# The non-local Fisher-KPP equation: traveling waves and steady states

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**Abstract**

We consider the Fisher-KPP equation with a nonlocal saturation effect defined through an interaction kernel $\phi(x)$ and investigate the possible differences with the standard Fisher-KPP equation. Our first concern is the existence of steady states. We prove that if the Fourier transform $\hat{\phi}(\xi)$ is positive or if the length $\sigma$ of the nonlocal interaction is short enough, then the only steady states are $u \equiv 0$ and $u \equiv 1$. Our second concern is the study of traveling waves. We prove that this equation admits traveling wave solutions that connect $u = 0$ to an unknown positive steady state $u_\infty(x)$, for all speeds $c \geq c^*$. The traveling wave connects to the standard state $u_\infty(x) \equiv 1$ under the aforementioned conditions: $\hat{\phi}(\xi) > 0$ or $\sigma$ is sufficiently small. However, the wave is not monotonic for $\sigma$ large.

## 1 Introduction and main results

We investigate the non-local Fisher-KPP equation

$$u_t - \Delta u = \mu u(1 - \phi \star u), \qquad x \in \mathbb{R}^d, \tag{1.1}$$

where $\phi$ is a given convolution kernel and

$$\phi \star u(x) = \int_{\mathbb{R}^d} u(x-y)\phi(y)dy.$$

Our main interest is to understand when solutions of the non-local Fisher equation behave qualitatively differently from those of the classical Fisher equation

$$u_t - \Delta u = \mu u(1-u), \tag{1.2}$$

that corresponds to $\phi(x) = \delta(x)$. Let us briefly recall that the only non-negative bounded steady solutions of (1.2) are the constants $u \equiv 0$ and $u \equiv 1$, and that for any $c \geq c_* = 2\sqrt{\mu}$ the local Fisher-KPP equation (1.2) admits traveling wave solutions of the form $u(t,x) = U(x - ct)$ with the boundary conditions $U(-\infty) = 1$, $U(+\infty) = 0$, and a monotonically decreasing in $x$ profile $U(x)$.

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The nonlocal Fisher equation arises in several areas. In ecology it takes into account a nonlocal saturation, thanks to the term $\phi\star u$, or nonlocal competition effects as in [7, 15]. It was also proposed as a simple model of adaptive dynamics in [8, 9] – there, $x$ represents a phenotypical trait of the population. A population of individuals with trait $x$ faces competition from all its counterparts. If this competition does not depend on the trait, then $\phi \equiv 1$. But if the competition is higher for populations with closer similarities, the kernel $\phi$ is localized. Other types of nonlocal terms may arise, see [5, 16] for dispersal by jumps rather than Brownian motion, or because of time delay, for instance, see [4, 6, 19, 11, 12, 17, 18].

Throughout the paper we assume that the convolution kernel satisfies the properties

$$\phi \geq 0, \quad \phi(0) > 0, \qquad \nabla\phi \in C_b(\mathbb{R}^d) \qquad \int_{\mathbb{R}} \phi(x)dx = 1, \quad \int_{\mathbb{R}} x^2\phi(x)dx < +\infty. \tag{1.3}$$

These assumptions are not necessarily optimal, in particular, regularity of $\phi$ and positivity at $x = 0$ may be relaxed but they are sufficient for our purposes, biologically reasonable and simplify some of the technicalities.

In order to quantify the range of the nonlocal interaction one may use the kernel of the form

$$\phi_\sigma(x) = \frac{1}{\sigma^d}\phi\left(\frac{x}{\sigma}\right). \tag{1.4}$$

If $\sigma \to 0$, then $\phi_\sigma \star u \to u$ and we are back to the classical Fisher-KPP equation. Up to a rescaling, the equation associated with the kernel $\phi_\sigma$ and the growth rate $\mu$ is equivalent to the equation associated with the kernel $\phi$, $\sigma = 1$, and the growth rate $\mu\sigma^2$. In other words, small interaction length of the nonlocal effect is equivalent to a small growth rate for a fixed interaction length. We will alternatively use parameters $\mu$ and $\sigma$, depending on which one is more convenient.

We are mainly interested in the steady states of this equation, positive solutions of

$$-\Delta u = \mu u(1 - \phi \star u), \tag{1.5}$$

and in the traveling wave solutions in a direction $e \in \mathbb{S}^{d-1}$, connecting the steady state $u = 0$ to a uniformly positive state, possibly identically equal to 1 as in the classical Fisher case. More precisely, we look for a pair $(c, u)$ where $c \in \mathbb{R}$ is the traveling wave speed and the function $u(x)$ satisfies

$$-cu' - u'' = \mu u(1 - \psi \star u), \quad \liminf_{x\to-\infty} u(x) > 0, \quad u(+\infty) = 0, \qquad x \in \mathbb{R}, \tag{1.6}$$

where $\psi = \int_{e^\perp} \phi$. The boundary condition at $x = -\infty$ appears because of the nonlocal effect: in general, it is possible that (1.6) may admit strictly positive solutions:

$$-cv' - v'' = \mu v(1 - \psi \star v), \quad \inf_{x\in\mathbb{R}} v(x) > 0, \tag{1.7}$$

other than $v \equiv 1$ that may serve as limit states as $x \to -\infty$. Moreover, as we recall below, $v \equiv 1$ is an unstable solution of (1.7) for some kernels $\phi$ and $\mu > 0$. Then one would not expect the traveling wave solution to converge to $v \equiv 1$ as $x \to -\infty$ but rather to a non-uniform stable solution of (1.7). This is one major difference with the classical Fisher-KPP equation (1.2) which has no non-trivial steady positive solutions other than $v \equiv 1$.

The non-local Fisher equation has been first introduced by Britton in [4]. He carried out a bifurcation analysis and observed that the uniform steady state $u \equiv 1$ may bifurcate to periodic steady states, standing waves or periodic traveling waves. A perturbative proof for the existence of traveling waves was given by Gourley in [11], under the assumption that the nonlocal interaction length $\sigma$ in (1.4) is sufficiently small. This result was also established by Wang, Li and Ruan in [17] for Gaussian probability densities but still with $\sigma$ small. These authors also proved in [18] that if the reaction term in (1.1) is replaced by a bistable nonlinearity with nonlocal saturation, then this modified equation admits a stable traveling wave, which is unique up to translation. Spatio-temporal nonlocal terms were considered in [1] and [17], where it was shown that for small delays and short interaction lengths solutions behave qualitatively as in the classical Fisher-KPP case.

This model was also investigated numerically in [1, 9, 11], and numerical simulations exhibit a much richer behavior than the aforementioned rigorous perturbative results close to $\sigma = 0$ indicate. In agreement with the theoretical predictions in [4, 11], it was shown that the steady state $u \equiv 1$ may be unstable for some particular values of the parameters. This may lead to non-monotonic traveling waves and behavior qualitatively different from that of the classical Fisher-KPP equation. More precisely, numerical studies show that for $\sigma$ very small, traveling wave is still monotonic and connects $u \equiv 0$ to $u \equiv 1$. As $\sigma$ increases, the wave loses its monotonicity and for sufficiently large $\sigma$ it links $u \equiv 0$ to a periodic steady state instead of $u \equiv 1$. This kind of bifurcation was related in [9] to the emergence of stable mutations in the population, which gives a nice way to model adaptive evolution mathematically. In [8], the authors studied this equation for a very small diffusion, equivalent to $\mu$ large here. They showed that several steady states may arise that usually (but nor always) concentrate around Dirac masses when $\hat{\phi}$ changes sign, but the change of sign of the Fourier transform is certainly not the only character of $\phi$ in this regime.

The aforementioned existence results for traveling waves [11, 17] rely on various perturbation methods starting from the classical Fisher-KPP equation. This is why the existence of traveling waves was proved only for small $\mu$ (or, equivalently, small $\sigma$). It is more natural to try to prove existence of traveling waves that connect $u \equiv 0$ to $u \equiv 1$ as soon as $u \equiv 1$ is stable (which is always true when $\mu$ is small enough). Here, we prove existence of traveling wave solutions of (1.1) that connect $u \equiv 0$ to $u \equiv 1$ under a hypothesis that implies the stability of $u \equiv 1$: the Fourier transform of $\phi$ is positive. This condition does not depend on $\mu$ (or $\sigma$). For example, our result establishes existence of traveling waves for Gaussian kernels for all $\mu > 0$ and not only for small $\mu$ as in [17].

In the general case when the state $u \equiv 1$ may be unstable, there might exist other steady states that can be connected to $u \equiv 0$ through a traveling wave. In that situation we also establish existence of a traveling wave that connects $u \equiv 0$ but to an a priori unknown uniformly positive state. We also show that if $\mu$ is small or $\hat{\phi}$ is positive, then the positive steady state $u \equiv 1$ is the unique possible positive end state. Finally, we show that waves connecting $u \equiv 0$ to $u \equiv 1$ (that we prove to exist if $\hat{\phi} > 0$), may not be monotonic for large values of $\mu$ – this should be contrasted with the monotonicity of the classical Fisher-KPP traveling waves.

### Steady state solutions

Obviously, there always exist two homogeneous steady states: $u \equiv 0$ and $u \equiv 1$. The state $u \equiv 0$ is always unstable since $\mu > 0$, but depending upon the kernel $\phi$, the state $u \equiv 1$ may be linearly stable or unstable. For instance, $u \equiv 1$ is stable for $\mu$ sufficiently small but may become unstable when $\mu$ is large. In that case, numerical computations [11] have shown that there might exist non-trivial periodic solutions of equation (1.5). We now present some conditions that ensure uniqueness of the solutions $u \equiv 1$ and $u \equiv 0$ in the class of bounded positive solutions when $u \equiv 1$ is stable. We first consider the case $d = 1$, with a small $\mu$, which is a small perturbation of the local Fisher-KPP equation.

**Theorem 1.1** *($d = 1$) There exists $\mu_0 > 0$ such that for all $\mu \leq \mu_0$, equation (1.5) has no bounded nonnegative solutions except $u(x) \equiv 0$ and $u(x) \equiv 1$.*

The smallness assumption in Theorem 1.1 may be removed if the Fourier transform $\hat{\phi}(\xi)$ is everywhere positive, as in the case of a Gaussian kernel $\phi(x)$.

**Theorem 1.2** *($d = 1$) Assume that $\hat{\phi}(\xi) > 0$ for all $\xi \in \mathbb{R}$. Then $u \equiv 1$ and $u \equiv 0$ are the only bounded nonnegative solutions of (1.5) for all $\mu > 0$.*

The Fourier transform above is defined as

$$\hat{\phi}(\xi) = \int_{\mathbb{R}} \phi(x) e^{-i\xi x} dx.$$

Given $L = (L_1, \ldots, L_d) \in \mathbb{R}^d$, we say that a solution is $L$-periodic if it is $L_j$-periodic in every coordinate $x_j$. We also use the notation

$$k \in \mathbb{Z}^d/L \iff k_j = \bar{k}_j/L_j \text{ with } \bar{k}_j \in \mathbb{Z}.$$

Following [9], we define the linear stability condition for the state $u \equiv 1$ with respect to $L$-periodic perturbations as

$$4\pi^2 k^2 + \mu\hat{\phi}\,(2\pi k) > 0 \qquad \forall k \in \mathbb{Z}^d/L. \tag{1.8}$$

A similar but stronger condition is that $u \equiv 1$ is linearly stable for all periodic perturbations, that is,

$$\xi^2 + \mu\hat{\phi}\,(\xi) > 0 \qquad \forall \xi \in \mathbb{R}^d. \tag{1.9}$$

Thus, we define the critical growth rate

$$\mu_c = \begin{cases} \displaystyle\max_{\xi > 0} \frac{|\xi|^2}{\max\{0, -\hat{\phi}(\xi)\}} & \text{if there exists } \xi \in \mathbb{R}^d \text{ such that } \hat{\phi}(\xi) < 0, \\ +\infty & \text{if } \hat{\phi}(\xi) \geq 0 \text{ for all } \xi \in \mathbb{R}^d. \end{cases}$$

The state $u \equiv 1$ is linearly unstable under some periodic perturbations if and only if $\mu > \mu_c$.
**Example 1.** If the kernel is a Gaussian probability density

$$\phi(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-x^2}{2\sigma^2}\right),$$

then $\hat{\phi}(\xi) = \exp(-\xi^2\sigma^2/2)$ and for all $k \in \mathbb{Z}$, one has :

$$\left|\frac{2\pi k}{L}\right|^2 + \mu\hat{\phi}\left(\frac{2\pi k}{L}\right) > 0.$$

Thus, in this case, the equilibrium state $u \equiv 1$ is always linearly stable.
**Example 2.** Consider the kernel $\phi(x) = \frac{1}{2a}\mathbb{I}_{[-a,a]}(x)$. In this case, one easily computes $\hat{\phi}(\xi) = \sin(\xi a)/(\xi a)$. Hence, $\mu_c < +\infty$, and for $\mu > \mu_c$ the state $u \equiv 1$ is linearly unstable.

The following result holds in an arbitrary dimension $d \geq 1$.

**Theorem 1.3** *Assume that $\hat{\phi}(2\pi k) \geq 0$ for all $k \in \mathbb{Z}^d/L$, then $u(x) \equiv 0$ and $u \equiv 1$ are the only nonnegative $L$-periodic bounded solutions of (1.5) for all $\mu > 0$.*

## Traveling wave solutions

We now consider the traveling wave solutions defined by (1.6). In terms of existence, the situation is close to that of the classical Fisher-KPP equation

**Theorem 1.4 (Existence of traveling wave)** *There exists a traveling wave solution $(c, u)$ to (1.6) for all $c \geq c^* = 2\sqrt{\mu}$ and there exists no such traveling wave solution $(c, u)$ with speed $c < 2\sqrt{\mu}$.*

The minimal speed $c = 2\sqrt{\mu}$ is the same as for the classical local Fisher-KPP equation. This means that the speed of propagation is determined only by the instability of the state 0, the nonlocal term does not play any role here.

We can show that the left limit is $u(-\infty) = 1$ in the following two cases.

**Theorem 1.5 (Small $\mu$)** *For any $\phi$ there exists $\mu_0 > 0$ so that traveling waves satisfy $u(-\infty) = 1$ for all $c \geq c^* = 2\sqrt{\mu}$ and $0 < \mu < \mu_0$.*

**Theorem 1.6 (The case $\hat{\phi}(\xi) > 0$)** *When $\hat{\phi}(\xi) > 0$ for all $\xi \in \mathbb{R}$ the traveling waves satisfy $u(-\infty) = 1$ for all $\mu > 0$ and all $c \geq c^* = 2\sqrt{\mu}$.*

The next theorem shows that even if a traveling wave approaches $u \equiv 1$ as $x \to -\infty$, it cannot be monotonic in $x$ when $\mu$ is sufficiently large.

**Theorem 1.7 (Non-monotonicity)** *Assume that $\phi(x)$ is continuous, $\phi(0) > 0$ and*

$$\int_{\mathbb{R}} \phi(x) e^{\lambda x} dx < +\infty \tag{1.10}$$

*for all $\lambda \in \mathbb{R}$. There exists $\mu_0 > 0$ and $C_0 > 0$ so that for all $\mu \geq \mu_0$ and $2\sqrt{\mu} := c^* \leq c \leq C_0 \mu$, the traveling wave $(c, u)$ cannot satisfy $u(-\infty) = 1$ and reach this state monotonically at $-\infty$.*

We do not know if the traveling waves constructed in Theorem 1.4 always connect $u(+\infty) = 0$ to the state $u(-\infty) = 1$. Numerical computations shown in Figure 1.1 and those in [9] suggest that, if $\mu$ is large and $\phi(\xi_0) < 0$ for some $\xi_0$, the traveling wave built in Theorem 1.4 may connect $u(+\infty) = 0$ to a periodic solution as $x \to -\infty$, but this remains an open problem. Some possible numerical scenarios for the traveling wave are depicted below in Figure 1.1. Let us also mention here that we construct traveling waves using the method introduced in [3], as a limit of solutions of approximating problems on intervals $(-a_n, a_n)$ with $a_n \to +\infty$. The nature of this procedure leads, at least heuristically, to construction of stable waves. Therefore, we believe that traveling waves are stable even if the left limit is a non-uniform periodic state, or if the left state $u = 1$ is approached non-monotonically as in the context of Theorem 1.7.

Throughout this paper, we denote by $C$ and $K$ constants that only depend on $\mu$ and $\phi$ and that are locally bounded with respect to $\mu \geq 0$.

The paper is organized as follows. In Section 2 we prove Theorems 1.1-1.3 on triviality of steady states. Traveling waves are constructed in Section 3 where Theorem 1.4 is proved. The uniformity of the limit on the left in Theorems 1.5 and 1.6 is established in Section 4. Finally, non-monotonicity of traveling waves for large $\mu$ is established in Section 5.

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## 2 Steady states

In this section we prove Theorems 1.1-1.3 on non-existence of non-trivial steady states. We begin with some preliminary a priori estimates for the steady solutions that are used later on.

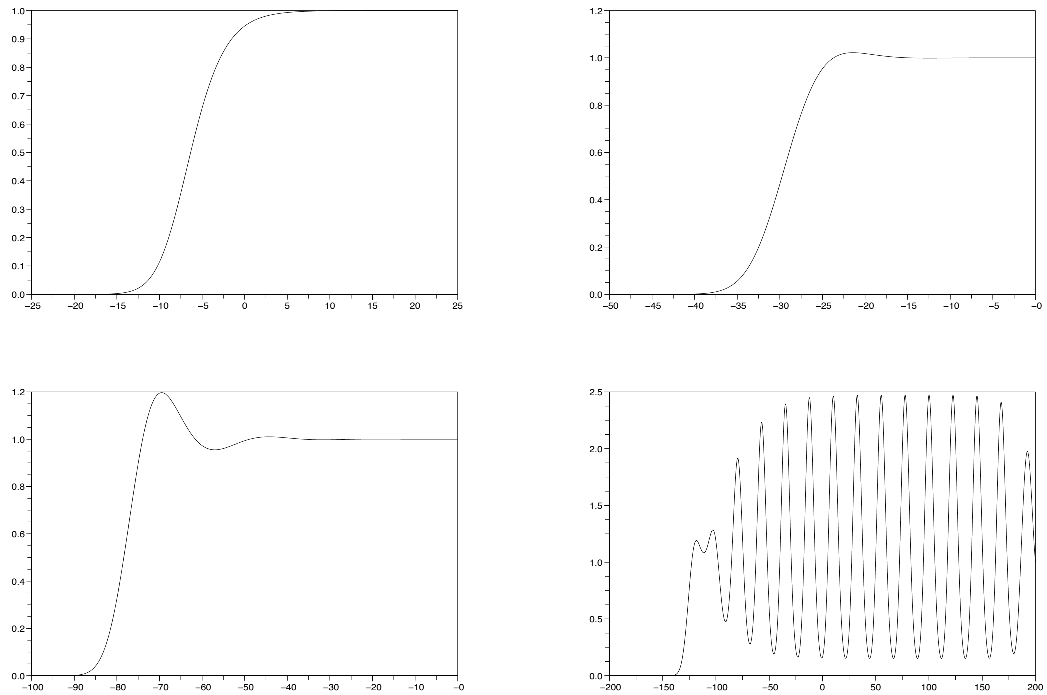


Figure 1.1: Traveling waves solutions to (1.6) with increasing values of $\mu$ obtained by numerical simulations. In accordance with Theorem 1.7, one observes first a monotonic Fisher-KPP like regime, then an overshoot appears and finally oscillatory waves, still linking the state 0 to 1. For $\mu$ large enough, the wave seems to connect $u = 0$ to a periodic solution. These are obtained with the convolution kernel $\phi$ equal to an indicator function of an interval.

## 2.1 A positive lower bound

First, we show that any steady solution of (1.5) in $\mathbb{R}^d$ is bounded away from zero from below.

**Lemma 2.1** *Any non-negative bounded solution $u \not\equiv 0$ of (1.5) in $\mathbb{R}^d$, $d \geq 1$, is bounded away from zero:*

$$\inf_{x \in \mathbb{R}^d} u(x) > 0. \tag{2.1}$$

**Proof.** We assume that

$$\inf_{x \in \mathbb{R}^d} u(x) = 0 \tag{2.2}$$

and look for a contradiction. The maximum principle implies that $u(x)$ can not attain its minimum at a point where it is equal to zero. Hence, there exists a sequence $x_k$, $|x_k| \to +\infty$ such that $u(x_k) \to 0$ as $k \to +\infty$. We claim that in this case for any $R > 0$ we have

$$\lim_{k \to +\infty} \sup_{x \in B(x_k;R)} u(x) = 0. \tag{2.3}$$

Indeed, consider the function $v_k(x) = u(x+x_k)$, then, after extracting a subsequence, $v_k(x)$ converges locally uniformly to a limit $v(x)$, due to the standard elliptic regularity estimates [10]. The function $v(x)$ satisfies (1.5) and attains its minimum $v = 0$ at $x = 0$. Therefore, the maximum principle implies that $v(x) \equiv 0$ and thus (2.3) holds.

Now, fix $\varepsilon > 0$ to be prescribed later, set $M = \|u\|_{L^\infty(\mathbb{R})}$ and take $R$ so large that

$$\int_{|y|\geq R/2} \phi(y)dy \leq \frac{\varepsilon}{2M}. \tag{2.4}$$

Choose $k$ so large that $0 < u(x) \leq \varepsilon/2$ in $B(x_k; R)$. Then we have, for $x \in B(x_k; R/2)$,

$$0 \leq \phi \star u(x) = \int\limits_{|x-y|\leq R/2} \phi(x-y)u(y)dy + \int\limits_{|x-y|\geq R/2} \phi(x-y)u(y)dy \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2M}M \leq \varepsilon.$$

Hence, inside the ball $B(x_k; R/2)$ the function $u(x)$ satisfies

$$-\Delta u \geq \mu(1-\varepsilon)u, \quad u(x) \geq 0. \tag{2.5}$$

Now, choose $R$ so large that, in addition to (2.4), the principle eigenvalue $\lambda_1$ of the Dirichlet Laplacian on $B(0; R/2)$ is smaller than $\mu(1-\varepsilon)$:

$$-\Delta\psi = \lambda_1\psi, \quad \psi = 0 \text{ on } \partial B(0; R/2),\ \psi > 0 \text{ in } B(0; R/2). \tag{2.6}$$

Multiplying (2.5) by $\psi_k = \psi(x - x_k)$ and (2.6) by $u$ and subtracting we obtain

$$(\mu(1-\varepsilon) - \lambda_1) \int_{B(x_k;R/2)} u\psi_k \leq -\int_{B(x_k;R/2)} \psi_k \Delta u + \int_{B(x_k;R/2)} u\Delta\psi_k = \int_{\partial B(x_k;R/2)} u\frac{\partial\psi_k}{\partial n} \leq 0,$$

as $\partial\psi_k/\partial n < 0$ on $B(x_k; R/2)$. This is a contradiction as by assumption $\lambda_1 < \mu(1-\varepsilon)$. Therefore, (2.2) is impossible and hence $\inf_x u(x) > 0$. $\Box$

## 2.2 Explicit upper and lower bounds for small growth rates

We now obtain explicit bounds on the solution when the parameter $\mu$ is small that show that non-zero steady states are close to $u \equiv 1$.

**Lemma 2.2** *There exists $\mu_0 > 0$ and a constant $K > 0$, which depends only on the kernel $\phi$ but not on $\mu \in (0, \mu_0)$, such that for all $\mu \in [0, \mu_0]$, any bounded positive solution $u(x)$ of (1.5) in $\mathbb{R}$ satisfies $0 < 1 - \mu K \leq u(x) \leq 1 + \mu K$.*

**Proof.** Let $M = \sup_{x\in\mathbb{R}} u(x)$, if $M \leq 1$ then $u(x)$ satisfies

$$-u''(x) = \mu u(1 - \phi \star u) \geq 0,$$

and, as $u(x)$ is bounded it has to be a constant, $u(x) \equiv M$. Then (1.5) implies that $M = 1$ since $u(x) > 0$. Therefore, we may assume that $M > 1$. First, suppose that $u(x)$ attains its maximum at some point $x_0$: $u(x_0) = M$. Then (1.5) implies that

$$\phi \star u(x_0) \leq 1. \tag{2.7}$$

On the other hand, because $u \geq 0$,

$$u'' = -\mu u(1 - \phi \star u) \geq -\mu u \geq -\mu M.$$

Considering Taylor's expansion around $x_0$ we deduce the following lower bound for $u(x)$:

$$u(x) = u(x_0) + u'(x_0)(x - x_0) + \frac{u''(\xi)}{2}(x - x_0)^2 \geq M - \frac{\mu M}{2}(x - x_0)^2,$$

with some $\xi$ between $x$ and $x_0$. It follows from (2.7) that

$$1 \geq M - \frac{M\mu}{2}\int \phi(x_0-y)(y-x_0)^2dy = M - \frac{M\mu}{2}\int \phi(y)y^2dy,$$

and thus

$$M \leq \frac{1}{1-\mu I_2}, \qquad I_2 = \frac{1}{2}\int \phi(y)y^2dy. \tag{2.8}$$

Then the conclusion for the supremum part of Lemma 2.2 follows in the case when $u(x)$ attains its maximum.

If $u(x)$ does not attain its maximum, then, without loss of generality, we may assume that there exists a sequence $x_n \to +\infty$ such that $u(x_n) \geq M - \mu/n^2$, $u''(x_n) \leq 0$ and $u'(x_n) \geq 0$. We set $y = x_n + \mu/(u'(x_n)n)$. Then we have

$$M > u(y) = u(x_n) + u'(x_n)(y-x_n) + \frac{u''(\xi)}{2}(y-x_n)^2 \geq M - \frac{\mu}{n^2} + \frac{\mu}{n} - \frac{M\mu^3}{2n^2(u'(x_n))^2},$$

so, for $n$ sufficiently large, we have

$$\frac{\mu^2 M}{2n^2(u'(x_n))^2} \geq \frac{1}{2n},$$

and thus $0 \leq u'(x_n) \leq \sqrt{\mu^2 M/n}$. Now, we proceed as before. We Taylor-expand around $x_n$ to get

$$u(x) = u(x_n) + u'(x_n)(x-x_n) + \frac{u''(\xi)}{2}(x-x_n)^2 \geq M - \frac{\mu\sqrt{M}}{\sqrt{n}}|x-x_n| - \frac{\mu M}{2}(x-x_n)^2.$$

As $u''(x_n) \leq 0$, we have $\phi \star u(x_n) \leq 1$, and thus

$$\begin{aligned} 1 &\geq M - \frac{\mu\sqrt{M}}{\sqrt{n}}\int \phi(x_n-y)|y-x_n|dy - \frac{\mu M}{2}\int \phi(x_n-y)(y-x_n)^2dy \\ &= M - \mu M\left(I_2 + \frac{I_1}{\sqrt{Mn}}\right), \end{aligned}$$

with

$$I_1 = \int \phi(y)|y|dy.$$

Passing to the limit $n \to +\infty$ we recover the upper bound (2.8) for $M$.

Next, we set $m = \inf_{x\in\mathbb{R}} u(y)$ – recall that, according to Lemma 2.1, we have $m > 0$. Again, assume that $u(x)$ attains its minimum at a point $x_1$: $u(x_1) = m$. Equation (1.5) implies an upper bound

$$u'' = -\mu u(1-\phi\star u) \leq \mu u(\phi\star u) \leq \mu M^2,$$

and, in addition, that $\phi \star u(x_1) \geq 1$. Using the Taylor expansion around $x_1$ we see that

$$u(x) \leq m + \frac{\mu M^2}{2}(x-x_1)^2.$$

Therefore, we have

$$1 \leq \phi\star u(x_1) \leq m + \frac{\mu M^2}{2}\int \phi(x-y)(x-y)^2dy \leq m + \sigma^2 M^2 I_2,$$

thus, using (2.8), we obtain

$$m \geq 1 - \frac{\mu I_2}{(1-\mu I_2)^2} \geq 1 - K\mu.$$

The case when $u(x)$ does not attain its minimum is treated similarly to that for the supremum. $\square$

### 2.3 An upper bound for all $\mu$

As an aside we note that the proof of Lemma 2.2 can be improved to obtain a uniform upper bound for steady states for arbitrary large $\mu$.

**Proposition 2.3** *For any $\mu > 0$ there exists a constant $M_\mu > 0$ such that any bounded non-negative solution $u(x)$ of (1.5) in $\mathbb{R}$ satisfies $0 < u(x) \le M_\mu$.*

**Proof.** Let $M = \sup_{x\in\mathbb{R}} u(x)$ then, as in the previous argument for any $n > 0$ we may choose a point $x_n$ so that $u(x_n) \ge M - 1/n^2$, $u''(x_n) > -\mu M$ and $|u'(x_n)| \le \sqrt{\mu M/n}$. Using the Taylor expansion near $x_n$ we observe that

$$u(x) \ge \left(M - \sqrt{\frac{\mu M}{n}}|x - x_n| - \frac{\mu M}{2}(x - x_n)^2\right)_+ .$$

As $u''(x_n) < 0$ we should have $\phi \star u(x_n) \le 1$. The above bound implies that

$$1 \ge \int \phi(y)\left(M - \sqrt{\frac{\mu M}{n}}|y| - \frac{\mu M}{2}y^2\right)_+ dy.$$

Passing to the limit as $n \to +\infty$ we conclude that

$$M \le \left(\int \phi(y)\left(1 - \frac{\mu}{2}y^2\right)_+ dy\right)^{-1},$$

and the conclusion of Proposition 2.3 follows. □

### 2.4 Proof of Theorem 1.1

Assume that $v$ is a bounded nonnegative solution of

$$-v'' = \mu v(1 - \phi \star v), \quad x \in \mathbb{R}.$$

We first choose $\mu_0$ as in Lemma 2.2 and set $M = \sup_{x\in\mathbb{R}} v(x) \le M' = 1 + \mu I_2$.

Here it is more convinient to use the range of the nonlocal kernel as parameter instead of the growth rate $\mu$. In other words, we set $\sigma = \sqrt{\mu}$ and we define $u(x) = v(\frac{x}{\sigma})$ and $\phi_\sigma(x) = \frac{1}{\sigma}\phi(\frac{x}{\sigma})$. The function $u$ satisfies

$$-u'' = u(1 - \phi_\sigma \star u), \quad x \in \mathbb{R}. \tag{2.9}$$

We need to show that $u \equiv 0$ or $u \equiv 1$ for $\sigma$ small enough. Let $R > 0$, multiply (2.9) by $(u - 1)$ and integrate in $x$ between $(-R)$ and $R$. We obtain

$$\int_{-R}^{R} |u_x|^2 dx - (u-1)u_x\Big|_{-R}^{R} = -\int_{-R}^{R} u(1-u)(1 - \phi_\sigma \star u)dx \tag{2.10}$$

$$= -\int_{-R}^{R} u(1-u)(1 - u + u - \phi_\sigma \star u)dx = -\int_{-R}^{R} u(1-u)^2 dx - \int_{-R}^{R} u(1-u)(u - \phi_\sigma \star u)dx.$$

The standard elliptic regularity estimates imply that $\sup_{x\in\mathbb{R}}|u_x|\le C<+\infty$. As $|u-1|\le C\mu=C\sigma^2$ for $\sigma$ sufficiently small, by Lemma 2.2, and $M$ is uniformly bounded as well, it follows that

$$\int_{-R}^{R}|u_x|^2dx+\int_{-R}^{R}u(1-u)^2dx\le C\sigma^2-\int_{-R}^{R}u(1-u)(u-\phi_\sigma\star u)dx \tag{2.11}$$
$$\le C\sigma^2+C\left(\int_{-R}^{R}u(1-u)^2dx\right)^{1/2}\left(\int_{-R}^{R}|u-\phi_\sigma\star u|^2dx\right)^{1/2}.$$

The next step is to prove, for $\sigma\in(0,\sigma_0)$, the estimate

$$\int_{-R}^{R}|u-\phi_\sigma\star u|^2dx\le C\sigma^2\int_{-R}^{R}|u_x|^2dx+C\sigma. \tag{2.12}$$

To do so, we introduce a smooth cut-off function $\theta_R(x)=\Theta(|x|-R)/\delta)$ with $\Theta(x)=1$ for $x<-2$ and $\Theta(x)=0$ for $x>-1$ and a small parameter $\delta>0$ to be chosen. We decompose $u$ as $u=u_1+u_2$, with $u_1=\theta_R u$, $u_2=(1-\theta_R)u$, to get

$$\int_{-R}^{R}|u-\phi_\sigma\star u|^2dx\le 2\int_{-R}^{R}|u_1-\phi_\sigma\star u_1|^2dx+2\int_{-R}^{R}|u_2-\phi_\sigma\star u_2|^2dx.$$

Note that

$$\int_{-R}^{R}|u_1-\phi_\sigma\star u_1|^2dx\le\int_{\mathbb{R}}|u_1-\phi_\sigma\star u_1|^2dx\int|1-\hat\phi(\sigma\xi)|^2|\hat u_1(\xi)|^2d\xi\le C\sigma^2\int_{\mathbb{R}}|u_{1,x}|^2dx$$
$$\le C\sigma^2\int_{-R}^{R}|u_x|^2dx+\frac{C\sigma^2}{\delta^2}\delta\le C\sigma^2\int_{-R}^{R}|u_x|^2dx+\frac{C\sigma^2}{\delta}. \tag{2.13}$$

The other term may be estimated as

$$\int_{-R}^{R}|u_2-\phi_\sigma\star u_2|^2dx \tag{2.14}$$
$$\le 2\int_{-R}^{R}|u_2|^2dx+2\int_{-R}^{R}|\phi_\sigma\star u_2|^2dx\le C\delta+C\int_{-R}^{R}|\phi_\sigma\star\chi_{|x|>R-2\delta}|^2dx,$$

where $\chi_S$ is the characteristic function of a set $S$. However, for $z<R-2\delta$ we have, using the Chebyshev inequality

$$\phi_\sigma\star\chi_{x>R-2\delta}(z)=\int_{R-2\delta}^{\infty}\phi\left(\frac{z-y}{\sigma}\right)\frac{dy}{\sigma}=\int_{-\infty}^{(z-R+2\delta)/\sigma}\phi(y)dy\le\frac{C\sigma^2}{1+(z-R+2\delta)^2},$$

hence,

$$\int_{-R}^{R}|\phi_\sigma\star\chi_{x>R-2\delta}|^2dx\le C\sigma^4\int_{-\infty}^{R-2\delta}\frac{dz}{(1+(z-R+2\delta)^2)^2}+C\delta=C(\sigma^4+\delta).$$

The term involving $\chi_{x<-R+2\delta}$ in (2.14) may be estimated similarly, thus

$$\int_{-R}^{R}|u_2-\phi_\sigma\star u_2|^2dx\le C(\sigma^4+\delta). \tag{2.15}$$

Choosing $\delta=\sigma$ in (2.13) and (2.15) we arrive at the bound (2.12).

The third step is to conclude that, still for $\sigma$ small enough, there is a constant $C$ such that

$$\int_{-\infty}^{\infty} |u_x|^2 dx + \mu \int_{-\infty}^{\infty} u(1-u)^2 dx \leq C\sqrt{\sigma}. \tag{2.16}$$

To show it, we deduce from (2.11) and (2.12) that

$$\int_{-R}^{R} |u_x|^2 dx + \int_{-R}^{R} u(1-u)^2 dx \leq C\sigma^2 + C\sigma^{1/2} \left( \int_{-R}^{R} u(1-u)^2 dx \right)^{1/2}$$
$$+ C\sigma \left( \int_{-R}^{R} u(1-u)^2 dx \right)^{1/2} \left( \int_{-R}^{R} |u_x|^2 dx \right)^{1/2}$$

and thus

$$\int_{-R}^{R} |u_x|^2 dx + \int_{-R}^{R} u(1-u)^2 dx \leq \frac{C\sigma^2 + C\sigma}{1 - C\sigma}. \tag{2.17}$$

Thus the functions $u(1-u)^2$, $|u_x|^2$ and $|u - \phi_\sigma \star u|^2$ are integrable. We may now conclude the proof. We return to (2.10) but now integrate from $L_n$ to $R_n$ with $L_n \to -\infty$ and $R_n \to +\infty$ chosen so that $u_x(L_n), u_x(R_n) \to 0$ as $n \to +\infty$ – this is possible since $u(x)$ is a smooth bounded function. Then we obtain

$$\int_{L_n}^{R_n} |u_x|^2 dx - (u-1)u_x\Big|_{L_n}^{R_n} = -\int_{L_n}^{R_n} u(1-u)^2 dx - \int_{L_n}^{R_n} u(1-u)(u - \phi_\sigma \star u) dx.$$

Passing to the limit $n \to +\infty$ we get

$$\int_{\mathbb{R}} |u_x|^2 dx + \int_{\mathbb{R}} u(1-u)^2 dx \leq \sqrt{M} \left( \int_{\mathbb{R}} u(1-u)^2 dx \right)^{1/2} \left( \int_{\mathbb{R}} |u - \phi_\sigma \star u|^2 dx \right)^{1/2}. \tag{2.18}$$

As $u - \phi_\sigma \star u$ and $u_x$ are $L^2(\mathbb{R})$ functions, the Fourier theory gives us the upper bound

$$\int_{\mathbb{R}} |u - \phi_\sigma \star u|^2 dx \leq C\sigma^2 \int_{\mathbb{R}} |u_x|^2 dx,$$

and we get

$$\int_{\mathbb{R}} |u_x|^2 dx + \int_{\mathbb{R}} u(1-u)^2 dx \leq C\sigma \left( \int_{\mathbb{R}} u(1-u)^2 dx \right)^{1/2} \left( \int_{\mathbb{R}} |u_x|^2 dx \right)^{1/2}$$

and we conclude that for $\sigma > 0$ sufficiently small we have

$$\int |u_x|^2 dx = \int u(1-u)^2 dx = 0,$$

which means that $u(x) \equiv 1$. □

## 2.5 Proof of Theorem 1.3

Before proving Theorem 1.2 we consider the periodic case because it is much easier and explains the main idea in the proof of Theorem 1.2.

Let $u(x) \neq 0$, $x \in \mathbb{R}^d$ be an $L$-periodic nonnegative bounded solution of

$$-\Delta u = \mu u(1 - \phi \star u) = -\mu u(\phi \star v), \tag{2.19}$$

with $v = u - 1$.

We claim that there are three remarkable identities (the second is not used in this proof but we record it here for completeness and future reference):

$$\int_{\mathbb{T}_L} u(1 - \phi \star u) = 0, \tag{2.20}$$

$$\frac{1}{\mu}\int_{\mathbb{T}_L} |\nabla u|^2 + \int_{\mathbb{T}_L} v(\phi \star v)dx = -\int_{\mathbb{T}_L} v^2(\phi \star v)dx, \tag{2.21}$$

$$\frac{1}{\mu}\int_{\mathbb{T}_L} \frac{|\nabla u|^2}{u^2}dx + \int_{\mathbb{T}_L} v(\phi \star v)dx = 0. \tag{2.22}$$

First, we obtain (2.20) by integration over $\mathbb{T}_L$ of equation (2.19). In order to get (2.21) we multiply (2.19) by $u - 1$ and integrate over $\mathbb{T}_L$, using (2.19), as follows:

$$\frac{1}{\mu}\int\limits_{\mathbb{T}_L} |\nabla u|^2 dx = -\int\limits_{\mathbb{T}_L} u(u-1)(\phi \star v)dx - \int\limits_{\mathbb{T}_L} (1+v)v(\phi \star v)dx.$$

To obtain (2.22), we recall that by Lemma 2.1, $u > 0$. We divide equation (2.19) by $u$ and integrate over $\mathbb{T}_L$. Then, we add (2.20).

To conclude, decompose $v$ and $\phi \star v$ into the Fourier series

$$v(x) = \sum_{k \in \mathbb{Z}^d/L} v_k e^{2\pi i k \cdot x}, \qquad \phi \star v(x) = \sum_{\xi \in \mathbb{Z}^d/L} g_k e^{2\pi i k \cdot x},$$

with

$$g_k = \frac{1}{|\mathbb{T}_L|}\int_{\mathbb{T}_L} \phi \star v(x) e^{-2\pi i k \cdot x} dx = \frac{1}{|\mathbb{T}_L|}\int_{\mathbb{R}}\int_{\mathbb{T}_L} \phi(x-y) v_k e^{2\pi i k \cdot y} e^{-2\pi i k \cdot x} dx dy = v_k \hat{\phi}\,(2\pi k)\,.$$

Therefore, we arrive at

$$\int_{\mathbb{T}_L} v(x)(\phi \star v)(x)dx = |\mathbb{T}_L| \sum_{k \in \mathbb{Z}^d/L} g_k v_{-k} |\mathbb{T}_L| \sum_{k \in \mathbb{Z}^d/L} \hat{\phi}\,(2\pi k)\,|v_k|^2.$$

Using this in (2.22) we obtain

$$\frac{1}{\mu}\int_{\mathbb{T}_L} \frac{|\nabla u|^2}{u^2}dx + |\mathbb{T}_L| \sum_{k \in \mathbb{Z}^d/L} \hat{\phi}\,(2\pi k)\,|v_k|^2 = 0. \tag{2.23}$$

Hence, if $\hat{\phi}(2\pi k) \geq 0$ for all $k \in \mathbb{Z}^d/L$, then $\nabla u = 0$ and thus $u(x) \equiv 1$ or $u(x) \equiv 0$. $\square$

### 2.6 Proof of Theorem 1.2

Our goal is to establish an identity similar to (2.22) for a general bounded solution, without the periodicity assumption. We set $\mu = 1$ without loss of generality in this proof. Let $u \neq 0$, $u \geq 0$, be a solution to

$$-u'' = u(1 - \phi \star u). \tag{2.24}$$

We set $v = u - 1$, multiply (2.24) by $v/u$ and integrate between $(-L_n)$ and $R_n$ chosen so that $L_n, R_n \to +\infty$ as $n \to +\infty$, and $|u_x(-L_n)| + |u_x(R_n)| \leq 1/n$:

$$-\int_{-L_n}^{R_n} \frac{u_x^2 v}{u^2} dx + \int_{-L_n}^{R_n} \frac{u_x v_x}{u} dx - \frac{u_x v}{u}\Big|_{-L_n}^{R_n} = -\int_{-L_n}^{R_n} v(\phi \star v) dx$$

so that, as $v = u - 1$,

$$\frac{u_x v}{u}\Big|_{-L_n}^{R_n} - \int_{-L_n}^{R_n} \frac{u_x^2}{u} dx + \int_{-L_n}^{R_n} \frac{u_x^2}{u^2} dx + \int_{-L_n}^{R_n} \frac{u_x^2}{u} dx + \int_{-L_n}^{R_n} v(\phi \star v) dx = \int_{-L_n}^{R_n} \frac{u_x^2}{u^2} dx + \int_{-L_n}^{R_n} v(\phi \star v) dx.$$

Therefore, we have

$$\int_{-L_n}^{R_n} \frac{u_x^2}{u^2} dx + \int_{-L_n}^{R_n} v(\phi \star v) dx \leq \frac{C}{n}, \tag{2.25}$$

which is the analog of (2.22) in the non-periodic case. We claim that

$$\liminf_{n \to +\infty} \int_{-L_n}^{R_n} v(x)(\phi \star v)(x) dx \geq 0. \tag{2.26}$$

Then, as a consequence of (2.25) we obtain

$$\limsup_{n \to +\infty} \int_{-L_n}^{R_n} \frac{|u_x|^2}{u^2} dx = 0,$$

and thus $u$ is constant. As $u > 0$ we conclude that $u \equiv 1$. Therefore, it remains only to show that (2.26) holds.

First, note that if $v(x) \in L^2(\mathbb{R})$ then we have

$$\int_{-\infty}^{\infty} v(\phi \star v) dx = \int_{-\infty}^{\infty} \hat{\phi}(\xi) |\hat{v}(\xi)|^2 d\xi \geq 0,$$

since $\hat{\phi}(\xi) > 0$, and thus (2.26) holds trivially. Therefore, we may assume that $v(x)$ is in $L^\infty(\mathbb{R})$, but not in $L^2(\mathbb{R})$. In addition, the standard elliptic regularity results and Proposition 2.3 imply that $v(x) \in C_b^m(\mathbb{R})$ for all $m \geq 0$ – all derivatives of $v(x)$ are uniformly bounded:

$$\|v\|_{C_b^m} \leq M_m. \tag{2.27}$$

We will use these properties together with the equation for $v(x)$ to show that not only (2.26) holds but, actually,

$$\liminf_{n \to +\infty} \int_{-L_n}^{R_n} v(x)(\phi \star v)(x) dx = +\infty. \tag{2.28}$$

In order to prove (2.28) we introduce a smooth cut-off function $\psi_n(x)$ such that $0 \leq \psi_n(x) \leq 1$, and

$$\psi_n(x) = \begin{cases} 1, & -L_n \leq x \leq R_n, \\ 0, & x \leq -L_n - 1, \text{ or } x \geq R_n + 1, \end{cases}$$

and set $v_n(x) = \psi_n(x) v(x)$. In particular, we have

$$\lim_{n \to +\infty} \int_{-\infty}^{\infty} |v_n|^2 dx = +\infty, \tag{2.29}$$

as $\|v\|_{L^2(\mathbb{R})} = +\infty$. Let us write

$$\int_{-L_n}^{R_n} (\phi\star v)v dx = \int_{-L_n}^{R_n} (\phi\star v)v_n dx \int_{-\infty}^{\infty} (\phi\star v)v_n - \int_{-\infty}^{-L_n} (\phi\star v)v_n dx - \int_{R_n}^{\infty} (\phi\star v)v_n dx = I + II_1 + II_2. \quad (2.30)$$

The last two terms on the right side of (2.30) are uniformly bounded in $n$. For instance, we have:

$$\left|\int_{R_n}^{+\infty} (\phi\star v)v_n dx\right| \le K\int_{R_n}^{R_n+1} |\phi\star v|dx \le K'. \quad (2.31)$$

Here and below we denote by $K$, $K'$, etc. various constants which depend on the constants $M_m$ in (2.27) and the function $\phi$ but are independent from $n$.

We rewrite the first term in (2.30) as

$$I = \int_{-\infty}^{\infty} (\phi\star v)v_n \int_{-\infty}^{\infty} (\phi\star v_n)v_n + \int_{-\infty}^{\infty} (\phi\star(v-v_n))v_n = I_1 + I_2. \quad (2.32)$$

Again, $I_2$ is bounded uniformly in $n$:

$$|I_2| \le \int_{-\infty}^{\infty} |\phi\star(v-v_n)||v_n| \le 2M_0^2 \int_{-L_n-1}^{R_n+1} [|\phi\star\chi_{[-L_n-1,-L_n]}| + |\phi\star\chi_{[R_n,R_n+1]}|] \le K.$$

Hence, (2.28) holds if and only if

$$\liminf_{n\to+\infty} \int_{-\infty}^{\infty} (\phi\star v_n)v_n dx = +\infty, \quad (2.33)$$

and this is what we will show now. Let us choose a function $g \in \mathcal{S}(\mathbb{R})$ such that its Fourier transform satisfies $0 \le \hat{g}(\xi) \le 1$, and

$$\hat{g}(\xi) = \begin{cases} 1, & |\xi| \le 1, \\ 0, & |\xi| \ge 2. \end{cases}$$

Then we may split

$$\int_{-\infty}^{\infty} (\phi\star v_n)v_n dx \int_{-\infty}^{\infty} \hat{\phi}(\xi)|\hat{v}_n(\xi)|^2 d\xi \quad (2.34)$$

$$= \int_{-\infty}^{\infty} \hat{\phi}(\xi)\hat{g}\left(\frac{\xi}{R}\right)|\hat{v}_n(\xi)|^2 d\xi + \int_{-\infty}^{\infty} \hat{\phi}(\xi)\left[1-\hat{g}\left(\frac{\xi}{R}\right)\right]|\hat{v}_n(\xi)|^2 d\xi \ge \int_{-\infty}^{\infty} \hat{\phi}(\xi)\hat{g}\left(\frac{\xi}{R}\right)|\hat{v}_n(\xi)|^2 d\xi.$$

To bound this from below, observe that, with $g_R(x) = Rg(Rx)$:

$$\int_{-\infty}^{\infty} \left[1-\hat{g}\left(\frac{\xi}{R}\right)\right]|\hat{v}_n(\xi)|^2 d\xi = \int_{-\infty}^{\infty} |v_n|^2 dx - \int_{-\infty}^{\infty} v_n(x)(g_R\star v_n)(x)dx$$

$$= \int_{-\infty}^{\infty} v_n(x)\left[v_n(x) - \int g_R(y)v_n(x-y)dy\right]dx. \quad (2.35)$$

We now choose $R$ sufficiently large so that for all $n > n_0$ we have

$$\int\limits_{-\infty}^{\infty} \left[1 - \hat{g}\left(\frac{\xi}{R}\right)\right] |\hat{v}_n(\xi)|^2 d\xi \leq \frac{1}{3}\int_{-\infty}^{\infty} |\hat{v}_n(\xi)|^2 d\xi \frac{1}{3}\int_{-\infty}^{\infty} |v_n(x)|^2 dx. \tag{2.36}$$

This is done as follows. Note that, as

$$\int g_R(x)dx = \int g(x)dx = 1,$$

equation (2.35) implies that

$$\int\limits_{-\infty}^{\infty} \left[1 - \hat{g}\left(\frac{\xi}{R}\right)\right] |\hat{v}_n(\xi)|^2 d\xi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v_n(x)g_R(y)\left[v_n(x) - v_n(x-y)\right] dxdy$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v_n(x)g(y)\left[v_n(x) - v_n\left(x - \frac{y}{R}\right)\right] dxdy. \tag{2.37}$$

Choose $r_0$ so that

$$\int_{|y|\geq r_0} |g(y)|dy \leq \frac{1}{120}$$

and split the $y$-integral in the right side of (2.37) as

$$\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} v_n(x)g(y)\left[v_n(x) - v_n\left(x - \frac{y}{R}\right)\right] dxdy \int\limits_{|y|\leq r_0}\int\limits_{-\infty}^{\infty} v_n(x)g(y)\left[v_n(x) - v_n\left(x - \frac{y}{R}\right)\right] dxdy$$

$$+\int\limits_{|y|\geq r_0}\int\limits_{-\infty}^{\infty} v_n(x)g(y)\left[v_n(x) - v_n\left(x - \frac{y}{R}\right)\right] dxdy = P_n + Q_n.$$

The second term is estimated as

$$|Q_n| \leq \int\limits_{|y|\geq r_0} |g(y)| \int\limits_{-\infty}^{\infty} |v_n(x)|\left[|v_n(x)| + \left|v_n\left(x - \frac{y}{R}\right)\right|\right] dxdy$$

$$\leq \int\limits_{|y|\geq r_0} |g(y)| \int\limits_{-\infty}^{\infty} \left[|v_n(x)|^2 + \frac{1}{2}|v_n(x)|^2 + \frac{1}{2}\left|v_n\left(x - \frac{y}{R}\right)\right|^2\right] dxdy$$

$$2\int\limits_{|y|\geq r_0} |g(y)|dy \left(\int\limits_{-\infty}^{\infty} |v_n(x)|^2 dx\right) \leq \frac{1}{60}\int\limits_{-\infty}^{\infty} |v_n(x)|^2 dx.$$

The term $P_n$ is bounded in the following way:

$$|P_n| \leq \int\limits_{|y|\leq r_0}\int\limits_{-\infty}^{\infty} |v_n(x)g(y)|\left|v_n(x) - v_n\left(x - \frac{y}{R}\right)\right| dxdy$$

$$\leq \int\limits_{|y|\leq r_0} |g(y)| \int\limits_{-\infty}^{\infty} |v_n(x)| \left(\int_{x-r_0/R}^{x+r_0/R} |v_n'(z)|dz\right) dxdy$$

$$\leq C\left(\frac{r_0}{R}\right)^{1/2} \int\limits_{-\infty}^{\infty} |v_n(x)| \left(\int_{x-r_0/R}^{x+r_0/R} |v_n'(z)|^2 dz\right)^{1/2} dx.$$

Using the Cauchy-Schwarz inequality and changing the order of integration this may be estimated as

$$|P_n| \le C\left(\frac{r_0}{R}\right)^{1/2}\left(\int\limits_{-\infty}^{\infty}|v_n(x)|^2dx\right)^{1/2}\left(\int\limits_{-\infty}^{\infty}\int\limits_{x-r_0/R}^{x+r_0/R}|v_n'(z)|^2dzdx\right)^{1/2}$$
$$C\left(\frac{r_0}{R}\right)\left(\int\limits_{-\infty}^{\infty}|v_n(x)|^2dx\right)^{1/2}\left(\int\limits_{-\infty}^{\infty}|v_n'(z)|^2dz\right)^{1/2}. \tag{2.38}$$

We now recall relation (2.25), which implies, as $u(x)$ is bounded from above by $u(x) \le M$, as in Proposition 2.3:

$$\int_{-\infty}^{\infty}|v_n'|^2dx \le C + \int_{-L_n}^{R_n}|v'|^2dx \le C + M^2\int_{-L_n}^{R_n}\frac{|u'|^2}{u^2}dxC - C\int_{-L_n}^{R_n}v(\phi\star v)dx + \frac{C}{n}$$
$$\le C_1 + C\int_{-\infty}^{\infty}|v_n(\phi\star v_n)|dx + \frac{C}{n}C_1 + C\int_{-\infty}^{\infty}\hat\phi|\hat v_n|^2d\xi + \frac{C}{n} \le C + C\int_{-\infty}^{\infty}|v_n|^2dx. \tag{2.39}$$

Using this in (2.38) we get, for $n > n_0$ sufficiently large,

$$|P_n| \le C\left(\frac{r_0}{R}\right)\left(\int\limits_{-\infty}^{\infty}|v_n(x)|^2dx\right)^{1/2}\left(1+\int\limits_{-\infty}^{\infty}|v_n(z)|^2dz\right)^{1/2} \le C\left(\frac{r_0}{R}\right)\left(\int\limits_{-\infty}^{\infty}|v_n(x)|^2dx\right). \tag{2.40}$$

We used (2.29) in the last step. Therefore, if we choose $R$ so that $Cr_0/R < 1/60$, (2.36) holds.

As a consequence of (2.36) we have

$$\int\limits_{-\infty}^{\infty}\hat g\left(\frac{\xi}{R}\right)|\hat v_n(\xi)|^2\int\limits_{-\infty}^{\infty}|\hat v_n(\xi)|^2d\xi - \int\limits_{-\infty}^{\infty}\left[1-\hat g\left(\frac{\xi}{R}\right)\right]|\hat v_n(\xi)|^2d\xi \ge \frac{1}{2}\int_{-\infty}^{\infty}|v_n(x)|^2dx, \tag{2.41}$$

for $R \ge R_0$ sufficiently large. Inserting this in (2.34) leads to

$$\int\limits_{-\infty}^{\infty}(\phi\star v_n)v_ndx \ge \int\limits_{-\infty}^{\infty}\hat\phi(\xi)\hat g\left(\frac{\xi}{R_0}\right)|\hat v_n(\xi)|^2d\xi \ge K(R_0)\int\limits_{-\infty}^{\infty}\hat g\left(\frac{\xi}{R_0}\right)|\hat v_n(\xi)|^2d\xi \tag{2.42}$$
$$\ge \frac{K(R_0)}{2}\int_{-\infty}^{\infty}|v_n(x)|^2dx,$$

with

$$K(R_0) = \inf_{|\xi|\le 2R_0}\hat\phi(\xi).$$

Now, it follows from (2.42) and (2.29) that (2.33) holds and thus so does (2.28). The proof of Theorem 1.2 is now complete. □

# 3 Existence of traveling waves

Here we consider existence of traveling waves connecting the steady state $u \equiv 0$ to a uniformly positive state. Recall that a traveling wave $u(x)$ moving with the speed $c \in \mathbb{R}$ is a bounded solution of the equation

$$-cu_x = u_{xx} + \mu u(1-\phi\star u), \tag{3.1}$$

with the boundary conditions

$$\liminf_{x\to-\infty} u(x) > 0, \quad u(+\infty) = 0. \tag{3.2}$$

We will now prove Theorem 1.4 which asserts existence of traveling wave solutions of (3.1)-(3.2) for all $c \geq c^* = 2\sqrt{\mu}$. The main steps of the proof are as follows. As usual, we first construct the traveling wave that moves with the minimal speed $c^* = 2\sqrt{\mu}$. This is done by considering a sequence of approximating problems on intervals $(-a, a)$ (see (3.3) below), and then passing to the limit $a \to +\infty$. In order to obtain a non-trivial limit one usually has to fix a normalization for $u^a$ at $x = 0$. Here, this is done as follows: first, we show that any solution of (3.3) is uniformly bounded from above by a constant $K_0$ which is independent of $c \in \mathbb{R}$, and that the speed $c$ is also uniformly bounded from above and below, uniformly in the normalization at $x = 0$. Next, we show that strictly positive global solutions of (3.3) with a bounded speed $c$ and an upper bound $K_0$ for $u$ are uniformly bounded from below by a constant $\varepsilon$. We set $u(0) = \varepsilon/2$. The rest of the proof is rather standard: we use the above a priori bounds and the Leray-Schauder degree theory to find a solution of (3.3) on a finite interval and then use the same a priori bounds to pass to the limit $a \to +\infty$. This concludes the proof for the speed $c = c^*$. Existence of traveling waves for speeds $c > c^*$ comes from an argument using sub- and super-solutions, as well as additional a priori uniform estimates that are required because the super-solution is exponentially growing as $x \to -\infty$. The sub- and super-solution part of the argument is similar to what was done in [2, 13, 14].

## 3.1 A priori bounds for the finite domain problem

We will first study the approximating problem on a finite interval $(-a, a)$ for an unknown function $u^a(x)$ and an unknown speed $c^a$:

$$\begin{aligned} &-c^a u^a_x = u^a_{xx} + \mu g(u^a)(1 - \phi \star \bar{u}^a), \quad -a \leq x \leq a, \\ &u^a(-a) = 1, \quad u^a(a) = 0, \\ &u^a(0) = \varepsilon/2, \end{aligned} \tag{3.3}$$

with the number $\varepsilon > 0$ to be specified later, as explained above. Here $\bar{u}^a$ is an extension of $u^a$ to the whole line:

$$\bar{u}^a(x) = \begin{cases} u^a, & -a \leq x \leq a, \\ 0, & x > a, \\ 1, & x < -a, \end{cases} \tag{3.4}$$

which is needed to define the convolution, and

$$g(u) = \begin{cases} 0 \text{ if } u \leq 0, \\ u \text{ if } u \geq 0. \end{cases} \tag{3.5}$$

Note that (3.3), without the normalization at $x = 0$ may have a solution for any $c^a \in \mathbb{R}$. The additional condition $u^a(0) = \varepsilon/2$ is needed to ensure compactness of the family $(c^a, u^a)$ as $a \to +\infty$, which is the limit we will take to obtain solutions on the whole line.

### An upper bound for the solution

We first prove existence of a uniform bound on the possible solutions of (3.3). We will drop the superscript $a$ below to simplify the notation whenever it causes no confusion.

**Lemma 3.1** *There exist $a_0 > 0$ and $K_0 > 0$ so that any solution to (3.3) satisfies*

$$0 \leq u(x) \leq K_0 \tag{3.6}$$

*for all $x \in [-a, a]$ and all $a > a_0$, where the constants $K_0$ and $a_0$ depend only on $\mu$ and $\phi$.*

**Proof.** First, if $u$ is a solution of (3.3) that attains a negative minimum at some point $x_m$, then $x_m$ is an interior point of $(-a,a)$ and

$$-u'' + cu' = 0$$

in a neighborhood of $x_m$. This would imply that $u \equiv u(x_m) < 0$ which is a contradiction. Thus, $u(x) \geq 0$ for all $x \in (-a,a)$, and $g(u) = u$. In particular, any solution of (3.3) actually solves

$$\begin{aligned} &-c^a u^a_x = u^a_{xx} + \mu u^a(1 - \phi \star \bar{u}^a), \quad u^a(x) > 0 \quad -a \leq x \leq a, \\ &u^a(-a) = 1, \quad u^a(a) = 0, \quad u^a(0) = \varepsilon/2. \end{aligned} \tag{3.7}$$

We argue as in the proof of Lemma 2.2 to prove the uniform upper bound. Set

$$K_0 = \max_{x \in (-a,a)} u(x) = u(x_M),$$

and assume $K_0 > 1$ so that $u(x)$ attains its maximum at an interior point $x_M \in (-a,a)$. We want to prove that $K_0$ is bounded by a constant that does not depend on $a$. Assume first that $c < 0$ (the same argument works for $c > 0$ but considering $x > x_M$ below, and the case $c = 0$ was considered in Lemma 2.2). On the one hand, the maximum principle implies that $\phi \star u(x_M) \leq 1$. On the other hand, because $u \geq 0$, we have

$$-cu' - u'' \leq \mu u \leq \mu K_0,$$

so that, as $c < 0$:

$$\left(u' e^{-|c|x}\right)' \geq -\mu K_0 e^{-|c|x}.$$

Integrating from $x < x_M$ to $x_M$, we find, since $u'(x_M) = 0$:

$$u'(x) \leq \mu K_0 \frac{1 - e^{|c|(x - x_M)}}{|c|}.$$

Integrating again, we obtain for $x \leq x_M$,

$$\begin{aligned} u(x) &\geq K_0 \left[1 + \mu \frac{x - x_M}{|c|} + \mu \frac{1 - e^{|c|(x - x_M)}}{c^2}\right] = K_0 \left[1 - \mu (x - x_M)^2 B(|c|(x_M - x))\right] \\ &\geq K_0 \left[1 - \mu \frac{(x - x_M)^2}{2}\right]. \end{aligned}$$

Here we have defined, for $y \geq 0$:

$$0 \leq B(y) := \frac{e^{-y} + y - 1}{y^2} \leq \frac{1}{2}. \tag{3.8}$$

Notice that, because $u(-a) = 1$, this implies that

$$1 \geq K_0 \left(1 - \mu \frac{(x_M + a)^2}{2}\right). \tag{3.9}$$

Choose $x_0 < \sqrt{2/\mu}$ so that

$$\int_0^{x_0} \phi(y) \left(1 - \mu \frac{y^2}{2}\right)_+ dy > 0.$$

It follows from (3.9) that either we have $K_0 \leq (1 - \mu x_0^2/2)^{-1}$ (and we are done), or $x_M > -a + x_0$. In the latter case, because $u \geq 0$, we have

$$1 \geq \phi \star u(x_M) \geq \int_0^{x_0} \phi(y) u(x_M - y) dy \geq K_0 \int_0^{x_0} \phi(y) \left(1 - \mu \frac{y^2}{2}\right)_+ dy.$$

This shows that $K_0$ is a priori smaller than a constant depending on $\phi$ and locally bounded with respect to $\mu$. This proves Lemma 3.1. □

### An upper bound for the speed

The next step is the following upper bound for the speed.

**Lemma 3.2** *For any normalization parameter $\varepsilon > 0$ in (3.7) there exists $a_0(\varepsilon) > 0$ so that for all $a > a_0(\varepsilon)$, any nonnegative solution of (3.7) satisfies an upper bound for the speed*

$$c \leq 2\sqrt{\mu}. \tag{3.10}$$

**Proof.** As $u \geq 0$, the function $u$ satisfies the inequality

$$-cu_x \leq u_{xx} + \mu u. \tag{3.11}$$

Let us assume that $c > 2\sqrt{\mu}$ and define a family of functions $\psi_A(x) = Ae^{-\sqrt{\mu}x}$: they satisfy

$$-c\psi_A' - \psi_A'' > \mu\psi_A. \tag{3.12}$$

Note that, since $u(x) \in L^\infty(-a, a)$, when $A > 0$ is sufficiently large (depending on the solution) we have $u(x) < \psi_A(x)$ while for $A < 0$ we have $u(x) > \psi_A(x)$. Therefore, we can define

$$A_0 = \inf\{A : \quad \psi_A(x) > u(x) \text{ for all } x \in [-a, a]\}.$$

It follows that there exists $x_0 \in [-a, a]$ so that $\psi_{A_0}(x_0) = u(x_0)$ and $A_0 > 0$. However, (3.11) and (3.12) imply that $x_0$ can not be an interior point of the interval $(-a, a)$. As $A_0 > 0$, it is impossible that $x_0 = a$, hence $x_0 = -a$. As a consequence, $\psi_{A_0}(-a) = 1$, thus $A_0 = e^{-a}$. However, then $u(0) \leq \psi_{A_0}(0) = e^{-a} < \varepsilon/2$ which is a contradiction to the normalization $u(0) = \varepsilon/2$ in (3.18) when $a > \ln 2 - \ln \varepsilon$. Hence, $c > 2\sqrt{\mu}$ is impossible for $a$ sufficiently large. □

### A lower bound for the speed

Now, we prove a lower bound for the speed $c$.

**Lemma 3.3** *There exist $a_0 > 0$ and $K > 0$ independent of the normalization parameter $\varepsilon \in (0, 1/4)$ in (3.7) so that any solution to (3.7) satisfies $c > -K$ for all $a > a_0$.*

**Proof.** Suppose that $c < -1$. We first prove that the derivative $u'$ is bounded by $K/|c|$ on an interval $(-a, a - K_0)$ with the constants $K_0$ and $K$ independent of $a$:

$$-\frac{K}{|c|} \leq u'(x) \leq \frac{K}{|c|} \tag{3.13}$$

for all $x \in (-a, a - K_0)$. Note that the function $Q(x) = -\mu u(1 - \phi \star u)$ is uniformly bounded by Lemma 3.1. We write

$$\left(u'e^{cx}\right)' = Q(x)e^{cx},$$

and thus for $x > y$,

$$u'(x)e^{cx} = u'(y)e^{cy} + \int_y^x Q(z)e^{cz}dz, \tag{3.14}$$

so that

$$u'(x) \geq u'(y)e^{|c|(x-y)} - \|Q\|_\infty \frac{e^{|c|(x-y)}}{|c|}. \tag{3.15}$$

Setting $x = a$ in (3.15) we conclude that

$$u'(y) \leq 2\|Q\|_\infty/|c| \tag{3.16}$$

for all $y \in (-a, a)$, as $u'(a) \leq 0$.

The other inequality coming from (3.14), still for $x > y$, gives

$$u'(x) \leq u'(y)e^{|c|(x-y)} + \|Q\|_\infty \frac{e^{|c|(x-y)}}{|c|}.$$

This shows that for some constants $K_0$ and $K$ independent of $a$ we have

$$u'(y) \geq -2\|Q\|_\infty/|c|, \tag{3.17}$$

for all $y \leq a - K_0$. Otherwise we would have $u'(x) \leq -\|Q\|_\infty \frac{e^{|c|(x-y)}}{|c|}$ for all $x \in (y, a)$ and this cannot hold for a bounded function $0 \leq u(y) \leq K$ and $u(a) = 0$ on a too long interval $(y, a)$.

The bounds on $u'$ in (3.13) mean that for $c$ very negative, $u$ is locally close to a constant. Now, we argue by contradiction and suppose that $|c| > |c_0|$. Let $x_0 < 0$ be the first point to the left of $x = 0$ where $u(x_0) = 3/4$. We claim that for $|c| > |c_0|$ sufficiently large the function $u(x)$ is monotonically decreasing on the interval $[x_0, a - K_0]$. Indeed, let $y$ be a point where $u(y) \leq 3/4$. If $u$ achieves a local minimum at $y$ then, from (3.7) we see that $\phi \star u(y) \geq 1$. This contradicts the fact that $u(y) \leq 3/4$ since the bounds (3.13) imply that, if $|c_0|$ is sufficiently large, and $\phi \star u(y) \geq 1$ then

$$u(y) \geq 1 - \frac{K'}{|c|},$$

with a constant $K'$ which depends only on $\phi$ and $\mu$. This is because there exists a finite distance $l$ which depends only on the function $\phi$ and the upper bound $K$ for the function $u$ such that $\phi \star u(y) \geq 1$ implies that there exists a point $y'$ such that $|y - y'| \leq 1$ and $u(y') \geq 1$. Hence, no local minimum can be attained at a point $y$ where $u(y) \leq 3/4$ provided that $c$ is sufficiently negative. This means that $u'(x) \leq 0$ at all points $x$ where $u(x) \leq 3/4$ (otherwise there would be a first local minimum to the left of $x$, where $u$ would be less than $3/4$, a contradiction). In the same way $u$ cannot decrease on an interval $(x_0, x_1)$, $x_1 > x_0$ and achieve a local minimum at $x_1$. Since $x_0$ is not a local minimum, we find that the function $u(x)$ is decreasing on the whole interval $(x_0, a - K_0)$. If $c < 0$ is very negative, using (3.13), we conclude that $u(x) \geq 1/4$ on a very long interval $(x_0, x_0 + R)$ with $R \geq K|c|$, and hence (3.7) implies that $u(x)$ is uniformly strictly concave on the interval $(x_0 + R/4, x_0 + 3R/4)$ (recall, again, that we have assumed that $c < 0$). This is a contradiction to the fact that $1/4 \leq u(x) \leq 3/4$ on $(x_0, x_0 + R)$, which proves Lemma 3.3. □

**A uniform infimum lower bound on the steady states**

**Lemma 3.4** *For all $K > 0$ and $\alpha < \beta$, there exists $\varepsilon = \varepsilon(K, \alpha, \beta) > 0$ such that if $u$ is a solution of* (3.1) *with $c \in [\alpha, \beta]$, $0 < u \leq K$ and $\inf_{x\in\mathbb{R}} u(x) > 0$, then $\inf_{x\in\mathbb{R}} u(x) > \varepsilon$.*

**Proof.** Assume that there exists a sequence $u_n$ of solutions to (3.1) with $c = c_n$ such that $\beta_n := \inf_{\mathbb{R}} u_n > 0$ for all $n$, with $\beta_n \to 0$ as $n \to +\infty$, and set $v_n = u_n/\beta_n$. As $\inf_{\mathbb{R}} v_n = 1$ for all $n$, one may assume that there exists $x_n \in \mathbb{R}$ such that $v_n(x_n) \leq (1 + \frac{1}{n})$. Consider the shifted functions $w_n(x) = v_n(x + x_n)$ which satisfy

$$-w_n'' - c_n w_n' = \mu w_n(1 - \phi \star \tilde{u}_n),$$

where $\tilde{u}_n = u(x + x_n)$. As $\sup_{x\in\mathbb{R}} u_n(x) \leq K$, the coefficients above are uniformly bounded with respect to $n$, hence, the elliptic interior estimates [10] yield that one can extract some subsequence which converges locally uniformly to a function $w_\infty(x)$, and $c_n \to \bar{c} \in [\alpha, \beta]$ Moreover, as $\tilde{u}_n(0) \leq$

$\beta_n(1+1/n)$, the Harnack inequality applied to $\tilde{u}_n(x)$ implies that $\tilde{u}_n(x) \to 0$ locally uniformly in $x$. Hence, the function $w_\infty(x)$ satisfies

$$-w_\infty'' - \bar{c}w_\infty' = \mu w_\infty.$$

Moreover, one has $w_\infty(0) = 1$ and $w_\infty \geq 1$. The strong maximum principle thus gives $w_\infty \equiv 1$ which is a contradiction since $\mu \neq 0$. □

**The normalized problem**

We may now set the normalization at $x = 0$ for the approximating problem (3.7). Lemmas 3.2 and 3.3 imply that the speed $c$ satisfies a priori bounds $\alpha < c < \beta$ when $a$ is large with the constants $\alpha$ and $\beta = 2\sqrt{\mu}$ which do not depend on the choice of $\varepsilon > 0$. Hence, we may set $\varepsilon = \varepsilon(K_0, \alpha, \beta)$ as in Lemma 3.4, where $K_0$ has been defined in Lemma 3.1. Consider the normalized (at $x = 0$) problem

$$\begin{aligned} &-c^a u_x^a = u_{xx}^a + \mu u^a(1 - \phi \star \bar{u}^a), \quad u^a(x) \geq 0, \quad -a \leq x \leq a, \\ &u^a(-a) = 1, \quad u^a(a) = 0, \\ &u^a(0) = \varepsilon/2. \end{aligned} \tag{3.18}$$

**Proposition 3.5** *For all $\varepsilon > 0$, there exist $a_0 > 0$ and $K > 0$ so that (3.18) admits a solution for every $a > a_0$, which in addition satisfies the a priori bounds*

$$\begin{aligned} &|c| + \|u\|_{C^2(-a,a)} \leq K, \\ &c \leq 2\sqrt{\mu}, \end{aligned} \tag{3.19}$$

*for all $a > a_0$.*

**Proof.** With the a priori bounds in Lemmas 3.2-3.3 in hand, the proof of Proposition 3.5 is standard using the Leray-Schauder topological degree argument [3]. We provide the details for reader's convenience. Given a nonnegative function $v(x)$ defined on $(-a, a)$ with $v(-a) = 1$, $v(a) = 0$, we consider a family of problems

$$\begin{cases} -cZ_x^\tau = Z_{xx}^\tau + \tau g(v)\mu v(1 - \phi \star \bar{v}), \quad -a \leq x \leq a, \\ Z^\tau(-a) = 1, \quad Z^\tau(a) = 0. \end{cases} \tag{3.20}$$

with the parameter $\tau \in [0, 1]$ and $\bar{v}$ an extension of $v$ to the whole line as in (3.4).

We introduce a map $\mathcal{K}_\tau : (c, v) \to (\theta^\tau, Z^\tau)$ as the solution operator of the linear system (3.20). The number $\theta^\tau$ is defined by

$$\theta^\tau = \frac{\varepsilon}{2} - v(0) + c.$$

The operator $\mathcal{K}_\tau$ is a mapping of the Banach space $X = \mathbb{R} \times C^{1,\alpha}(-a, a)$, equipped with the norm $\|(c, v)\|_X = \max(|c|, \|v\|_{C^{1,\alpha}(-a,a)})$, onto itself. A solution $\mathbf{q}^\tau = (c, u)$ of (3.18) is a fixed point of $\mathcal{K}_\tau$ with $\tau = 1$ and satisfies $\mathcal{K}_1\mathbf{q}^1 = \mathbf{q}^1$, and vice versa: a fixed point of $\mathcal{K}_1$ provides a solution to (3.18). Hence, in order to show that (3.18) has a traveling front solution it suffices to show that the kernel of the operator $\mathcal{F}_1 = \mathrm{Id} - \mathcal{K}_1$ is not trivial. The operator $\mathcal{K}_\tau$ is compact and depends continuously on the parameter $\tau \in [0, 1]$. Thus the Leray-Schauder topological degree theory can be applied. Let us introduce a ball $B_M = \{\|(c, v)\|_X \leq M\}$. Then Lemmas 3.2-3.3 (with $\mu$ replaced by $\tau\mu$) show that the operator $\mathcal{F}_\tau$ does not vanish on the boundary $\partial B_K$ with $K$ sufficiently large for any $\tau \in [0, 1]$. It remains only to show that the degree $\deg(\mathcal{F}_1, B_K, 0)$ in $\bar{B}_K$ is not zero. However, the homotopy invariance property of the degree implies that $\deg(\mathcal{F}_\tau, B_K, 0) = \deg(\mathcal{F}_0, B_K, 0)$ for all $\tau \in [0,1]$. Moreover, the degree at $\tau = 0$ can be computed explicitly as the operator $\mathcal{F}_0$ is given by

$$\mathcal{F}_0(c,v) = \left(v(0) - \frac{\varepsilon}{2}, v - v_0^c\right).$$

Here the function $v_0^c(x)$ solves

$$\frac{d^2 v_0^c}{dx^2} + c\frac{dv_0^c}{dx} = 0, \quad v_0^c(-a) = 1, \quad v_0^c(a) = 0$$

and is given by

$$v_0^c(x) = \frac{e^{-cx} - e^{-ca}}{e^{ca} - e^{-ca}}.$$

The mapping $\mathcal{F}_0$ is homotopic to

$$\Phi(c,v) = (v_0^c(0) - \frac{\varepsilon}{2}, v - v_0^c)$$

that in turn is homotopic to

$$\tilde{\Phi}(c,v)\left(v_0^c(0) - \frac{\varepsilon}{2}, v - v_0^{c_*^0}\right),$$

where $c_*^0$ is the unique number so that $v_0^{c_*}(0) = \varepsilon/2$. The degree of the mapping $\tilde{\Phi}$ is the product of the degrees of each component. The last one has degree equal to one, and the first to $(-1)$, as the function $v_0^c(0)$ is decreasing in $c$. Thus $\deg\mathcal{F}_0 = -1$ and hence $\deg\mathcal{F}_1 = -1$ so that the kernel of $\mathrm{Id} - \mathcal{K}_1$ is not empty. This finishes the proof of Proposition 3.5. $\square$

## 3.2 Solution on the whole line

Having constructed a solution $(c^a, u^a)$ of (3.18) which satisfies the a priori estimates (3.19), we now pass to the limit $a_n \to +\infty$. The aforementioned bounds imply that passing to a subsequence $a_n \to +\infty$ we obtain a speed $c \in \mathbb{R}$, $|c| \le K$ and a positive function $u \in C_b^2(\mathbb{R})$, $0 \le u \le K$ which satisfy

$$\begin{aligned} -cu_x &= u_{xx} + \mu u(1 - \phi \star u), \quad 0 \le u(x) \le K, \\ u(0) &= \varepsilon/2. \end{aligned} \tag{3.21}$$

We now have to verify that $u(x)$ satisfies the boundary conditions (3.2) at infinity and that $c = 2\sqrt{\mu}$. This will finish the proof of Theorem 1.4 for $c = c^* = 2\sqrt{\mu}$ and is done in several steps.

### Monotonicity on the right

We first show that $u(x)$ is monotonically decreasing on the right. This will be the consequence of the two following lemmas.

**Lemma 3.6** *Let $u$ satisfy (3.21). Then there exists a sequence $x_n$, so that $|x_n| \to +\infty$ and $u(x_n) \to 0$ as $n \to +\infty$.*

**Proof.** If $\inf_{\mathbb{R}} u > 0$, then as $u \le K$, we know from Lemma 3.4 that $\inf_{\mathbb{R}} u \ge \varepsilon$, which would contradict the normalization $u(0) = \varepsilon/2$. $\square$

We will assume without loss of generality that $x_n \to +\infty$.

**Lemma 3.7** *Assume that $u$ satisfies (3.21), and that there exists a sequence $x_n \to +\infty$, so that $u(x_n) \to 0$ as $n \to +\infty$. Then there exists $R_0$ so that $u(x)$ is monotonically decreasing for $x > R_0$ and $\lim_{x\to+\infty} u(x) = 0$.*

**Proof.** Let us assume that the statement is false. As $u(x_n) \to 0$ as $n \to +\infty$ and $u(x)$ is not eventually monotonic, there exists another sequence $z_n \to +\infty$ so that $u(x)$ attains a local minimum at $z_n$ and $u(z_n) \to 0$. It follows from (3.21) that

$$\phi \star u(z_n) \geq 1. \tag{3.22}$$

On the other hand, as $u(x)$ is bounded in $C^2(\mathbb{R})$, the Harnack inequality implies that for any $R > 0$ and any $\delta > 0$ there exists $N$ so that $u(x) \leq \delta$ for all $x \in (z_n - R, z_n + R)$. This, however, contradicts (3.22) when $R$ is sufficiently large and $\delta$ is sufficiently small. □

### Characterization of the minimal speed

We now prove that $c = 2\sqrt{\mu}$ and that there exists no traveling wave solution of speed $c < 2\sqrt{\mu}$. As we already know that $c \leq 2\sqrt{\mu}$, this will be a direct consequence of the next lemma.

**Lemma 3.8** *Assume that $u$ is a positive bounded solution of (3.21) such that $\liminf_{x\to+\infty} u(x) = 0$, then $c \geq 2\sqrt{\mu}$.*

**Proof.** Take a sequence $x_n \to +\infty$ such that $u(x_n) \to 0$ and set $v_n(x) = u(x + x_n)/u(x_n)$. As $u$ is bounded and satisfies (3.21), the Harnack inequality implies that the sequence $v_n$ is locally uniformly bounded. This function satisfies:

$$-v_n'' - cv_n' = \mu v_n(1 - \phi \star \tilde{u}_n) \text{ in } \mathbb{R},$$

where $\tilde{u}_n(x) = u(x + x_n)$. The Harnack inequality implies that the shifted functions $\tilde{u}(x_n)$ converge to zero locally uniformly in $x$. Thus one may assume, up to extraction of a subsequence, that the sequence $v_n$ converges to a function $v$ that satisfies:

$$-v'' - cv' = \mu v \text{ in } \mathbb{R}. \tag{3.23}$$

Moreover, $v$ is positive since it is nonnegative and $v(0) = 1$. Equation (3.23) admits such a solution if and only if $c \geq 2\sqrt{\mu}$, which ends the proof. □

### The limit on the left

Lastly, we show that $u(x)$ is strictly positive on the left. This will conclude the proof of Theorem 1.4 for $c = c^*$.

**Lemma 3.9** *Assume that $u(x)$ satisfies (3.21) with $c > 0$. Then we have*

$$\liminf_{x\to-\infty} u(x) > 0. \tag{3.24}$$

**Proof.** Let us assume that there exists a sequence $y_n \to -\infty$ such that $u(y_n) \to 0$. Then using the arguments as in the proof of Lemmas 3.7 and 3.8 we would be able to show that $c < 0$ which is a contradiction. □

### 3.3 Traveling waves with speeds $c > c^*$

The last part in the proof of Theorem 1.4 is to take $c > 2\sqrt{\mu}$ and construct a traveling wave moving with speed $c$. The proof also goes through an approximating problem on a finite interval but solution for the latter problem is constructed with different boundary values and using sub- and super-solutions rather than with the a priori bounds. The key point is that though the super-solution is an unbounded exponential, the solution itself is bounded uniformly in the interval size $2a$: see Lemma 3.10.

#### Sub- and super-solutions

To begin we need a pair of sub- and super-solutions. As a super-solution we take the exponential

$$\bar{q}_c(x) = e^{-\lambda_c x},$$

with $\lambda_c > 0$ being the smallest root of

$$\lambda_c^2 - c\lambda_c + \mu = 0.$$

The function $\bar{q}_c$ satisfies

$$-c\bar{q}_c' = \bar{q}_c'' + \mu\bar{q}_c \geq \bar{q}_c'' + \mu\bar{q}_c(1 - \phi \star \bar{q}_c).$$

Next, we look for a sub-solution $r_c$ which would satisfy

$$-cr_c' \leq r_c'' + \mu r_c - \mu\bar{q}_c(\phi \star \bar{q}_c).$$

Note that

$$\phi \star \bar{q}_c = \int \phi(y)e^{-\lambda_c(x-y)}dy = Z_c e^{-\lambda_c x},$$

with

$$Z_c = \int \phi(y)e^{\lambda_c y}dy.$$

We take $r_c$ of the form

$$r_c(x) = \frac{1}{A}e^{-\lambda_c x} - e^{-(\lambda_c+\varepsilon)x},$$

with $\varepsilon > 0$ small chosen so that

$$\gamma_c = c(\lambda_c + \varepsilon) - (\lambda_c + \varepsilon)^2 - \mu > 0,$$

and $A > 1$ to be chosen below. Note that $r_c(x) > 0$ only on the set $\{A < e^{\varepsilon x}\}$, that is, for $x > (\ln A)/\varepsilon$. Then we have

$$\begin{aligned} -cr_c' - r_c'' - \mu r_c + \mu\bar{q}_c(\phi \star \bar{q}_c) &= \left[-c(\lambda_c + \varepsilon) + (\lambda_c + \varepsilon)^2 + \mu\right] e^{-(\lambda_c+\varepsilon)x} + Z_c\mu\bar{q}_c e^{-\lambda_c x} \\ &= -\gamma_c e^{-(\lambda_c+\varepsilon)x} + Z_c\mu e^{-2\lambda_c x} \\ &= e^{-(\lambda_c+\varepsilon)x}\left[-\gamma_c + Z_c\mu e^{-(\lambda_c-\varepsilon)x}\right] < 0 \end{aligned}$$

for all

$$x > \frac{1}{\lambda_c - \varepsilon}\ln\left(\frac{Z_c\mu}{\gamma_c}\right),$$

which includes the set $\{r_c(x) > 0\}$, provided that $\varepsilon < \lambda_c$ and $A$ is sufficiently large. We set

$$\bar{r}_c(x) = \max(0, r_c(x)).$$

This function still satisfies

$$-c\bar{r}_c' - \bar{r}_c'' \leq \mu\bar{r}_c - \mu\bar{q}_c\phi \star \bar{q}_c,$$

but in the sense of distributions.

**The finite domain problem**

Given $c > 2\sqrt{\mu}$ consider an approximating problem on the interval $(-a,a)$:

$$-cu' = u'' + \mu u(1 - \phi \star u), \quad u(\pm a) = \bar{r}_c(\pm a). \tag{3.25}$$

Define a convex set of functions $R_a = \{u \in C(-a,a) : \quad \bar{r}_c(x) \le u(x) \le \bar{q}_c(x)$, and consider the mapping $\Phi_a$ which maps a function $u_0 \in C(-a,a)$ to the solution of

$$-cu' = u'' + \mu u_0(1 - \phi \star u_0), \quad u(\pm a) = \bar{r}_c(\pm a). \tag{3.26}$$

The map $\Phi_a$ is compact. We claim that it leaves the set $R_a$ invariant. Indeed, given $u_0 \in R_a$, the function $\bar{q}_c$ satisfies the inequality

$$-c\bar{q}_c' - \bar{q}_c'' = \mu \bar{q}_c \ge \mu u_0 \ge \mu u_0(1 - \phi \star u_0) = -cu' - u'',$$

and $u(\pm a) = \bar{r}_c(\pm a) \le \bar{q}_c(\pm a)$. It follows from the maximum principle that $u(x) \le \bar{q}_c(x)$ for all $x \in (-a,a)$. On the other hand, the function $\bar{r}_c(x)$ satisfies

$$-c\bar{r}_c' - \bar{r}_c'' \le \mu \bar{r}_c - \mu \bar{q}_c \phi \star \bar{q}_c \le \mu u_0(1 - \phi \star u_0) = -cu' - u'',$$

and $u(\pm a) = \bar{r}_c(\pm a)$. It follows that $u(x) \ge \bar{r}_c(x)$ for all $x \in (-a,a)$ and thus the set $R_a$ is invariant.

The Schauder fixed point theorem now implies that the mapping $\Phi_a$ has a fixed point $u_a$ in $R_a$ which, in addition, satisfies $\bar{r}_c(x) \le u_a(x) \le \bar{q}_c(x)$. Now, we show that $u_a(x)$ is uniformly bounded.

**Lemma 3.10** *There exists a constant $K_0$ which does not depend on $c > c_* = 2\sqrt{\mu}$ so that any solution of (3.25) satisfies $0 \le u_a(x) \le K_0$ for all $a > 1$ and all $x \in (-a,a)$.*

**Proof.** The proof is exactly as that of Lemma 3.1: set

$$K_0 = \max_{x \in (-a,a)} u_a(x) = u_a(x_M),$$

and assume $K_0 > 1$. Then $u_a(x)$ attains its maximum at a point $x_M \in (-a,a)$, and the maximum principle implies that $\phi \star u_a(x_M) \le 1$. Also, we have

$$-cu_a' - u_a'' \le \mu K_0,$$

so that:

$$\left(u_a' e^{cx}\right)' \ge -\mu K_0 e^{cx}.$$

Integrating from $x_M$ to $x > x_M$, we find, since $u'(x_M) = 0$:

$$u_a'(x) \ge -\frac{\mu K_0}{c}\left(1 - e^{-c(x-x_M)}\right).$$

Hence, for $x \ge x_M$, we obtain

$$u_a(x) \ge K_0 \left[1 - \mu \frac{x - x_M}{c} + \mu \frac{1 - e^{-c(x-x_M)}}{c^2}\right] = K_0 \left[1 - \mu (x - x_M)^2 B(c(x - x_M))\right]$$
$$\ge K_0 \left[1 - \mu \frac{(x - x_M)^2}{2}\right],$$

with $B(y)$ as in (3.8). Since $u_a(a) \le e^{-\lambda_c a}$, this implies that if $K_0 > 1$ then $x_M < a - l_\mu$, with $l_\mu$ which depends on $\mu$. In the latter case, we have

$$1 \ge \phi \star u(x_M) \ge \int_{-l_\mu}^0 \phi(y) u(x_M - y) dy \ge K_0 \int_{-l_\mu}^0 \phi(y) \left(1 - \mu \frac{y^2}{2}\right)_+ dy.$$

This shows that $K_0$ is bounded by a constant which depends neither on $c$ nor on $a$. This proves Lemma 3.10. □

**Passing to the limit $a_n \to +\infty$**

As a consequence of Lemma 3.10 the family $u_a(x)$ is uniformly bounded in $C^{2,\alpha}(\mathbb{R})$ and we may pass to the limit $a \to +\infty$, possibly along a subsequence. In this limit, we have $u_a \to u$, and $u(x)$ is a solution of

$$-cu' = u'' + \mu u(1 - \phi \star u),$$

which satisfies $\bar{r}_c(x) \le u(x) \le \min(K_0, \bar{q}_c(x))$. In particular, we have $\lim_{x\to+\infty} u(x) = 0$. If we had $\liminf_{x\to-\infty} u(x) = 0$, then Lemma 3.8 applied to $u(-x)$ would give $c \le -2\sqrt{\mu}$, which is a contradiction. Thus $\liminf_{x\to-\infty} u(x) > 0$ and the proof of Theorem 1.4 is complete. □

# 4 Convergence on the left

In this section, we assume that $\hat{\phi}(\xi) > 0$ for all $\xi \in \mathbb{R}$ or that $\mu$ is small and we prove that any traveling wave has the left limit $u(-\infty) = 1$ under either of these conditions.

### Triviality of uniformly positive "traveling waves"

We first show that even with $c \neq 0$ the only uniformly positive bounded solution of

$$\begin{aligned} &-cu_x = u_{xx} + \mu u(1 - \phi \star u), \quad 0 \le u(x) \le K, \\ &u(0) = \varepsilon/2. \end{aligned} \tag{4.1}$$

is $u \equiv 1$ if $\hat{\phi}(\xi) > 0$ for all $\xi \in \mathbb{R}$ or if $\mu$ is small.

**Lemma 4.1** **($\hat{\phi}(\xi) > 0$)** *Let $u \in C_b^2(\mathbb{R})$ satisfy (4.1). Assume that $0 \le u(x) \le K$, $\inf_{x\in\mathbb{R}} u(x) \ge \alpha > 0$ and that $\hat{\phi}(\xi) > 0$ for all $\xi \in \mathbb{R}$. Then $u(x) = 1$ for all $x \in \mathbb{R}$.*

**Proof.** The proof is very similar to that of Theorem 1.2. Let $u(x) \ge \alpha > 0$ be a solution to (3.21). Set $v = u - 1$, multiply this equation by $v/u$ and integrate between $L_n$ and $R_n$ chosen as before: $L_n \to -\infty$, $R_n \to +\infty$ as $n \to +\infty$, $|u_x(-L_n)| + |u_x(R_n)| \le 1/n$:

$$-\int_{-L_n}^{R_n} \frac{u'^2 v}{u^2} dx + \int_{-L_n}^{R_n} \frac{u'v'}{u} dx - \left[\frac{u'v}{u} - cu + c\ln(u)\right]\Big|_{-L_n}^{R_n} = -\int_{-L_n}^{R_n} v(\phi \star v) dx$$

so that

$$\int_{-L_n}^{R_n} \frac{u'^2}{u^2} dx + \int_{-L_n}^{R_n} v(\phi \star v) dx \left[\frac{u'v}{u} + c\ln(u) - cu\right]\Big|_{-L_n}^{R_n}$$

Therefore, we have

$$\int_{-L_n}^{R_n} \frac{u'^2}{u^2} dx + \int_{-L_n}^{R_n} v(\phi \star v) dx \le \frac{C}{n} + |c||G_n|, \tag{4.2}$$

where $G_n = [\ln(u) - u]\Big|_{-L_n}^{R_n}$. As in the proof of Theorem 1.2 we claim that

$$\liminf_{n\to+\infty} \int_{-L_n}^{R_n} v(x)(\phi \star v)(x) dx \ge 0. \tag{4.3}$$

This, together with (4.2) leads to

$$\limsup_{n\to+\infty} \int_{-L_n}^{R_n} \frac{|u_x|^2}{u^2} dx \le |c| \limsup_{n\to+\infty} |G_n| := K.$$

As a consequence of the boundedness of $u$, we have

$$\int_{\mathbb{R}} |u'|^2 dx \leq CK.$$

Elliptic regularity and the standard translation arguments imply that then $u(x)$ tends to two constants $u_+$ and $u^-$ as $x \to \pm\infty$. These constants have to be solutions of (3.21) and thus $u^{\pm} = 1$, since $u(x) \geq \alpha > 0$. This implies, in turn, that $G_n \to 0$ as $n \to +\infty$ and thus $K = 0$ and $u(x) \equiv 1$.

Therefore, it remains only to show that (4.3) holds in the case $c \neq 0$. However, it is straightforward to verify that the proof of (2.26) still applies. The only minor modification required is an addtional term of the form $C\tilde{G}_n$ in (2.39) (this is the only place where the equation for $v(x)$ is used in the proof of (2.26)) in the expression for

$$\int_{-L_n}^{R_n} \frac{u'^2}{u^2} dx.$$

It is easy to check that this modification is harmless and (4.3) holds. This completes the proof of Lemma 4.1. □

**Lemma 4.2 (Small $\mu$)** *There exists $\mu_0 > 0$ (that does not depend on the speed $c \in \mathbb{R}$) such that for all $0 < \mu \leq \mu_0$, if $u$ is a bounded solution of (3.21) such that $\inf_{x\in\mathbb{R}} u(x) \geq \alpha > 0$, then $u(x) = 1$ for all $x \in \mathbb{R}$.*

**Proof.** We use the same method as in the proof of Theorem 1.1. First of all, using the same rescaling as in the proof of Theorem 1.1, we assume that $\mu = 1$ and use as a new parameter the range of the nonlocal kernel $\sigma$. The (rescaled) function $u$ then satisfies

$$-u'' - cu' = u(1 - \phi_\sigma \star u) \text{ in } \mathbb{R}. \tag{4.4}$$

We need to prove that $u \equiv 1$ for $\sigma$ small enough. First, arguing as in the proof of Lemmas 3.1 and 3.10 we may show that there exists a constant $K_0$ which does not depend on the speed $c$ so that if $u(x)$ attains a local maximum at a point $x_M$ then $u(x_M) \leq K_0$. If $M = \sup_{x\in\mathbb{R}} u(x) > K_0$ then $u(x)$ approaches $M$ monotonically either as $x \to -\infty$ or $x \to +\infty$. The standard translation argument and elliptic regularity imply that then $u \equiv M$ is a solution of (4.4), hence $M \equiv 1$. We conclude that in any case $0 < \alpha \leq u(x) \leq K_0$ for all $x \in \mathbb{R}$, with $K_0$ independent of $c$. We also have a bound $K_1 := \sup |u_x| < +\infty$ from the standard elliptic regularity estimates, with the constant $K_1$ that may depend on $c$.

Next, multiply (4.4) by $(u-1)$ and integrate in $x$ between $(-R)$ and $R$. We obtain

$$\int_{-R}^{R} |u_x|^2 dx - (u-1)u_x\Big|_{-R}^{R} - \frac{c}{2}((u-1)^2(R) - (u-1)^2(-R)) = -\int_{-R}^{R} u(1-u)^2 dx - \int_{-R}^{R} u(1-u)(u-\phi_\sigma \star u) dx. \tag{4.5}$$

It follows that

$$\int_{-R}^{R} |u_x|^2 dx + \int_{-R}^{R} u(1-u)^2 dx - \frac{c}{2}((u-1)^2(R) - (u-1)^2(-R))$$
$$\leq 2K_1(K_0 - 1) + \sqrt{K_0}\left(\int_{-R}^{R} u(1-u)^2 dx\right)^{1/2}\left(\int_{-R}^{R} |u - \phi_\sigma \star u|^2 dx\right)^{1/2}.$$

We know from the proof of Theorem 1.1 that, for $\sigma \in (0,\sigma_0)$, one has

$$\int_{-R}^{R} |u-\phi_\sigma \star u|^2 dx \le C\sigma^2 \int_{-R}^{R} |u_x|^2 dx + C\sigma. \tag{4.6}$$

Thus, using the same computations as in the proof of Theorem 1.1, one has, still for $\sigma$ small enough:

$$\int_{-R}^{R} |u_x|^2 dx + \int_{-R}^{R} u(1-u)^2 dx - \frac{c}{2}((u-1)^2(R) - (u-1)^2(-R)) \le C(1+K_1). \tag{4.7}$$

This implies that

$$\int_{-\infty}^{\infty} |u_x|^2 dx < +\infty$$

and thus elliptic regularity and the usual translation arguments imply that $u(x)$ converges to two constants $u_+$ and $u_-$ as $x \to \pm\infty$. As $\inf_{\mathbb{R}} u > 0$, these constants are positive. Moreover, these constants have to satisfies equation (4.4). Thus

$$u(+\infty) = u(-\infty) = 1.$$

We can now conclude the proof. We return to (4.5) but now integrate from $L_n$ to $R_n$ with $L_n \to -\infty$ and $R_n \to +\infty$ chosen so that $u_x(L_n), u_x(R_n) \to 0$ as $n \to +\infty$ – this is possible since $u(x)$ is a smooth bounded function. Then we obtain

$$\begin{aligned} &\int_{L_n}^{R_n} u_x^2 dx - (u-1)u_x\Big|_{L_n}^{R_n} - \frac{c}{2}((u-1)^2(R_n) - (u-1)^2(L_n)) \\ &= -\int_{L_n}^{R_n} u(1-u)^2 dx - \int_{L_n}^{R_n} u(1-u)(u-\phi_\sigma \star u) dx. \end{aligned} \tag{4.8}$$

Passing to the limit $n \to +\infty$ we get

$$\int_{\mathbb{R}} u_x^2 dx + \int_{\mathbb{R}} u(1-u)^2 dx \le \sqrt{K_0} \left( \int_{\mathbb{R}} u(1-u)^2 dx \right)^{1/2} \left( \int_{\mathbb{R}} |u-\phi_\sigma \star u|^2 dx \right)^{1/2}. \tag{4.9}$$

As $(u-\phi_\sigma \star u)$ and $u_x$ are $L^2(\mathbb{R})$ functions, the Fourier theory gives us the upper bound

$$\int_{\mathbb{R}} |u-\phi_\sigma \star u|^2 dx \le C\sigma^2 \int_{\mathbb{R}} |u_x|^2 dx$$

we get

$$\int_{\mathbb{R}} |u_x|^2 dx + \int_{\mathbb{R}} u(1-u)^2 dx \le C\sqrt{K_0}\sigma \left( \int_{\mathbb{R}} u(1-u)^2 dx \right)^{1/2} \left( \int_{\mathbb{R}} |u_x|^2 dx \right)^{1/2}$$

and thus we conclude that for $\sigma > 0$ sufficiently small we have

$$\int |u_x|^2 dx = \int u(1-u)^2 dx = 0,$$

and thus $u(x) \equiv 1$. $\square$

**The left limit**

Finally, we show that if either $\hat{\phi}(\xi) > 0$ for all $\xi \in \mathbb{R}$ or $\mu \in (0, \mu_0)$, the left limit is $u(-\infty) = 1$.

**Lemma 4.3** *Assume that $u$ is a bounded solution of (3.21) with $\liminf_{x\to-\infty} u(x) \geq \alpha > 0$ and assume that $u \equiv 1$ is the only bounded function with a positive infimum that satisfies equation (3.21). Then $\lim_{x\to-\infty} u(x) = 1$.*

**Proof.** Set $\alpha = \liminf_{x\to-\infty} u(x)$, take a sequence $r_n \to +\infty$, and consider the functions $v_n(x) = u(x - r_n)$. The uniform bounds on the function $u(x)$ imply that we can extract a subsequence $n_k \to +\infty$ so that the sequence $v_{n_k}(x)$ converges locally uniformly to a limit $\bar{v}(x)$, which satisfies

$$-c\bar{v}' = \bar{v}'' + \bar{v}(1 - \phi \star \bar{v}), \quad \inf_x \bar{v} \geq \alpha > 0.$$

The uniqueness hypothesis implies that $\bar{v}(x) \equiv 1$. It follows that the whole sequence $v_n(x)$ converges locally uniformly to $\bar{v}(x) \equiv 1$ as $n \to +\infty$ and the conclusion of Lemma 4.3 follows. □

This completes the proof of Theorems 1.5 and 1.6.

# 5 Non-monotonic waves

We now show that there is a range of parameters where we both know that there exists a traveling wave connecting the states $u \equiv 0$ and $u \equiv 1$, and that no monotonic traveling wave of this type may exist. First, we need a couple of definitions. We assume in this section that $\phi(x)$ is continuous, $\phi(0) > 0$ and $\phi(x)$ decays sufficiently fast at infinity so that the Laplace transform of $\phi$ is defined for all $\gamma \in \mathbb{R}$,

$$\Phi(\gamma) = \int_{-\infty}^{\infty} \phi(x) e^{-\gamma x} dx = \lim_{R\to+\infty} \Phi(\gamma, R), \tag{5.1}$$

where we have set

$$\Phi(\gamma_0, R) = \int_{y>-R} \phi(y) e^{-\gamma_0 y} dy.$$

Define the function

$$\mu_*(c) = \sup_{s>0} \frac{s(c+s)}{\Phi(s)}, \tag{5.2}$$

then $\mu_*(c)$ is increasing in $c$ for $c > 0$. Note that $\Phi(s) \geq Ce^{\alpha|s|}$ for some $\alpha > 0$, and thus

$$\frac{c}{\Phi(1)} \leq \mu_*(c) \leq \sup_{s>0} \left(\frac{s^2}{\Phi(s)}\right) + c\left(\sup_{s>0} \frac{s}{\Phi(s)}\right) = M_0 + cM_1.$$

In particular, the inverse function $\bar{c}(\mu) = \mu_*^{-1}(\mu)$ satisfies $\bar{c}(\mu) = O(\mu)$ for large $\mu$, and for $\mu$ large enough $2\sqrt{\mu} < \bar{c}(\mu)$. We have the following proposition, which precises Theorem 1.7.

**Proposition 5.1** *Assume that $\mu$ satisfies $2\sqrt{\mu} := c^*(\mu) < \bar{c}(\mu)$. Then for $c^*(\mu) \leq c \leq \bar{c}(\mu)$, the problem*

$$-cu' = u'' + \mu u(1 - \phi \star u), \quad u(-\infty) = 1, \; u(+\infty) = 0, \tag{5.3}$$

*does not admit a traveling wave solution which is monotonic on any interval of the form $(-\infty, R_0)$. In particular, for $\hat{\phi}(\xi) > 0$ for all $\xi \in \mathbb{R}$ and $\mu$ large enough, traveling waves with speeds $c \in [2\sqrt{\mu}, \bar{c}(\mu))$ exist and are not monotonic.*

**Proof.** Existence of a traveling wave with the speeds $c \geq c_* = 2\sqrt{\mu}$ follows from Theorem 1.4. Hence, all we need to show is that when $\mu$ is sufficiently large this wave may not be monotonic. The main idea of the proof is to translate far to the left, linearize around $u = 1$ and show that the linearized equation admits no monotonic in $x$ solution, a property which is elementary if we knew that all solution are of the form $e^{zx}$ for some $z \in \mathbb{C}$.

Let us assume for the sake of contradiction that (5.3) admits a monotonically, say decreasing, solution $u(x)$ and write $u(x) = 1 - v(x)$. The function $v(x)$ is monotonically increasing and satisfies

$$v'' + cv' + \mu v(\phi \star v) = \mu\phi \star v, \quad v(-\infty) = 0, \ v(+\infty) = 1. \tag{5.4}$$

Consider the points $x_n \to -\infty$ such that $v(x_n) = 1/n$ and set

$$w_n(x) = \frac{v(x + x_n)}{v(x_n)}.$$

The function $w_n(x)$ is increasing and satisfies

$$w_n'' + cw_n' + \mu g_n(x) w_n = \mu\phi \star w_n, \quad w_n(0) = 1, \tag{5.5}$$

with $0 \leq g_n(x) := (\phi \star v)(x + x_n) \leq 1$.

We now show that the sequence $w_n(x)$ converges to a limit $\bar{w}(x)$ as $n \to +\infty$. First, we prove that the sequence $w_n(x)$ is bounded in $C^{2,\alpha}_{loc}(\mathbb{R})$. Without loss of generality let us assume that $\phi(x) > 0$ on the interval $(-2, 2)$. It suffices to show that $w_n(1) \leq M_0$ with some constant $M_0$ which is independent of $n$. To see that, multiply (5.5) by a non-negative test function $\psi(x) \in C^2_c(-1, 0)$, such that $\psi(x) > 0$ for all $x \in (-1, 0)$, then we get

$$\int_{-1}^{0} [\psi'' w_n - c\psi' w_n + \mu g_n \psi w_n] dx = \mu \int_{-1}^{0} \left( \int_{\mathbb{R}} \phi(x - y)\psi(x) w_n(y) dy \right) dx. \tag{5.6}$$

As $0 \leq w_n(x) \leq 1$ for $x \leq 0$, and $0 \leq g_n(x) \leq 1$, the left side of (5.6) can be bounded from above by

$$\int_{-1}^{0} |\psi'' w_n - c\psi' w_n + \mu g_n \psi w_n| dx \leq C_\psi,$$

with the constant $C_\psi$ which does not depend on $n$. On the other hand, the right side of (5.6) may be bounded from below by

$$\mu \int_{-3/4}^{-1/4} \left( \int_{\mathbb{R}} \phi(x - y)\psi(x) w_n(y) dy \right) dx \geq C_\psi \int_{-3/4}^{-1/4} \left( \int_{1}^{+\infty} \phi(x - y) w_n(y) dy \right) dx \geq C'_\psi w_n(1).$$

As $w_n(x) \leq w_n(0) = 1$, it follows that $0 \leq w_n(1) \leq \beta_0$ with a constant $\beta_0(\mu, c)$ independent of $n$. The same argument leads to an estimate

$$w_n(x + 1) \leq \beta_0 w_n(x), \text{ for all } x \in \mathbb{R}. \tag{5.7}$$

Therefore, the right side of (5.5) is locally uniformly bounded which gives $C^2_{loc}(\mathbb{R})$ bounds for $w_n$. Finally, differentiating (5.5) in $x$ we get the local $C^{2,\alpha}_{loc}(\mathbb{R})$ bounds.

We may now pass to the (strong) limit $n \to +\infty$, for a subsequence if necessary, and, as $g_n(x) \to 0$ locally uniformly from the previous step, conclude that $w_n(x)$ converges locally uniformly as $n \to +\infty$ to a solution $\bar{w}(x)$ of the linearized problem

$$\begin{cases} \bar{w}'' + c\bar{w}' = \mu\phi \star \bar{w}, \quad \bar{w}(0) = 1, \\ \bar{w} \geq 0, \quad \bar{w}'(x) \geq 0, \quad \bar{w}(x) \to 0 \quad as \ x \to -\infty. \end{cases} \tag{5.8}$$

It follows from differentiating (5.8) that $z(x) = \bar{w}'(x)$ satisfies

$$z'' + cz' \geq 0,$$

and $z(x) \to 0$ as $x \to -\infty$, at least along some subsequence. Thus, integrating between $-\infty$ and $x$ we get $z'(x) \geq 0$. Thus, the function $\bar{w}(x)$ is actually convex.

Moreover, the function $\bar{w}(x)$ satisfies the same estimate (5.7): $\bar{w}(x+1) \leq \beta_0 \bar{w}(x)$ and, since it is monotonically increasing, it follows that there are constants $A > 0$ and $\gamma_0 > 0$ so that

$$\bar{w}(x) \geq Ae^{\gamma_0 x}, \quad \text{for all } x \leq 0. \tag{5.9}$$

Hence, we may define

$$\bar{\gamma} = \inf\{\gamma : \text{ there exists } A > 0 \text{ so that } \bar{w}(x) \geq Ae^{\gamma x} \text{ for all } x \leq 0.\}$$

We will now show that no such solution may exist in the stated range of $c$ and $\mu$. As $\mu > \mu_*(c)$, we may find $R$ so that

$$\mu > \mu_*(c, R) := \sup_{s>0} \frac{s(c+s)}{\Phi(s, R)}.$$

Let $\delta_0 = \dfrac{1}{R} \ln(\mu/\mu_*(c, R)) > 0$ and take $\gamma_0 \in (\bar{\gamma}, \bar{\gamma} + \delta_0/2)$ so that

$$\gamma_0 - \delta_0 < \bar{\gamma}. \tag{5.10}$$

Since $\gamma_0 > \bar{\gamma}$, (5.9) holds with some $A > 0$, thus the function $\bar{w}$ satisfies

$$\bar{w}'' + c\bar{w}' \geq A\mu \int_{y>-R} \phi(y) e^{\gamma_0(x-y)} dy A\mu\Phi(\gamma_0, R) e^{\gamma_0 x}, \quad x \leq -R.$$

Integrating on the interval $(-\infty, x)$, with $x < -R$ gives

$$\bar{w}'(x) + c\bar{w} \geq \frac{A\mu\Phi(\gamma_0, R)}{\gamma_0} e^{\gamma_0 x}, \quad x < -R.$$

Integrating again leads to the inequality

$$\bar{w}(x) \geq \frac{A\mu\Phi(\gamma_0, R)}{\gamma_0(\gamma_0 + c)} e^{\gamma_0 x} \geq \frac{A\mu}{\mu_*(c, R)} e^{\gamma_0 x} = Ae^{\delta_0 R + \gamma_0 x}, \qquad \forall x \leq -R. \tag{5.11}$$

We may iterate the argument so that for all integers $N \geq 1$ we have

$$\bar{w}(-NR) > Ae^{-(\gamma_0 - \delta_0)NR}.$$

Consider now any $y \leq 0$ and the non-negative integer $N_y$ such that

$$-(N_y + 1)R < y \leq -N_y R.$$

One has

$$\bar{w}(y) \geq \bar{w}(-(N_y + 1)R) \geq Ae^{-(\gamma_0 - \delta_0)(N_y + 1)R} \geq A' e^{(\gamma_0 - \delta_0)y},$$

with $A' = Ae^{-(\gamma_0 - \delta_0)R}$. Therefore, $\gamma_0 - \delta_0 > \bar{\gamma}$ which contradicts (5.10). Hence, such a monotonic solution $\bar{w}$ may not exist. □

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