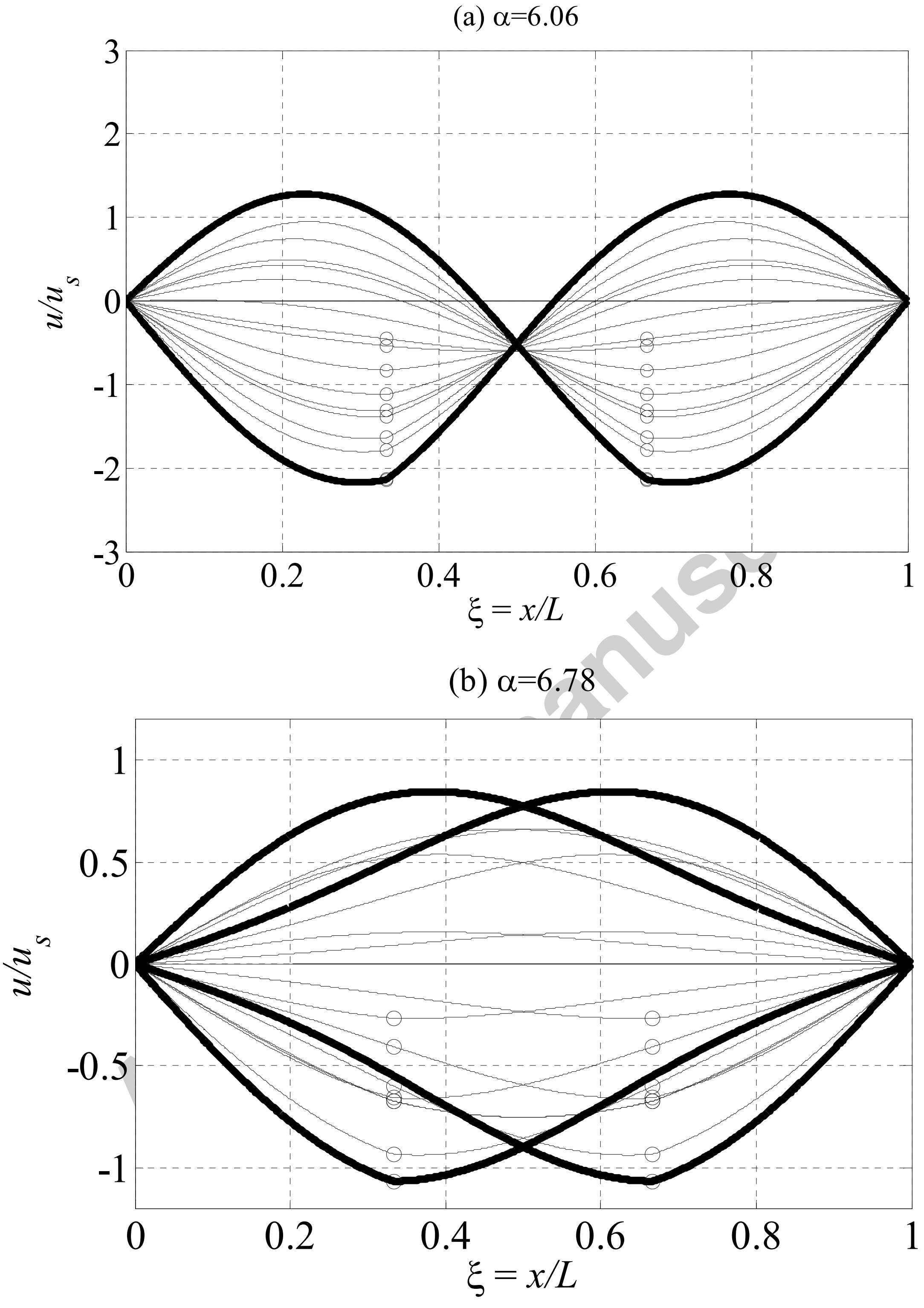


Figure 9

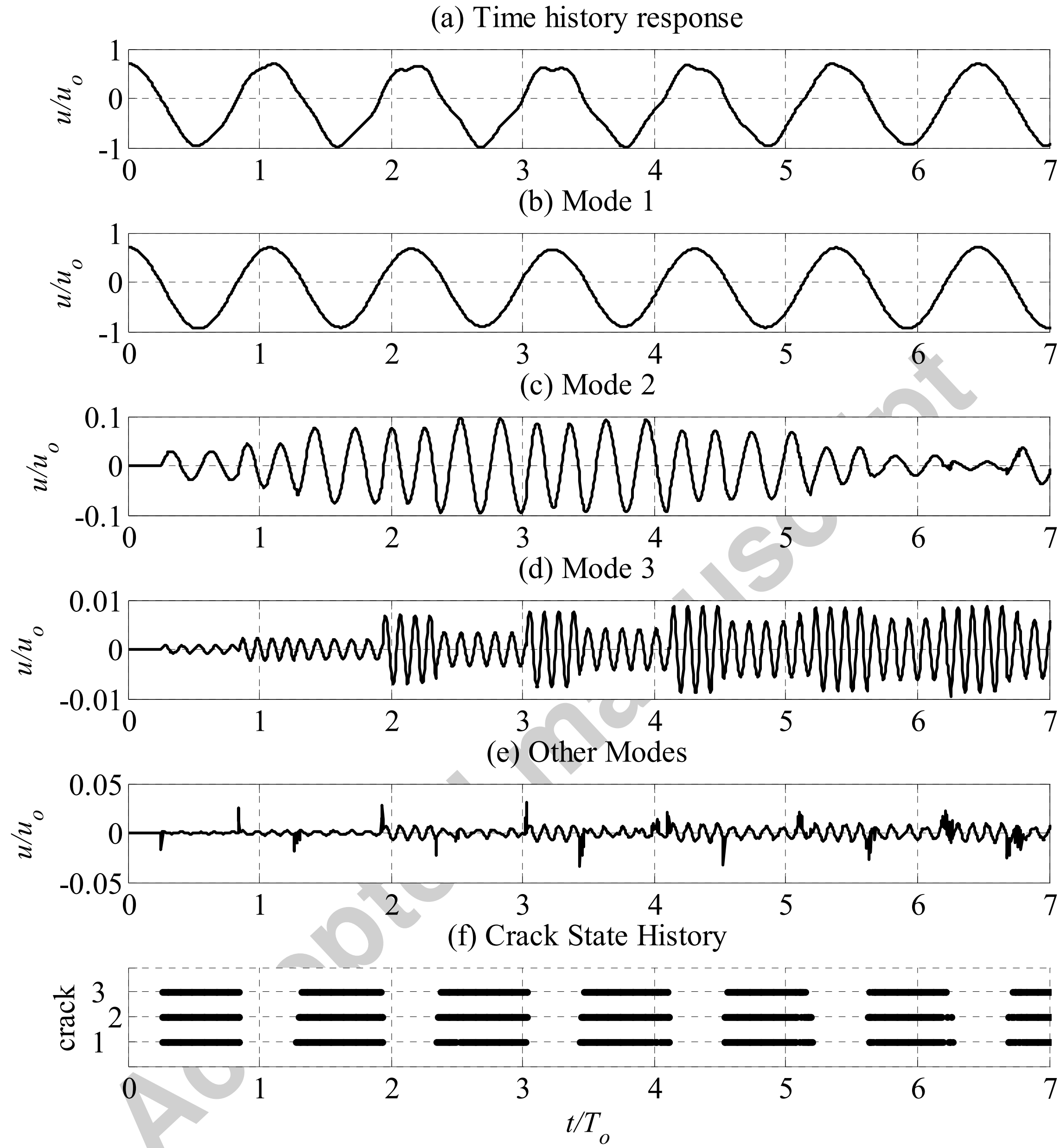

(a) Time history response
u/u_o
(b) Mode 1
(c) Mode 2
(d) Mode 3
(e) Other Modes
(f) Crack State History
crack
t/T_o

Figure 10

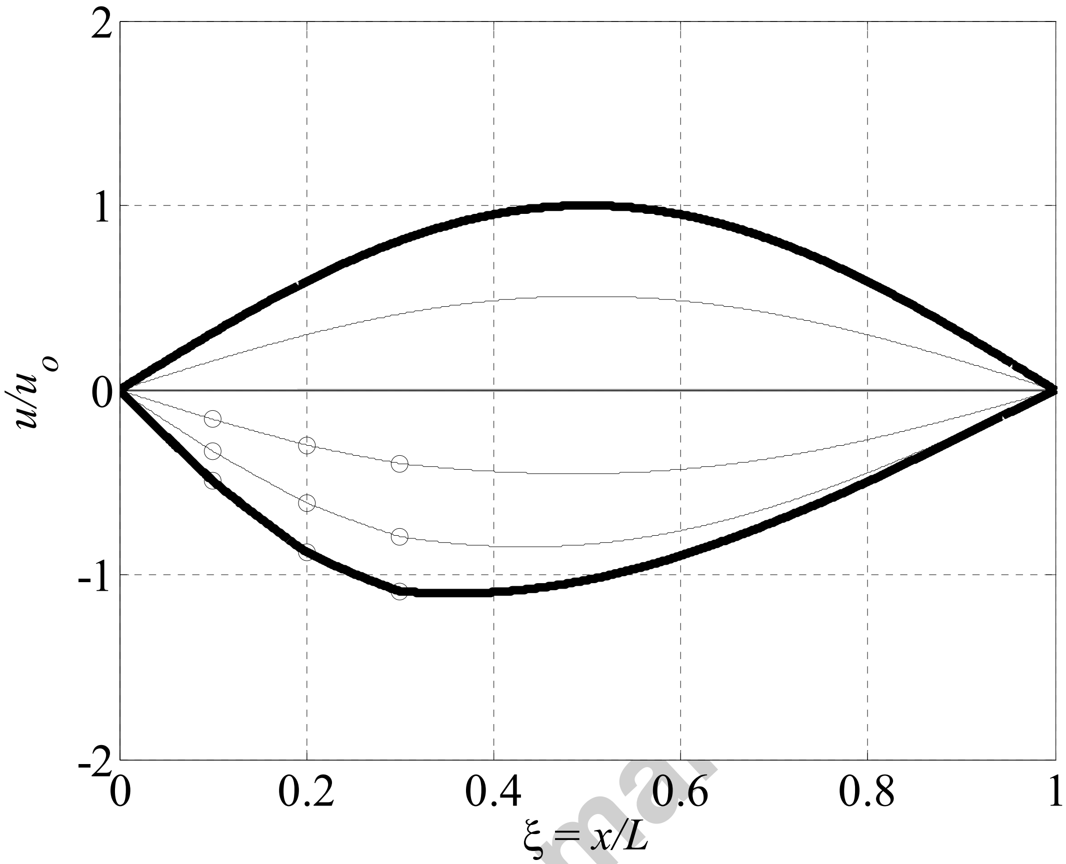


Figure 11

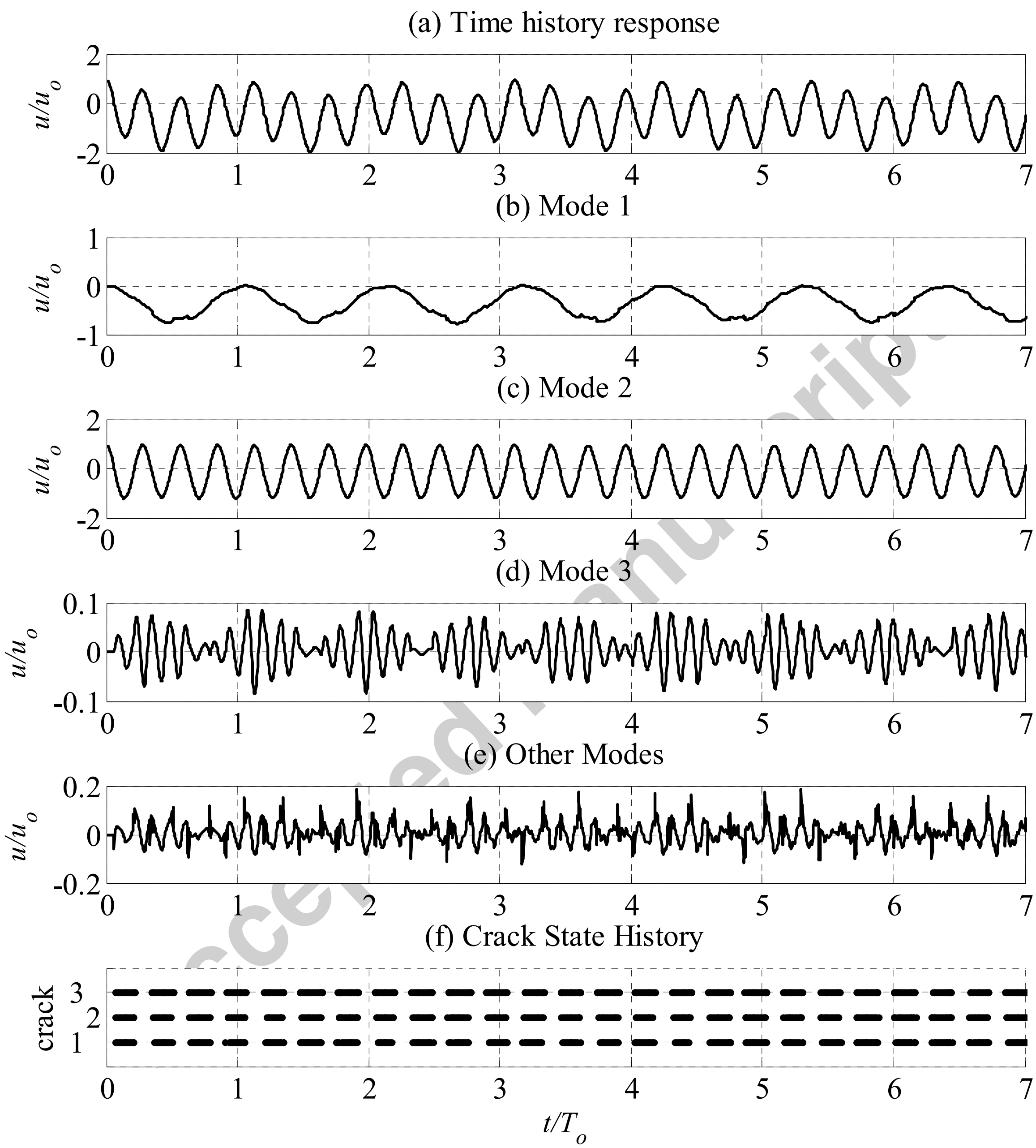

(a) Time history response
u/u_o
2
0
-2
0 1 2 3 4 5 6 7
(b) Mode 1
u/u_o
1
0
-1
0 1 2 3 4 5 6 7
(c) Mode 2
u/u_o
2
0
-2
0 1 2 3 4 5 6 7
(d) Mode 3
u/u_o
0.1
0
-0.1
0 1 2 3 4 5 6 7
(e) Other Modes
u/u_o
0.2
0
-0.2
0 1 2 3 4 5 6 7
(f) Crack State History
crack
3
2
1
0 1 2 3 4 5 6 7
t/T_o

Figure 12

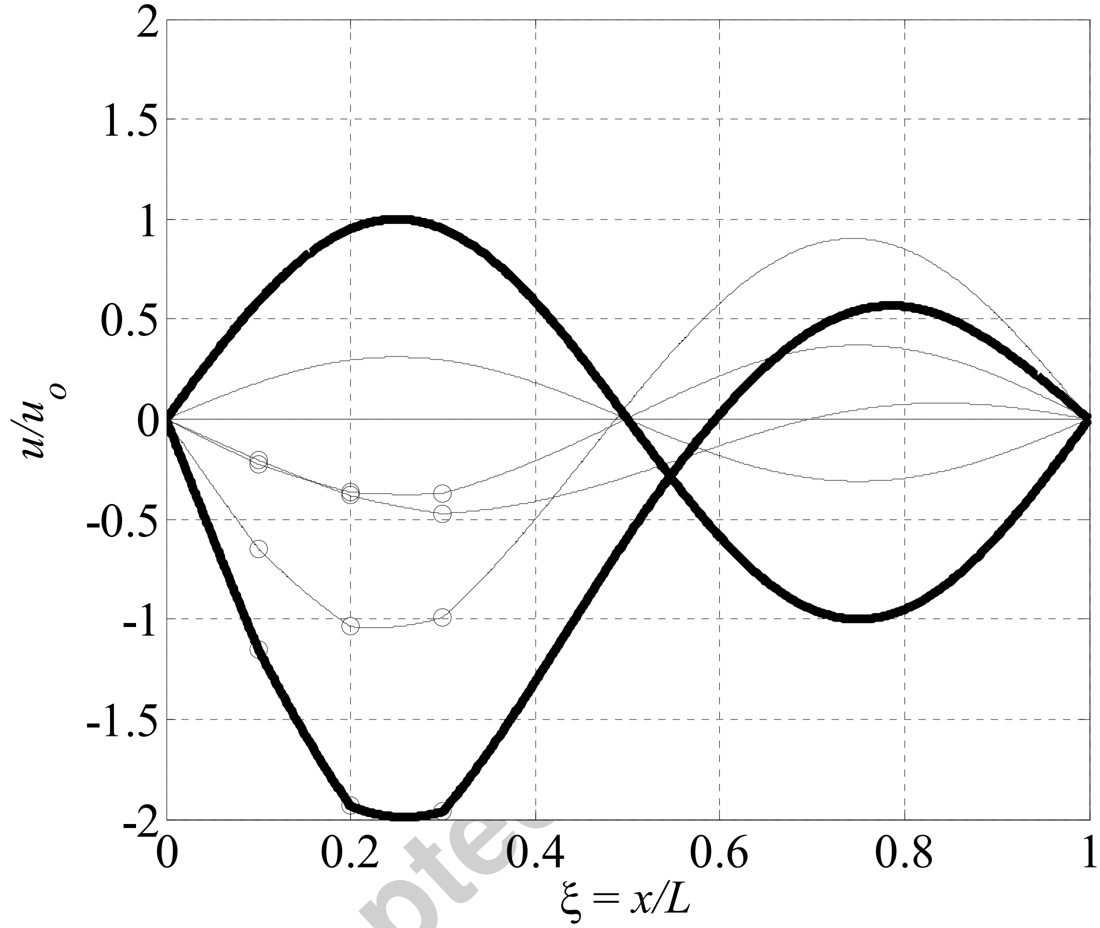


Figure 13

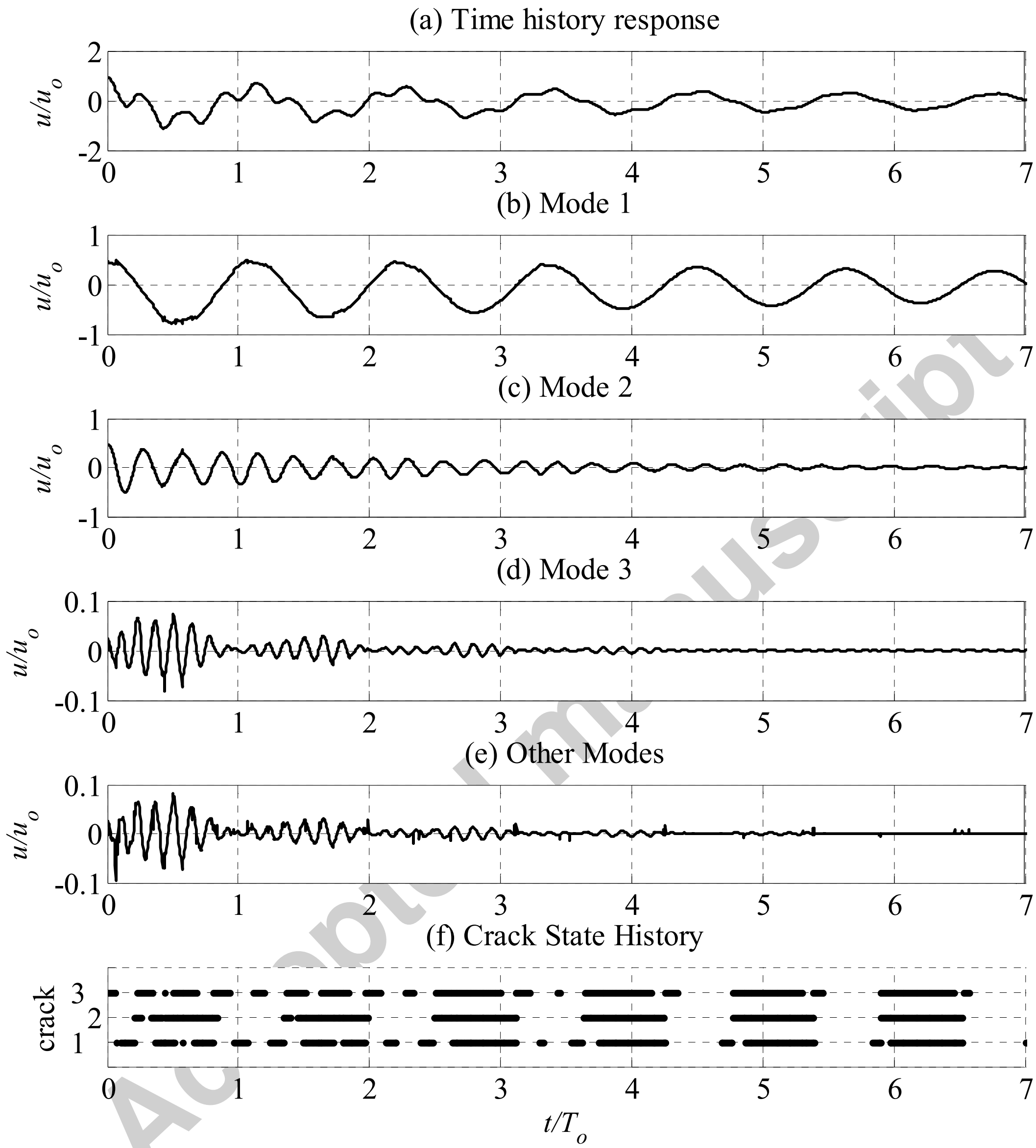

(a) Time history response
u/u_o
2
0
-2
(b) Mode 1
1
0
-1
(c) Mode 2
(d) Mode 3
0.1
-0.1
(e) Other Modes
(f) Crack State History
crack
3
2
1
0 1 2 3 4 5 6 7
t/T_o

Figure 14

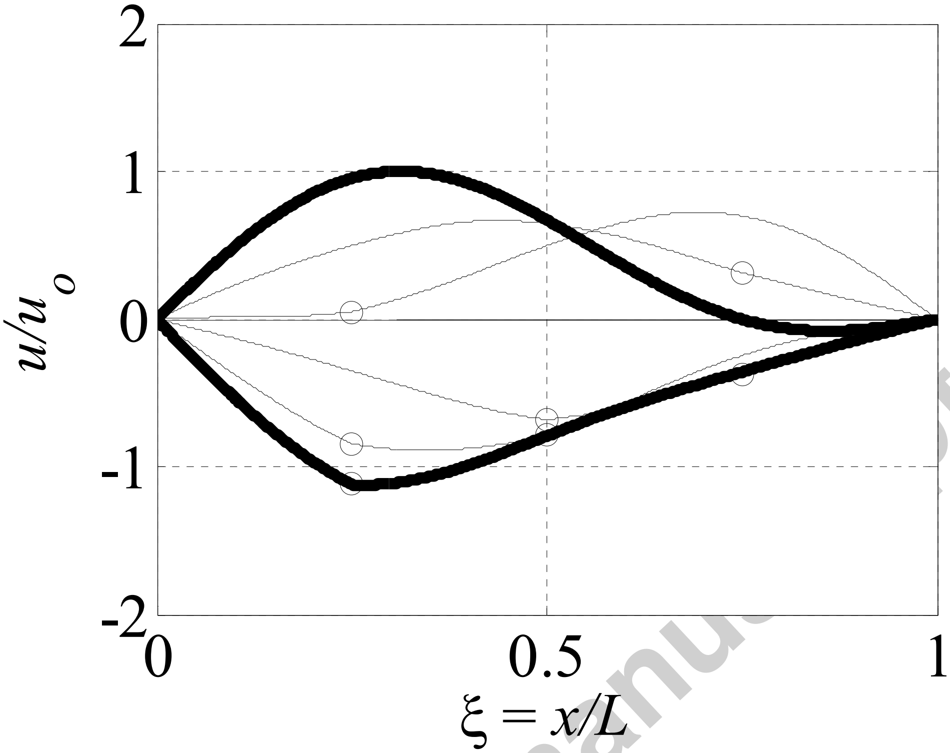


Figure 15

Configuration	Crack 1			Crack 2		
	Depth, d (mm)	Position (mm)	Intensity, λ	Depth, d (mm)	Position (mm)	Intensity, λ
1	4	50	0.0179	8	500	0.0564
2	6	50	0.033	8	500	0.0564
3	6	50	0.033	8	350	0.0564

Table 1