The nonlinear flexural-torsional behaviour of straight slender elastic beams with arbitrary cross sections

Eindhoven University of Technology Research Reports

EINDHOVEN UNIVERSITY OF TECHNOLOGY

Department of Mechanical Engineering
Division of Mechanical Engineering Fundamentals
Eindhoven The Netherlands

The Nonlinear Flexural-Torsional Behaviour of Straight Slender Elastic Beams with Arbitrary Cross Sections

Ir. G.M. van Erp

EUT Report WFW 87.050
ISBN 90-6808-004-0
ISSN 0167-9708

This work was supported by research grants made available by
The Netherlands Technology Foundation (STW)

Eindhoven
september 1987

Contents pag.

Abstract 4

Notation 5

1. Introduction 8

2. The potential energy functional 10

2.1 Three dimensional potential energy expression 10

2.2 Linear elastic isotropic materials 14

2.3 Technical beam theory 15

3. Kinematics of straight slender elastic beams 17

3.1 The undeformed configuration G(0) 17

3.2 The deformed configuration G(t) 18

4. The warping function 25

5. The strain energy U 27

6. Load potential 29

7. The potential energy functional 32

8. The rotation tensor **R** 33

9. Curvatures 37

10. Special theories 40

10.1 General 40

10.2 Beams exhibiting moderate deflections and large rotations 40

10.3 Beams exhibiting moderate deflections and moderate or small rotations 41

10.4 Beams exhibiting small deflections and large rotations 41

10.5 Beams exhibiting small deflections and moderate or small rotations 42

10.6 Second order theory 43

11. Comparison with other results 44

11.1 General 44

11.2 Thin-walled open sections 44

11.3 The rotation tensor 45

11.4 A.A. Ghobarah and Tso 46
'A non-linear thin-walled beam theory'

11.5 K. Roik, J. Carl and J. Lindner 49
'Biegetorsionsprobleme gerader dünnwändiger stäbe'

11.6 A. Rosen and P. Friedmann 51
'The nonlinear behaviour of elastic slender straight beams undergoing small strains and moderate rotations'

11.7 M.M. Attard 52
'Nonlinear theory of non-uniform torsion of thin-walled open beams'

11.8 Z.M. Elias 54
'Theory and methods of structural analysis'

11.9 Bending in one plane 55

12. Conclusions 57

13. Acknowledgement 58

14. References 59

Appendix A 62

B 65

C 67

D 69

E 70

F 71

G 74

H 76

I 80

J 82

Abstract

A potential energy functional for the non-linear flexural-torsional behaviour of straight slender elastic beams with arbitrary cross sections is derived. The result is generally applicable to situations where the strains are small and the Bernoulli hypotheses are valid.
To postpone the approximations as long as possible, a dyactic notation is used. Special theories obtained from the general theory by neglecting specific terms, are compared with results obtained by other researchers.

Notation

$\vec{a}$	vector
$\mathbf{A}$	second order tensor
$\lvert\vec{a}\rvert$	length of $\vec{a}$
$\lvert\mathbf{A}\rvert$	norm of $\mathbf{A}$
$\mathbf{A}^c$	conjugate of $\mathbf{A}$
$\mathbf{A}^{-1}$	inverse of $\mathbf{A}$
${}^4\mathbf{A}$	fourth order tensor
$\vec{a}\cdot\vec{b}$	inner product of two vectors
$\vec{a}*\vec{b}$	cross product of two vectors
$\vec{a}\vec{b}$	dyadic product of two vectors
$\mathbf{A}\cdot\vec{a}$	inner product of a tensor and a vector
$\mathbf{A}\cdot\mathbf{B}$	inner product of two tensors
$\mathbf{A}:\mathbf{B}$	double inner product of two tensors
$tr(\mathbf{A})$	trace of $\mathbf{A}$
$det(\mathbf{A})$	determinant of $\mathbf{A}$
A	undeformed cross sectional area
d	linear dimension of the cross section
D, D^*	geometric constants
E	Young's modulus
$\vec{e}_i$	unit base vector
e_{ij}	cartesian components of the Green-Lagrange strain tensor
$\vec{e}$	unit vector
$\mathbf{F}$	deformation gradient tensor
f	function representing the warping displacements
G	shear modulus
G(0)	undeformed configuration
G(t)	deformed configuration
G_1, G_2	potentials
g	warping amplitude
H	higher-order torsion constant
$\mathbf{I}$	unit tensor
$\vec{i}$	unit vector
I_2, I_3	moments of inertia about the y and z axis respectively
I_s	polar moment of inertia about the shear center

J	jacobian
$\vec{k}$	axial vector
l	length
$\vec{n}$	unit normal vector
$\vec{p}$	boundary force vector
p_i, P_i	cartesion component of force vector
$\mathbf{P}_2$	2nd Piola-Kirchoff stress tensor
P_{ij}	cartesian components of $\mathbf{P}_2$
p	material point
$\mathbf{Q}$	rotation tensor
$\vec{q}$	bodyforce per unit mass
q_i	cartesian component of distributed load vector
$\mathbf{R}$	rotation tensor
$\underline{R}$	rotation matrix
R_{ij}	cartesian components of R
$\vec{r}(\vec{x},t)$	beam axis
$\tilde{r}$	radial distance of a point on the cross section to the shear center
S	surface
s	arc length
subscript s	shear center
t	deformed state
U	strain energy
u_s, v_s, w_s	displacement components of the shear center in the direction $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ respectively
$\bar{U}$	average displacement of the cross section
$\vec{U}(\vec{x})$	displacement field
V	work done by the applied loads
V_0, V_t	volume in configuration G(0) respectively G(t)
$\vec{w}(\vec{x})$	weighting function
$\vec{x}$	position vector
x	coordinate along beam axis
y, z	coordinate along the principal centroidal axes
$\tilde{y}$, $\tilde{z}$	coordinate along y and z axes through the shear center
$\alpha, \beta, \phi, \varphi, \gamma$	angles
$\vec{\alpha}_o$	$= \tilde{y}\vec{e}_2 + \tilde{z}\vec{e}_3$

$\alpha(t)$	jacobian
$\beta_2, \beta_3, \beta_\psi$	geometric constants
$\vec{\gamma}$	axial vector
Γ	warping constant
δ	symbol for first variation or infinitesimal quantity
ε	strain
$\vec{\varrho}$	axial vector
ϱ	mass density
π	potential energy functional
ϕ	strain energy function
ψ	warping function
$\vec{\eta}$	unit vector
$\vec{\nabla}$	gradient operator
Ω	potential energy of the loads
$\vec{\omega}$, $\vec{\mu}$	axial vectors
χ_i	curvature components
$\vec{\chi}$	curvature vector
$O(\ )$	order of magnitude
${}^4\mathbf{L}$	fourth order elastic material tensor
$(')$	differentiation with respect to x
σ	Cauchy stress tensor

1. Introduction

One of the research projects in the Faculty of Mechanical Engineering at Eindhoven University of Technology is entitled; 'Lateral-torsional buckling of aluminium beams with complex cross sections'.
A characteristic feature of aluminum is its low Young's modulus which can cause moderate deflections prior to buckling. To incorporate these deflections into a buckling and/or post-buckling analysis, a non-linear beam theory is needed.
The two approaches to deriving such a theory are the equilibrium and the energy methods. Since in this report only elastic material is considered, preference is given to the potential energy formulation.
The potential energy functional in terms of strains, for finite displacement problems is usually obtained from the one used in the linear theory by replacing the linear strain expressions with non-linear ones. Since this is not the correct way to derive such an expression, this report starts with the derivation of the potential energy functional from the three-dimensional theory of elasticity.
During the past 15 years several articles on non-linear flexural-torsional behaviour of beams have been published. In the majority of these articles, a potential energy functional in terms of displacements and rotations is used, which is mostly derived in the following manner. Firstly, the displacement field containing the components of the rotation matrix is determined, then, this field is used to calculate the non-linear strain components and, finally, these strains are used to determine the potential energy functional.
Deriving the potential energy functional in this manner, without introducing various approximations is almost impossible, because both the components of the rotation matrix and the strain expressions in terms of displacements are lengthy and complicated in their exact form. Therefore, most of the articles on non-linear flexural-torsional behaviour of beams restrict themselves to a special class of deformation as a consequence of the approximations made.
In this report, a coordinate-free dyadic notation is used to postpone the approximations as long as possible. This enables a potential energy functional and curvature expressions to be derived that are generally

applicable to situations where the strains are small and the Bernoulli hypotheses are valid.
It is shown that the special theories of the articles mentioned can be derived easily from these general expression by neglecting some specific terms.
Rotations in non-linear beam theory are mostly described in terms of Euler-angles or modified Euler-angles. This results in a 'geometric-torsion' expression which is asymmetric in the transverse displacement components and therefore often leads to confusion. In this report, the rotations are described in a special way, resulting in a 'geometric-torsion' expression which is 'skew'-symmetric in these components, as would be expected.
Since in this report beams with arbitrary cross sections are being considered, the warping displacements are described in a more general way than is usual for beams with thin-walled open cross sections. An expression similar to the one proposed by Reissner [15] for non-uniform linear torsion is used. The 'normalized' warping ($\psi(y,z)$) is described with the well-known Saint Venant warping function, while the amplitude of the warping is taken to be a function $g(x)$ yet to be determined.
Readers who are not familiar with the dyadic notation used in this report should refer to appendix A, [23] or any other appropriate textbook on this subject.

2. The potential energy functional

2.1 Three dimensional potential energy expression

2.1.1 Consider the deformation of a body from an undeformed state G(0) to a deformed state G(t), with no restriction as to the amount of deformation. Assuming inertia effects to be negligible, it follows from the momentum conservation law that in every state and at each material point the following equilibrium equation must be satisfied:

$$(\vec{\nabla}\cdot\sigma_t) + \varrho_t\vec{q}_t = 0 \quad , \quad \forall\ \vec{x}\in V_t \tag{2.1}$$

where: σ is the Cauchy stress tensor; $\vec{\nabla}$ is a gradient operator; ϱ_t is the mass-density; $\vec{q}_t$ represents the body force per unit mass and $\vec{x}$ is the position vector.

From the moment of momentum conservation law it follows that the Cauchy stress tensor is symmetric, hence

$$\sigma_t = \sigma_t^c \quad , \quad \forall\ \vec{x} \in V_t \tag{2.2}$$

where: σ^c is the conjugate of σ

Simultaneously the stress distribution and the deformation have to satisfy the constitutive equation, the strain-displacement relationship, the kinematic boundary conditions and the dynamic boundary conditions.

2.1.2 Since, generally speaking, an exact solution of the above equations cannot be found, attention is focused on the determination of an approximate solution. Equations (2.1) and (2.2) are not very suitable for this purpose and thus an integral formulation is used. According to the principle of weighted residuals, the equilibrium equation (2.1) is equivalent to the requirement that the following integral equation is satisfied for every allowable weighting function $\vec{w}(\vec{x})$

$$\int_{V_t} [(\vec{\nabla}\bullet\sigma_t) + \varrho_t\vec{q}_t]\bullet\vec{w}\, dv = 0 \qquad (2.3)$$

Integrating by parts and applying Gauss' theorem yields:

$$\int_{V_t} \sigma_t:(\vec{\nabla}\vec{w})dv = \int_{V_t} \varrho_t\vec{q}_t\bullet\vec{w}\, dv + \int_{S_t} \vec{p}_t\bullet\vec{w}\, dS \qquad (2.4)$$

where: $\vec{p}_t = \vec{n}_t\bullet\sigma_t$

$\vec{p}_t$ is the stress vector at a point of the boundary and $\vec{n}_t$ is the unit outward normal at that point.

2.1.3 If the weighting functions are considered to be virtual displacement functions $\delta\vec{u}(\vec{x})$, the so-called weak form of the weighted residual formulation (2.4) is transformed into the principal of virtual work.

$$\int_{V_t} \sigma_t:(\vec{\nabla}\delta\vec{u})\, dv = \int_{V_t} \varrho_t\vec{q}_t\bullet\delta\vec{u}\, dv + \int_{S_t} \vec{p}_t\bullet\delta\vec{u}\, dS \qquad (2.5)$$

Excluding mixed boundary conditions, the boundary S_t of a body can be subdivided into a part S_t^u where the displacements are prescribed and a part S_t^p where the load $\vec{p}_t$ is prescribed to be $\tilde{\vec{p}}_t$. Since $\delta\vec{u} = \vec{0}$ on S_t^u, (2.5) can be replaced by:

$$\int_{V_t} \sigma_t:(\vec{\nabla}\delta\vec{u})dv = \int_{V_t} \varrho_t\vec{q}_t\bullet\, \delta\vec{u}\, dv + \int_{S_t^p} \tilde{\vec{p}}_t\bullet\delta\vec{u}\, dS \qquad (2.6)$$

when (2.6) is satisfied for all admissible $\delta\vec{u}(\vec{x})$, σ satisfies the equation of equilibrium (2.1) and the dynamic boundary conditions $\vec{p}_t=\tilde{\vec{p}}_t$ on S_t^p.

2.1.4 Equation (2.6) is formulated in the deformed state of the body. This state however is not known a priori and therefore (2.6) is often transformed such that it refers to the known undeformed stat G(0). (2.6) then changes to:

$$\int_{V_0} (\mathbf{P}_{2t}:\delta\mathbf{E})dv = \int_{V_0} \varrho_0\vec{q}_t\bullet\delta\vec{u}\ dv + \int_{S_0^p} \alpha_t\ \tilde{\vec{p}}_t\bullet\delta\vec{u}\ dS \qquad (2.7)$$

* $\mathbf{P}_{2t}$ is the 2nd Piola-Kirchhoff stress tensor, related to the Cauchy stress tensor by: $\mathbf{P}_{2t} = \det(\mathbf{F}_t)\mathbf{F}_t^{-1}\bullet\sigma_t\bullet\mathbf{F}_t^{-c}$ (2.8)

* $\delta\mathbf{E}$ is the first variation of the Green-lagrange strain tensor $\mathbf{E}_t$.

$$\mathbf{E} = \frac{1}{2}\,[\vec{\nabla}_0\vec{u}_t + (\vec{\nabla}_0\vec{u}_t)^c + \vec{\nabla}_0\vec{u}_t\bullet(\vec{\nabla}_0\vec{u}_t)^c] \qquad (2.9)$$

$$\delta\mathbf{E} = \frac{1}{2}\,[\vec{\nabla}_0\delta\vec{u} + (\vec{\nabla}_0\delta\vec{u})^c + \vec{\nabla}_0\vec{u}_t\bullet(\vec{\nabla}_0\delta\vec{u})^c + \vec{\nabla}_0\delta\vec{u}\bullet(\vec{\nabla}_0\vec{u}_t)^c]$$

* $\varrho_0 = (\det \mathbf{F}_t)\varrho_t$ (mass conservation law)

* α_t is a jacobian

* $\mathbf{F} = \vec{\nabla}_0\vec{x}$ is the deformation gradient tensor

Note that the 2nd Piola-Kirchhoff stress is not a stress in the physical sense; rather, it is a mathematically convenient stress measure obtained through a transformation process.

2.1.5 In the case of elastic material and negligible temperature effects, $\mathbf{P}_2$ is related to Φ, the strain energy function, by:

$$\mathbf{P}_2 = \frac{d\Phi}{d\mathbf{E}} \quad , \quad \Phi = \int_0^{\mathbf{E}} \mathbf{P}_2:d\mathbf{E} \qquad (2.10)$$

With (2.10) the first integral of (2.7) may be written as:

$$\int_{V_0} (\mathbf{P}_{2t}:\delta\mathbf{E})dv = \int_{V_0} \frac{d\Phi_t}{d\mathbf{E}}:\delta\mathbf{E}\ dv = \delta \int_{V_0} \Phi_t\ dv \qquad (2.11)$$

For conservative body forces $\varrho_0\vec{q}_t$ and surface forces $\alpha_t\tilde{\vec{p}}_t$ it is possible te define potentials G_1 and G_2 by

$$G_1 = \int_0^{\vec{u}_t} \varrho_0 \, \vec{q}_t \bullet d\vec{u} \quad ; \quad G_2 = \int_0^{\vec{u}_t} \alpha_t \, \tilde{\vec{p}}_t \bullet d\vec{u} \tag{2.12}$$

and reversily

$$\varrho_0 \, \vec{q}_t = \frac{dG_1}{d\vec{u}} \quad , \quad \alpha_t \, \tilde{\vec{p}}_t = \frac{dG_2}{d\vec{u}} \tag{2.13}$$

The variations of G_1 and G_2 due to variations of the displacements are given by:

$$\delta G_1 = \frac{dG_1}{d\vec{u}} \bullet \delta\vec{u} = \varrho \vec{q}_t \bullet \delta\vec{u} \quad ; \quad \delta G_2 = \frac{dG_2}{d\vec{u}} \bullet \delta\vec{u} = \alpha_t \bar{\vec{p}}_t \bullet \delta\vec{u} \tag{2.14}$$

2.1.6 Substitution of (2.11) and (2.14) into (2.7) yields:

$$\int_{V_0} \delta\phi_t \, dv - \int_{V_0} \delta G_1 \, dv - \int_{S_0} \delta G_2 \, dS =$$

$$\delta[\int_{V_0} \phi_t dv - \int_{V_0} G_1 dv - \int_{S_0} G_2 dS] = 0 \quad \forall \, \delta\vec{u} \tag{2.15}$$

The expression in brackets is the potential energy functional π for finite deformations. Eqn. (2.15) expresses that π_t is stationary with respect to all admissible variations of the displacements, only if the equilibrium equation (2.1) and the dynamic boundary conditions are satisfied.

In the case of 'dead loads' the body forces $\varrho_0 \vec{q}_t$ and surface forces $\alpha_t \bar{p}_t$ are independent of the displacements, (2.12) then becomes

$$G_1 = \varrho_0 \, \vec{q} \bullet \vec{u}_t \text{ and } G_2 = \alpha_t \, \tilde{\vec{p}}_0 \bullet \vec{u}_t \tag{2.16}$$

The potential energy functional π_t then changes to:

$$\pi_t = \int_{V_0} \phi_t \, dv - \int_{V_0} \varrho_0 \vec{q} \bullet \vec{u}_t \, dv - \int_{S_0^p} \tilde{\vec{p}}_0 \bullet \vec{u}_t \, dS \qquad (2.17)$$

2.2 Linear elastic isotropic materials

2.2.1 As already mentioned in section 2.5, the 2nd Piola Kirchhoff stress tensor is a function of the Green-Lagrange strain tensor only, in the case of elastic material behaviour and negligible temperature effects.

$$\mathbf{P}_2 = \mathbf{P}_2(\mathbf{E}) \qquad (2.18)$$

If attention is restricted to isotropic materials and small strains, the actual material behaviour can be approximated reasonably well by the linear relationship:

$$\mathbf{P}_2 = 2G[\mathbf{E} + \frac{\nu}{1-2\nu} \, \mathrm{tr}\,(\mathbf{E})\mathbf{I}] = {}^4\mathbf{L}:\mathbf{E} \qquad (2.19)$$

* G is the shear modulus
* ν is the Poisson ratio
* $\mathbf{I}$ is the unit tensor
* ${}^4\mathbf{L}$ is an isotropic fourth-order tensor defined by:

$$ {}^4\mathbf{L} = G[{}^4\mathbf{I} + {}^4\mathbf{I}^c + \frac{2\nu}{1-2\nu} \, \mathbf{II}] \qquad (2.20)$$

* ${}^4\mathbf{I}$, ${}^4\mathbf{I}^c$ and $\mathbf{II}$ are fourth order tensors, respectivily defined by:

$$ {}^4\mathbf{I}:\mathbf{A} = \mathbf{A} \quad ; \quad {}^4\mathbf{I}^c:\mathbf{A} = \mathbf{A}^c \quad , \quad \mathbf{II}:\mathbf{A} = \mathrm{tr}(\mathbf{A})\mathbf{I} \qquad (2.21)$$

Substituting (2.19) into (2.10) and (2.11) yields:

$$\phi_t = \int_0^{\mathbf{E}_t} ({}^4\mathbf{L}:\mathbf{E}):d\mathbf{E} = \frac{1}{2} \, ({}^4\mathbf{L}:\mathbf{E}_t):\mathbf{E}_t \qquad (2.22)$$

Combination of (2.22) and (2.17) renders the potential energy functional for linear elastic isotropic materials with small strains loaded by 'dead loads'

$$\pi_t = \int_{V_0} \frac{1}{2} ({}^4\mathbf{L}:\mathbf{E}_t):\mathbf{E}_t dv - \int_{V_0} \varrho_0 \vec{q}\cdot\vec{u}_t \, dv - \int_{S_p^0} \tilde{\vec{p}}_0\cdot\vec{u}_t \, dS \qquad (2.23)$$

The remaining part of this paper is restricted to linear elastic isotropic materials with small strains loaded by dead loads.

2.3 Technical beam theory

2.3.1 The object of beam theory is to reduce a three dimensional problem to an approximate 1-dimensional one, by making simplifying assumptions with respect to stress distribution, displacement field and constitutive equations. The variational process will then lead to the proper equations of equilibrium and boundary conditions for the problem at hand, except that it now has certain internal constraints (as a result of stress and displacement assumptions) and behaves according to the assumed constitutive law.

2.3.2 The main assumptions made in beam theory are:

* The cross section does not distort in its plane during deformation.
* The cross sections are orthogonal to the beam axis after deformation
* Transverse normal stresses are assumed to be small compared to the other normal stress component and may therefore be neglected.

These assumptions, however, are not consistent with the three-dimensional theory discussed in the previous section.
To show this, a cartesian coordinate system (x, y, z) is choosen as reference system, such that the x-axis coincide with the axis of the undeformed straight beam. Assuming the transverse normal stresses to be negligible, will, in the case of small strains, lead to:

$$P_{2yy} = P_{2zz} = 0 \qquad (2.24)$$

where P_{2yy} and P_{2zz} are the 2nd Piola-Kirchhoff stress components in respectively y and z direction.
If $\mathbf{P}_2 = {}^4\mathbf{L}:\mathbf{E}$ is used as constitutive equation, relation (2.24) requires (see (2.19))

$$e_{yy} = e_{zz} = -\nu\, e_{xx} \tag{2.25}$$

where e_{ij} are the cartesian components of the Green-lagrange strain tensor. Equation (2.25), however, is in contradiction with the assumption of an undeformable cross section.

2.3.3 To overcome this problem, the constitutive equations in the case of beam theory are taken to be:

$$\begin{aligned} P_{2xx} &= E e_{xx} \\ P_{2xy} &= 2\,G\,e_{xy} \\ P_{2xz} &= 2\,G\,e_{xz} \end{aligned} \tag{2.26}$$

Where E is Young's modulus

With these constitutive equations the three-dimensional potential energy expression (2.5) changes to:

$$\pi_t = \int_{V_0} \tfrac{1}{2}[E\, e_{xx}^2 + 4G\, e_{xy}^2 + 4G\, e_{xz}^2]\, dv - \int_{V_0} \varrho_0 q_i u_i\, dv$$

$$- \int_{S_p^0} \tilde{p}_{0i}\, u_i\, dS \quad (i = 1, 2, 3) \tag{2.27}$$

In this equation the summation convention is applied, which means that repetition of an index in a term denotes a summation with respect to that index over its range.

The maximum error resulting from the assumptions made in beam theory is order of magnitude $(d/L)^2$, where d denotes a linear dimension of the cross section and L is the length of the beam.

3. Kinematics of straight slender elastic beams

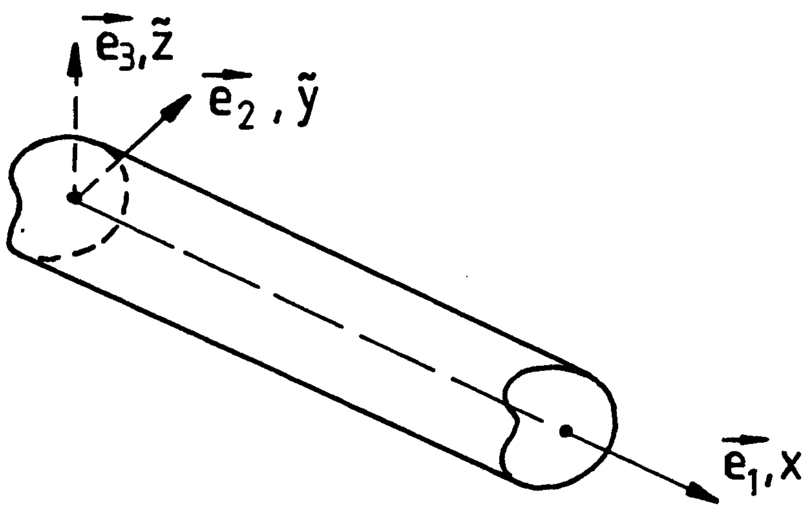


fig. 1.

3.1 The undeformed configuration G(O)

3.1.1 A straight slender prismatic beam is shown in fig. 1. The beam is made of a homogeneous, linear elastic isotropic material. Every material point p of this beam is described by a rectangular cartesian system of coordinates $(x, \tilde{y}, \tilde{z})$. The coordinate x coincides with the elastic axis of the beam, defined as the line which connects the shear centers of the cross sections of the beam.

3.1.2 It is assumed that the elastic axis is a straight line before deformation. With x representing length along this axis, it can be represented by:

$$\vec{r}_0 = x\vec{e}_1 \quad ; \quad x \in [0, L] \tag{3.1}$$

In the undeformed state the cross section is oriented such that $\vec{e}_2$ and $\vec{e}_3$ are parallel to the principal axes.
The position of an arbitrary material point, before deformation is given by:

$$\vec{x}_0(x, \tilde{y}, \tilde{z}) = \vec{r}_0(x) + \vec{\alpha}_0(\tilde{y}, \tilde{z}) \tag{3.2}$$

where $\vec{\alpha}_0 = \tilde{y}\,\vec{e}_2 + \tilde{z}\,\vec{e}_3$, while $\tilde{y}$ and $\tilde{z}$ denote length along the axes $\vec{e}_2$ and $\vec{e}_3$.

3.1.3 Beside the coordinate axes $\tilde{y}$, $\tilde{z}$ a second set of coordinate axes is defined in the cross section, parallel to $\tilde{y}$, $\tilde{z}$ but with the origin located at the centroid (fig. 2).

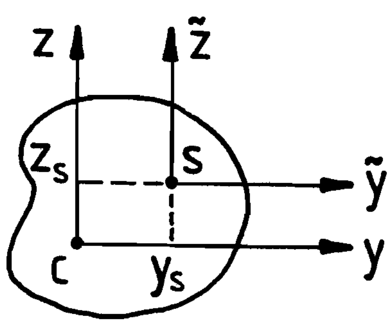


fig. 2.

$\tilde{y}$, $\tilde{z}$ and y, z are related by:

$$\tilde{y} = (y-y_s) \text{ and } \tilde{z} = (z-z_s) \tag{3.3}$$

y_s, z_s are respectively the y and z coordinate of the shear center.

3.1.4 The gradient operator with respect to G(0) can be written as:

$$\vec{\nabla}_0 = \vec{e}_1 \frac{\partial}{\partial x} + \vec{e}_2 \frac{\partial}{\partial y} + \vec{e}_3 \frac{\partial}{\partial z} \tag{3.4}$$

3.2 <u>The deformed configuration G(t)</u>

3.2.1 After deformation, the position vector of a material point $p(x, \tilde{y}, \tilde{z})$ is given by $\vec{x}(x, \tilde{y}, \tilde{z}, t)$. In order to determine $\vec{x}$ the following assumptions are made:

- The deformation of the beam can be looked upon as the result of two successive motions: first, a rigid translation and rotation of each cross section due to bending and warping free torsion, next a warping displacement perpendicular to the displaced cross sections.
- The cross section does not distort in its plane during deformation.
- Shear deformation due to transverse forces can be neglected.

The position vector $\vec{x}$ can now be expressed as:

$$\vec{x}(x, \tilde{y}, \tilde{z}, t) = \vec{r}(x, t) + \vec{\alpha}(x, \tilde{y}, \tilde{z}, t) + f(x, \tilde{y}, \tilde{z}, t)\vec{i}_1 \quad (3.5)$$

* $\vec{\alpha}(x, \tilde{y}, \tilde{z}, t) = \tilde{y}\ \vec{i}_2(x, t) + \tilde{z}\ \vec{i}_3(x, t)$
* $\vec{r} = \vec{r}(x, t)$ represents the beam axis in G(t)
* $\vec{i}_2$, $\vec{i}_3$ are unit vectors parallel to the principal axes of the cross section after the warping free motion see fig. 3.
* $f(x, \tilde{y}, \tilde{z}, t)\vec{i}_1$ represents small normal warping displacements ($\vec{i}_1 = \vec{i}_2 * \vec{i}_3$)

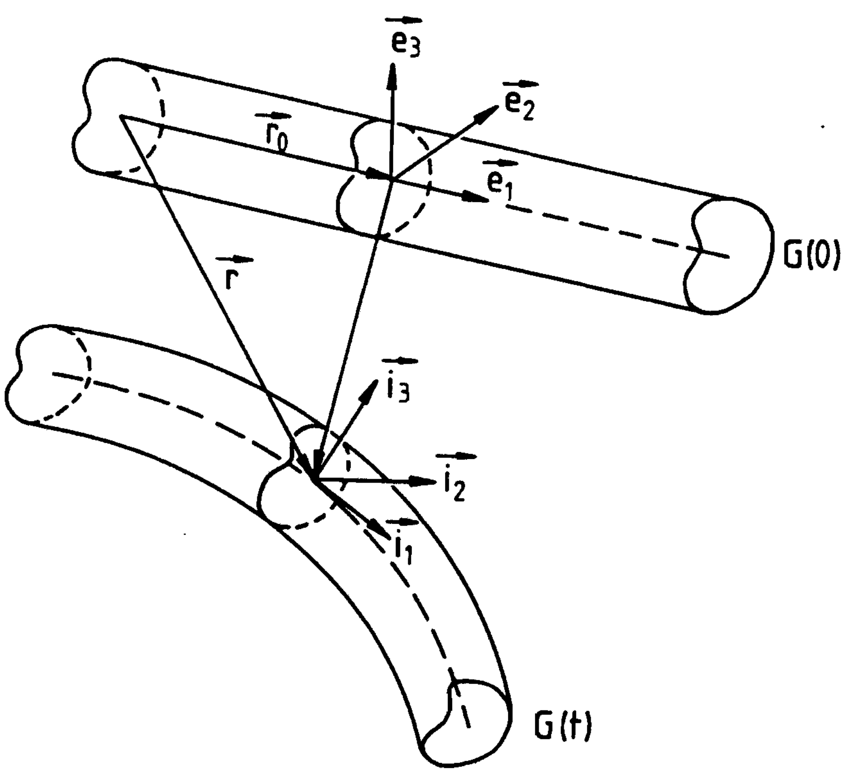


Fig. 3 Beam after warping free motion

3.2.2 The unit tangent vector to the elastic axis in G(t) at a point Q can be obtained from:

$$\frac{d\vec{r}}{ds}(Q) = \vec{n}(Q) \tag{3.6}$$

where s is the arc length along the deformed elastic axis

Differentiating $\vec{r}$ with respect to x in stead of s gives:

$$\frac{d\vec{r}}{dx} = \frac{ds}{dx}\frac{d\vec{r}}{ds} = \frac{ds}{dx}\vec{n} \quad \rightarrow \quad \frac{ds}{dx} = \left|\frac{d\vec{r}}{dx}\right| \tag{3.7}$$

If the displacement of a point on the elastic axis is described by its components u_s, v_s, w_s in the direction $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ respectively, $\vec{r}$ can be expressed as:

$$\vec{r} = (x + u_s)\,\vec{e}_1 + v_s\,\vec{e}_2 + w_s\,\vec{e}_3 \tag{3.8}$$

Substitution of this expression into (3.7) yields

$$\frac{d\vec{r}}{dx} = (1 + \varepsilon_s)\,\vec{n} = [(1 + u_s')\vec{e}_1 + v_s'\,\vec{e}_2 + w_s'\,\vec{e}_3] \tag{3.9}$$

where: (') stands for differentiation with respect to x

$$\text{and } \varepsilon_s = \left|\frac{d\vec{r}}{dx}\right| - 1 = [(1+u_s')^2 + v_s'^2 + w_s'^2]^{1/2} - 1 \tag{3.10}$$

3.2.3 If shear deformation due to transverse forces is neglected, the unit tangent vector $\vec{n}(Q)$ will be perpendicular to the cross section in every point Q.

$$\vec{n} = \vec{i}_1 = \vec{i}_2 * \vec{i}_3 \tag{3.11}$$

The triad $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ can be transformed into the triad $\vec{i}_1$, $\vec{i}_2$, $\vec{i}_3$ by means of a rigid translation and rotation.
The rigid rotation can be described by an orthogonal tensor $\mathbf{R}$.

$$\mathbf{R}\bullet\vec{e}_k = \vec{i}_k \quad (k = 1,\ 2,\ 3) \tag{3.12}$$

$$\mathbf{R} = \mathbf{R}(x,\ t) \quad ; \quad \mathbf{R}^C\bullet\mathbf{R} = \mathbf{R}\bullet\mathbf{R}^C = \mathbf{I} \quad ; \quad \det \mathbf{R} = 1$$

Where **I** is the unit tensor and $\mathbf{R}^C$ is the conjugate of **R**.

3.2.4 To analyse the beam deformation, that is to define the flexural and torsional curvatures of the beam axis, the derivatives of $\vec{i}_k$ with respect to s are studied.

$$\frac{d\vec{i}_k}{ds} = \frac{d\mathbf{R}}{ds}\bullet\vec{e}_k = \frac{d\mathbf{R}}{ds}\bullet\mathbf{R}^C\bullet\vec{i}_k \tag{3.13}$$

The orthogonality property of **R** implies that $\frac{d\mathbf{R}}{ds}\bullet\mathbf{R}^C$ is a skew tensor and therefore (3.13) may also be written as:

$$\frac{d\vec{i}_k}{ds} = \frac{d\mathbf{R}}{ds}\bullet\mathbf{R}^C\bullet\ \vec{i}_k = \vec{\varrho}\times\vec{i}_k \tag{3.14}$$

where $\vec{\varrho}$ is the axial vector of $\frac{d\mathbf{R}}{ds}\bullet\mathbf{R}^C$

According to the classical definition [12], the torsional and flexural beam curvatures are defined as the components of the vector $\vec{\varrho}$ with respect to the local basis $\vec{i}_1$, $\vec{i}_2$, $\vec{i}_3$

$$\vec{\varrho} = \chi_1\ \vec{i}_1 + \chi_2\ \vec{i}_2 + \chi_3\ \vec{i}_3 \tag{3.15}$$

where χ_1 is the torsional curvature and χ_2 and χ_3 are the flexural curvatures.
Substitution of (3.15) into (3.14) yields:

$$\frac{d\vec{i}_1}{ds} = \chi_3\, \vec{i}_2 - \chi_2\, \vec{i}_3$$

$$\frac{d\vec{i}_2}{ds} = -\chi_3\, \vec{i}_1 + \chi_1\, \vec{i}_3 \tag{3.16}$$

$$\frac{d\vec{i}_3}{ds} = \chi_2\, \vec{i}_1 - \chi_1\, \vec{i}_2$$

The components of $\vec{\varrho}$ can be obtained from:

$$\chi_1 = -\vec{i}_2 \bullet \frac{d\vec{i}_3}{ds} = \vec{i}_3 \bullet \frac{d\vec{i}_2}{ds} = \vec{e}_3 \bullet \mathbf{R}^c \bullet \frac{d\mathbf{R}}{ds} \bullet \vec{e}_2$$

$$\chi_2 = -\vec{i}_3 \bullet \frac{d\vec{i}_1}{ds} = \vec{i}_1 \bullet \frac{d\vec{i}_3}{ds} = \vec{e}_1 \bullet \mathbf{R}^c \bullet \frac{d\mathbf{R}}{ds} \bullet \vec{e}_3 \tag{3.17}$$

$$\chi_3 = -\vec{i}_1 \bullet \frac{d\vec{i}_2}{ds} = \vec{i}_2 \bullet \frac{d\vec{i}_1}{ds} = \vec{e}_2 \bullet \mathbf{R}^c \bullet \frac{d\mathbf{R}}{ds} \bullet \vec{e}_1$$

Differentiating $\vec{i}_k$ (k= 1, 2, 3) with respect to x instead of s yields:

$$\frac{d\vec{i}_k}{dx} = \frac{ds}{dx}\frac{d\vec{i}_k}{ds} = (1 + \varepsilon_s)\, \vec{\varrho} * \vec{i}_k = \vec{\chi} * \vec{i}_k \tag{3.18}$$

combination of (3.18) and (3.15) yields:

$$\vec{\chi} = (1 + \varepsilon_s)[\chi_1\, \vec{i}_1 + \chi_2\, \vec{i}_2 + \chi_3\, \vec{i}_3] \tag{3.19}$$

3.2.5 The deformation proces which carries G(0) into G(t) can be described completely in terms of the deformation gradient tensor **F**, which is given by:

$$\mathbf{F} = (\vec{\nabla}_0\, \vec{x})^c \tag{3.20}$$

Substitution of eqn. (3.4) into (3.20) yields:

$$\mathbf{F} = \frac{\partial\vec{x}}{\partial x}\vec{e}_1 + \frac{\partial\vec{x}}{\partial y}\vec{e}_2 + \frac{\partial\vec{x}}{\partial z}\vec{e}_3 \tag{3.21}$$

Substitution of (3.5) and (3.18) into (3.21) yields:

$$\begin{aligned}
\frac{\partial\vec{x}}{\partial x} &= (1+\varepsilon_s)\,\vec{i}_1 + (\vec{\chi}*\vec{\alpha}) + \frac{\partial f}{\partial x}\vec{i}_1 + f(\vec{\chi}*\vec{i}_1) \\
\frac{\partial\vec{x}}{\partial y} &= \vec{i}_2 + \frac{\partial f}{\partial y}\vec{i}_1 \\
\frac{\partial\vec{x}}{\partial z} &= \vec{i}_3 + \frac{\partial f}{\partial z}\vec{i}_1
\end{aligned} \tag{3.22}$$

The deformation gradient tensor can now be written as:

$$\begin{aligned}
\mathbf{F} = {} & [\varepsilon_s\,\vec{i}_1 + (\vec{\chi}*\vec{\alpha}) + \frac{\partial f}{\partial x}\vec{i}_1 + f(\vec{\chi}*\vec{i}_1)]\,\vec{e}_1 + \frac{\partial f}{\partial y}\vec{i}_1\,\vec{e}_2 + \\
& + \frac{\partial f}{\partial z}\vec{i}_1\,\vec{e}_3 + \mathbf{R}
\end{aligned} \tag{3.23}$$

where $\mathbf{R} = (\vec{i}_1\vec{e}_1 + \vec{i}_2\vec{e}_2 + \vec{i}_3\vec{e}_3)$

3.2.6 The Green-lagrange strain tensor is defined by:

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^c\bullet\mathbf{F} - \mathbf{I}) \tag{3.24}$$

The components of $\mathbf{E}$ which play a role in beam theory are: (see section 2.3)

$$\begin{aligned}
e_{11} &= \vec{e}_1\bullet\,\mathbf{E}\,\bullet\,\vec{e}_1 = \frac{1}{2}[(\mathbf{F}\bullet\vec{e}_1)\bullet(\mathbf{F}\bullet\vec{e}_1) - 1] \\
e_{12} &= \vec{e}_1\bullet\,\mathbf{E}\,\bullet\,\vec{e}_2 = \frac{1}{2}[(\mathbf{F}\bullet\vec{e}_1)\bullet(\mathbf{F}\bullet\vec{e}_2)] \\
e_{13} &= \vec{e}_1\bullet\,\mathbf{E}\,\bullet\,\vec{e}_3 = \frac{1}{2}[(\mathbf{F}\bullet\vec{e}_1)\bullet(\mathbf{F}\bullet\vec{e}_3)]
\end{aligned} \tag{3.25}$$

Using eqn. (3.23) and substituting

$$(\vec{\chi} * \vec{\alpha}) = (1 + \varepsilon_s)\ [\tilde{y}\chi_1\ \vec{i}_3 - \tilde{y}\chi_3\ \vec{i}_1 - \tilde{z}\chi_1\ \vec{i}_2 + \tilde{z}\chi_2\ \vec{i}_1] \qquad (3.26)$$

$$(\chi * \vec{i}_1) = (1 + \varepsilon_s)\ [-\ \chi_2\ \vec{i}_3 + \chi_3\ \vec{i}_2]$$

yields:

$$e_{11} = \varepsilon_s + \frac{1}{2}\varepsilon_s^2 + \frac{1}{2}(1+\varepsilon_s)^2[(\tilde{y}^2+\tilde{z}^2)\chi_1^2 + (\tilde{y}\chi_3 - \tilde{z}\chi_2)^2] + \frac{1}{2}(\frac{\partial f}{\partial x})^2$$
$$+\frac{1}{2}f^2(1+\varepsilon_s)^2[\chi_2^2+\chi_3^2] + (1+\varepsilon_s)^2[-\tilde{y}\chi_3+\tilde{z}\chi_2] + (1+\varepsilon_s)\frac{\partial f}{\partial x}$$
$$+(1+\varepsilon_s)\frac{\partial f}{\partial x}(-\tilde{y}\chi_3+\tilde{z}\chi_2) + (1+\varepsilon_s)^2 f\chi_1(-\tilde{y}\chi_2-\tilde{z}\chi_3) \qquad (3.27)$$

$$e_{12} = \frac{1}{2}[-(1+\varepsilon_s)\tilde{z}\chi_1 + (1+\varepsilon_s)f\chi_3 + (1+\varepsilon_s)\frac{\partial f}{\partial y} + (1+\varepsilon_s)\frac{\partial f}{\partial y}(-\tilde{y}\chi_3+\tilde{z}\chi_2) + \frac{\partial f}{\partial x}\frac{\partial f}{\partial y}]$$

$$e_{13} = \frac{1}{2}[(1+\varepsilon_s)\tilde{y}\chi_1 - (1+\varepsilon_s)f\chi_2 + (1+\varepsilon_s)\frac{\partial f}{\partial z} + (1+\varepsilon_s)\frac{\partial f}{\partial z}(-\tilde{y}\chi_3+\tilde{z}\chi_2) + \frac{\partial f}{\partial x}\frac{\partial f}{\partial z}]$$

Neglecting small terms according to the assumption of small strains yields (see appendix B)

$$e_{11} = \varepsilon_s + (-\tilde{y}\chi_3 + \tilde{z}\chi_2) + \frac{1}{2}(\tilde{y}^2 + \tilde{z}^2)\ \chi_1^2 + \frac{\partial f}{\partial x}$$

$$e_{12} = \frac{1}{2}\ [-\tilde{z}\chi_1 + \frac{\partial f}{\partial y}] \qquad (3.28)$$

$$e_{13} = \frac{1}{2}\ [\tilde{y}\chi_1 + \frac{\partial f}{\partial z}]$$

4. The warping function

4.1.1 Before proceeding, the warping function, which until now was treated in a general manner is studied more closely.
In the case of slender beams an assumption, analogous to the one used for the case of saint-Venant torsion is often introduced, whereby f is written as:

$$f(x, \tilde{y}, \tilde{z}, t) = \chi_1(x, t)\, \psi(\tilde{y}, \tilde{z}) \quad (4.1)$$

where: $\psi(\tilde{y}, \tilde{z})$ is the well known warping function which is used in the theory of uniform linear torsion [6].

However, by postulating that the amplitude of the warping displacement equals χ_1, it is also postulated that at a section where warping is prevented, the strains e_{12} and e_{13} have to vanish too (see 3.28). This is certainly not the case in reality and should in general not be assumed, since these strains are proportional to the stresses σ_{12} and σ_{13}. To overcome this problem, an expression similar to the one proposed by Reissner [15] for the case of non-uniform linear torsion is used in this report:

$$f(x, \tilde{y}, \tilde{z}, t) = g(x, t)\, \psi(\tilde{y}, \tilde{z}) \quad (4.2)$$

where g(x, t) is a function yet to be determined.

4.1.2 Reissner [15, 16, 17] showed that in the case of non-uniform linear torsion of thin walled beams with open cross sections, the practical improvement gained by working with (4.2) in stead of (4.1) will in general be negligible. For non-uniform linear torsion of thin walled beams with closed or partly closed cross sections, however, the more accurate equation (4.2) leads to results which are quite different from what would follow from a use of eqn. (4.1).
In the case of beams with an arbitrary cross section undergoing both bending and non-uniform torsion, eqn. (4.2) must also be expected to lead to more accurate results than eqn. (4.1).

In the following, the functions ψ and g will be refered to as 'the normalized warping' and 'the warping amplitude' respectively.

5. The strain energy U.

5.1.1 Substitution of (4.2) into (3.28) yields:

$$e_{11} = \varepsilon_s + (-\tilde{y}\chi_3 + \tilde{z}\chi_2) + \frac{1}{2}(\tilde{y}^2 + \tilde{z}^2)\,\chi_1^2 + \frac{dq}{dx}\,\psi \tag{5.1}$$

$$e_{12} = \frac{1}{2}\left(-\tilde{z}\chi_1 + g\,\frac{\partial\psi}{\partial y}\right) \tag{5.2}$$

$$e_{13} = \frac{1}{2}\left(\tilde{y}\chi_1 + g\,\frac{\partial\psi}{\partial z}\right) \tag{5.3}$$

Using relation (3.3): $\tilde{y}=(y-y_s)$ and $\tilde{z}=(z-z_s)$, e_{11} may also be written as:

$$e_{11} = \bar{\varepsilon} - y\chi_3 + z\chi_2 + \frac{1}{2}\,\tilde{r}^2\chi_1^2 + \frac{dq}{dx}\,\psi \tag{5.4}$$

where: $\bar{\varepsilon} = \varepsilon_s + y_s\,\chi_3 - z_s\chi_2$ and $\tilde{r}^2 = (\tilde{y}^2 + \tilde{z}^2)$

5.1.2 In beam theory the strain energy is given by: (see 2.27)

$$U = \int_{V_0} \frac{1}{2}\,[Ee_{11}^2 + 4Ge_{12}^2 + 4Ge_{13}^2]dv \tag{5.5}$$

Since y and z are coordinates along the principle axes, and the shear center is choosen as the pole of the normalized warping, the following identities hold

$$\int_A ydA = \int_A zdA = \int_A z\psi dA = \int_A y\psi dA = \int_A \psi dA = 0 \tag{5.6}$$

Where: A is the area of the cross section.

Substituting (5.2-5.4) into (5.5) yields:

$$U = \frac{1}{2} \int_0^L [EA\bar{\varepsilon}^2 + EI_2\chi_2^2 + EI_3\chi_3^2 + EH\chi_1^4 + E\Gamma(\frac{dg}{dx})^2 +$$

$$\chi_1^2(EI_s\bar{\varepsilon} + EI_2\beta_2\chi_2 - EI_3\beta_3\chi_3 + E\Gamma\beta\psi \frac{dg}{dx}) + \qquad (5.7)$$

$$G(\chi_1^2 I_s + 2\chi_1 gD^* + g^2 D)]dx$$

where: $I_2 = \int_A z^2 dA$; $I_3 = \int_A y^2 dA$; $\Gamma = \int_A \psi^2 dA$

$$I_s = \int_A \tilde{r}^2 dA \quad ; \quad H = \frac{1}{4}\int_A \tilde{r}^4 dA \quad ; \quad D^* = \int_A (\tilde{y}\,\frac{\partial\psi}{\partial z} - \tilde{z}\,\frac{\partial\psi}{\partial y})dA \qquad (5.8)$$

$$D = \int_A [(\frac{\partial\psi}{\partial y})^2 + (\frac{\partial\psi}{\partial z})^2]dA \quad ; \quad \beta_2 = \frac{1}{I_2}\int_A (y^2+z^2)z dA - 2\, z_s$$

$$\beta_3 = \frac{1}{I_3}\int_A (y^2+z^2)y dA - 2\, y_s \quad ; \quad \beta_\psi = \frac{1}{\Gamma}\int_A (y^2+z^2)\psi dA$$

* If the warping amplitude is taken to be $\chi_1(x)$ in stead of $g(x)$ (see chapter 4) the term $G(\chi_1^2 I_s + 2\chi_1 gD^* + g^2 D)$ changes to $GI_t\chi_1^2$.

6. Load potential

6.1.1 It is assumed that the beam is subjected to dead weight surface tractions $p_1 \vec{e}_1 + p_2 \vec{e}_2 + p_3 \vec{e}_3$ at both ends ($x = 0$ and $x = L$) and $q_2\vec{e}_2$, $q_3\vec{e}_3$ per unit length.
($p_i = p_i(x, \tilde{y}, \tilde{z})$ and $q_\alpha = q_\alpha(x)$; $i = 1, 2, 3$ and $\alpha = 2, 3$)

Beside these 'external' tractions the beam is loaded by a force $\bar{q}\,\vec{e}_3$ acting at the centroid, representing the weight of the beam.

6.1.2 The potential energy Ω of the applied loads is the negative of the work done by the loads as the structure is deformed. The total work of the applied loads may be written as:

$$V = \sum_{i=1}^{3} \left[\int_A p_i u_i dA \right]_{x=0;x=L} + \int_0^L [q_2 u_2 + q_3 u_3 + \bar{q} u_3(c)]dx \quad (6.1)$$

Where: $u_3(c)$ is the displacement component of the centroid in $\vec{e}_3$ direction.

6.1.3 The displacement vector $\vec{u}$ of a material point $p(x, \tilde{y}, \tilde{z})$ is given by:

$$\vec{u}(p) = \vec{x}(p) - \vec{x}_0(p) \quad (6.2)$$

where: $$\vec{x}_0(p) = x\,\vec{e}_1 + \tilde{y}\,\vec{e}_2 + \tilde{z}\,\vec{e}_3 \quad (6.3)$$

and $$\vec{x}(p) = (x + u_s)\vec{e}_1 + v_s\,\vec{e}_2 + w_s\,\vec{e}_3 + \tilde{y}\,\vec{i}_2 + \tilde{z}\,\vec{i}_3 + g\psi\vec{i}_1 \quad (6.4)$$

Substitution of (6.3) and (6.4) into (6.2) yields:

$$\vec{u}(p) = u_s\vec{e}_1 + v_s\,\vec{e}_2 + w_s\,\vec{e}_3 + \tilde{y}(\vec{i}_2-\vec{e}_2) + \tilde{z}(\vec{i}_3-\vec{e}_3) + g\psi\vec{i}_1 \quad (6.5)$$

$\vec{i}_1, \vec{i}_2, \vec{i}_3$ can be expressed in terms of $\vec{e}_1, \vec{e}_2, \vec{e}_3$ as:

$$\vec{i}_1 = R_{11}\,\vec{e}_1 + R_{21}\,\vec{e}_2 + R_{31}\,\vec{e}_3$$

$$\vec{i}_2 = R_{12}\,\vec{e}_1 + R_{22}\,\vec{e}_2 + R_{32}\,\vec{e}_3 \qquad (6.6)$$

$$\vec{i}_3 = R_{13}\,\vec{e}_1 + R_{23}\,\vec{e}_2 + R_{33}\,\vec{e}_3$$

Where: $R_{kL} = \vec{e}_k \cdot \vec{i}_L$ (L, k = 1, 2, 3) (6.7)

The components of $\vec{u}(p)$ in the direction $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ can now be expressed as:

$$u_1 = \bar{u} + y\,R_{12} + z\,R_{13} + g\psi\,R_{11}$$

$$u_2 = v_s + \tilde{y}\,(R_{22} - 1) + \tilde{z}\,R_{23} + \underline{g\psi\,R_{21}} \qquad (6.8)$$

$$u_3 = w_s + \tilde{y}\,R_{32} + \tilde{z}\,(R_{33} - 1) + \underline{g\psi\,R_{31}}$$

where: $\bar{u} = u_s - y_s\,R_{12} - z_s\,R_{13}$

The underlined terms in (6.8) may be neglected according to the assumption of small strains (see appendix J)

6.1.4 Substitution of (6.8) into (6.1) yields:

$$V = [P_1\bar{u} + P_2 v_s + P_3 w_s + M_2 R_{13} + (-M_3)R_{12} + BgR_{11} + (-M_{t2})R_{23} + M_{t3}R_{32} + (R_{22}-1)\int_A P_2\tilde{y}dA + (R_{33}-1)\int_A P_3\tilde{z}dA]_{x=0;x=L}$$

$$+ \int_0^L [q_2 v_s + q_2\tilde{y}(R_{22}-1) + (-m_{t2})R_{23} + q_3 w_s + m_{t3}R_{32} + q_3\tilde{z}(R_{33}-1) + \bar{q}(w_s - y_s R_{32} - (R_{33}-1)z_s)]dx \qquad (6.9)$$

where: $P_1 = \int_A p_1 dA$; $P_2 = \int_A p_2 dA$; $P_3 = \int_A p_3 dA$

$$M_2 = \int_A (p_1 z)\,dA \quad ; \quad M_3 = \int_A (-p_1 y)\,dA \quad ; \quad B = \int_A (p_1 \psi)\,dA$$

$$M_{t2} = \int_A (-p_2 \tilde{z})\,dA \quad ; \quad M_{t3} = \int_A (p_3 \tilde{y})\,dA \tag{6.10}$$

$$m_{t2} = (-q_2 \tilde{z}) \quad ; \quad m_{t3} = q_3 \tilde{y}$$

7. The potential energy functional

7.1.1 The total potential energy is the sum of the strain energy U and the potential energy of the loads Ω

$$\pi = U + \Omega = U - V \tag{7.1}$$

Substitution of (5.7) and (6.9) into (7.1) yields:

$$
\begin{aligned}
\pi = \frac{1}{2} \int_0^L [& EA\bar{\varepsilon}^2 + EI_2\chi_2^2 + EI_3\chi_3^2 + EH\chi_1^4 + E\Gamma\left(\frac{dg}{dx}\right)^2 + \\
& + \chi_1^2(EI_s\bar{\varepsilon} + EI_2\beta_2\chi_2 - EI_3\beta_3\chi_3 + E\Gamma\beta_\psi\frac{dg}{dx}) \\
& G(\chi_1^2 I_s + 2\chi_1 gD^* + g^2 D)]dx \\
& - [P_1\bar{u} + P_2 v_s + P_3 w_s + M_2R_{13} + (-M_3)R_{12} + BgR_{11} + \\
& + (-M_{t2})R_{23} + M_{t3}R_{32} + (R_{22}-1)\int_A p_2\tilde{y}dA + \\
& + (R_{33}-1)\int_A p_3\tilde{z}dA]_{x=0;x=L} \\
& - \int_0^L [q_2 v_s + q_2\tilde{y}(R_{22}-1) + (-m_{t2})R_{23} + q_3 w_s + m_{t3}R_{32} + \\
& + q_3\tilde{z}(R_{33}-1) + \bar{q}(w_s - y_sR_{32} - (R_{33}-1)z_s]dx
\end{aligned}
\tag{7.2}
$$

where: $\bar{\varepsilon} = \varepsilon_s + y_s\chi_3 - z_s\chi_2$ (see 5.4)

and : $\varepsilon_s = [(1+u_s')^2 + v_s'^2 + w_s'^2]^{1/2} - 1$ (see 3.10)

* If the warping amplitude is taken to be $\chi_1(x)$ in stead of $g(x)$ (see chapter 4), the term $G(\chi_1^2 I_s + 2\chi_1 gD^* + g^2 D)$ changes to $GI_t\chi_1^2$.

8. The rotation tensor R.

8.1.1 To express the strain energy and load potential in terms of displacements, the rotation tensor **R** is studied more closely.
The rigid rotation which transforms the triad $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ at a material point Q of the undeformed axis to the triad $\vec{i}_1$, $\vec{i}_2$, $\vec{i}_3$ at Q of the deformed axis, is represented by a rotation tensor **R**.
In beam theory this rotation is often described in terms of so-called 'modified-Euler' angles: The rotation tensor is then broken down as:

$$\mathbf{R} = \mathbf{R}_\gamma \bullet \mathbf{R}_\beta \bullet \mathbf{R}_\alpha \tag{8.1}$$

where $\mathbf{R}_\alpha$, $\mathbf{R}_\beta$, $\mathbf{R}_\gamma$ are respectively rotations of magnitude α, β, γ about the reference axes $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$.
This method, however, has some disadvantages (to be discussed in chapter 11) and therefore an alternative method is used in this report.

8.1.2 An axis with unit vector $\vec{\eta}$, perpendicular to $\vec{e}_1$ and $\vec{i}_1$ is choosen as rotation axis. When $\vec{e}_1$ and $\vec{i}_1$ are inclined by an angle ϕ, $\vec{\eta}$ can be written as:

$$\vec{\eta} = (\sin\phi)^{-1}\ \vec{e}_1 * \vec{i}_1 \tag{8.2}$$

8.1.3 A rotation tensor **Q**, representing a rotation over an angle γ, of an arbitrary vector $\vec{a}$, about an axis with unit vector $\vec{e}$ can be expressed as: (see appendix C)

$$\mathbf{Q} \bullet \vec{a} = \cos\gamma \vec{a} + (1-\cos\gamma)\vec{e}\vec{e} \bullet \vec{a} + \sin\gamma \vec{e} * \vec{a} \tag{8.3}$$

The rotation tensor $\mathbf{R}_1$, which maps $\vec{e}_1$ on $\vec{i}_1$ can thus be written as:

$$\mathbf{R}_1 \bullet \vec{a} = \cos\phi \vec{a} + (1-\cos\phi)\vec{\eta}\vec{\eta} \bullet \vec{a} + \sin\phi \vec{\eta} * \vec{a} \tag{8.4}$$

where: $\cos\phi = \vec{e}_1 \bullet \vec{i}_1$ and $\sin\phi \vec{\eta} = \vec{e}_1 * \vec{i}_1$

8.1.4 When $\vec{e}_1$ is mapped on $\vec{i}_1$ by $\mathbf{R}_1$, $\mathbf{R}_1\bullet\vec{e}_2$ and $\mathbf{R}_1\bullet\vec{e}_3$ will generally not coincide with respectively $\vec{i}_2$ and $\vec{i}_3$.
Coincidence of $\mathbf{R}_1\bullet\vec{e}_2$ and $\mathbf{R}_1\bullet\vec{e}_3$ with $\vec{i}_2$ and $\vec{i}_3$ can be achieved by an additional rotation α about $\vec{i}_1$.
It is, however, also possible to rotate $\vec{e}_2$ and $\vec{e}_3$ about $\vec{e}_1$ first, such that if $\mathbf{R}_1$ is applied to the rotated basis, $\mathbf{R}_1\bullet\vec{e}_2$ and $\mathbf{R}_1\bullet\vec{e}_3$ coincide with $\vec{i}_2$ and $\vec{i}_3$.
According to (8.3), the rotation tensors representing a rotation about $\vec{i}_1$ and about $\vec{e}_1$, can respectively be written as:

$$\mathbf{R}_2\bullet\vec{a} = \cos\alpha\vec{a} + (1-\cos\alpha)\,\vec{i}_1\vec{i}_1\bullet\vec{a} + \sin\alpha\;\vec{i}_1*\vec{a} \qquad (8.5)$$

and

$$\tilde{\mathbf{R}}_2\bullet\vec{a} = \cos\beta\vec{a} + (1-\cos\beta)\vec{e}_1\vec{e}_1\bullet\vec{a} + \sin\beta\vec{e}_1*\vec{a} \qquad (8.6)$$

It is emphasized here that in general $\alpha\neq\beta$.

8.1.5 The rotation tensor $\mathbf{R}$ representing the total rigid rotation of $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ to $\vec{i}_1$, $\vec{i}_2$, $\vec{i}_3$ can now be written as:

$$\mathbf{R} = \mathbf{R}_2\bullet\mathbf{R}_1 \qquad (8.7)$$

or

$$\mathbf{R} = \mathbf{R}_1\bullet\tilde{\mathbf{R}}_2 \qquad (8.8)$$

8.1.6 When $\mathbf{R}$ is written as $\mathbf{R} = \mathbf{R}_2\bullet\mathbf{R}_1$, substitution of (8.4) and (8.5) yields: (see appendix D)

$$\begin{aligned}\mathbf{R}\bullet\vec{a} = {} & \cos\phi\,\cos\alpha\;\vec{a} + \cos\phi\;\sin\alpha\;\vec{i}_1*\vec{a} \\ & + [(1-\cos\phi)(\cos\alpha\;\vec{\eta}\vec{\eta} + \frac{\sin\alpha}{\sin\phi}\,\vec{e}_1\vec{\eta} - \sin\alpha\;\text{cotan}\phi\;\vec{i}_1\vec{\eta}) \\ & - (\cos\alpha\;\vec{e}_1\vec{i}_1 - \sin\alpha\;\sin\phi\;\vec{\eta}\vec{i}_1) + \vec{i}_1\vec{e}_1]\bullet\vec{a}\end{aligned} \qquad (8.9)$$

where: $\cos\phi = \vec{e}_1 \bullet \vec{i}_1$ and $\sin\phi\vec{\eta} = \vec{e}_1 * \vec{i}_1$

Writing **R** as $\mathbf{R}=\mathbf{R}_1 \bullet \tilde{\mathbf{R}}_2$ yields, (see appendix E)

$$\mathbf{R}\bullet\vec{a} = [\cos\phi \; \mathbf{I} - \vec{e}_1\vec{i}_1 + (1-\cos\phi)\vec{\eta}\vec{\eta}]\bullet(\cos\beta\vec{a} + \sin\beta\vec{e}_1 * \vec{a}) + $$

$$+ \vec{i}_1\vec{e}_1\bullet\vec{a} \tag{8.10}$$

where: $\cos\phi = \vec{e}_1 \bullet \vec{i}_1$ and $\sin\phi\vec{\eta} = \vec{e}_1 * \vec{i}_1$

8.1.7 According to (3.9) and (3.11) $\vec{i}_1$ can be written as:

$$\vec{i}_1 = (1+\varepsilon_s)^{-1} \; [(1+u_s')\vec{e}_1 + v_s' \; \vec{e}_2 + w_s' \; \vec{e}_3] \tag{8.11}$$

If ε_s is neglected compared to unity in accordance with the assumption of small strains (8.11) reduces to:

$$\vec{i}_1 = [(1 + u_s') \; \vec{e}_1 + v_s' \; \vec{e}_2 + w_s'\vec{e}_3] \tag{8.12}$$

Substitution of (8.12) into (8.9) and representing $\mathbf{R} = \mathbf{R}_2\bullet\mathbf{R}_1$ in terms of its cartesian components yields: (see appendix F)

$$\underline{R} = \underline{R}_2\underline{R}_1 =$$

$$\begin{bmatrix} 1+u_s' & -w_s'\sin\alpha\left((1+u_s')+b-b(1+u_s')^2\right)-v_s'\cos\alpha & v_s'\sin\alpha\left((1+u_s')+b-b(1+u_s')^2\right)-w_s'\cos\alpha \\ v_s' & (1+u_s')\cos\alpha+b\left(w_s'\cos\alpha+w_s'v_s'(1+u_s')\sin\alpha\right)-v_s'w_s'\sin\alpha & -\sin\alpha\left((1+u_s')^2+b(1+u_s')v_s'^2+w_s'^2\right)-bv_s'w_s'\cos\alpha \\ w_s' & \sin\alpha\left((1+u_s')^2+b(1+u_s')w_s'^2+v_s'^2\right)-bw_s'v_s'\cos\alpha & (1+u_s')\cos\alpha+b\left(v_s'^2\cos\alpha-(1+u_s')w_s'v_s'\sin\alpha\right)+v_s'w_s'\sin\alpha \end{bmatrix} \tag{8.13}$$

where: $(1+u_s') = \cos\phi = \sqrt{(1-v_s'^2-w_s'^2)}$ and $b = \dfrac{1-\cos\phi}{\sin^2\phi}$

8.1.8 Substitution of (8.12) into (8.10) and representing $\mathbf{R} = \mathbf{R}_1 \bullet \tilde{\mathbf{R}}_2$ in terms of its cartesian components yields: (see appendix G)

$$\underline{R} = \underline{R}_1 \tilde{\underline{R}}_2 =$$

$$\begin{bmatrix} 1+u_s' & -v_s'\cos\beta - w_s'\sin\beta & -w_s'\cos\beta + v_s'\sin\beta \\ v_s' & (1+u_s')\cos\beta + b\left(w_s'^2\cos\beta - w_s'v_s'\sin\beta\right) & -(1+u_s')\sin\beta + b\left(-w_s'v_s'\cos\beta - w_s'^2\sin\beta\right) \\ w_s' & (1+u_s')\sin\beta + b\left(-w_s'v_s'\cos\beta + v_s'^2\sin\beta\right) & (1+u_s')\cos\beta + b\left(v_s'^2\cos\beta + w_s'v_s'\sin\beta\right) \end{bmatrix} \quad (8.14)$$

where: $(1+u_s') = \cos\phi = \sqrt{(1-v_s'^2-w_s'^2)}$ and $b = \dfrac{1-\cos\phi}{\sin^2\phi}$

8.1.9 From a kinematical point of view both representations are equivalent and there is no preference for one of them. But since the representation $\mathbf{R} = \mathbf{R}_1 \bullet \tilde{\mathbf{R}}_2$ leads to simpler expressions then $\mathbf{R} = \mathbf{R}_2 \bullet \mathbf{R}_1$, the representation $\mathbf{R} = \mathbf{R}_1 \bullet \tilde{\mathbf{R}}_2$ will be used in the following.

9. Curvatures

9.1.1 According to eqn. (3.17) the curvatures χ_1, χ_2, χ_3 can be obtained from:

$$\chi_1 = (\vec{e}_3 \bullet \mathbf{R}^c \bullet \frac{d\mathbf{R}}{dx} \bullet \vec{e}_2)$$

$$\chi_2 = (\vec{e}_1 \bullet \mathbf{R}^c \bullet \frac{d\mathbf{R}}{dx} \bullet \vec{e}_3) \qquad (9.1)$$

$$\chi_3 = (\vec{e}_2 \bullet \mathbf{R}^c \bullet \frac{d\mathbf{R}}{dx} \bullet \vec{e}_1)$$

where the term $\frac{1}{1+\varepsilon_s}$ is replaced by unity according to the assumption of small strains.

The orthogonality property of $\mathbf{R}$ implies that $\mathbf{R}^c \bullet \frac{d\mathbf{R}}{dx}$ is a skew tensor and therefore $\mathbf{R}^c \bullet \frac{d\mathbf{R}}{dx} \bullet \vec{a}$ can also be written as:

$$\mathbf{R}^c \bullet \frac{d\mathbf{R}}{dx} \bullet \vec{a} = \vec{\chi} * \vec{a} \qquad (9.2)$$

9.1.2 The rotation tensor $\mathbf{R}$, which is considered here, consists of two successive rotations; $\mathbf{R} = \mathbf{R}_1 \bullet \tilde{\mathbf{R}}_2$.
$\mathbf{R}^c \bullet \frac{d\mathbf{R}}{dx} \bullet \vec{a}$ may therefore also be written as:

$$\mathbf{R}^c \bullet \frac{d\mathbf{R}}{dx} \bullet \vec{a} = \tilde{\mathbf{R}}_2^c \bullet \mathbf{R}_1^c \bullet (\frac{d\mathbf{R}_1}{dx} \bullet \tilde{\mathbf{R}}_2 + \mathbf{R}_1 \bullet \frac{d\tilde{\mathbf{R}}_2}{dx}) \bullet \vec{a} \qquad (9.3)$$

$$= (\tilde{\mathbf{R}}_2^c \bullet \vec{\mu} + \vec{\gamma}) * \vec{a}$$

where $\vec{\mu}$ and $\vec{\gamma}$ are the axial vectors of respectively $\mathbf{R}_1^c \bullet \frac{d\mathbf{R}_1}{dx}$ and $\tilde{\mathbf{R}}_2^c \bullet \frac{d\tilde{\mathbf{R}}_2}{dx}$
Combining (9.2) and (9.3) yields:

$$\vec{\chi} = (\tilde{\mathbf{R}}_2^c \bullet \vec{\mu} + \vec{\gamma}) \qquad (9.4)$$

9.1.3 The axial vector $\vec{k}$ of the skew tensor $Q^c \bullet \frac{dQ}{dx}$, when Q is given by

$$Q \bullet \vec{a} = \cos\gamma \vec{a} + (1-\cos\gamma)\vec{e}\vec{e} \bullet \vec{a} + \sin\gamma \vec{e} * \vec{a}$$

can be written as: (see appendix H)

$$\vec{k} = [\frac{d\gamma}{dx}\vec{e} + (1-\cos\gamma)(\frac{d\vec{e}}{dx} * \vec{e}) + \sin\gamma \frac{d\vec{e}}{dx}] \qquad (9.5)$$

9.1.4 Using eqn. (9.5) $\vec{\gamma}$ and $\vec{\mu}$ can respectively be expressed as: (see appendix I)

$$\vec{\gamma} = \beta' \vec{e}_1 \qquad (9.6)$$

$$\vec{\mu} = \frac{1-\cos\phi}{\sin^2\phi}(w_s' v''_s - w_s'' v'_s)\vec{e}_1$$

$$+ [-w_s'' - \frac{1-\cos\phi}{\sin^2\phi\cos\phi}(v_s' v_s'' w_s' + w_s'^2 w_s'')]\vec{e}_2 \qquad (9.7)$$

$$+ [v_s'' + \frac{1-\cos\phi}{\sin^2\phi\cos\phi}(v_s'^2 v_s'' + w_s' w_s'' v_s')]\vec{e}_3$$

where $\cos\phi = (1-v_s'^2-w_s'^2)^{1/2}$ and $\sin^2\phi = (v_s'^2 + w_s'^2)$

9.1.5 For the determiniation of $\vec{\chi}$ the following expressions are needed:

$$\tilde{R}_2^c \bullet \vec{e}_1 = \vec{e}_1 \qquad (9.8)$$

$$\tilde{R}_2^c \bullet \vec{e}_2 = \cos\beta \vec{e}_2 - \sin\beta \vec{e}_3 \qquad (9.9)$$

$$\tilde{R}_2^c \bullet \vec{e}_3 = \cos\beta \vec{e}_3 + \sin\beta \vec{e}_2 \qquad (9.10)$$

Substitution of (9.4) and (9.6-9.10) into (9.1) yields:

$$\chi_1 = [\beta' + \frac{1-\cos\phi}{\sin^2\phi}(w_s'v_s'' - w_s''v_s')] \tag{9.11}$$

$$\chi_2 = \cos\beta\ [-w_s'' - \frac{1-\cos\phi}{\sin^2\phi\cos\phi}(v_s'v_s''w_s' + w_s'^2 w_s'')] +$$

$$+ \sin\beta\ [v_s'' + \frac{1-\cos\phi}{\sin^2\phi\cos\phi}(v_s'^2 v_s'' + w_s'w_s''v_s')] \tag{9.12}$$

$$\chi_3 = \cos\beta\ [v_s'' + \frac{1-\cos\phi}{\sin^2\phi\cos\phi}(v_s'^2 v_s'' + w_s'w_s''v_s')] +$$

$$+ \sin\beta\ [w_s'' + \frac{1-\cos\phi}{\sin^2\phi\cos\phi}(v_s'v_s''w_s' + w_s'^2 w_s'')] \tag{9.13}$$

where: $\cos\phi = (1-v_s'^2 - w_s'^2)^{1/2}$ and $\sin^2\phi = (v_s'^2 + w_s'^2)$

10. Special theories

10.1 General

10.1.1 In the derivation of the rotation matrix and the curvature expressions, the only approximation applied so far, is the replacement of terms of order $(1+\varepsilon)$ by unity, in accordance with the assumption of small strains. This means that the expressions obtained, are valid for beams exhibiting deflections and/or rotations of arbitrary magnitude. In many practical problems however the magnitude of the deflections and rotations is limited and expressions as accurate as (8.14) and (9.11-9.13) are not needed. Therefore, simplified expressions suited for certain classes of problems are derived in the following, by neglecting higher order terms

10.2 Beams exhibiting moderate deflections and large rotations

10.2.1 For this specific class of problems, terms quadratic in the derivatives of v_s and/or w_s, should be retained compared to one. This implies:

$$\cos\phi = 1-\frac{1}{2}v'^2-\frac{1}{2}w'^2 \; ; \; \frac{1-\cos\phi}{\sin^2\phi} = \frac{1}{2} \; ; \; \frac{1-\cos\phi}{\sin^2\phi\cos\phi} = \frac{1}{2} + O(2) \qquad (10.1)$$

Substitution of (10.1) into (8.14) and (9.11-9.13) yields:

$$\underline{R} = \begin{bmatrix} 1-\frac{1}{2}v_s'^2-\frac{1}{2}w_s'^2 & -v_s'\cos\beta-w_s'\sin\beta & -w_s'\cos\beta+v_s'\sin\beta \\ v_s' & (1-\frac{1}{2}v_s'^2)\cos\beta-\frac{1}{2}w_s'v_s'\sin\beta & -(1-\frac{1}{2}v_s'^2)\sin\beta-\frac{1}{2}w_s'v_s'\cos\beta \\ w_s' & (1-\frac{1}{2}w_s'^2)\sin\beta-\frac{1}{2}w_s'v_s'\cos\beta & (1-\frac{1}{2}w_s'^2)\cos\beta+\frac{1}{2}w_s'v_s'\sin\beta \end{bmatrix} \qquad (10.2)$$

$$\chi_1 = \beta'+\frac{1}{2}(w_s'v_s''-w_s''v_s')$$

$$\chi_2 = \cos\beta[-w_s''-\frac{1}{2}(w_s'^2w_s''+v_s'v_s''w_s')] + \sin\beta[v_s''+\frac{1}{2}(v_s'^2v_s''+w_s'w_s''v_s')] \quad (10.3)$$

$$\chi_3 = \cos\beta[v_s''+\frac{1}{2}(v_s'^2v_s''+w_s'w_s''v_s')] + \sin\beta[w_s''+\frac{1}{2}(v_s'v_s''w_s'+w_s'^2w_s'')]$$

The strain ε_s is now given by: $\varepsilon_s = u_s' + \frac{1}{2} v_s'^2 + \frac{1}{2} w_s'^2$

10.3 Beams exhibiting moderate deflections and moderate or small rotations

10.3.1 The rotation matrix and curvatures for problems with moderate deflections and moderate rotations can be obtained from (10.2) and (10.3) when in these expressions cosβ and sinβ are replaced by:

$$\cos\beta = 1-\frac{1}{2}\beta^2 \text{ and } \sin\beta = \beta-\frac{1}{6}\beta^3 \quad (10.4)$$

In the case of problems with moderate deflections and small rotations cosβ and sinβ may be replaced by:

$$\cos\beta = 1 \quad \text{and} \quad \sin\beta = \beta \quad (10.5)$$

10.4 Beams exhibiting small deflections and large rotations.

10.4.1 For this specific class of problems only terms linear in the derivatives of the v_s and w_s have to be retained in the rotation tensor and curvatures. This implies:

$$\cos\phi = 1 \quad \text{and} \quad \frac{1-\cos\phi}{\sin^2\phi\,\cos\phi} \approx \frac{1-\cos\phi}{\sin^2\phi} = \frac{1}{2} \quad (10.6)$$

Substition of (10.6) into (8.14) and (9.11-9.13) yields

$$\underline{R} = \begin{bmatrix} 1 & -v_s'\cos\beta - w_s'\sin\beta & -w_s'\cos\beta + v_s'\sin\beta \\ v_s' & \cos\beta & -\sin\beta \\ w_s' & \sin\beta & \cos\beta \end{bmatrix} \tag{10.7}$$

$$\begin{aligned} \chi_1 &= \beta' \\ \chi_2 &= -\cos\beta\ w_s'' + \sin\beta\ v_s'' \\ \chi_3 &= \cos\beta\ v_s'' + \sin\beta\ w_s'' \\ \varepsilon_s &= u_s' \end{aligned} \tag{10.8}$$

10.5 <u>Beams exhibiting small deflections and moderate or small rotations</u>;

10.5.1 The rotation matrix and curvatures for problems with small deflections and moderate rotations can be obtained from (10.7) and (10.8) when $\cos\beta$ and $\sin\beta$ are replaced by:

$$\cos\beta = 1-\frac{1}{2}\beta^2 \quad \text{and} \quad \sin\beta = \beta-\frac{1}{6}\beta^3 \tag{10.9}$$

If small deflections and small rotations are considered, only first order terms in β ; β' and the derivatives of v_s and w_s have to be taken into account. (8.14) and (9.11-9.13) then reduce to the following well known forms:

$$\underline{R} = \begin{bmatrix} 1 & -v_s' & -w_s' \\ v_s' & 1 & -\beta \\ w_s' & \beta & 1 \end{bmatrix} \quad \text{and} \quad \begin{aligned} \chi_1 &= \beta' \\ \chi_2 &= -w_s'' \\ \chi_3 &= v_s'' \\ \varepsilon_s &= u_s' \end{aligned} \tag{10.10}$$

10.6 Second order theory

10.6.1 Another theory, sometimes used [7, 20], is the so-called second order theory. In this theory deflections and rotations are assumed to be relatively small but in stead of using (10.10), second order terms in β and/or the derivatives of β, v_s and w_s are retained in $\underline{R}$, the curvatures and ε_s. When $\cos\beta$ and $\sin\beta$ in (10.2) and (10.3) are respectively replaced by: $1-\frac{1}{2}\beta^2$ and β the following rotation matrix and curvatures can be derived (all terms $\geq O(3)$ are neglected).

$$\underline{R} = \begin{bmatrix} 1-\frac{1}{2}v_s'^2-\frac{1}{2}w_s'^2 & -\beta w_s' - v_s' & -w_s' + \beta v_s' \\ v_s' & 1-\frac{1}{2}v_s'^2-\frac{1}{2}\beta^2 & -\beta-\frac{1}{2}w_s'v_s' \\ w_s' & \beta-\frac{1}{2}w_s'v_s' & 1-\frac{1}{2}w_s'-\frac{1}{2}\beta^2 \end{bmatrix} \qquad (10.11)$$

$$\chi_1 = \beta' + \frac{1}{2}(w_s'v_s''-w_s''v_s') \qquad \chi_3 = v_s'' + \beta w_s''$$

$$\chi_2 = -w_s'' + \beta v_s'' \qquad \varepsilon_s = u_s' + \frac{1}{2}v_s'^2 + \frac{1}{2}w_s'^2 \qquad (10.12)$$

Terms $\geq O(3)$ in β and the derivatives of β, v_s and w_s are also neglected, in the expressions for e_{11}, e_{12}, e_{13} and π_t.

This second order theory is derived to calculate the buckling loads of structures. The approximations used in the derivation are mathematically seen not consistent, but since the buckling displacements are infinitely small at the moment of buckling these approximations are allowed and the buckling loads calculated with this theory are correct.
However, due to the inconsitency in the approximations this theory should not be used as a general non linear beam theory.

11. Comparison with other results.

11.1.1 General

During the last 15 years several articles, dealing with non-linear flexural torsional behaviour of beams, appeared in the literature. In this chapter the theory developed in the previous sections is compared with the theories used in some of these articles.
However, before this comparison is started, some general items are discussed.

11.2 Thin-walled open sections.

11.2.1 In the case of thin-walled open sections, the normalized warping displacements are mostly described in terms of the so-called sectorial area ω, while the average displacement of the cross sections is written as: [2, 13, 20]

$$\bar{u} = u_D - y_D R_{12} - z_D R_{13} \tag{11.1}$$

where D is the sectorial origin.
$R_{12} = - v_s' \cos\beta - w_s' \sin\beta$ and $R_{13} = - w_s' \cos\beta + v_s' \sin\beta$
In this report, however, preference is given to the equivalent, but more general expression of normalized warping, $\phi(\tilde{y}, \tilde{z})$, while the average displacement of the cross section is written as:

$$\bar{u} = u_s - y_s R_{12} - z_s R_{13} \tag{11.2}$$

It can easily be seen that (11.1) represents the displacement of the centroid as no warping occurs and that (11.2) represents the same displacement.
Representing the average displacement of the cross section by (11.2) in stead of (11.1) has some advantages, as will be shown later. (section 11.7.2)

11.2.2 If the shear center is not a material point of the cross section, as is often the case for thin-walled open sections, it is still considered to follow all the deformations of the cross section, as if it was a real material point of that cross-section.

11.2.3 To enable a comparison of the results obtained in this report with other results, the terms in the articles, representing the warping displacement and the average displacement of the cross section, are converted to the notation used in this report.

11.3 The rotation tensor.

11.3.1 As already mentioned in section 8.11 the rotations in beam theory are often described in terms of modified Euler angles. Expressing these angles in terms of transverse displacements, χ_1 may be written as: [11, 14]

$$\chi_1 = \varphi' + \left(v_s'' w_s' + \frac{v_s' w_s' w_s''}{1-w_s'^2}\right) \frac{1}{(1-v_s'^2-w_s'^2)^{1/2}} \tag{11.3}$$

where φ is an Euler angle.
Egn. (11.3) is asymmetric in v_s and w_s.
That this may lead to confusion is demonstrated by the following remarks, made by M. Attard in his article about non-linear non-uniform torsion [2].

'when a prismatic beam experiences biaxial bending, the axis of shear centres becomes a space curve and the cross sections experience a twist known as 'geometric torsion' due to the bending curvatures.
The inclusion of geometric torsion is important in determining the equations for the lateral buckling of shafts under torsion. Attempts have been made by Goodier [9] and Rosen and Friedmann [21] to derive expressions for the geometric torsion in terms of the transverse displacements of the beam axis. The expression derived is doubtful because of its lack of symmetry and the inconsistent order of approximation made in the derivations".

(Remark; in both articles, refered to by Attard, a second order approximation for χ_1 is used: $\chi_1=\varphi'+v''w'$)

11.3.2 If the method of this paper is used to describe the rotations, χ_1 is given by:

$$\chi_1 = \beta' + \frac{1-(1-v_s'^2-w_s'^2)^{1/2}}{v_s'^2+w_s'^2}[v_s''w_s' - w_s''v_s'] \quad (11.4)$$

This eqn. is skew symmetric in v_s and w_s as would be expected. When only moderate or small deformations are considered, (11.4) reduces to:

$$\chi_1 = \beta' + \frac{1}{2}(v_s''w_s' - w_s''v_s') \quad (11.5)$$

11.4 A.A. Ghobarah and W.K. Tso. [8]

'A non-linear thin-walled beam theory'
Int. J. Mech. Sc., 1971, vol. 13, pp. 1025-1038.

11.4.1 Starting from the non-linear theory of elastic prismatic shells, a non-linear theory for thin-walled beams of open section is formulated by making special assumptions and neglecting terms of high order of smallness. Treating torsional deformational quantities as of order δ, where $\delta<1$, and flexural deformational quantities as of order δ^2, terms are retained in the energy expressions such that the resulting equations of equilibrium contain terms up to an order δ^3. Thus, products of torsional deformations and products of torsional deformation and flexural deformation are retained in the final expressions. However, products of flexural deformations are neglected, being treated as terms of order δ^4 or higher. Therefore, the theory takes into account the non-linear nature of torsional deformations and the coupling between torsional and flexural deformations. However, the theory is 'linear' in bending deformations in the sense that it does not take into account large deformation due to flexure.

To simplify the algebra, Ghobarah and Tso replaced the sinβ and cosβ functions by the two term approximation $(\beta-\frac{1}{6}\beta^3)$ and $(1-\frac{1}{2}\beta^2)$ respectively.

To make a comparison with the results obtained in this paper possible, the terms representing the normalized warping displacements in the article of Ghobarah and Tso, are replaced by the equivalent but more general expression $\psi(y, z)$.

11.4.2 Although the theory of Ghobarah and Tso is not written in terms of rotation matrices and curvatures, it can easily be converted to this form. The converted results are given by:

$$u_1 = \bar{u} + y\,R_{12} + z\,R_{13} + \beta'\psi\,R_{11}$$

$$u_2 = v_s + \tilde{y}\,(R_{22} - 1) + \tilde{z}\,R_{23} \qquad (11.6)$$

$$u_3 = w_s + \tilde{y}\,R_{32} + \tilde{z}\,(R_{33}-1)$$

where $\bar{u}$ is the average displacement of the cross section.

$$e_{11} = \bar{\varepsilon} - y\chi_3 + z\chi_2 + \frac{1}{2}\tilde{r}^2\chi_1^2 + \beta''\psi \qquad (11.7)$$

where $\bar{\varepsilon} = \varepsilon_s + y_s\chi_3 - z_s\chi_2$

$$\underline{R} = \begin{bmatrix} 1 & -v_s'-w_s'\beta & -w_s'+v_s'\beta \\ v_s' & 1-\frac{1}{2}\beta^2 & -(\beta-\frac{1}{6}\beta^3) \\ w_s' & \beta-\frac{1}{6}\beta^3 & 1-\frac{1}{2}\beta^2 \end{bmatrix} \qquad (11.8)$$

$$\chi_1 = \beta'$$

$$\chi_2 = -w_s'' + v_s'' \beta \qquad (11.9)$$

$$\chi_3 = v_s'' + w_s'' \beta$$

$$\varepsilon_s = u_s'$$

11.4.3 Equation (11.6) and (11.7) are the same as equation (6.8) and (5.4) derived in this report, except that the amplitude of the warping displacements is replaced by β'.

The rotation matrix, curvatures and strain ε_s obtained in this report for beams exhibiting small deflections and moderate rotations are given by: (see section 10.4 and 10.5).

$$\underline{R} = \begin{bmatrix} 1 & -v_s'(1-\frac{1}{2}\beta^2)-w_s'(\beta-\frac{1}{6}\beta^3) & -w_s'(1-\frac{1}{2}\beta^2)+v_s'(\beta-\frac{1}{6}\beta^3) \\ v_s' & (1-\frac{1}{2}\beta^2) & -(\beta-\frac{1}{6}\beta^3) \\ w_s' & (\beta-\frac{1}{6}\beta^3) & (1-\frac{1}{2}\beta^2) \end{bmatrix} \qquad (11.10)$$

$$\chi_1 = \beta'$$

$$\chi_2 = -(1-\frac{1}{2}\beta^2)w_s'' + (\beta-\frac{1}{6}\beta^3)v_s'' \qquad (11.11)$$

$$\chi_3 = (1-\frac{1}{2}\beta^2)v_s'' + (\beta-\frac{1}{6}\beta^3)w_s''$$

$$\varepsilon_s = u_s'$$

If (11.10) and (11.11) are compared with (11.8) and (11.9) it becomes clear that due to the fact that Ghobarah and Tso neglected all terms of order δ^4, an inconsistency in the representation of $\sin\beta$ and $\cos\beta$ is introduced.

Because of this, the theory is not valid for cross sectional rotations as large as 45^o, as claimed by the authors, but only for rotations as large 15^o.

The strain energy expression used by Ghobarah and Tso is derived from the strain energy expression of thin elastic shells given by [12]. Due to the different strain energy expressions used, a comparison of the final potential energy expressions is abandoned.

11.5 K. Roik, J. Carl and J. Lindner [20]

'Biegetorsions probleme gerader dünnwändiger stäbe.
Ernst & Sohn, 1972.

11.5.1 In this book a second order theory is presented. If the results are converted to the notation used in this paper, they can be written as:

$$\begin{aligned} u_1 &= \bar{u} + yR_{12} + zR_{13} + \chi_1\psi \\ u_2 &= v_s + \tilde{y}(R_{22}-1) + \tilde{z}R_{23} \\ u_3 &= w_s + \tilde{y}R_{32} + \tilde{z}(R_{33}-1) \end{aligned} \qquad (11.12)$$

where: $\bar{u} = u_s - y_sR_{12} - z_sR_{13}$.

$$e_{11} = \bar{\varepsilon} - y\chi_3 + z\chi_2 + \frac{1}{2}\tilde{r}^2\varphi'^2 + \chi_1'\psi \qquad (11.13)$$

where: $\bar{\varepsilon} = \varepsilon_s + y_s\chi_3 - z_s\chi_2$

$$\underline{R} = \begin{bmatrix} 1-\frac{1}{2}v_s'^2-\frac{1}{2}w_s'^2 & -\varphi w_s'-v_s' & -w_s'+\varphi v_s' \\ v_s' & 1-\frac{1}{2}v_s'^2-\frac{1}{2}\varphi^2 & -\varphi-v_s'w_s' \\ w_s' & \varphi & 1-\frac{1}{2}w_s'-\frac{1}{2}\varphi^2 \end{bmatrix} \qquad (11.14)$$

$$\chi_1 = \varphi' + w'_s v''_s - v'_s w''_s \qquad \chi_3 = v''_s + \varphi\, w''_s$$

$$\chi_2 = -w''_s + \varphi\, v''_s \qquad \varepsilon_s = u'_s + \frac{1}{2} v'^{\,2}_s + \frac{1}{2} w'^{\,2}_s \qquad (11.15)$$

11.5.2 Comparison of (11.14) with (6.8) shows that beside the replacement of g by χ_1, R_{11} is replaced by unity. This is in accordance with the fact that only terms quadratic in φ and/or the derivatives of φ, v_s and w_s are retained in the second order theory.

11.5.3 Roik et. al. derived the rotation matrix and curvatures by means of Euler angles. It can be shown that in case of a second order approximation φ and β (see (11.3) and (11.4)) are related by:

$$\varphi = \beta - \frac{1}{2} v'_s w'_s \qquad (11.16)$$

If (11.16) is substituted into (11.14), and terms $\geq O(3)$ are neglected, (11.14) changes to the rotation matrix of the second order theory, derived in this report (10.11)

Substitution of (11.16) into (11.15) and (11.13) yields:

$$\chi_1 = \beta' + \frac{1}{2} v''_s w'_s - \frac{3}{2} v'_s w''_s \qquad \chi_3 = v''_s + \beta w''_s$$

$$\chi_2 = - w''_s + \beta v''_s \qquad \varepsilon_s = u'_s + \frac{1}{2} v'^{\,2}_s + \frac{1}{2} w'^{\,2}_s \qquad (11.17)$$

$$e_{11} = \bar{\varepsilon} - y\chi_3 + z\chi_2 + \frac{1}{2} \tilde{r}^2 \beta'^{\,2} + \chi'_1 \psi \qquad (11.18)$$

Comparison of (11.17) with (10.12) shows that only the expression for χ_1, derived by Roik et.al. differs from the expressions derived in this report.
The reason for this can be found in the fact that Roik et. al. represents the rotation of the triad by a vector although this rotation is finite. This, however, is incorrect.

11.5.4 The strain expressions derived by Roik et. al. are the same as those derived in this report (5.1-5.4) except that g is replaced by χ_1 and terms $\geq$ O(3) are neglected.

Eqn. (2.27) is also used by Roik et.al. to determine the final potential energy expression. Only terms $\leq$ O(2) are retained in this expression.

11.6 A. Rosen and P. Friedmann [21]

'The nonlinear behavior of elastic slender straight beams undergoing small strains and moderate rotations'
Jounal of Applied Mechanics, vol. 46, (1979), pp. 161-168.

11.6.1 The non-linear behaviour of slender initially straight beams is investigated. A set of equilibrium equations is derived. The investigation is restricted to cases where u_s', $v_s'^2$, $w_s'^2$ and $\varphi^2$are negligible compared to unity. The converted results are given by:

$$\underline{R} = \begin{bmatrix} 1 & -v_s'-w_s'\varphi & -w'+\varphi v_s' \\ v_s' & 1 & -\varphi-v_s'w_s' \\ w_s' & \varphi & 1 \end{bmatrix} \qquad \begin{aligned} \chi_1 &= \varphi' + v_s''w_s' \\ \chi_2 &= -w_s'' + \varphi v_s'' \\ \chi_3 &= v_s'' + \varphi w_s'' \\ \varepsilon_s &= u_s' + \frac{1}{2} v_s'^2 + \frac{1}{2} w_s'^2 \end{aligned} \tag{11.19}$$

$$\begin{aligned} \varepsilon_{11} &= \varepsilon_s - \tilde{y}\chi_3 + \tilde{z}\chi_2 + \frac{\partial f}{\partial x} \\ \varepsilon_{12} &= \frac{1}{2} (-\tilde{z}\chi_1 + \frac{\partial f}{\partial y} + f\chi_3) \\ \varepsilon_{13} &= \frac{1}{2} (\tilde{y}\chi_1 + \frac{\partial f}{\partial z} - f\chi_2) \end{aligned} \tag{11.20}$$

11.6.2 The rotations and curvatures are derived by means of Euler angels. Substitution of (11.16) into (11.19) yields:

$$\underline{R} = \begin{bmatrix} 1 & -v_s'-w_s'\beta & -w'+\beta v_s' \\ v_s' & 1 & -\beta-\frac{1}{2}v_s'w_s' \\ w_s' & \beta-\frac{1}{2}v_s'w_s' & 1 \end{bmatrix} \qquad (11.21)$$

$$\chi_1 = \beta' + \frac{1}{2}\,(v_s''w_s' - w_s''v_s')$$

$$\chi_2 = -w_s'' + \beta\, v_s'' \qquad (11.22)$$

$$\chi_3 = v_s'' + \beta\, w_s''$$

$$\varepsilon_s = u_s' + \frac{1}{2}\,v_s'^2 + \frac{1}{2}\,w_s'^2$$

When in the rotation matrix (10.2) and curvature expressions (10.3), sinβ and cosβ are replaced by respectively β and 1, and $v_s'^2$, $w_s'^2$ and β^2 are neglected compared to unity, (10.2) and (10.3) reduce respectively to the converted equations (11.21) and (11.22).
The strain expressions derived in this paper (B.2) reduce to the equations derived by Rosen and Friedmann, when terms quadratic in χ are neglected.

Due to the type of approximations applied by Rosen and Friedmann, their theory is valid for small rotations and deflections which are somewhat larger than those of the linear theory.

11.7 <u>M.M. Attard</u> [2]

'Nonlinear theory of non-uniform torsion of thin-walled open beams' thin walled structures, 4, 1986, pp. 101-134.

11.7.1 A set of displacement relationschips for a straight prismatic thin-walled open beam of polygonal cross section is developed. The results are claimed to be applicable to situations where displacements are finite, the cross section does not distort, strains are small and flexural displacements are small to moderate while cross sectional twist can be large.

11.7.2 When the results are converted to the notation used in this report, they can be written as:

$$\underline{R} = \begin{bmatrix} 1 & -v_s'\cos\beta - w_s'\sin\beta & -w_s'\cos\beta + v_s'\sin\beta \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{bmatrix} \qquad (11.23)$$

$$\chi_1 = \beta'$$

$$\chi_2 = -\cos\beta w_s'' + \sin\beta v_s'' \qquad (11.24)$$

$$\chi_3 = \cos\beta v_s'' + \sin\beta w_s''$$

$$\varepsilon_s = u_s' + \frac{1}{2} v_s'^2 + \frac{1}{2} w_s'^2$$

$$e_{11} = \bar{u}' - y\chi_3 + z\chi_2 + 1/2\tilde{r}^2\chi_1^2 + \chi_1'\psi + \frac{1}{2}(R_{12}^2 + R_{13}^2) + (y_s R_{13} - z_s R_{12})\beta'$$

$$e_{12} = \frac{1}{2}(-\tilde{z} + \frac{\partial\psi}{\partial y})\chi_1 \qquad (11.25)$$

$$e_{13} = \frac{1}{2}(\tilde{y} + \frac{\partial\psi}{\partial z})\chi_1$$

where $\bar{u}$ is given by: $\bar{u} = u_D - y_D R_{12} - z_D R_{13}$ (see section 11.2)

When $\bar{u}$ is expressed as: $\bar{u} = u_s - y_s R_{12} - z_s R_{13}$, e_{11} may be written as:

$$e_{11} = \bar{\varepsilon} - y\chi_3 + z\chi_2 + \frac{1}{2}\tilde{r}^2\chi_1^2 + \chi_1'\psi \qquad (11.26)$$

where $\bar{\varepsilon} = \varepsilon_s + y_s\chi_3 + z_s\chi_2$

11.7.3 If these results are compared with the results obtained in this report, it shows that there is an inconsistence in the theory of Attard. Although bending deformations are considered finite, the influence of geometric torsion is neglected. That Attard is aware of this inconsistence shows the remark he has made in his article, which is copied in section 11.3 of this report.
Due to this inconsistence, the theory is only valid for small deflections and large rotations

11.8 Z.M. Elias [7]

'Theory and methods of structural analysis'
John Wiley & Sons (1986).

11.8.1 In this book a second order theory is derived. The strain energy expression and the curvatures are compared with the expressions derived in this report. The results obtained by Elias can be written as:

$$U = \frac{1}{2}\int_0^L [EA\bar{\varepsilon}^2 + EI_2\chi_2^2 + EI_3\chi_3^2 + EH\chi_1^4 + E\Gamma(\chi_1')^2 + GI_t\chi_1^2$$

$$+ \chi_1^2(EI_s\bar{\varepsilon} + EI_2\beta_2\chi_2 - EI_3\beta_3\chi_3)]dx \qquad (11.27)$$

where $\bar{\varepsilon} = \varepsilon_s + y_s\chi_3 - z_s\chi_2$

$$\chi_1 = \beta' + \frac{1}{2}(v_s''w_s' - w_s''v_s')$$

$$\chi_2 = -w_s'' + \beta v_s'' \qquad (11.28)$$

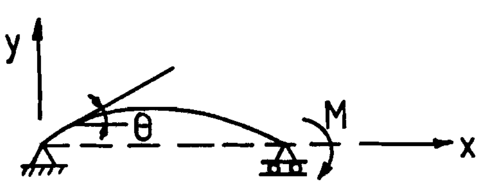
$$\chi_3 = v_s'' + \beta w_s''$$

$$\varepsilon_s = u_s' + \frac{1}{2}v_s'^2 + \frac{1}{2}w_s'^2$$

11.8.2 When g(x) in (5.7) is replaced by $\chi_1(x)$, comparison of (11.27) with (5.7) shows that the term $E\Gamma\beta_\psi\chi_1'\chi_1^2$ is missing in the strain energy expression of Elias. This term, however, must be taken into account when the cross-section is not double symmetric.

The curvature expressions (11.28) are exactly the same as those derived in this paper for the second order theory.
It is emphasized here that Elias is the only one, the author is aware of who has also found the 'skew'-symmetric expression for the geometric torsion. This expression is derived in a totally different manner, than it is in this paper.

11.9 Bending in one plane



11.9.1 When ε is neglected compared to unity, the following relations can be derived:

$$\vec{i}_1 = (1+u')\vec{e}_1 + v'\vec{e}_2 \tag{11.29}$$

$$\sin\theta = v' \quad \rightarrow \quad \cos\theta = (1-v'^2)^{1/2} \tag{11.30}$$

$$\cos\theta \frac{d\theta}{dx} = v'' \tag{11.31}$$

$$\chi \approx \frac{d\theta}{dx} = v''(1-v'^2)^{-1/2} \tag{11.32}$$

The rotation matrix can be written as:

$$\underline{R} = \begin{bmatrix} \sqrt{(1-v'^2)} & -v' & 0 \\ v' & \sqrt{(1-v'^2)} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{11.33}$$

Substituting $\beta = w = 0$ into (8.14) and (9.11-9.13) yields exactly the same results.

11.9.2 In the linear theory terms quadratic in v' are neglected, this implies:

$$\underline{R} = \begin{bmatrix} 1 & -v' & 0 \\ v' & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{and } \chi = v'' \tag{11.34}$$

Substituting $\beta = w = 0$ into (10.10) yields exactly the same result.

11.9.3 In the case of moderate deflections, terms quadratic in v' are retained. this implies:

$$\underline{R} = \begin{bmatrix} 1 - \frac{1}{2} v'^2 & -v' & 0 \\ v' & 1-\frac{1}{2} v'^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \chi = v'' + \frac{1}{2} v'' v'^2 \tag{11.35}$$

Substituting $\beta = w = 0$ into (10.2) and (10.3) yields the same result.

12. Conclusions

A potential energy functional for non-linear flexural-torsional behaviour of straight elastic beams with arbitrary cross sections, has been presented. The functional is generally applicable to situations where the strains are small and the Bernoulli hypotheses are valid. It has been shown that the potential energy expressions for special theories can be derived from this general expression in a consistent manner, by neglecting some specific terms.

Using a special representation of the rotation tensor, a skew symmetric expression for the 'geometric-torsion', has been derived.

13. Acknowledgement

The author is indebted to Dr.ir. C.M. Menken and Dr.ir. F.E. Veldpaus of Eindhoven University of Technology, for their helpful suggestions and criticisms.
This work was supported by research grants made available by the Netherlands Technology Foundation (STW)

14. References

[1] Ascione, L. and Grimaldi, A.: On the stability and postbuckling behaviour of elastic beams, Thin-Walled Struc., 1, 1983, pp. 325-351.

[2] Attard, M.M.: Non linear theory of non uniform torsion of thin walled open beams, Thin-Walled Struc., 4, 1986, pp. 101-134.

[3] Benscoter, S.U.: A theory of torsion bending for multicell beams, Journal of Applied Mechanics, 21, 1954, pp. 25-34.

[4] Besseling, J.F.: Non-linear theory for elastic beams and rods and its finite element representation, Techn. Hogeschool Delft, Delft Univ. of Techn., 1981

[5] Brekelmans, W.A.M.: Toepasbare Sterkteleer, Reader of Dept. of Mech. Eng., Eindhoven Univ. of Techn., the Netherlands, 1978.

[6] Dym, C.L. and Shames : Solid Mechanics: a variational approach, McGraw-Hill Kogakuska, 1973.

[7] Elias Z.M.: Theory and methods of structural analysis, John Wiley & Sons Inc., 1986.

[8] Ghobarah, A.A. and Tso, W.K.: A non-linear thin-walled beam theory, Int. Journal of Mechanical Science, Vol. 13, 1971, pp. 1025-1038.

[9] Goodier, J.N.: On combined flexure and torsion and the flexural buckling of a twisted bar, Quarterly of Applied Mathematics, 2(2) (1944) pp. 93-101.

[10] Gregory, M.: Elastic torsion of members of thin-walled open cross sections, Australian Journal of Applied Science, 12 (2), june 1961, pp. 174-193.

[11] Grimaldi, A. and Pignataro, M.: Postbuckling behaviour of thin-walled open cross section compression members, J. Struct. Mech, 7(2), 1979, 143-159.

[12] Love, A.E.H.: A treatise on the mathematical theory of elasticity, 4th edn., New York, Dover Publ. Inc. 1944.

[13] Murray, N.W.: Introduction to the theory of thin-walled structures, Oxford University Press, New York, 1984.

[14] Peters, A.P. and Ormiston P.A.: The effects of second order blade bending on the angle of attack of hingeless rotor blades, Journal of the American Helicopter Soc., vol. 18, No. 4, 1973, pp. 45-48.

[15] Reissner, E.: On Non-uniform torsion of cylindrical rods, Journal of Mathematics and Physics, vol. 31, 1952, pp. 214-221.

[16] Reissner, E.: On torsion with variable twist, Österreichisches Ingenieur Archiv, 9, 1955, pp. 218-224.

[17] Reissner, E.: Note on torsion with variable twist, Journal of Applied Mechanics, june 1956, pp. 315-316.

[18] Roberts, T.M.: Second order strains and instability of thin walled bars of open cross section, Int. J. Mech. Sci, vol 23, 1981, pp. 297-306.

[19] Roberts, T.M. and Azizian, Z.G.: Non linear analysis of thin walled bars of open cross section, Int. J. Mech. Sci, Vol. 25, no. 8, 1983, pp. 565-577,

[20] Roik, K., Carl, J., and Lindner, J.: Biegetorsionsprobleme gerader dünnwandige Stäbe., W. Ernst & Sohn, 1972.

[21] Rosen, A. and Friedmann P.: The non-linear behaviour of elastic slender straight beams undergoing small strains and moderate rotations, Journal of Applied Mechanics, Vol. 46, March 1979, pp. 161-168.

[22] Trahair, N.S. and Woolcock, S.T.: Effect of major axis curvature on I-beam stability, Journal of the Engineering Mechanics Division, EMI, Febr. 1979, pp. 85-98.

[23] Veldpaus, F.E.: Inleiding continuumsmechanica, Reader of Dept. of Mech. Eng., Eindhoven University of Technology, The Netherlands, 1985.

[24] Vacharajittiphan, P., Woolcock, S.T. and Trahair, N.S.: Effect of In-plane deformation on lateral buckling, J. Struct. Mech, 3(1), 1974, pp. 29-60.

[25] Woolcock, S.T. and Trahair, N.S.: Post-buckling behaviour of determinate beams, Journal of the Engineering Mechanics Division, EM2, apr. 1974, pp 151-171.

Appendix A

A.1. Throughout this report a coordinate-free dyadic notation is used. In this notation $\vec{a}$ symbolizes a vector. The length of the vector is denoted by its norm $|\vec{a}|$, and its direction is obtained by dividing the vector by its length (normalization).

$$\vec{e} = \vec{a}/|\vec{a}| \tag{A.1}$$

Between the two vectors $\vec{a}$ and $\vec{b}$, the inner product is defined by:

$$\vec{a}\bullet\vec{b} = |\vec{a}||\vec{b}| \cos\phi \tag{A.2}$$

where ϕ is the smallest angle between the two vectors

A.2. A most important aspect is the existence of a dyadic product of two vectors, which creates a dyad or second order tensor

$$\mathbf{A} = \vec{a}\vec{b} \tag{A.3}$$

It can be proven that every tensor can be written as the sum of, at most, three dyadic products:

$$\mathbf{A} = \vec{a}_i\vec{b}_i \tag{A.4}$$

in which expression the summation convention has been used.

The cummutative law does not apply for the dyadic product. Interchanging the vectors yields the conjugate of a dyad.

$$\mathbf{A}^c = (\vec{a}_i\vec{b}_i)^c = \vec{b}_i\vec{a}_i \tag{A.5}$$

A.3. In this report, frequent use is also made of the double inner product between two tensors. This double inner product is a repeated application of the inner product:

$$\mathbf{A}:\mathbf{B} = (\vec{a}_i\vec{b}_i):(\vec{c}_j\vec{d}_j) = (\vec{b}_i\bullet\vec{c}_j)(\vec{a}_i\bullet\vec{d}_j) \tag{A.6}$$

An important tensor is the unit tensor **I**, which satisfies the property:

$$\mathbf{I}\bullet\vec{a} = \vec{a} \quad \text{for every } \vec{a} \tag{A.7}$$

A.4. The tensors discussed so far are second order tensors or dyads. Higher order tensors can be obtained by (summation of) dyadic products of more than two vectors or, alternatively as dyadic products of tensors. A fourth order tensor may thus be defined as:

$$^4\mathbf{D} = \mathbf{AB} = \vec{a}\vec{b}\vec{c}\vec{d} \tag{A.8}$$

Some important fourth order tensors are $^4\mathbf{I}$, $^4\mathbf{I}^c$ and $\mathbf{II}$ which are respectively defined by:

$$^4\mathbf{I}:\mathbf{A} = \mathbf{A} \quad , \quad {}^4\mathbf{I}^c:\mathbf{A} = \mathbf{A}^c \quad , \quad \mathbf{II}:\mathbf{A} = \mathrm{tr}(\mathbf{A}) = \mathbf{A}:\mathbf{I} \tag{A.9}$$

where tr(**A**) stand for the trace of **A**.

A.5. Finally, use is made of the gradient operator $\vec{\nabla}$. This gradient operator has the property that it defines the derivative of a quantity along a line in space. If such a line is defined by the parameter representation

$$\vec{x} = \vec{x}(\lambda) \tag{A.10}$$

where $\vec{x}$ is the position vector, it follows that the derivative of a quantity u along this line is obtained as:

$$du/d\lambda = (d\vec{x}/d\lambda)\bullet(\vec{\nabla}u) \tag{A.11}$$

Note that u may be a vector or tensor quantity as well.

A.6. The dyadic notation is readily converted into a cartesian notation by introduction of a set of orthonormal base vectors $\vec{e}_i$. Orthonormality yields for the inner product of two base vectors:

$$\vec{e}_i \bullet \vec{e}_j = \delta_{ij} \tag{A.12}$$

hwere δ_{ij} is the Kronecker delta.

With these base vectors, vectors and tensors may be represented in terms of their Cartesian components:

$$\vec{a} = a_i \vec{e}_i \quad , \quad \mathbf{A} = A_{ij} \vec{e}_i \vec{e}_j \tag{A.13}$$

From (A.12) and (A.13), equivalence of inner products and double inner products in the two notations is readily obtained:

$$\vec{a} \bullet \vec{b} = (a_i \vec{e}_i) \bullet (b_j \vec{e}_j) = a_i b_j \ \delta_{ij} = a_i b_i \tag{A14}$$

$$\mathbf{A} : \mathbf{B} = A_{ij} \vec{e}_i \vec{e}_j : B_{kl} \ \vec{e}_k \vec{e}_l = A_{ij} B_{ji} \tag{A.15}$$

The equivalent of the gradient operator is readily obtained from (A.11) by observing that $\vec{x} = x_i \vec{e}_i$ and taking x_i as the parameter λ:

$$\frac{\partial u}{\partial x_i} = \vec{e}_i \bullet \vec{\nabla} u \tag{A.16}$$

Multiplication by $\vec{e}_i$ and summation yields alternatively:

$$\vec{\nabla} u = \vec{e}_i \ \frac{\partial u}{\partial x_i} \tag{A.17}$$

Hence, the cartesian notation is readily derived from the more general dyadic notation used in this paper.

For additional information about this notation, the reader should refer to [23] (in Dutch) or any other appropriate textbook on this subject.

Appendix B

B.1. If $(1+\varepsilon_s)$ and $(1+\varepsilon_s)^2$ are replaced by unity, in accordance with the assumption of small strains, (3.27) reduces to:

$$
\begin{aligned}
e_{11}&=\varepsilon_s+[1-(-\tilde{y}\chi_3+\tilde{z}\chi_2)](-\tilde{y}\chi_3+\tilde{z}\chi_2)+(\tilde{y}^2+\tilde{z}^2)\chi_1^2+\frac{\partial f}{\partial x}[1 \\
&\quad +(-\tilde{y}\chi_3+\tilde{z}\chi_2)+\frac{1}{2}\frac{\partial f}{\partial x}]+f(-\tilde{y}\chi_1\chi_2-\tilde{z}\chi_1\chi_3)+\frac{1}{2}f^2(\chi_2^2+\chi_3^2) \\
e_{12}&=(-\tilde{z}\chi_1+f\chi_3+\frac{\partial f}{\partial y}[1+(-\tilde{y}\chi_3+\tilde{z}\chi_2)+\frac{\partial f}{\partial x}]) \\
e_{13}&=(\tilde{y}\chi_1-f\chi_2+\frac{\partial f}{\partial z}[1+(-\tilde{y}\chi_3+\tilde{z}\chi_2)+\frac{\partial f}{\partial x}])
\end{aligned}
\qquad \text{B.1)}
$$

The terms $\tilde{y}\chi_3$ and $\tilde{z}\chi_2$ represent strains due to bending, while $\frac{\partial f}{\partial x}$ represents strain due to warping. Therefore, $[1-(-\tilde{y}\chi_3+\tilde{z}\chi_2)]$, $[1+(-\tilde{y}\chi_3+\tilde{z}\chi_2)+\frac{1}{2}\frac{\partial f}{\partial x}]$ and $[1+(-\tilde{y}\chi_3+\tilde{z}\chi_2)+\frac{\partial f}{\partial x}]$ may also be replaced by unity. this implies:

$$
\begin{aligned}
e_{11}&=\varepsilon_s+(-\tilde{y}\chi_3+\tilde{z}\chi_2)+(\tilde{y}^2+\tilde{z}^2)\chi_1^2+f(-\tilde{y}\chi_1\chi_2-\tilde{z}\chi_1\chi_3)+\frac{1}{2}f^2(\chi_2^2+\chi_3^2)+\frac{\partial f}{\partial x} \\
e_{12}&=[-\tilde{z}\chi_1+f\chi_3+\frac{\partial f}{\partial y}] \\
e_{13}&=[\tilde{y}\chi_1-f\chi_2+\frac{\partial f}{\partial z}]
\end{aligned}
\qquad \text{(B.2)}
$$

B.2. In all the strain expressions, f appears in more than one term. To see if it is neccessary to retain all these terms, the order of magnitude of these terms is determined.

$$
\begin{aligned}
&\left|\frac{\partial f}{\partial x}\right|=O(f/L)\le O(\varepsilon); \quad \left|\frac{\partial f}{\partial y}\right|=O(f/d) \quad ; \quad \left|\frac{\partial f}{\partial z}\right|=O(f/d) \\
&\chi_2^2\le O[(\varepsilon/d)^2] \; ; \; \chi_3^2\le O[(\varepsilon/d)^2] \; ; \; |\tilde{y}\chi_1|\le O(\varepsilon) \; ; \; |\tilde{z}\chi_1|\le O(\varepsilon)
\end{aligned}
\qquad \text{(B.3)}
$$

where d denotes a linear dimension of the cross section.

The order of magnitude of the other terms can now be written as:

$$|f(-\tilde{y}\chi_1\chi_2 - \tilde{z}\chi_1\chi_3)| \leq O(f\ \varepsilon^2/d) \tag{B.4}$$

$$f^2(\chi_2^2 + \chi_3^2) \leq O[f^2(\varepsilon/d)^2] \tag{B.5}$$

with $|f| \leq O(\varepsilon L)$ this can be written as:

$$f^2(\chi_2^2 + \chi_3^2) \leq O[\varepsilon^4(L/d)^2] \tag{B.6}$$

$$|f\chi_3| \leq O(f\varepsilon/d) \tag{B.7}$$

$$|f\chi_2| \leq O(f\varepsilon/d) \tag{B.8}$$

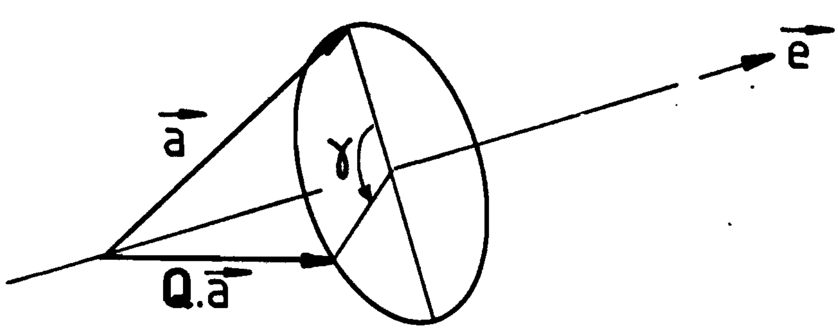
If in each strain equation warping terms, which are $O(\varepsilon^n)$ $n \geq 1$ smaller or larger than another warping term in that equation, are neglected, (B.2) reduces to:

$$\begin{aligned} e_{11} &= \varepsilon_s + (-\tilde{y}\chi_3 + \tilde{z}\chi_2) + (\tilde{y}^2 + \tilde{z}^2)\ \chi_1^2 + \frac{\partial f}{\partial x} \\ e_{12} &= (-\tilde{z}\chi_1 + \frac{\partial f}{\partial y}) \\ e_{13} &= (\tilde{y}\chi_1 + \frac{\partial f}{\partial z}) \end{aligned} \tag{B.9}$$

Appendix C

A rotation tensor **Q**, representing a rotation γ, of an arbitrary vecotr $\vec{a}$, about an axis with unit vector $\vec{e}$ can, according to eqn. (8.3), be expressed as:

$$\mathbf{Q}\bullet\vec{a} = \cos\gamma\ \vec{a} + (1-\cos\gamma)\vec{e}\vec{e}\bullet\vec{a} + \sin\gamma\ \vec{e}*\vec{a} \qquad (C.1)$$



To show this, the vector $\vec{a}$ is written as:

$$\vec{a} = (\vec{a}\bullet\vec{e})\vec{e} + [\vec{a}-(\vec{a}\bullet\vec{e})\vec{e}] \qquad (C.2)$$

where: $(\vec{a}\bullet\vec{e})\vec{e}$ is the projection of $\vec{a}$ in the direction $\vec{e}$

The vector $[\vec{a}-(\vec{a}\bullet\vec{e})\vec{e}]$ is perpendicular to $\vec{e}$
$\mathbf{Q}\bullet\vec{a}$ may now also be written as:

$$\mathbf{Q}\bullet\vec{a} = (\vec{a}\bullet\vec{e})\vec{e} + \mathbf{Q}\bullet[\vec{a}-(\vec{a}\bullet\vec{e})\vec{e}] \qquad (C.3)$$

To calculate the result of $\mathbf{Q}\bullet[\vec{a}-(\vec{a}\bullet\vec{e})\vec{e}]$, a unit vector $\vec{b}$ perpendicular to $\vec{e}$ and a unit vector $(\vec{e}*\vec{b})$ are introduced. The components of $[\vec{a}-(\vec{a}\bullet\vec{e})\vec{e}]$ in the direction $\vec{b}$ and $(\vec{e}*\vec{b})$ are respectively:

$$(\vec{a}\bullet\vec{b})\vec{b} \text{ and } [\vec{a}\bullet(\vec{e}*\vec{b})](\vec{e}*\vec{b}) \qquad (C.4)$$

If $[\vec{a}-(\vec{a}\cdot\vec{e})\vec{e}]$ is rotated γ about $\vec{e}$, then the components of $Q\cdot[\vec{a}-(\vec{a}\cdot\vec{e})\vec{e}]$ in the direction $\vec{b}$ and $(\vec{e}*\vec{b})$ become respectively:

$$[\cos\gamma(\vec{a}\cdot\vec{b}) - \sin\gamma(\vec{a}\cdot(\vec{e}*\vec{b}))]\vec{b}$$
$$[\cos\gamma(\vec{a}\cdot(\vec{e}*\vec{b})) + \sin\gamma(\vec{a}\cdot\vec{b})](\vec{e}*\vec{b}) \qquad (C.5)$$

The final result of $Q\cdot\vec{a}$ can now be written as:

$$Q\cdot\vec{a} = \vec{e}\vec{e}\cdot\vec{a} + \cos\gamma[\vec{b}\vec{b} + (\vec{e}*\vec{b})(\vec{e}*\vec{b})]\cdot\vec{a}$$
$$+ \sin\gamma[(\vec{e}*\vec{b})(\vec{b}\cdot\vec{a}) - \vec{b}((\vec{e}*\vec{b})\cdot\vec{a})] \qquad (C.6)$$

Making use of the identity: $\vec{k}*(\vec{m}*\vec{n}) = (\vec{k}\cdot\vec{n})\vec{m} - (\vec{k}\cdot\vec{m})\vec{n}$ (C.7)

and the fact that $[\vec{e}\vec{e} + \vec{b}\vec{b} + (\vec{e}*\vec{b})(\vec{e}*\vec{b})] = I$ (C.8)

eqn. (C.6) can be written as:

$$Q\cdot\vec{a} = \cos\gamma\vec{a} + (1-\cos\gamma)\vec{e}\vec{e}\cdot\vec{a} + \sin\gamma\ \vec{e}*\vec{a} \qquad (C.9)$$

Appendix D

D.1. $$\mathbf{R}\cdot\vec{a} = \mathbf{R}_2\cdot\mathbf{R}_1\cdot\vec{a} \qquad (D.1)$$

According to (8.4) $\mathbf{R}_1\cdot\vec{a}$ can be written as:

$$\mathbf{R}_1\cdot\vec{a} = \cos\phi\vec{a} + (1-\cos\phi)\vec{\eta}\vec{\eta}\cdot\vec{a} + \sin\phi\vec{\eta}*\vec{a} \qquad (D.2)$$

where: $\cos\phi = \vec{e}_1\vec{i}_1$ and $\sin\phi\vec{\eta} = \vec{e}_1*\vec{i}_1$

Using (8.5) $\mathbf{R}_2\cdot\vec{a}$ may be written as:

$$\mathbf{R}_2\cdot\vec{a} = \cos\alpha\vec{a} + (1-\cos\alpha)\vec{i}_1\vec{i}_1\cdot\vec{a} + \sin\alpha\ \vec{i}_1*\vec{a} \qquad (D.3)$$

D.2. Substitution of (D.2) and (D.3) into (D.1) yields:

$$\mathbf{R}_2\cdot\mathbf{R}_1\cdot\vec{a}=\cos\phi\mathbf{R}_2\cdot\vec{a}+[(1-\cos\phi)(\vec{\eta}\cdot\vec{a})\mathbf{R}_2\cdot\vec{\eta}+\sin\phi\mathbf{R}_2\cdot(\vec{\eta}*\vec{a})] \qquad (D.4)$$

Using the identity: $\vec{a}*(\vec{b}*\vec{c})=(\vec{a}\cdot\vec{c})\vec{b} - (\vec{b}\cdot\vec{a})\vec{c}$ (D.5)

$\mathbf{R}_2\cdot\vec{\eta}$ and $\sin\phi\mathbf{R}_2\cdot(\vec{\eta}*\vec{a})$ may be written as:

$$\mathbf{R}_2\cdot\vec{\eta} = \cos\alpha\vec{\eta} + \frac{\sin\alpha}{\sin\phi}\ \vec{e}_1 - \sin\alpha\ \mathrm{cotan}\phi\ \vec{i}_1 \qquad (D.6)$$

$$\begin{aligned}\sin\phi\mathbf{R}_2\cdot(\vec{\eta}*\vec{a}) = \mathbf{R}_2\cdot[(\vec{e}_1*\vec{i}_1)*\vec{a}] = \vec{i}_1\vec{e}_1\cdot\vec{a} - [\cos\alpha\vec{e}_1\vec{i}_1 + \\ + (1-\cos\alpha)\cos\phi\ \vec{i}_1\vec{i}_1 - \sin\alpha\sin\phi\vec{\eta}\vec{i}_1]\cdot\vec{a}\end{aligned} \qquad (D.7)$$

D.3. Substitution of (D.6) and (D.7) into (D.4) finally yields.

$$\begin{aligned}\mathbf{R}_2\cdot\mathbf{R}_1\cdot\vec{a} = \cos\phi\cos\alpha\ \vec{a} + \cos\phi\sin\alpha\ \vec{i}_1*\vec{a} \\ [(1-\cos\phi)(\cos\alpha\vec{\eta}\vec{\eta} + \frac{\sin\alpha}{\sin\phi}\ \vec{e}_1\vec{\eta} - \sin\alpha\ \mathrm{cotan}\phi\ \vec{i}_1\vec{\eta}) \\ -(\cos\alpha\ \vec{e}_1\vec{i}_1 - \sin\alpha\sin\phi\vec{\eta}\vec{i}_1) + \vec{i}_1\vec{e}_1]\cdot\vec{a}\end{aligned} \qquad (D.8)$$

Appendix E

$$\mathbf{R}\bullet\vec{a} = \mathbf{R}_1\bullet\tilde{\mathbf{R}}_2\bullet\vec{a} \tag{E.1}$$

E.1. According to (8.4) $\mathbf{R}_1\bullet\vec{a}$ can be written as:

$$\mathbf{R}_1\bullet\vec{a} = \cos\phi\vec{a} + (1-\cos\phi)\vec{\eta}\vec{\eta}\bullet\vec{a} + \sin\phi\vec{\eta}*\vec{a} \tag{E.2}$$

where: $\cos\phi = \vec{e}_1\bullet\vec{i}_1$ and $\sin\phi\vec{\eta} = \vec{e}_1*\vec{i}_1$

According to (8.6) $\tilde{\mathbf{R}}_2\bullet\vec{a}$ may be written as:

$$\tilde{\mathbf{R}}_2\bullet\vec{a} = \cos\beta\vec{a} + (1-\cos\beta)\ \vec{e}_1\vec{e}_1\bullet\vec{a} + \sin\beta\ \vec{e}_1*\vec{a} \tag{E.3}$$

E.2 Substitution of (E.2) and (E.3) into (E.1) yields:

$$\begin{aligned}\mathbf{R}_1\bullet\tilde{\mathbf{R}}_2\bullet\vec{a} &= (\mathbf{R}_1\bullet\vec{e}_1)\vec{e}_1\bullet\vec{a} + \mathbf{R}_1\bullet[\cos\beta\ \vec{e}_1*(\vec{a}*\vec{e}_1)+\sin\beta\ \vec{e}_1*\vec{a}] \\ &= \vec{i}_1\vec{e}_1\bullet\vec{a} + \mathbf{R}_1\bullet[\vec{e}_1*(\cos\beta(\vec{a}*\vec{e}_1)+\sin\beta\vec{a})]\end{aligned} \tag{E.4}$$

$$\mathbf{R}_1\bullet(\vec{e}_1*\vec{a}) = [\cos\phi\mathbf{I} - \vec{e}_1\vec{i}_1 + (1-\cos\phi)\vec{\eta}\vec{\eta}]\bullet(\vec{e}_1*\vec{a}) \tag{E.5}$$

Substitution of (E.5) into (E.4) yields:

$$\begin{aligned}\mathbf{R}_1\bullet\tilde{\mathbf{R}}_2\bullet\vec{a} &= [\cos\phi\mathbf{I}-\vec{e}_1\vec{i}_1+(1-\cos\phi)\vec{\eta}\vec{\eta}]\bullet \\ &\quad (\cos\beta\vec{a}+\sin\beta\vec{e}_1*\vec{a})+\vec{i}_1\vec{e}_1\bullet\vec{a}\end{aligned} \tag{E.6}$$

Appendix F

F.1. According to (8.9) $R_2 \bullet R_1 \bullet \vec{a}$ can be written as:

$$R_2 \bullet R_1 \bullet \vec{a} = \cos\phi \cos\alpha \, \vec{a} + \cos\phi \sin\alpha \, \vec{i}_1 * \vec{a} +$$

$$[(1-\cos\phi)(\cos\alpha \vec{\eta}\vec{\eta} + \frac{\sin\alpha}{\sin\phi} \vec{e}_1 \vec{\eta} - \sin\alpha \cot\mathrm{an}\phi \, \vec{i}_1 \vec{\eta}) \tag{F.1}$$

$$- (\cos\alpha \, \vec{e}_1 \vec{i}_1 - \sin\alpha \sin\phi \vec{\eta} \vec{i}_1) + \vec{i}_1 \vec{e}_1] \bullet \vec{a}$$

where: $\cos\phi = \vec{e}_1 \bullet \vec{i}_1$ and $\sin\phi \vec{\eta} = \vec{e}_1 * \vec{i}_1$

F.2. The unit vector $\vec{i}_1$ is given by

$$\vec{i}_1 = (1+\varepsilon_s)^{-1} [(1+u_s') \, \vec{e}_1 + v_s' \, \vec{e}_2 + w_s' \, \vec{e}_3] \tag{F.2)}$$

With this expression $\cos\phi$ and $\vec{\eta}$ can be written as:

$$\cos\phi = (1+\varepsilon_s)^{-1} (1+u_s) \tag{F.3}$$

$$\vec{\eta} = [(1+\varepsilon_s)\sin\phi]^{-1} (v_s' \, \vec{e}_3 - w_s' \, \vec{e}_2) \tag{F.4}$$

Using eqns. (F.2-F.4) the terms in (F.1) may be written as:

$$(1-\cos\phi)\cos\alpha \vec{\eta}\vec{\eta} = \bar{b}[v_s'^2 \vec{e}_3 \vec{e}_3 - v_s' w_s' \vec{e}_3 \vec{e}_2 - w_s' v_s' \vec{e}_2 \vec{e}_3 + w_s'^2 \vec{e}_2 \vec{e}_2] \tag{F.5}$$

$$(1-\cos\phi)\frac{\sin\alpha}{\sin\phi} \vec{e}_1 \vec{\eta} = \bar{b}(1+\varepsilon_s)\sin\alpha[v_s' \vec{e}_1 \vec{e}_3 - w_s' \vec{e}_1 \vec{e}_2] \tag{F.6}$$

$$(1-\cos\phi)\sin\alpha \cot\mathrm{an}\phi \, \vec{i}_1 \vec{\eta} = \bar{b}\sin\alpha(\frac{1+u_s'}{1+\varepsilon_s})[(1+u_s')(v_s' \vec{e}_1 \vec{e}_3 - w_s' \vec{e}_1 \vec{e}_2) +$$

$$+ v_s'^2 \vec{e}_2 \vec{e}_3 - v_s' w_s' \, \vec{e}_2 \vec{e}_2 + w_s' v_s' \, \vec{e}_3 \vec{e}_3 - w_s'^2 \, \vec{e}_3 \vec{e}_2] \tag{F.7}$$

$$\cos\alpha \, \vec{e}_1 \vec{i}_1 = \frac{\cos\alpha}{(1+\varepsilon_s)}[(1+u_s')\vec{e}_1 \vec{e}_1 + v_s' \, \vec{e}_1 \vec{e}_2 + w_s' \, \vec{e}_1 \vec{e}_3] \tag{F.8}$$

$$\sin\alpha \sin\phi \vec{\eta} \vec{i}_1 = \frac{\sin\alpha}{(1+\varepsilon_s)^2}[(1+u_s')(v_s' \vec{e}_3 \vec{e}_1 - w_s' \vec{e}_2 \vec{e}_1) + v_s'^2 \vec{e}_3 \vec{e}_2$$

$$- w_s'v_s'\vec{e}_2\vec{e}_2 + w_s'v_s'\vec{e}_3\vec{e}_3 - w_s'^2\vec{e}_2\vec{e}_3] \quad (F.9)$$

$$\vec{i}_1\vec{e}_1 = \frac{1}{1+\varepsilon_s}[(1+u_s')\vec{e}_1\vec{e}_1 + v_s'\,\vec{e}_2\vec{e}_1 + w_s'\,\vec{e}_3\vec{e}_1] \quad (F.10)$$

where: $\bar{b} = \dfrac{(1-\cos\phi)}{(1+\varepsilon_s)^2\sin^2\phi}$

F.3. When (F.2-F.10) are substituted into (F.1); $\mathbf{R}\cdot\vec{e}_1$, $\mathbf{R}\cdot\vec{e}_2$ and $\mathbf{R}\cdot\vec{e}_3$ can be written as:

$$\mathbf{R}_2\cdot\mathbf{R}_1\cdot\vec{e}_1 = \frac{1}{1+\varepsilon_s}[(1+u_s')\,\vec{e}_1 + v_s'\,\vec{e}_2 + w_s'\,\vec{e}_3] \quad (F.11)$$

$$\mathbf{R}_2\cdot\mathbf{R}_1\cdot\vec{e}_2 = [-\frac{1+u_s'}{(1+\varepsilon_s)^2}\sin\alpha w_s' - \bar{b}\sin\alpha(1+\varepsilon_s)w_s' + \bar{b}\sin\alpha\frac{(1+u_s')^2}{1+\varepsilon_s}w_s'$$

$$-\frac{\cos\alpha}{1+\varepsilon_s}v_s']\vec{e}_1$$

$$+[\frac{1+u_s'}{1+\varepsilon_s}\cos\alpha + \bar{b}\cos\alpha w_s'^2 + \bar{b}\sin\alpha\frac{1+u_s'}{1+\varepsilon_s}w_s'v_s'$$

$$-\frac{\sin\alpha}{(1+\varepsilon_s)^2}w_s'v_s']\vec{e}_2$$

$$+[(\frac{1+u_s'}{1+\varepsilon_s})^2\sin\alpha - \bar{b}\cos\alpha w_s'v_s' + \quad (F.12)$$

$$+\bar{b}\sin\alpha\frac{1+u_s'}{1+\varepsilon_s}w_s'^2 + \frac{\sin\alpha}{(1+\varepsilon_s)^2}v_s'^2]\vec{e}_3$$

$$\mathbf{R}_2 \cdot \mathbf{R}_1 \cdot \vec{e}_3 = [\frac{1+u'_s}{(1+\varepsilon_s)^2} \sin\alpha v'_s + \bar{b}\sin\alpha(1+\varepsilon_s)v'_s - \bar{b}\sin\alpha \frac{(1+u'_s)^2}{1+\varepsilon_s} v'_s$$

$$- \frac{\cos\alpha}{1+\varepsilon_s} w'_s]\vec{e}_1$$

$$+ [-(\frac{1+u'_s}{1+\varepsilon_s})^2 \sin\alpha - \bar{b}\cos\alpha\, v'_s w'_s - \bar{b}\sin\alpha\, (\frac{1+u'_s}{1+\varepsilon_s}) v'^2_s$$

$$- \frac{\sin\alpha}{(1+\varepsilon_s)^2} w'^2_s]\vec{e}_2$$

$$+ [\frac{1+u'_s}{1+\varepsilon_s} \cos\alpha + \bar{b}\cos\alpha v'^2_s \qquad \text{(F.13)}$$

$$- \bar{b}\sin\alpha \frac{1+u'_s}{1+\varepsilon_s} w'_s v'_s + \frac{\sin\alpha}{(1+\varepsilon_s)^2} v'_s w'_s]\vec{e}_3$$

The cartesian components of the rotation tensor $\mathbf{R} = \mathbf{R}_2 \cdot \mathbf{R}_1$ can now be obtained from:

$$R_{ij} = \vec{e}_i \cdot \mathbf{R} \cdot \vec{e}_j \qquad \text{(F.14)}$$

Appendix G

G.1. According to (8.10) $\mathbf{R}_1\bullet\tilde{\mathbf{R}}_2\bullet\vec{a}$ can be written as:

$$\mathbf{R}_1\bullet\tilde{\mathbf{R}}_2\bullet\vec{a} = [\cos\phi\mathbf{I} - \vec{e}_1\vec{i}_1+(1-\cos\phi)\vec{\eta}\vec{\eta}]\bullet(\cos\beta\vec{a}+\sin\beta\vec{e}_1*\vec{a}) + \quad (G.1)$$

$$+ \vec{i}_1\vec{e}_1\bullet\vec{a}$$

where: $\cos\phi = \vec{e}_1\bullet\vec{i}_1$ and $\sin\phi\vec{\eta} = \vec{e}_1*\vec{i}_1$

G.2. The unit vector $\vec{i}_1$ is given by:

$$\vec{i} = (1+\varepsilon_s)^{-1}[(1+u_s')\vec{e}_1+v_s'\vec{e}_2 + w_s'\vec{e}_3] \quad (G.2)$$

with this expression $\cos\phi$ and $\vec{\eta}$ can be written as:

$$\cos\phi = (1+\varepsilon_s)^{-1}(1+u_s') \quad (G.3)$$

$$\vec{\eta} = [(1+\varepsilon_s)\sin\phi]^{-1}(v_s' \ \vec{e}_3-w_s'\vec{e}_2) \quad (G.4)$$

Using (G.2-G.4) the terms in (G.1) may be written as:

$$\cos\phi\mathbf{I} = \frac{1+u_s'}{1+\varepsilon_s} (\vec{e}_1\vec{e}_1 + \vec{e}_2\vec{e}_2 + \vec{e}_3\vec{e}_3) \quad (G.5)$$

$$\vec{e}_1\vec{i}_1 = \frac{1}{1+\varepsilon_s} [(1+u_s')\vec{e}_1\vec{e}_1 + v_s' \ \vec{e}_1\vec{e}_2 + w_s' \ \vec{e}_1\vec{e}_3] \quad (G.6)$$

$$(1-\cos\phi)\vec{\eta}\vec{\eta} = \bar{b}[v_s'^2\vec{e}_3\vec{e}_3 - w_s'v_s' \ \vec{e}_3\vec{e}_2 - w_s'v_s' \ \vec{e}_2\vec{e}_3 + w_s'^2\vec{e}_2\vec{e}_2] \quad (G.7)$$

$$\vec{i}_1\vec{e}_1 = \frac{1}{1+\varepsilon_s} [(1+u_s') \ \vec{e}_1\vec{e}_1 + v_s' \ \vec{e}_2\vec{e}_1 + w_s' \ \vec{e}_3\vec{e}_1] \quad (G.8)$$

where $\bar{b} = \dfrac{(1-\cos\phi)}{(1+\varepsilon_s)^2\sin^2\phi}$

G.3. When (G.2-G.8) are substituted into (G.1), $\mathbf{R}\bullet\vec{e}_1$, $\mathbf{R}\bullet\vec{e}_2$, and $\mathbf{R}\bullet\vec{e}_3$ can be written as:

$$\mathbf{R}_1 \bullet \tilde{\mathbf{R}}_2 \bullet \vec{e}_1 = (1+\varepsilon_s)^{-1}[(1+u_s')\vec{e}_1 + v_s'\vec{e}_2 + w_s'\vec{e}_3] \tag{G.9}$$

$$\begin{aligned}\mathbf{R}_1 \bullet \tilde{\mathbf{R}}_2 \bullet \vec{e}_2 = {} & [- \frac{\cos\beta}{1+\varepsilon_s} v_s' - \frac{\sin\beta}{1+\varepsilon_s} w_s']\vec{e}_1 \\ & + [\frac{1+u_s'}{1+\varepsilon_s} \cos\beta + \bar{b}(w_s'^2\cos\beta - w_s'v_s'\sin\beta)]\, \vec{e}_2 \\ & + [\frac{1+u_s'}{1+\varepsilon_s} \sin\beta + \bar{b}(v_s'^2\sin\beta - w_s'v_s' \cos\beta)]\, \vec{e}_3\end{aligned} \tag{G.10}$$

$$\begin{aligned}\mathbf{R}_1 \bullet \tilde{\mathbf{R}}_2 \bullet \vec{e}_3 = {} & [- \frac{\cos\beta}{1+\varepsilon_s} w_s' + \frac{\sin\beta}{1+\varepsilon_s} v_s']\vec{e}_1 \\ & + [- \frac{1+u_s'}{1+\varepsilon_s} \sin\beta + \bar{b}(-w_s'v_s'\cos\beta - w_s'^2\sin\beta)]\, \vec{e}_2 \\ & + [\frac{1+u_s'}{1+\varepsilon_s} \cos\beta + \bar{b}(v_s'^2\cos\beta + w_s'v_s' \sin\beta)]\, \vec{e}_3\end{aligned} \tag{G.11}$$

The cartesian components of the rotation tensor $\mathbf{R} = \mathbf{R}_1 \bullet \tilde{\mathbf{R}}_2$ can now be obtained from:

$$R_{ij} = \vec{e}_i \bullet \mathbf{R} \bullet \vec{e}_j \tag{G.12}$$

Appendix H

H.1. Derivation of the axial vector of the skew tensor $Q^c \cdot \frac{dQ}{dx}$ when Q is given by:

$$Q \cdot \vec{a} = \cos\gamma \vec{a} + (1-\cos\gamma)\vec{e}\vec{e} \cdot \vec{a} + \sin\gamma \; \vec{e} * \vec{a} \tag{H.1}$$

where: γ is the angle of rotation and $\vec{e}$ is an unit vector in the direction of the rotation axis.

The rotation tensor Q is a function of the variables γ and $\vec{e}$, this implies:

$$dQ = \frac{\partial Q}{\partial \gamma} d\gamma + d\vec{e} \cdot \frac{\partial Q}{\partial \vec{e}} \tag{H.2}$$

The tensor $Q^c \cdot \frac{dQ}{dx}$ may thus be written as:

$$Q^c \cdot \frac{dQ}{dx} = Q^c \cdot \frac{\partial Q}{\partial \gamma} \frac{d\gamma}{dx} + Q^c \cdot \left(\frac{d\vec{e}}{dx} \cdot \frac{\partial Q}{\partial \vec{e}}\right) \tag{H.3}$$

H.2. The first term to be considered is: $Q^c \cdot \frac{\partial Q}{\partial \gamma} \frac{d\gamma}{dx}$. When a unit vector $\vec{b}$, perpendicular to $\vec{e}$ is introduced, then Q, Q^c and $Q^c \cdot \frac{\partial Q}{\partial \gamma}$ may respectively be written as:

$$Q = \cos\gamma I + (1-\cos\gamma)\vec{e}\vec{e} + \sin\gamma \; [(\vec{e} * \vec{b})\vec{b} - \vec{b}(\vec{e} * \vec{b})] \tag{H.4}$$

$$Q^c = \cos\gamma I + (1-\cos\gamma)\vec{e}\vec{e} + \sin\gamma \; [(\vec{e} * \vec{b})\vec{b} - \vec{b}(\vec{e} * \vec{b})] \tag{H.5}$$

$$\frac{\partial Q}{\partial \gamma} = - \sin\gamma I + \sin\gamma \; \vec{e}\vec{e} + \cos\gamma \; [(\vec{e} * \vec{b})\vec{b} - \vec{b}(\vec{e} * \vec{b})] \tag{H.6}$$

$$Q^c \cdot \frac{\partial Q}{\partial \gamma} = -\sin\gamma Q^c + \sin\gamma (Q^c \cdot \vec{e})\vec{e} + \cos\gamma [(Q^c \cdot (\vec{e} * \vec{b})\vec{b} - (Q^c \cdot \vec{b})(\vec{e} * \vec{b})] \tag{H.7}$$

Calculating the unknown terms in (H.7) yields:

$$Q^c \cdot \vec{e} = \vec{e} \tag{H.8}$$

$$Q^c\cdot(\vec{e}*\vec{b}) = \cos\gamma(\vec{e}*\vec{b}) + \sin\gamma\vec{b} \tag{H.9}$$

$$Q^c\cdot\vec{b} = \cos\gamma\ \vec{b} - \sin\gamma(\vec{e}*\vec{b}) \tag{H.10}$$

Substitution of (H.8-H.10) and (H.5) into (H.7) yields:

$$Q^c\cdot\frac{\partial Q}{\partial\gamma} = -\sin\gamma\cos\gamma I - \sin\gamma\vec{e}\vec{e} + \cos\gamma\sin\gamma\vec{e}\vec{e} + \sin^2\gamma(\vec{e}*\vec{b})\vec{b}$$
$$- \sin^2\gamma\ \vec{b}(\vec{e}*\vec{b}) + \sin\gamma\vec{e}\vec{e} + \cos^2\gamma(\vec{e}*\vec{b})\vec{b} + \cos\gamma\sin\gamma\vec{b}\vec{b} \tag{H.11}$$
$$- \cos^2\gamma\vec{b}(\vec{e}*\vec{b}) + \sin\gamma\cos\gamma(\vec{e}*\vec{b})(\vec{e}*\vec{b})$$

Using the identity: $[\vec{e}\vec{e} + \vec{b}\vec{b} + (\vec{e}*\vec{b})(\vec{e}*\vec{b})] = I$ (H.12)

eqn. (H.11) yields:

$$Q^c\cdot\frac{\partial Q}{\partial\gamma}\cdot\vec{a} = [(\vec{e}*\vec{b})\vec{b} - \vec{b}(\vec{e}*\vec{b})]\cdot\vec{a} = \vec{e}*\vec{a} \tag{H.13}$$

The term $Q^c\cdot\frac{\partial Q}{\partial\gamma}\frac{d\gamma}{dx}\cdot\vec{a}$ may now be written as:

$$Q^c\cdot\frac{\partial Q}{\partial\gamma}\frac{d\gamma}{dx}\cdot\vec{a} = \frac{d\gamma}{dx}\ \vec{e}*\vec{a} \tag{H.14}$$

H.3. The next term to be considered is $Q^c\cdot(\frac{d\vec{e}}{dx}\cdot\frac{\partial Q}{\partial\vec{e}})$

Writing $(d\vec{e}\cdot\frac{\partial Q}{\partial\vec{e}})\cdot\vec{a}$ as $Q(\gamma,\ \vec{e}+d\vec{e})\cdot\vec{a} - Q(\gamma,\vec{e})\cdot\vec{a}$ yields: (H.15)

$$(d\vec{e}\cdot\frac{\partial Q}{\partial\vec{e}})\cdot\vec{a} = (1-\cos\gamma)[(d\vec{e})\vec{e} + \vec{e}d\vec{e}]\cdot\vec{a} + \sin\gamma\ d\vec{e}*\vec{a} \tag{H.16}$$

where $d\vec{e}\cdot\vec{e} = 0$

$Q^c\cdot(\frac{d\vec{e}}{dx}\cdot\frac{\partial Q}{\partial\vec{e}})\cdot\vec{a}$ may now be written as:

$$Q^c\cdot(\frac{d\vec{e}}{dx}\cdot\frac{\partial Q}{\partial\vec{e}})\cdot\vec{a} = Q^c\cdot[(1-\cos\gamma)(\frac{d\vec{e}}{dx}\vec{e} + \vec{e}\frac{d\vec{e}}{dx})\cdot\vec{a} + \sin\gamma\ \frac{d\vec{e}}{dx}*\vec{a} \tag{H.17}$$

H.4. At this point the following three orthonormal unit vectors are introduced.

$$\frac{1}{L}\frac{d\vec{e}}{dx}\ ,\ (\vec{e}*\frac{1}{L}\frac{d\vec{e}}{dx}),\ \vec{e} \quad \text{where: } L = |\frac{d\vec{e}}{dx}| \tag{H.18}$$

With these orthonormal unit vectors, $\sin\gamma\,\frac{d\vec{e}}{dx}*\vec{a}$ may be written as:

$$\sin\gamma\frac{d\vec{e}}{dx}*\vec{a} = L\sin\gamma\,\frac{1}{L}\frac{d\vec{e}}{dx}*\vec{a} = L\sin\gamma[\{(\vec{e}*\frac{1}{L}\frac{d\vec{e}}{dx})*\vec{e}\}*\vec{a}]$$

$$= \sin\gamma[\vec{e}(\vec{e}*\frac{d\vec{e}}{dx})-(\vec{e}*\frac{d\vec{e}}{dx})\vec{e}]\bullet\vec{a} \tag{H.19}$$

Substitution of (H.19) into (H.17) yields:

$$Q^c\bullet(\frac{d\vec{e}}{dx}\bullet\frac{\partial Q}{\partial\vec{e}})\bullet\vec{a} = \{(1-\cos\gamma)[(Q^c\bullet\frac{d\vec{e}}{dx})\vec{e} + (Q^c\bullet\vec{e})\frac{d\vec{e}}{dx}] + \\ + \sin\gamma[Q^c\bullet\vec{e}(\vec{e}*\frac{d\vec{e}}{dx}) - Q^c\bullet(\vec{e}*\frac{d\vec{e}}{dx})\vec{e}]\}\bullet\vec{a} \tag{H.20}$$

Calculating the unknown terms in (H.20) yields:

$$Q^c\bullet\vec{e} = \vec{e} \tag{H.21}$$

$$Q^c\bullet\frac{d\vec{e}}{dx} = \cos\gamma\,\frac{d\vec{e}}{dx} - \sin\gamma(\vec{e}*\frac{d\vec{e}}{dx}) \tag{H.22}$$

$$Q^c\bullet(\vec{e}*\frac{d\vec{e}}{dx}) = \cos\gamma\,(\vec{e}*\frac{d\vec{e}}{dx}) + \sin\gamma\,\frac{d\vec{e}}{dx} \tag{H.23}$$

Substitution of (H.21-H.23) into (H.20) yields

$$Q^c\bullet(\frac{d\vec{e}}{dx}\bullet\frac{\partial Q}{\partial\vec{e}})\bullet\vec{a} = (1-\cos\gamma)[\vec{e}\frac{d\vec{e}}{dx} + \cos\gamma\frac{d\vec{e}}{dx}\vec{e} - \sin\gamma(\vec{e}*\frac{d\vec{e}}{dx})\vec{e})] +$$

$$+ \sin\gamma[\vec{e}(\vec{e}*\frac{d\vec{e}}{dx})-(\cos\gamma(\vec{e}*\frac{d\vec{e}}{dx})\vec{e} + \sin\gamma\frac{d\vec{e}}{dx}\vec{e})]\bullet\vec{a}$$

$$= (1-\cos\gamma)[\vec{e}\frac{d\vec{e}}{dx} - \frac{d\vec{e}}{dx}\vec{e}] + \sin\gamma[\vec{e}(\vec{e}*\frac{d\vec{e}}{dx})-(\vec{e}*\frac{d\vec{e}}{dx})\vec{e}]\bullet\vec{a}$$

$$= [(1-\cos\gamma)(\frac{d\vec{e}}{dx}*\vec{e}) + \sin\gamma\frac{d\vec{e}}{dx}]*\vec{a} \qquad (H.24)$$

H.5. The axial vector of the skew tensor $Q^c\bullet\frac{dQ}{dx}$ can now be obtained by substitution of (H.24) and (H.14) in (H.3)

$$Q^c\bullet\frac{dQ}{dx}\bullet\vec{a} = [\frac{d\gamma}{dx}\vec{e} + (1-\cos\gamma)(\frac{d\vec{e}}{dx}*\vec{e}) + \sin\gamma\frac{d\vec{e}}{dx}]*\vec{a} \qquad (H.25)$$

H.6. The axial vector of the skew tensor $\frac{dQ}{dx}\bullet Q^c$ can be derived in a similar way:

$$\frac{dQ}{dx}\bullet Q^c\bullet\vec{a} = [\frac{d\gamma}{dx}\vec{e} + (1-\cos\gamma)(\vec{e}*\frac{d\vec{e}}{dx}) + \sin\gamma\frac{d\vec{e}}{dx}]*\vec{a} \qquad (H.26)$$

Appendix I: Derivation of $\vec{\gamma}$ and $\tilde{R}_2^c \cdot \vec{\mu}$

I.1. $\vec{\gamma}$ is the axial vector of the skew tensor $\tilde{R}_2^c \cdot \frac{d\tilde{R}_2}{dx}$

where: $\tilde{R}_2 \cdot \vec{a} = \cos\beta\vec{a} + (1-\cos\beta)\vec{e}_1\vec{e}_1 \cdot \vec{a} + \sin\beta\vec{e}_1 * \vec{a}$ (I.1)

According to (9.5), $\vec{\gamma}$ is given by:

$$\vec{\gamma} = [\frac{d\beta}{dx}\vec{e}_1 + (1-\cos\beta)(\frac{d\vec{e}}{dx} * \vec{e}_1) + \sin\gamma \frac{d\vec{e}_1}{dx}] \quad (I.2)$$

Since $\frac{d\vec{e}_1}{dx} = 0$, eqn (I.2) reduces to:

$$\vec{\gamma} = \frac{d\beta}{dx}\vec{e}_1 \quad (I.3)$$

I.2. $\vec{\mu}$ is the axial vector of the skew tensor $R_1^c \cdot \frac{dR_1}{dx}$

where: $R_1 \cdot \vec{a} = \cos\phi\vec{a} + (1-\cos\phi)\vec{\eta}\vec{\eta} \cdot \vec{a} + \sin\phi\vec{\eta} * \vec{a}$, (I.4)

$\cos\phi = \vec{e}_1 \cdot \vec{i}_1$ and $\sin\phi\vec{\eta} = \vec{e}_1 * \vec{i}_1$

the unit vector $\vec{i}_1$ is given by eqn (8.11)

$$\vec{i}_1 = \frac{1}{1+\varepsilon_s}[(1+u_s')\vec{e}_1 + v_s'\vec{e}_2 + w_s'\vec{e}_3] \quad (I.5)$$

When ε_s is neglected compared to unity, according to the assumption of small strains, (I.5) reduces to

$$\vec{i}_1 = [(1+u_s')\vec{e}_1 + v_s'\vec{e}_2 + w_s'\vec{e}_3] \quad (I.6)$$

This implies: $(1+u_s')^2 = 1-v_s'^2 - w_s'^2$ (I.7)

Using eqn (I.7), $\cos\phi$ and $\sin\phi\vec{\eta}$ may be written as:

$$\cos\phi = \vec{e}_1 \cdot \vec{i}_1 = (1+u_s') = \sqrt{(1-v_s'^2 - w_s'^2)} \qquad (I.8)$$

$$\sin\phi\vec{\eta} = (v_s'\vec{e}_3 - w_s'\vec{e}_2) \qquad (I.9)$$

Combination of (I.7) and (I.8) yields:

$$\sin^2\phi = 1-\cos^2\phi = (v_s'^2 + w_s'^2) \qquad (I.10)$$

Differentiating cosϕ with respect to x yields:

$$\frac{d\cos\phi}{dx} = -\sin\phi\frac{d\phi}{dx} = \frac{-(v_s'v_s''+w_s'w_s'')}{\sqrt{(1-v_s'^2-w_s'^2)}} = \frac{-(v_s'v_s''+w_s'w_s'')}{\cos\phi} \qquad (I.11)$$

I.3. According to (9.5) $\vec{\mu}$ is given by:

$$\vec{\mu} = [\frac{d\phi}{dx}\vec{\eta} + (1-\cos\phi)(\frac{d\vec{\eta}}{dx}*\vec{\eta}) + \sin\phi\frac{d\vec{\eta}}{dx}] \qquad (I.12)$$

Using (I.9) and (I.11), the terms needed in (I.12) can be determined

$$\frac{d\phi}{dx} = (\sin\phi\cos\phi)^{-1}\,(v_s'v_s'' + w_s'w_s'') \qquad (I.13)$$

$$\frac{d\vec{\eta}}{dx} = (\sin\phi)^{-1}(v_s''\vec{e}_3-w_s''\vec{e}_2)-(\sin^2\phi)^{-1}(v_s'\vec{e}_3-w_s'\vec{e}_2)\cos\phi\frac{d\phi}{dx} \qquad (I.14)$$

$$\frac{d\vec{\eta}}{dx}*\vec{\eta} = (\sin^2\phi)^{-1}(w_s'v_s'' - w_s''v_s')\vec{e}_1 \qquad (I.15)$$

Substitution of (I.13-I.15) into (I.12) yields:

$$\begin{aligned}\vec{\mu} = &\frac{(1-\cos\phi)}{\sin^2\phi}[w_s'v_s'' - w_s''v_s']\vec{e}_1 \\ &+ [-w_s'' - \frac{(1-\cos\phi)}{\sin^2\phi\cos\phi}(v_s'v_s''w_s' + w_s'^2w_s'')]\vec{e}_2 \\ &+ [v_s'' + \frac{1-\cos\phi}{\sin^2\phi\cos\phi}(v_s'^2v_s'' + w_s'w_s''v_s')]\vec{e}_3\end{aligned} \qquad (I.16)$$

where: $\cos\phi = (1-v_s'^2-w_s'^2)^{1/2}$ and $\sin^2\phi = (v_s'^2+w_s'^2)$

Appendix J

J.1. According to (3.9-3.11) $\vec{i}_1$ is given by:

$$\vec{i}_1 = (1+\varepsilon_s)^{-1}[(1+u_s')\vec{e}_1+v_s'\vec{e}_2 + w_s'\vec{e}_3] \tag{J.1}$$

Substitution of this expression into (6.7) yields:

$$R_{11} = (1+\varepsilon_s)^{-1}(1+u_s') \tag{J.2}$$

$$R_{21} = (1+\varepsilon_s)^{-1} v_s' \tag{J.3}$$

$$R_{31} = (1+\varepsilon_s)^{-1}w_s' \tag{J.4}$$

J.2. In appendix B it is shown that the order of magnitude of $g\phi \equiv f$ may be written as:

$$|g\phi| \lesssim O(\varepsilon L) \tag{J.5}$$

The order of magnitude of v_s' and w_s' may respectively be written as:

$$\begin{aligned} |v_s'| &= O(v_s/L) \\ |w_s'| &= O(w_s/L) \end{aligned} \tag{J.6}$$

Combination of (J.5) and (J.6) yields:

$$\begin{aligned} |g\phi R_{21}| &\lesssim O\ (\varepsilon v_s) \\ |g\phi R_{31}| &\lesssim O\ (\varepsilon w_s) \end{aligned} \tag{J.7}$$

Because v_s and w_s also appear in the expression of u_2 respectively u_3, $g\phi R_{21}$ and $g\phi R_{31}$ may be neglected.