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The Non-relativistic Limit of $\mathscr{P}(\varphi)_2$ Quantum Field Theories: Two-Particle Phenomena

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Abstract. It is proved that for two-particle phenomena the $\mathscr{P}(\varphi)_2$ quantum field theories with speed of light c converge to non-relativistic quantum mechanics with a δ function potential in the limit $c\to\infty$.

I. Introduction

In this paper we are concerned with the general question of how relativistic quantum mechanics with speed of light c is approximated by non-relativistic quantum mechanics in the limit $c\to\infty$. Only a few rigorous results of this nature exist. For example, for a single particle in an external field, the relation between the Dirac equation and the Schrödinger equation is understood. ([12], and earlier references.)

Specifically we consider $\mathscr{P}(\varphi)_2$ quantum field theory models with speed of light c, denoted $\mathscr{P}(\varphi)_{2,c}$. According to the folklore the $c\to\infty$ limit should produce a multiparticle Schrödinger theory with δ-function potentials. For $(\varphi^4)_{2,c}$ the argument goes as follows. Set

$$\omega_c(p)=(p^2c^2+m^2c^4)^{1/2} \qquad p\in\mathbb{R}^1$$

$$\varphi_c(x)=(2\pi)^{-1/2}\int e^{-ipx}c(2\omega_c(p))^{-1/2}(a^*(p)+a(-p))dp\ ,$$

where m is the single particle mass and a^*, a are the usual creation and annihilation operators. The Hamiltonian for the theory has the form

$$H_c=\int a^*(p)\omega_c(p)a(p)dp+\lambda\int :\varphi_c^4(x):dx\ .$$

As $c\to\infty$ all creation and annihilation processes are somehow kinematically suppressed. If we also ignore the "zitterbewegung" term mc^2 in $\omega_c(p)=mc^2+(2m)^{-1}p^2+\mathcal{O}(c^{-2})$, then in some vague sense we have

$$H_\infty=\int a^*(p)(2m)^{-1}p^2a(p)dp+\frac{1}{2}\left(\frac{3\lambda}{m^2}\right)\int a^*(x)a^*(y)\delta(x-y)a(x)a(y)dxdy\ .$$

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This corresponds to non-relativistic bosons interacting with a two body potential $V(x)=3\lambda m^{-2}\delta(x)$.

In trying to establish precise results one must decide for which objects in the theory the limit $c\to\infty$ should exist. It is evident that $\lim_{c\to\infty} H_c = H_\infty$ is too much to ask for. On the other hand, at least the physically measurable quantities should have the correct non-relativistic limit. This is essentially what we show, but restricted to two particle interactions.

The main result is the following. Let $\mathscr{P}^{\pm}(\varphi)=\lambda(\mathscr{R}(\varphi)\pm\varphi^4)$ where $\mathscr{R}$ is an even polynomial with no second or fourth order terms.

Theorem. *The two particle scattering amplitude and the two particle binding energies for the $\mathscr{P}^{\pm}(\varphi)_{2,c}$ quantum field theory converge to the corresponding objects for a $\pm 3\lambda/m^2\delta(x)$ potential as $c\to\infty$.*

The proof of these results depends on the fact that for c large the dimensionless coupling constant $\lambda/m^2 c$ is small, and so we are in the weak coupling regime which is relatively well understood [9, 2, 6]. In particular one has the Bethe-Salpeter equation at one's disposal [16, 8, 17, 4]. The results essentially follow by showing that the Bethe-Salpeter equation (one might better say Bethe-Salpeter identity) converges to the resolvent identity. To obtain this one must shift energies by mc^2 and restrict to wave functions independent of relative energy (i.e. depending only on relative momentum).

The plan of attack is the following. In Section II we define the non-relativistic model. In Section III we develop the weakly coupled $\mathscr{P}(\varphi)_2$ model with $c=1$. The results here are the basis for the study of the large c $\mathscr{P}(\varphi)_{2,c}$ models in Section IV.

II. The Non-relativistic Model

In this section we define non-relativistic quantum mechanics for a δ function potential. To describe two spinless bosons of mass m in a world with one space dimension we take for the Hilbert space $L_2^+(\mathbb{R}^1)$, where $\mathbb{R}^1$ corresponds to relative momentum and L_2^+ means even functions in L_2 corresponding to Bose statistics. The Hamiltonian has the form $H=H_0+V$ where H_0 is multiplication by p^2/m (the reduced mass is $m/2$) and V denotes a potential function $V(p)$ and also the bilinear form with kernel $(2\pi^{-1/2}V(p+q)$[1]. We are concerned with the case of constant V, and take $V=V_\alpha$ with $V_\alpha(p)=(2\pi)^{-1/2}\alpha$, $\alpha\in\mathbb{R}^1$. This corresponds to multiplication by $\alpha\delta(x)$ in configuration space.

As is well known $H_\alpha=H_0+V_\alpha$ defines a self-adjoint operator on $L_2^+(\mathbb{R}^1)$ (e.g. [7, 15]). This can be approached as follows. Consider the Hilbert spaces

$$\mathscr{H}=L_2^+(\mathbb{R}^1,(p^2+1)^{-1}dp)$$
$$\mathscr{H}^*=L_2^+(\mathbb{R}^1,(p^2+1)dp)$$

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[1] A tempered distribution $\mathscr{O}(p,q)\in\mathscr{S}'(\mathbb{R}^2)$ is said to be the kernel of the continuous bilinear form $\mathscr{O}$ on $\mathscr{S}(\mathbb{R}^1)\times\mathscr{S}(\mathbb{R}^1)$ given by

$$\langle\chi,\mathscr{O}\psi\rangle=\int\bar{\chi}(p)\mathscr{O}(p,q)\psi(q)dpdq\,.$$

By the nuclear theorem any such bilinear form has a unique kernel which are dual with the pairing given by the Lebesgue inner product. Then both H_0 and V_α define bounded symmetric bilinear forms in $\mathscr{H}^* \times \mathscr{H}^*$ and hence operators in $\mathscr{L}(\mathscr{H}^*, \mathscr{H})$. Thus $H_\alpha = H_0 + V_\alpha$ is well defined in $\mathscr{L}(\mathscr{H}^*, \mathscr{H})$. Furthermore V_α is a small form perturbation of H_0, and so H_α restricted to $\{\psi \in \mathscr{H}^* : H_\alpha \psi \in L_2^+(\mathbb{R}^1)\}$ is a self-adjoint operator.

The binding energies $E<0$ are the eigenvalues of H_α on $L_2^+(\mathbb{R}^1)$. These coincide with the eigenvalues of H_α on $\mathscr{H}^*$ and hence with the solutions of the implicit eigenvalue problem on $\mathscr{H}$

$$V_\alpha(H_0 - E)^{-1}\psi = -\psi .$$

The operator $V_\alpha(H_0 - E)^{-1} \in \mathscr{L}(\mathscr{H})$ is compact; in fact it is rank one with range equal to the constant functions. For $\psi =$ constant we have

$$V_\alpha(H_0 - E)^{-1}\psi = K_\alpha(E)\psi \qquad K_\alpha(E) = \frac{\alpha}{2} m^{1/2}(-E)^{-1/2} .$$

Thus E is an eigenvalue if and only if $K_\alpha(E) = -1$. If α is positive there are no solutions, while if α is negative there is the unique solution

$$E_B(\alpha) = -\tfrac{1}{4}\alpha^2 m .$$

We now note the resolvent identity in $\mathscr{L}(\mathscr{H}, \mathscr{H}^*)$

$$(H_\alpha - E)^{-1} = (H_0 - E)^{-1}(1 + V_\alpha(H_0 - E)^{-1})^{-1}$$

valid in the cut plane $\{E \in \mathbb{C} : E \notin \mathbb{R}^+, E \neq E_B(\alpha) \text{ if } \alpha < 0\}$ (Fredholm theorem). We also define the T operator $\mathbb{T}_\alpha(E) \in \mathscr{L}(\mathscr{H}^*, \mathscr{H})$ in the same region by

$$\mathbb{T}_\alpha(E) = (1 + V_\alpha(H_0 - E)^{-1})^{-1} V_\alpha .$$

Actually we have $\mathbb{T}_\alpha(E) = (1 + K_\alpha(E))^{-1} V_\alpha$ and so $\mathbb{T}_\alpha(E)$ can be analytically continued across the cut onto a two sheeted manifold. For $\alpha > 0$ there is a pole on the second sheet at $E = E_B(\alpha)$. The kernel $\mathbb{T}_\alpha(E, p, q)$ has the same analyticity in E, and is constant in p, q:

$$\mathbb{T}_\alpha(E, p, q) = (1 + K_\alpha(E))^{-1}(2\pi)^{-1}\alpha .$$

Finally we consider the scattering operator $\mathbb{S}_\alpha$ on $L_2^+(\mathbb{R}^1, dp)$. According to the Lipmann-Schwinger equation the kernel of $\mathbb{S}_\alpha$ is given by

$$\mathbb{S}_\alpha(p, q) = \delta(p - q) - 2\pi i \mathbb{T}_\alpha\left(\frac{p^2}{m} + i0^+, p, q\right)\delta\left(\frac{p^2}{m} - \frac{q^2}{m}\right) .$$

The verification of this equation as an identity in $\mathscr{S}'(\mathbb{R}^2)$ away from $p = 0$ for a class of potentials including the δ function will be presented elsewhere (for similar results see [13, 19]). For the present we take this as the definition of $\mathbb{S}_\alpha$. We further define

$$k_\alpha(p) = K_\alpha\left(\frac{p^2}{m} + i0^+\right) = \tfrac{1}{2} i\alpha m |p|^{-1} .$$

For even test functions $\delta(p - q) = \delta(p + q)$ and so

$$\delta(p^2/m - q^2/m) = \frac{m}{|p|}\delta(p - q) .$$

Thus away from $p=0$ we have

$$\mathbb{S}_\alpha(p,q)=\left(\frac{1-k_\alpha(p)}{1+k_\alpha(p)}\right)\delta(p-q)\,.$$

Scattering consists of a phase shift.

III. Weakly Coupled $\mathscr{P}(\varphi)_2$ Models

III.1. The Models

A $\mathscr{P}(\varphi)_2$ model for a self-interacting boson field may be defined in terms of its Schwinger functions $\mathfrak{S}=\mathfrak{S}_{\lambda,m,\sigma}$ which are formally given by

$$\mathfrak{S}(x_1,\ldots,x_n)=\frac{\int q(x_1)\ldots q(x_n)\exp(-\int :\mathscr{P}(q(x)):dx)d\mu(q)}{\int \exp(-\int :\mathscr{P}(q(x)):dx)d\mu(q)}\,, \tag{3.1}$$

where $q\in\mathscr{S}'(\mathbb{R}^2)$, $d\mu=d\mu_m$ is the Gaussian measure with mean zero and covariance $(-\Delta+\mathrm{m}^2)^{-1}$ and $\mathscr{P}=\mathscr{P}^{\pm}_{\lambda,\sigma}$ is an even polynomial of the form

$$\begin{aligned}&\mathscr{P}^{\pm}_{\lambda,\sigma}(q)=\lambda(\mathscr{R}(q)\pm q^4)+\sigma^2q^2\\&\mathscr{R}(q)=\sum_{n=3}^{N}a_{2n}q^{2n}\,,\qquad a_{2N}>0\,.\end{aligned} \tag{3.2}$$

With $+q^4$ we also allow $\mathscr{R}=0$. We do not consider polynomials lacking a quartic term (which are trivial for our purposes).

The Schwinger functions $\mathfrak{S}_{\lambda,m,\sigma}\in\mathscr{S}'(\mathbb{R}^{2n})$ may be constructed using the cluster expansion of Glimm et al. [9] provided λ/m^2 and σ/m are sufficiently small. By analytic continuation one obtains a family of Wightman distributions $\mathscr{W}_{\lambda,m,\sigma}$ satisfying the Wightman axioms and by reconstruction a quantum field theory [18, 14]. The energy-momentum spectrum has isolated single particle states of mass $m_*=m_*(\lambda,m,\sigma)$. We make a finite mass renormalization, taking $\sigma=\sigma_*(\lambda)$ so $m=m_*(\lambda,m,\sigma_*(\lambda))$ [6]. Then (m,σ) are supressed, writing $\mathfrak{S}_\lambda=\mathfrak{S}_{\lambda,m,\sigma_*(\lambda)}$, etc. The truncated Schwinger function has a Fourier transform of the form

$$\tilde{\mathfrak{S}}^T_\lambda(p_1,\ldots,p_n)=\delta(\textstyle\sum p_i)\mathring{H}_\lambda(p_1,\ldots,p_n)\,, \tag{3.3}$$

where $\mathring{H}_\lambda$ is a bounded real analytic function in $\left\{p\in\mathbb{R}^{2n}:\sum_{i=1}^{n}p_i=0\right\}$. (Here and in the following, "$\circ$" means "Euclidean".)

III.2. The Bethe-Salpeter Equation

We now discuss the (Wick-rotated) Bethe-Salpeter equation, mostly following Spencer and Zirilli [17]. We define $S_\lambda(p)=\mathring{H}_\lambda(p,-p)$ and

$$\begin{aligned}&\mathring{Q}_\lambda(k,p,q)=(2\pi)^{-1}S_\lambda\left(p+\frac{k}{2}\right)S_\lambda\left(-p+\frac{k}{2}\right)(\delta(p+q)+\delta(p-q))\\&\mathring{H}_\lambda(k,p,q)=(2\pi)^{-1}\mathring{H}_\lambda\left(p+\frac{k}{2},\,-p+\frac{k}{2},\,-q-\frac{k}{2},\,+q-\frac{k}{2}\right)\\&\mathring{R}_\lambda(k,p,q)=\mathring{Q}_\lambda(k,p,q)+\mathring{H}_\lambda(k,p,q)\,.\end{aligned} \tag{3.4}$$

Then $\mathring{R}_\lambda$ is the four point function truncated only in the (1, 2), (3, 4) channel, k is a center of mass variable for this channel, and (p, q) are relative variables. By $\mathring{Q}_\lambda(k)$, $\mathring{H}_\lambda(k)$, $\mathring{R}_\lambda(k)$ we denote bilinear forms with kernels $\mathring{Q}_\lambda(k, p, q)$, etc. We are mainly concerned with $Q_\lambda(\varkappa) \equiv \mathring{Q}_\lambda((i\varkappa, 0))$, $H_\lambda(\varkappa) \equiv \mathring{H}_\lambda((i\varkappa, 0))$, $R_\lambda(\varkappa) \equiv \mathring{R}_\lambda((i\varkappa, 0))$, defined initially for $\varkappa$ imaginary.

Consider the Hilbert spaces

$$\begin{aligned} \mathscr{K} &= L_2^+(\mathbb{R}^2, (p^2+1)^{-2}dp) \\ \mathscr{K}^* &= L_2^+(\mathbb{R}^2, (p^2+1)^{2}dp) \,. \end{aligned} \tag{3.5}$$

For λ sufficiently small the Lehman spectral formula for the two point function takes the form [9]

$$S_\lambda(p) = Z_\lambda^2 (p^2+m^2)^{-1} + \int_{(3m-\varepsilon)^2}^{\infty} (p^2+a^2)^{-1} d\varrho_\lambda(a) \tag{3.6}$$

and it follows that $Q_\lambda(\varkappa)$ defines a bounded bilinear form on $\mathscr{K} \times \mathscr{K}$, even for $|\mathrm{Re}\varkappa| < 2m$. By integration by parts in the functional integral (3.1) [8], one may also show that for $\mathrm{Re}\varkappa = 0$, $H_\lambda(\varkappa)$ defines a bilinear form on $\mathscr{K} \times \mathscr{K}$, and hence so does $R_\lambda(\varkappa)$. Corresponding to the forms we have operators $Q_\lambda(\varkappa)$, $H_\lambda(\varkappa)$, $R_\lambda(\varkappa)$ in $\mathscr{L}(\mathscr{K}, \mathscr{K}^*)$.

It is straightforward that $Q_\lambda(\varkappa)^{-1}$ exists and is in $\mathscr{L}(\mathscr{K}^*, \mathscr{K})$. We also have $\|H_\lambda(\varkappa)\| \leqq \mathcal{O}(\lambda)$ and so $R_\lambda(\varkappa)^{-1}$ exists for λ sufficiently small. Thus we may define $K_\lambda(\varkappa) \in \mathscr{L}(\mathscr{K}^*, \mathscr{K})$ by

$$K_\lambda(\varkappa) \equiv R_\lambda(\varkappa)^{-1} - Q_\lambda(\varkappa)^{-1} \tag{3.7}$$

and then we have the Bethe-Salpeter equation

$$R_\lambda(\varkappa) = Q_\lambda(\varkappa) - R_\lambda(\varkappa) K_\lambda(\varkappa) Q_\lambda(\varkappa) \,. \tag{3.8}$$

Spencer [16] shows that for λ sufficiently small, the kernel $K_\lambda(\varkappa, p, q)$ of $K_\lambda(\varkappa)$ is analytic and bounded in

$$\begin{aligned} &|\mathrm{Re}\varkappa| < 3m - \varepsilon \\ &|\mathrm{Im}\mathrm{p}_0|, |\mathrm{Im}\mathrm{q}_0| < \tfrac{3}{4}m - \varepsilon \equiv \delta_0 \\ &|\mathrm{Im}\mathrm{p}_1|, |\mathrm{Im}\mathrm{q}_1| < \tfrac{1}{4}m - \varepsilon \equiv \delta_1 \,. \\ &\varepsilon > 0 \end{aligned} \tag{3.9}$$

(Note: our (p, q) variables are half those of [16].)

Furthermore in the same domain, $K_\lambda(\varkappa, p, q)$ is C^∞ in $\lambda \geqq 0$ and the coefficients of the asymptotic series in λ are the usual two particle irreducible diagrams [4]. In first order there is one diagram, and for $\mathscr{P} = \mathscr{P}^\pm$ we have

$$K_\lambda(\varkappa, p, q) = \pm \frac{3\lambda}{\pi} + \mathcal{O}(\lambda^2) \,. \tag{3.10}$$

As a consequence of the analyticity, the operator $K_\lambda(\varkappa)$ has an analytic continuation to $|\mathrm{Re}\varkappa| < 2m$. Furthermore $(KQ)_\lambda(\varkappa) \equiv K_\lambda(\varkappa) Q_\lambda(\varkappa)$ is compact and analytic in this region. Therefore the implicit eigenvalue problem $(KQ)_\lambda(\varkappa)\psi = -\psi$ has solutions at only a discrete set of points, and the identity

$$R_\lambda(\varkappa) = Q_\lambda(\varkappa)(1 + (KQ)_\lambda(\varkappa))^{-1} \tag{3.11}$$

provides a meromorphic continuation of $R_\lambda(\varkappa)$ to $|\mathrm{Re}\,\varkappa| < 2m$ (Fredholm theorem). The poles of $R_\lambda(\varkappa)$ (= implicit eigenvalues) contain all two particle bound state masses [17].

At this point we remark that it is not necessary to stick with the Hilbert space $\mathscr{K} = L_2^+(\mathbb{R}^2, (p^2+1)^{-2}dp)$. Instead we could take, for example, the smaller spaces $\mathscr{K} = \mathscr{K}_\alpha$,

$$\mathscr{K}_\alpha = L_2^+(\mathbb{R}^2, (p^2+1)^{-\alpha}dp) \qquad 1 < \alpha < 2\,. \tag{3.12}$$

One easily shows that with new $\mathscr{K}$, $Q_\lambda(\varkappa)$ restricts to an element of $\mathscr{L}(\mathscr{K}, \mathscr{K}^*)$ that $K_\lambda(\varkappa)$ extends to an element of $\mathscr{L}(\mathscr{K}^*, \mathscr{K})$ (since the kernel is bounded), that $(KQ)_\lambda(\varkappa) \in \mathscr{L}(\mathscr{K})$ is compact with the same eigenvalues, and that $R_\lambda(\varkappa)$ restricted to $\mathscr{L}(\mathscr{K}, \mathscr{K}^*)$ is given by (3.11). [However we do not have $Q_\lambda^{-1} \in \mathscr{L}(\mathscr{K}^*, \mathscr{K})$.] Another possible choice we will use is

$$\mathscr{K} = L_2^+(\mathbb{R}^2, \pi^{-1}(p_0^2 + (p_1^2+1)^2)^{-1}dp) \tag{3.13}$$

One can also take $\mathscr{K} = L_2^+(\mathbb{R}^2, dp)$, however $K_\lambda(\varkappa)$ is no longer a bounded operator (as was erroneously stated in [3]).

We also consider the Sobolev-Hardy space A [17], consisting of even functions on $\mathbb{R}^2$ which have analytic continuations to the tube $\mathbb{R}^2 + iI$ where $I = (-\delta_0, \delta_0) \times (-\delta_1, \delta_1)$ and satisfying

$$\begin{aligned} \|\psi\| &= \sup_{\alpha \in I} \left[\int |w(p+i\alpha)\psi(p+i\alpha)|^2 dp\right]^{1/2} < \infty \\ w(p) &= (p^2 + 16m^2)^{-2/3}\,. \end{aligned} \tag{3.14}$$

We have the topological inclusions $Z \subset A \subset \mathscr{K}_{4/3} \subset \mathscr{S}'$ [where $Z = C_0^\infty(\mathbb{R}^2)^\sim$] and hence $\mathscr{S} \subset \mathscr{K}_{4/3}^* \subset A^* \subset Z'$. Using the boundedness and analyticity of $K_\lambda(\varkappa, p, q)$ one can show that $K_\lambda(\varkappa)$ extends to $\mathscr{L}(A^*, A)$. Furthermore $Q_\lambda(\varkappa) \in \mathscr{L}(A, A^*)$, $(KQ)_\lambda(\varkappa) \in \mathscr{L}(A)$ is compact with the same eigenvalues, and $R_\lambda(\varkappa) \in \mathscr{L}(A, A^*)$ and is given by (3.11).

For λ sufficiently small, the eigenvalue problem $(KQ)_\lambda(\varkappa)\psi = -\psi$ on A has been solved by the author and Eckmann [4]. For $\mathscr{P} = \mathscr{P}^+$ there are no solutions, while for $\mathscr{P} = \mathscr{P}^-$ there is one solution $\varkappa = m_B(\lambda)$ which is C^∞ in $\lambda \geqq 0$ and has the expansion

$$m_B(\lambda) = 2m - \frac{9}{4}\frac{\lambda^2}{m^3} + \mathcal{O}(\lambda^4)\,. \tag{3.15}$$

Correspondingly the $\mathscr{P}^+$ field theories have no bound states and the $\mathscr{P}^-$ field theories have one bound state of mass $m_B(\lambda)$.

III.3. The T-Operator

We now define an operator which will turn out to play a role analagous to the non-relativistic $\mathbb{T}_\alpha$. Let $\mathring{H}'_\lambda$ be the amputated Euclidean n-point function

$$\mathring{H}'_\lambda(p_1,\dots,p_n)=\prod_j (p_j^2+m^2)\mathring{H}_\lambda(p_1,\dots,p_n) \tag{3.16}$$

and

$$\mathring{T}_\lambda(k,p,q)=-(2\pi)\mathring{H}'_\lambda\left(p+\frac{k}{2},-p+\frac{k}{2},-q-\frac{k}{2},q-\frac{k}{2}\right). \tag{3.17}$$

By integration by parts $\mathring{T}_\lambda(k,p,q)$ is a bounded function and we let $\mathring{T}_\lambda(k)$ be the associated form on $\mathscr{K}^*\times\mathscr{K}^*$ [$\mathscr{K}$ given by (3.5)]. For $\varkappa$ imaginary we set $T_\lambda(\varkappa)=\mathring{T}_\lambda((i\varkappa,0))$.

Lemma 3.1. *$T_\lambda(\varkappa)\in\mathscr{L}(\mathscr{K}^*,\mathscr{K})$ is meromorphic in $|\mathrm{Re}\varkappa|<2m$ and is given by*

$$T_\lambda(\varkappa)=4(Q_0^{-1}Q_\lambda)(\varkappa)(1+(KQ)_\lambda(\varkappa))^{-1}K_\lambda(\varkappa)(Q_\lambda Q_0^{-1})(\varkappa). \tag{3.18}$$

Proof. It suffices to prove the identity for $\mathrm{Re}\varkappa=0$, then the right side provides the continuation with poles at implicit eigenvalues of $(KQ)_\lambda(\varkappa)$. We note that $\mathring{Q}_0(k)$ is multiplication by $\mathring{Q}_0(k,p)$ where

$$\mathring{Q}_0(k,p)=\pi^{-1}\left(\left(p-\frac{k}{2}\right)^2+m^2\right)^{-1}\left(\left(p+\frac{k}{2}\right)^2+m^2\right)^{-1}.$$

Hence we have $\mathring{T}_\lambda(k)=-4(\mathring{Q}_0^{-1}\mathring{H}_\lambda\mathring{Q}_0^{-1})(k)$ and hence $T_\lambda(\varkappa)=-4(Q_0^{-1}H_\lambda Q_0^{-1})(\varkappa)$. However since $H_\lambda(\varkappa)=R_\lambda(\varkappa)-Q_\lambda(\varkappa)$ we have

$$\begin{aligned}H_\lambda(\varkappa)&=Q_\lambda(\varkappa)((1+(KQ)_\lambda(\varkappa))^{-1}-1)\\&=-Q_\lambda(\varkappa)(1+(KQ)_\lambda(\varkappa))^{-1}(KQ)_\lambda(\varkappa)\quad\text{Q.E.D.}\end{aligned}$$

Lemma 3.2. (a) *$(Q_0^{-1}Q_\lambda)(\varkappa)\in\mathscr{L}(\mathscr{K})$ is multiplication by a function $(Q_0^{-1}Q_\lambda)(\varkappa,p)$ which is analytic and bounded in $|\mathrm{Re}\varkappa|<3m-\varepsilon$, $|\mathrm{Im}p_0|<\frac{3}{4}m-\varepsilon$, $|\mathrm{Im}p_1|<\frac{1}{4}m-\varepsilon$.*

(b) $\lim_{\lambda\to 0}(Q_0^{-1}Q_\lambda)(\varkappa,p)=1$ *uniformly in this region.*

Proof. We have

$$(\mathring{Q}_0^{-1}\mathring{Q}_\lambda)(k,p)=D_\lambda\left(p+\frac{k}{2}\right)D_\lambda\left(p-\frac{k}{2}\right), \tag{3.19}$$

where

$$\begin{aligned}D_\lambda(p)&=(p^2+m^2)S_\lambda(p)\\&=Z_\lambda^2+\int_{3m-\varepsilon}^{\infty}(p^2+m^2)(p^2+a^2)^{-1}d\varrho_\lambda(a).\end{aligned} \tag{3.20}$$

Thus $(Q_0^{-1}Q_\lambda)(\varkappa,p)=(\mathring{Q}_0^{-1}\mathring{Q}_\lambda)((i\varkappa,0),p)$ depends on $D_\lambda\left(p_0\pm\frac{i\varkappa}{2},p_1\right)$. This is analytic and bounded in the stated region since the denominator is bounded away from zero. The convergence follows from $Z_\lambda\to 1$, $\varrho_\lambda\to 0$. Q.E.D.

Corollary 3.3. *Lemma 3.1 holds for any of the spaces $\mathscr{K}, A$ given by* (3.12), (3.13), (3.14).

Proof. Lemma 3.2a shows that $(Q_0^{-1}Q_\lambda)(\varkappa)$ restricts to $\mathscr{L}(\mathscr{K})$ or $\mathscr{L}(A)$. The other operators are treated similarly. Q.E.D.

In the next lemma we explore the analytic structure of the kernel $T_\lambda(\varkappa, p, q)$ near the threshold $(2m, 0, 0)$ and find that it is meromorphic in $\varkappa$ on a two sheeted domain with branch point at $\varkappa = 2m$.

Lemma 3.4. *$T_\lambda(\varkappa, p, q)$ has the form $T_\lambda(\varkappa, p, q) = \hat{T}_\lambda((4m^2-\varkappa^2)^{1/2}, p, q)$ where $\hat{T}_\lambda(\zeta, p, q)$ is meromorphic in $|\zeta| < \frac{m}{8}$ and analytic in $|p|, |q| < m/8$. Furthermore, let $\zeta_B(\lambda) = (4m - m_B(\lambda)^2)^{1/2}$. Then we have $\hat{T}_\lambda(\zeta, p, q) = U_\lambda(\zeta, p, q) + V_\lambda(\zeta, p, q)$ where $U_\lambda(\zeta, p, q)$ and $(\zeta \pm \zeta_B(\lambda))V_\lambda(\zeta, p, q)$ (for $\mathscr{P} = \mathscr{P}^\pm$) are analytic and bounded in $|\zeta|, |p|, |q| < m/8$ with constants which are respectively $\mathcal{O}(\lambda)$, $\mathcal{O}(\lambda^2)$.*

Proof. Consider all operators relative to the A, A^* pairing. For $f \in A$, and $p \in \mathbb{R}^2 + iI$ define $\langle \varepsilon_p, f\rangle = f(p)$. Then $\varepsilon_p \in A^*$, ε_p is analytic in $\mathbb{R}^2 + iI$ and for $g \in \mathscr{S} \subset A^*$ we have $\int \bar{g}(p)\langle \varepsilon_p, f\rangle dp = \langle g, f\rangle$. Now we claim that for $|\mathrm{Re}\varkappa| < 2m$ (except a discrete set) and $p, q \in \mathbb{R}^2$

$$T_\lambda(\varkappa, p, q) = 4(Q_0^{-1}Q_\lambda)(\varkappa, p)\langle \varepsilon_p, (1+(KQ)_\lambda(\varkappa))^{-1}K_\lambda(\varkappa)\varepsilon_q\rangle \cdot (Q_0^{-1}Q_\lambda)(\varkappa, q)\,. \tag{3.21}$$

This is true because it holds in the sense of distributions by Lemma 3.1. This equation provides a continuation of $T_\lambda(\varkappa, p, q)$ to $|\mathrm{Re}\varkappa| < 2m$, $p, q \in \mathbb{R}^2 + iI$. In fact every factor except $(1+(KQ)_\lambda(\varkappa))^{-1}$ also continues to $|\mathrm{Re}\varkappa| < 3m - \varepsilon$ and hence in terms of $\zeta = (4m^2 - \varkappa^2)^{1/2}$ is analytic in $|\zeta| < \frac{m}{8}$. Furthermore these terms are bounded in $|\zeta|, |p|, |q| < m/8$ ($\|\varepsilon_p\|$ is bounded on compact sets) and we have a factor of λ from $\|K_\lambda(\varkappa)\| \leq \mathcal{O}(\lambda)$.

It remains to consider the factor $(1+(KQ)_\lambda\hat{}(\zeta))^{-1} \in \mathscr{L}(A)$ where $(KQ)_\lambda\hat{}(\zeta) = (KQ)_\lambda((4m^2-\zeta^2)^{1/2})$. In [4] it is shown that $(KQ)_\lambda\hat{}(\zeta) = \tau_{1,\lambda}(\zeta) + \tau_{2,\lambda}(\zeta)$ where $\tau_{1,\lambda}(\zeta)$ is a rank one operator with a pole at $\zeta = 0$ and satisfies $\|\zeta\tau_{1,\lambda}(\zeta)\| \leq \mathcal{O}(\lambda)$ while $\tau_{2,\lambda}(\zeta)$ is analytic near zero and satisfies $\|\tau_{2,\lambda}(\zeta)\| \leq \mathcal{O}(\lambda)$. Thus in

$$(1+KQ)^{-1} = (1+(1+\tau_2)^{-1}\tau_1)^{-1}(1+\tau_2)^{-1}$$

we may focus attention on the first factor. Since $(1+\tau_2)^{-1}\tau_1$ is rank one we have

$$(1+(1+\tau_2)^{-1}\tau_1)^{-1} = 1 - (1+\mathrm{Tr}(1+\tau_2)^{-1}\tau_1)^{-1}(1+\tau_2)^{-1}\tau_1$$

and this defines the division into U_λ, V_λ. We have immediately $|U_\lambda(\zeta, p, q)| \leq \mathcal{O}(\lambda)$. For the second term multiply the numerator and denominator by ζ. Then the numerator $\zeta(1+\tau_2)^{-1}\tau_1$ is holomorphic and bounded by $\mathcal{O}(\lambda)$ for a second factor of λ. The denominator $\zeta(1+\mathrm{Tr}((1+\tau_2)^{-1}\tau_1)$ is the function $H(\lambda, \zeta)$ of [4] which has a simple zero at $\zeta = \mp\zeta_B(\lambda)$ and satisfies $|(\zeta \pm \zeta_B(\lambda))H(\lambda, \zeta)^{-1}| \leq \mathcal{O}(1)$. Thus we have $|(\zeta \pm \zeta_B(\lambda))V_\lambda(\zeta, p, q)| \leq \mathcal{O}(\lambda^2)$. Q.E.D.

III.4. The S-Operator

We now consider the real time aspects of $\mathscr{P}(\varphi)_2$ theories. It is known that time ordered products $\tau_\lambda(p_1,\ldots,p_n)\in\mathscr{S}'(\mathbb{R}^{2n})$ and retarded products exist for these models, and hence so does the momentum analytic function $H_\lambda(k_1,\ldots,k_n)$, defined on the "axiomatic domain" in $\sum k_j=0$, whose boundary values are locally the $\tau_\lambda(p_1,\ldots,p_n)$. At Euclidean points these are the Schwinger functions [6]:

$$(i)^{n-1}H_\lambda(\mathring{p}_1,\ldots,\mathring{p}_n)=\mathring{H}_\lambda(p_1,\ldots,p_n)\,, \tag{3.22}$$

where for $p=(p_0,p_1)\in\mathbb{R}^2$ we denote $\mathring{p}=(ip_0,p_1)$. We also consider the amputated functions τ'_λ which are the boundary values of

$$H'_\lambda(k_1,\ldots,k_n)=\prod_j(k_j\cdot k_j-m^2)H_\lambda(k_1,\ldots,k_n) \tag{3.23}$$

(here $k\cdot k=k_0^2-k_1^2$). Then H'_λ and $\mathring{H}'_\lambda$ are also related by an equation like (3.22).

The LSZ formula [10] gives the scattering operator (S-matrix) in terms of the restriction of τ'_λ to the mass shell. This formula has been used by Eckmann et al. [6] to show that the scattering operator is a C^∞ function of $\lambda\geqq 0$ and hence that standard perturbation theory is asymptotic. We remark that in general for the LSZ formula one must require all velocities to be non-overlapping. However for two particle scattering it is sufficient to require that the initial and final velocities be separately non-overlapping [1]. This is fortunate since, as noted in the non-relativistic case, we are kinematically constrained to forward scattering on one spatial dimension.

In detail, let $\psi_\pm$ be the canonical injections of the Fock space

$$\mathscr{F}=\bigoplus_{n=0}^{\infty}[\otimes_s^n L_2(\mathbb{R}^1,dp)]$$

(note: Lesbesgue measure) into the physical Hilbert Space as given by the Haag-Ruelle scattering theory. Let Π be the projection of $L_2(\mathbb{R}^2,dp)$ onto $L_2(\mathbb{R}^1,dp)\otimes_s L_2(\mathbb{R}^1,dp)\subset\mathscr{F}$. We define the kernel of the S-matrix $S_\lambda\in\mathscr{S}'(\mathbb{R}^4)$ by

$$(\psi_+(\Pi g),\psi_-(\Pi f))=\int\bar{g}(p_1,p_2)S_\lambda(p_1,p_2,p_3,p_4)f(p_3,p_4)dp_1,\ldots,dp_4\,. \tag{3.24}$$

Then the LSZ formula says that away from $p_1=p_2$ and $p_3=p_4$ we have [with $\omega(p)=(p^2+m^2)^{1/2}$]

$$\begin{aligned}S_\lambda(p_1,\ldots,p_4)=(2!)^{-1}[&\delta(p_1-p_3)\delta(p_2-p_4)+\delta(p_1-p_4)\delta(p_2-p_3)\\&+Z_\lambda^{-4}(2\pi)^2\prod_j(2\omega(p_j))^{-1/2}\tau'_\lambda(\omega(p_1),p_1,\ldots,-\omega(p_4),-p_4)\\&\cdot\delta(p_1+p_2-p_3-p_4)\delta(\omega(p_1)+\ldots-\omega(p_4))]\,.\end{aligned} \tag{3.25}$$

We now restrict to small momenta in this formula. For the time ordered product this means we are interested in a center of mass energy $\varkappa$ in an interval $(2m,2m+\varepsilon)$ and all other momenta in a neighborhood of zero. In such a region it follows from Lemma 3.4 that the distribution τ'_λ is actually an analytic function. To see this consider $\tau'_\lambda(k, p, q) = \tau'_\lambda\left(p + \frac{k}{2}, \ldots\right)$, the boundary value of $H'_\lambda(k, p, q) = H'_\lambda\left(p + \frac{k}{2}, \ldots\right)$, and note that by Lorentz invariance it suffices to consider $\tau'_\lambda((\varkappa, 0), \mathrm{p}, \mathrm{q})$. Then by analytically continuing (3.17) we have

$$\begin{aligned} \tau'_\lambda((\varkappa, 0), p, q) &= H'_\lambda((\varkappa + i0^+, 0), p, q) \\ &= -i(2\pi)^{-1} T_\lambda(\varkappa + i0^+, (ip_0, p_1), (iq_0, q_1)) \,. \end{aligned} \tag{3.26}$$

To compare the scattering amplitude with the non-relativistic formula, we shift to center of mass and relative variables in S_λ, defining $S_\lambda(k, p; k', q)$ by $S_\lambda(k, p; k', q) = S_\lambda\left(p + \frac{k}{2}, \ldots, -q + \frac{k'}{2}\right)$. Then $S_\lambda(k, p; k', q) = S_\lambda(k, p, q)\delta(k - k')$ and we consider the center of mass at rest defining $S_\lambda(p, q) \in \mathscr{S}'(\mathbb{R}^2)$ by $S_\lambda(p, q) = S_\lambda(0, p, q)$. Then by (3.25), (3.26), for (p, q) small and away from zero

$$\begin{aligned} S_\lambda(p, q) = \delta(p - q) - 2\pi i Z_\lambda^{-4}(8\omega(p)\omega(q))^{-1} \\ \cdot T_\lambda(2\omega(p) + i0^+, (0, p), (0, q))\delta(2\omega(p) - 2\omega(q)) \,. \end{aligned} \tag{3.27}$$

IV. $\mathscr{P}(\varphi)_{2,c}$ Models as $c \to \infty$

IV.1. The Models

We define the $\mathscr{P}(\varphi)_{2,c}$ models in terms of their Schwinger functions $\mathfrak{S}_{\lambda, m, \sigma, c}$. These are given by a functional integral like (3.1) except that now $d\mu = d\mu_{m,c}$ is the Gaussian measure on $\mathscr{S}'(\mathbb{R}^2)$ with covariance

$$\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + m^2 c^2\right)^{-1} \tag{4.1}$$

and $\mathscr{P} = \mathscr{P}^\pm_{\lambda, \sigma, c}$ is the polynomial

$$\begin{aligned} \mathscr{P}^\pm_{\lambda, \sigma, c}(q) &= \lambda(\mathscr{R}_c(q) \pm q^4) + \sigma^2 c^2 q^2 \\ \mathscr{R}_c(q) &= \sum_{n=3}^{N} c^{-n+2} a_{2n} q^{2n} \,. \end{aligned} \tag{4.2}$$

With this choice we have the scaling relation (at least formally)

$$\begin{aligned} &\mathfrak{S}_{\lambda, m, \sigma, c}(t_1, x_1, \ldots, t_n, x_n) \\ &= \alpha^{n/2}\beta^{-n/2}\mathfrak{S}_{\lambda\alpha/\beta^3, m\alpha/\beta^2, \sigma\alpha/\beta^2, c\beta/\alpha}(\alpha t_1, \beta x_1, \ldots) \,. \end{aligned} \tag{4.3}$$

In particular with $\alpha = c^2$, $\beta = c$ we have

$$\begin{aligned} &\mathfrak{S}_{\lambda, m, \sigma, c}(t_1, x_1, \ldots, t_n, x_n) \\ &= c^{n/2}\mathfrak{S}_{\lambda/c, m, \sigma}(c^2 t_1, c x_1, \ldots) \,. \end{aligned} \tag{4.4}$$

For arbitrary (λ, m), c sufficiently large, and σ sufficiently small we take this as the definition of $\mathfrak{S}_{\lambda,m,\sigma,c}$. We further define $\mathfrak{S}_{\lambda,c}=\mathfrak{S}_{\lambda,m,\sigma_*(\lambda/c),c}$ and then

$$\mathfrak{S}_{\lambda,c}(t_1,x_1,\ldots,t_nx_n)=c^{n/2}\mathfrak{S}_{\lambda/c}(c^2t_1,cx_1,\ldots)\ . \tag{4.5}$$

We also define distributions $\mathscr{W}_{\lambda,c}$ by

$$\mathscr{W}_{\lambda,c}(t_1,x_1,\ldots,t_n,x_n)=c^{n/2}\mathscr{W}_{\lambda/c}(c^2t_1,cx_1,\ldots)\ . \tag{4.6}$$

Then the $\mathscr{W}_{\lambda,c}$ are the analytic continuations of $\mathfrak{S}_{\lambda,c}$ and satisfy the Wightman axioms for a two-dimensional Minkowski space with quadratic form

$$((t,x)\cdot(t,x))_c=c^2t^2-x^2\ .$$

By reconstruction we obtain a $\mathscr{P}(\varphi)_{2,c}$ quantum field theory. We still have single particles of mass m, i.e. the spectral measure $dE_{\lambda,c}(p_0,p_1)$ has support on the hyperbolas $p_0^2/c^2-p_1^2=mc^2$.

The momentum analytic function $H_{\lambda,c}(k_1,\ldots,k_n)$ and the amputated function $H'_{\lambda,c}(k_1,\ldots,k_n)$ are given by

$$\begin{aligned} H_{\lambda,c}(k_1,\ldots,k_n)&=c^{-5n/2+3}H_{\lambda/c}(k_{1,c},\ldots,k_{n,c})\\ H'_{\lambda,c}(k_1,\ldots,k_n)&=c^{-n/2+3}H'_{\lambda/c}(k_{1,c},\ldots,k_{n,c})\ , \end{aligned} \tag{4.7}$$

where for $k=(k^0,k^1)$ we define

$$k_c=(k^0/c^2,k^1/c)\ . \tag{4.8}$$

If we define $\mathring{Q}_{\lambda,c}, \mathring{R}_{\lambda,c}, \mathring{T}_{\lambda,c}$ in terms of $\mathring{H}_{\lambda,c}, \mathring{H}'_{\lambda,c}$ as before, then

$$\begin{aligned} \mathring{Q}_{\lambda,c}(k,p,q)&=c^{-7}\mathring{Q}_{\lambda/c}(k_c,p_c,q_c)\\ \mathring{R}_{\lambda,c}(k,p,q)&=c^{-7}\mathring{R}_{\lambda/c}(k_c,p_c,q_c)\\ \mathring{T}_{\lambda,c}(k,p,q)&=c\mathring{T}_{\lambda/c}(k_c,p_c,q_c)\ . \end{aligned} \tag{4.9}$$

These are the kernels of operators $\mathring{Q}_{\lambda,c}(k)$, etc. in $\mathscr{L}(\mathscr{K},\mathscr{K}^*)$, and we define $Q_{\lambda,c}(\varkappa)=\mathring{Q}_{\lambda,c}((i\varkappa,0))$, etc. If we further define

$$K_{\lambda,c}(\varkappa,p,q)=cK_{\lambda/c}(\varkappa/c^2,p_c,q_c) \tag{4.10}$$

and let $K_{\lambda,c}(\varkappa)\in\mathscr{L}(\mathscr{K}^*,\mathscr{K})$ be the operator with this kernel, then we have the Bethe-Salpeter equation

$$R_{\lambda,c}(\varkappa)=Q_{\lambda,c}(\varkappa)-R_{\lambda,c}(\varkappa)K_{\lambda,c}(\varkappa)Q_{\lambda,c}(\varkappa)\ .$$

IV.2. The $c\to\infty$ Limit

Now we are ready to discuss the non-relativistic limit. The following four theorems all say that some object for the $\mathscr{P}^{\pm}(\varphi)_{2,c}$ field theory converges to a corresponding object for the $\alpha\delta(x)$ model, $\alpha=\pm 3\lambda/m^2$, as defined in Section II.

Theorem 4.1. *Let c be sufficiently large.*

a) *$(KQ)_{\lambda,c}(\varkappa)\in\mathscr{L}(\mathscr{K})$ is compact and analytic in $|\mathrm{Re}\varkappa|<2mc^2$.*

b) *For $\mathscr{P}=\mathscr{P}^{\pm}$ the eigenvalue equation $(KQ)_{\lambda,c}(\varkappa)\psi=-\psi$ has respectively no solutions or one solution at $\varkappa=m_B(\lambda/c)c^2$.*

c) *The corresponding masses* ($\emptyset$ *or* $\{m_B(\lambda/c)\}$) *coincide with the two particle bound state masses for the* $\mathscr{P}^{\pm}$ *field theory.*

d) *Let* $E_{B,c}(\lambda)=m_B(\lambda/c)c^2-2mc^2$ *be the binding energy for the* $\mathscr{P}^-$ *bound state. Then with* $\alpha=-3\lambda/m^2$

$$\lim_{c\to\infty} E_{B,c}(\lambda)=E_B(\alpha)\ .$$

Proof. Define $\sigma_c\in\mathscr{L}(\mathscr{K})$ by

$$(\sigma_c\psi)(p)=c^{-3/2}\psi(p_c)\ .$$

This operator has a bounded inverse, namely $(\sigma_c)^{-1}=\sigma_{c^{-1}}$. Since

$$(KQ)_{\lambda,c}(\varkappa,p,q)=c^{-3}(KQ)_{\lambda/c}(\varkappa/c^2,p_c,q_c)$$

we have

$$(KQ)_{\lambda,c}(\varkappa)=\sigma_c(KQ)_{\lambda/c}(\varkappa/c^2)\sigma_c^{-1}\ . \tag{4.11}$$

Now a) follows immediately. Furthermore $(KQ)_{\lambda,c}(\varkappa)$ has eigenvalue -1 if and only if $(KQ)_{\lambda/c}(\varkappa/c^2)$ has eigenvalue -1, and so b) follows from the results quoted in § II.2. Part c) also follows from the same result for $c=1$. For Part d) we use (3.15) to obtain

$$\lim_{c\to\infty} E_{B,c}(\lambda)=-\frac{9}{4}\frac{\lambda^2}{m^3}=E_B(\alpha)\quad \text{Q.E.D.}$$

We can rephrase d) by saying that the implicit eigenvalues of $(KQ)_{\lambda,c}(E+2mc^2)$ converge to those of $V_\alpha(H_0-E)^{-1}$. The next theorem indicates why this should be true: the operators themselves converge. However, since they act on different Hilbert spaces, we must clarify what this statement means.

Until now the space $\mathscr{K}$ could be any of (3.5), (3.12), (3.13); now we only consider the last, namely $\mathscr{K}=L_2^+(\mathbb{R}^2,\pi^{-1}(p_0^2+(p_1^2+1)^2)^{-1}dp)$. The advantage of this choice is that with $\mathscr{H}=L_2^+(\mathbb{R}^1,(p_1^2+1)^{-1}dp_1)$ the map $i\in\mathscr{L}(\mathscr{H},\mathscr{K})$ defined by

$$(ig)(p_0,p_1)=g(p_1) \tag{4.12}$$

is an isometry. (Thus one could regard $\mathscr{H}$ as a subspace of $\mathscr{K}$.) The adjoint $i^*\in\mathscr{L}(\mathscr{K}^*,\mathscr{H}^*)$ is a partial isometry onto $\mathscr{H}^*$ and is given by

$$(i^*f)(p_1)=\int f(p_0,p_1)dp_0\ . \tag{4.13}$$

Theorem 4.2. *Let* $\alpha=\pm 3\lambda/m^2$ *and* $E<0$. *Then in the sense of strong operator convergence:*

a) $\lim_{c\to\infty} i^*Q_{\lambda,c}(E+2mc^2)i=(2m^2)^{-1}(H_0-E)^{-1}$ *in* $\mathscr{L}(\mathscr{H},\mathscr{H}^*)$.

b) $\lim_{c\to\infty} K_{\lambda,c}(E+2mc^2)=i(2m^2V_\alpha)i^*$ *in* $\mathscr{L}(\mathscr{K}^*,\mathscr{K})$.

c) $\lim_{c\to\infty} (KQ)_{\lambda,c}(E+2mc^2)i=iV_\alpha(H_0-E)^{-1}$ *in* $\mathscr{L}(\mathscr{H},\mathscr{K})$.

d) $\lim_{c\to\infty} i^*R_{\lambda,c}(E+2mc^2)i=(2m^2)^{-1}(H_\alpha-E)^{-1}$ *in* $\mathscr{L}(\mathscr{H},\mathscr{H}^*)$.

e) $\lim_{c\to\infty} T_{\lambda,c}(E+2mc^2)=i(8m^2\mathbb{T}_\alpha(E))i^*$ *in* $\mathscr{L}(\mathscr{K}^*,\mathscr{K})$.

For d), e) *we exclude* $E=E_B(\alpha)$ *if* $\mathscr{P}=\mathscr{P}^-$.

Proof.

a) We have

$$Q_{\lambda,c}(\varkappa,p,q)=\pi^{-1}S_{\lambda,c}\left(p_0+\frac{i\varkappa}{2},p_1\right)S_{\lambda,c}\left(p_0-\frac{i\varkappa}{2},p_1\right)\delta(p-q)$$

$$S_{\lambda,c}(p)=Z^2_{\lambda,c}((p_0/c)^2+p_1^2+m^2c^2)^{-1}+\int_{3m-\varepsilon}^{\infty}\left(\left(\frac{p_0}{c}\right)^2+p_1^2+a^2c^2\right)^{-1}d\varrho_{\lambda,c}(a)\,.$$

However $Z_{\lambda,c}=Z_{\lambda/c}\to 1$ and $\varrho_{\lambda,c}=\varrho_{\lambda/c}\to 0$ and

$$\lim_{c\to\infty}\left(c^{-2}\left(p_0\pm\frac{i}{2}(E+2mc^2)\right)^2+p_1^2+m^2c^2\right)^{-1}$$
$$=(2m)^{-1}\left(\left(\frac{p_1^2}{2m}-\frac{E}{2}\right)\pm ip_0\right)^{-1}$$

and so

$$\lim_{c\to\infty}Q_{\lambda,c}(E+2mc^2,p,q)=Q_\infty(E,p,q)$$

$$Q_\infty(E,p,q)=(4m^2\pi)^{-1}\left(\left(\frac{p_1^2}{2m}-\frac{E}{2}\right)^2+p_0^2\right)^{-1}\delta(p-q)\,.$$

Now $Q_\infty(E,p,q)$ is the kernel of a bilinear form on $\mathscr{K}\times\mathscr{K}$ and defines an operator $Q_\infty(E)\in\mathscr{L}(\mathscr{K},\mathscr{K}^*)$. Then $Q_{\lambda,c}(E+2mc^2)\to Q_\infty(E)$ strongly since this holds a dense set [say $\mathscr{S}(\mathbb{R}^2)$] and $\|Q_{\lambda,c}(E+2mc^2)\|$ is bounded. Finally we note that as bilinear forms on $\mathscr{H}\times\mathscr{H}$

$$i^*Q_\infty(E)i=(2m^2)^{-1}(H_0-E)^{-1}\,. \tag{4.14}$$

b) Using (3.10) we have

$$K_{\lambda,\infty}(p,q)\equiv\lim_{c\to\infty}K_{\lambda,c}(E+2mc^2,p,q)$$
$$=\lim_{c\to\infty}cK_{\lambda/c}(E/c^2+2m,p_c,q_c)$$
$$=\pm\frac{3\lambda}{\pi}\,.$$

If $K_{\lambda,\infty}\in\mathscr{L}(\mathscr{K}^*,\mathscr{K})$ is the operator with this kernel then $K_{\lambda,c}(E+2mc^2)\to K_{\lambda,\infty}$. The result now follows from

$$K_{\lambda,\infty}=i(2m^2V_\alpha)i^*\,. \tag{4.15}$$

c) It suffices to note

$$K_{\lambda,\infty}Q_\infty(E)i=iV_\alpha(H_0-E)^{-1}\,. \tag{4.16}$$

d) $\lim_{c\to\infty}i^*R_{\lambda,c}(E+2mc^2)i$

$$=\lim_{c\to\infty}i^*Q_{\lambda,c}(E+2mc^2)(1+(KQ)_{\lambda,c}(E+2mc^2))^{-1}i$$
$$=i^*Q_\infty(E)(1+K_{\lambda,\infty}Q_\infty(E))^{-1}i$$
$$=(2m^2)^{-1}(H_0-E)^{-1}(1+V_\alpha(H_0-E))^{-1}$$
$$=(2m^2)^{-1}(H_\alpha-E)^{-1}\,.$$

Note that $R_{\lambda,c}(E+2mc^2)$ has a pole at $E+2mc^2=m_B(\lambda/c)c^2$ which we avoid for c large by the assumption $E\neq E_B(\alpha)$.

e) We have the identity

$$T_{\lambda,c}(\varkappa)=4(Q_0^{-1}Q_\lambda)_c(\varkappa)(1+(KQ)_{\lambda,c}(\varkappa))^{-1}K_{\lambda,c}(\varkappa)(Q_\lambda Q_0^{-1})_c(\varkappa) \tag{4.17}$$

which follows by applying $\sigma_c[\cdot]\sigma_c^*$ to Lemma 3.1. Here $(Q_0^{-1}Q_\lambda)_c(\varkappa)\in\mathscr{L}(\mathscr{K})$ is interpreted as multiplication by $(Q_0^{-1}Q_{\lambda/c})(\varkappa/c^2,p_c)$ as given by Lemma 3.2a. Then $(Q_0^{-1}Q_\lambda)_c(E+2mc^2)$ is multiplication by $(Q_0^{-1}Q_{\lambda/c})(E/c^2+2m,p_c)$ and hence converges to the identity by Lemma 3.2b. Thus we have

$$\begin{aligned}&\lim_{c\to\infty}T_{\lambda,c}(E+2mc^2)\\&\quad=4(1+K_{\lambda,\infty}Q_\infty(E))^{-1}K_{\lambda,\infty}\\&\quad=8m^2i((1+V_\alpha(H_0-E))^{-1}V_\alpha)i^*\\&\quad=8m^2i\mathbb{T}_\alpha i^*\,.\quad \text{Q.E.D.}\end{aligned}$$

Next we study the convergence of the kernel of $T_{\lambda,c}(E+2mc^2)$ and enlarge the domain in E to include positive values. Let $\mathscr{D}$ be the two sheeted domain for $(-E)^{1/2}$ with $E_B(\alpha)$ deleted.

Theorem 4.3. *For c sufficiently large, $T_{\lambda,c}(E+2mc^2,p,q)$ is analytic in any compact set in $\{E\in\mathscr{D};p,q\in\mathbb{C}^2\}$ and is bounded there uniformly in c. Furthermore for $p,q\in\mathbb{R}^2$*

$$\lim_{c\to\infty}T_{\lambda,c}(E+2mc^2,(p_0,p_1),(q_0,q_1))=8m^2\mathbb{T}_\alpha(E,p_1,q_1)$$

uniformly on compact sets in $\mathscr{D}$.

Proof. By (4.9) we have

$$\begin{aligned}T_{\lambda,c}(E+2mc^2,p,q)&=cT_{\lambda/c}(E/c^2+2m,p_c,q_c)\\&=c\hat{T}_{\lambda/c}((4m^2-(E/c^2+2m)^2)^{1/2},p_c,q_c)\,.\end{aligned}$$

The analyticity follows by Lemma 3.4 since

$$(4m^2-(E/c^2+2m)^2)^{1/2}=(4m+E/c^2)^{1/2}(-E/c^2)^{1/2}$$

and the pole is avoided for c sufficiently large.

For the uniform bound we also use Lemma 3.4. The U term in immediately $\mathcal{O}(1)$, and for the V term we must bound

$$c(\lambda/c)^2|(4m^2-(E/c^2+2m)^2)^{1/2}\pm(4m^2-m_B(\lambda/c)^2)^{1/2}|^{-1}\,.$$

Rationalizing this expression, the numerator is $\mathcal{O}(c^{-1})$, and so this is bounded by a constant times

$$\begin{aligned}&=c^{-2}|-(E/c^2+2m)^2+m_B(\lambda/c)^2|^{-1}\\&=c^2|m_B(\lambda/c)c^2-E-2mc^2|^{-1}|m_B(\lambda/c)c^2+E+2mc^2|^{-1}\\&\leqq\mathcal{O}(1)\,.\end{aligned}$$

For the convergence we note that by Vitali's theorem it is sufficient to prove convergence for $p,q\in\mathbb{R}^2$ and $\operatorname{Re}E<0$ (first sheet). However we have convergence here in the sense of distributions in (p,q) by Theorem 4.2e, and for uniformly bounded analytic functions this implies pointwise convergence. Q.E.D.

Theorem 4.4. *Let $S_{\lambda,c}(p,q)\in\mathcal{S}'(\mathbb{R}^2)$ be the two body scattering amplitude for $\mathcal{P}^{\pm}(\varphi)_{2,c}$. Then away from $p,q=0$ with $\alpha=\pm 3\lambda/m^2$*

$$\lim_{c\to\infty} S_{\lambda,c}(p,q)=\mathbb{S}_\alpha(p,q)\ .$$

Proof. Here $S_{\lambda,c}(p,q)$ is the amplitude for relative momentum q to scatter to relative momentum p, defined from the full kernel $S_{\lambda,c}(p_1,\dots,p_4)$ as in §III.4. We have $S_{\lambda,c}(p,q)=c^{-1}S_{\lambda/c}(p/c,q/c)$ and (3.27) scales to become

$$S_{\lambda,c}(p,q)=\delta(p-q)-2\pi i(Z_{\lambda,c})^{-4}c^4(8\omega_c(p)\omega_c(q))^{-1}$$
$$\cdot\, T_{\lambda,c}(2\omega_c(p)+i0^+,(0,p),(0,q))\delta(2\omega_c(p)-2\omega_c(q))\ .$$

Then using $\omega_c(p)=mc^2+p^2/2m+\mathcal{O}(c^{-2}))$ and Theorem 4.3 we have

$$\lim_{c\to\infty} S_{\lambda,c}(p,q)=\delta(p-q)-2\pi i\mathbb{T}_\alpha(p^2/m+i0^+,p,q)\delta(p^2/m-q^2/m)$$
$$=\mathbb{S}_\alpha(p,q)\ ,\quad \text{Q.E.D.}$$

V. Concluding Remarks

1. We have not dealt specifically with the question of asymptotics. However by combining the methods of the present paper with those of [4] one can show that $R_{\lambda,c}(E+2mc^2)$, for example, is a C^∞ function of $1/c\geqq 0$. Thus $R_{\lambda,c}(E+2mc^2)$ has an asymptotic expansion in powers of $1/c$ with leading term $(H_\alpha-E)^{-1}$. There seems to be no obstacle to extending this type of result to the S-matrix.

2. We conjecture that the $2n$-point function:

$$\tau_{\lambda,c}(p_1+(mc^2,0),\dots,p_n+(mc^2,0),p_{n+1}-(mc^2,0),\dots,p_{2n}-(mc^2,0))$$

has a non trivial limit as $c\to\infty$. (Theorem 4.3 establishes this for the 4-point function.) The limit should be the $2n$- point function for a non-relativistic multiparticle system with δ-function potentials.

3. The methods of this paper should work for other models once one has control over the Bethe-Salpeter kernel. For Yukawa models we still expect to get a δ-function potential in the limit. This is consistent with a Yukawa potential of the form $c^2(p^2+mc^2)^{-1}$ which also converges to a constant. It is not clear whether the Yukawa potential plays any more fundamental role. For models with a massless particle exchange one presumably gets the Coulomb potential in the limit.

4. A related question to the present investigation is to reinstate $\hbar$ as a parameter and ask for the limit $\hbar\to 0$. One expects the quantum field theory to converge to a classical field theory. Some results in this direction for $\mathcal{P}(\varphi)_2$ have been obtained by Hepp [11] and Eckmann [5].

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