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The Nonlinear Evolution Equations Related to the Wadati-Konno-Ichikawa Spectral Problem

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The Wadati-Konno-Ichikawa (WKI) spectral problem is considered and the nonlinear evolution equations (NLEE) related to this problem are explicitly constructed. Together with the integrable NLEE already found by WKI new pure differential solvable NLEEs are derived. The properties of the integro-differential operator L generating all the considered NLEEs are studied. It is shown that the usually successful procedure used to derive the Bäcklund transformation does not work with these highly nonlinear evolution equations.

§1. Introduction

Many nonlinear two-dimensional evolution equations have been solved by the use of the spectral (or scattering) transform (ST) techniques.[1)~4)]

In general, according to the AKNS method,[3)] one considers the linear spectral problem

$$\Psi_x = U\Psi\,, \tag{1·1}$$

where Ψ and U are complex valued $N\times N$ matrices of x, t and of the spectral parameter λ.

The time dependence is fixed by imposing to Ψ to satisfy the auxiliary spectral equation

$$\Psi_t = V\Psi\,, \tag{1·2}$$

where V is an $N\times N$ matrix of x, t and λ.

The compatibility condition for (1·1) and (1·2) furnishes the so-called Lax representation[5)] for the nonlinear evolution equation (NLEE) one is looking for

$$U_t - V_x + [U, V] = 0\,. \tag{1·3}$$

In this paper, we consider the spectral problem proposed by Wadati, Konno and Ichikawa (WKI)[6)] with

$$U = -i\lambda\sigma_3 + \lambda P(x, t), \tag{1·4}$$

where σ_3 is the Pauli matrix and $P=\begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}$.

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Wadati, Konno, Ichikawa and Shimizu in a series of papers[6),7)] solve the direct and inverse problem for this spectral equation and find a hierarchy of NLEEs that can be solved by the use of the ST techniques.

These authors consider for V a polynomial of fourth order in λ without constant term.

We choose a more general V of the form

$$V=\sum_{j=0}^{n+s}\lambda^{n+1-j}V_j(P), \qquad (s,\ n=0,1,\cdots) \tag{1·5}$$

and we apply the method developed in Refs. 8) and 9).

Specifically, by equating to zero the matrix coefficients of the powers of λ in (1·3) we get a set of recurrence relations that are solved furnshing the explicit form of V and a set of NLEEs for P.

We collect the found NLEEs in different hierarchies according to the values of s.

It is convenient for any hierarchy to choose some specific equations, which we call basic, as representative of the all NLEEs in the hierarchy.

The more general NLEE in the hierarchy can be obtained by taking an arbitrary linear combination of the basic equations.

At $s=0$ we get the explicit form of the hierarchy of NLEEs already found by Wadati, Konno and Ichikawa.[6)]

At $s\geq 1$ we get new hierarchies of NLEEs.

We show that the hierarchy at $s=1$ contains pure partial differential equations and we write explicitly the first few of them.

From these equations, by a reduction procedure, we derive the following nonlinear evolution equation in one complex function $R(x,t)$:

$$R_tR_{xx}-R_{tx}R_x=a(t)R_x^2(1-R_x^2)^{1/2}+b(t)R_{xx}\,, \tag{1·6}$$

where $a(t)$ and $b(t)$ are arbitrary functions of t.

According to a common feature of all integrable NLEEs,[3),4),10)~13)] all the considered hierarchies are generated by the repeated applications of an integro-differential matrix operator L to a 2×2 matrix $K(P)$ function of P.

By using a method suggested by Chern and Peng[14)] we show that the differential character of the considered NLEEs at $s=0$ and $s=1$ is related to the fact that the recurrent integro-differential formula

$$K_{n+1}=LK_n \tag{1·7}$$

can be transformed into the recursion formula

$$K_{n+1}=F(K_n,\ K_{n-1},\ \cdots,\ K_0), \tag{1·8}$$

where F is a pure differential 2×2 matrix operator which is explicitly given.

The fact that, in spite of the integro-differential character of the operator L, the expression $L^k K$ which appears in the basic NLEEs is pure differential is also related to the existence of a conserved density $H=2(1-\sqrt{1-P^2})$ common to all the basic equations defining the WKI hierarchy at $s=0$.

It has been suggested by WKI[6] that the evolution equations of this hierarchy can be written in generalized Hamiltonian form.[15] Specifically, the conserved density H is the Hamiltonian of the first equation in the hierarchy which can be written as

$$P=J\frac{\delta H}{dP}, \tag{1·9}$$

where $\delta H/\delta P=\mathrm{diag}(\delta H/\delta q,\ \delta H/\delta r)$ and $J=i\sigma_2 D^2$ is a symplectic operator with σ_2 the Pauli matrix and $D=d/dx$.

We define a new operator

$$\tilde{L}=\sigma_1 L \sigma_1 \tag{1·10}$$

with σ_1 the Pauli matrix and we show that $\tilde{L}$ operates on a matrix Q as follows:

$$\tilde{L}Q=WQ+ZI[H'(P)(JQ)], \tag{1·11}$$

where W and Z are differential operators, $I=\int_{-\infty}^{x}dx'$ and $H'(P)$ is the Gateaux (or directional) derivative of the Hamiltonian H.

By using the representation (1·11) for the integro-differential operator $\tilde{L}$ we derive the differential character of the term $L^k K(P)$ in the NLEEs as a consequence of the fact that the Hamiltonian H is a common conserved density of all the equations in the WKI hierarchy.

In § 4 we try to generate the self-Bäcklund transformations (BT) for the equations considered via gauge transformations of the eigenmatrices Ψ.

The used procedure has been already used with success in the $N\times N$ AKNS case[8] and in the 2×2 Kaup-Newell[9] and other cases.[16)~18)]

However, in this case, one must add to the set of differential equations defining the BT the algebraic constraint

$$P^2=\bar{P}^2, \tag{1·12}$$

where $\bar{P}$ is the BT of P.

It is shown that this constraint, in general, is incompatible with the found BT.

We conclude that the high nonlinearity of the NLEEs implies that either these NLEEs do not have BTs at all or have BTs that must be obtained by a more involved procedure than usual.

§ 2. The solvable NLEEs related to the WKI spectral problem

We search for NLEEs related to the WKI spectral linear problem

$$\Psi_x = U\Psi \,, \tag{2·1}$$

where

$$U = -i\lambda\sigma_3 + \lambda P(x, t) \tag{2·2}$$

with

$$P(x, t) = \begin{pmatrix} 0 & q(x, t) \\ r(x, t) & 0 \end{pmatrix}. \tag{2·3}$$

The matrix $P(x, t)$ is supposed to decrease rapidly as $|x| \to \infty$.

Let the 2×2 matrix V in the auxiliary spectral problem

$$\Psi_t = V\Psi \tag{2·4}$$

to be of the form

$$V(P, \lambda) = \sum_{j=0}^{n+s} \lambda^{n+1-j} V_j(P). \quad (n, s = 0, 1, \cdots) \tag{2·5}$$

By equating to zero the coefficients of the powers of λ in the Lax representation

$$U_t - V_x + [U, V] = 0 \tag{2·6}$$

and by introducing for convenience V_{-1} and V_{n+s+1} we obtain

$$V_{-1} = 0 \,, \tag{2·7}$$

$$V_{jx} = [P, V_{j+1}] - i[\sigma_3, V_{j+1}] + \delta_{nj} P_t \,, \quad (j = -1, 0, \cdots, n+s) \tag{2·8}$$

$$V_{n+s+1} = 0 \,. \tag{2·9}$$

If the diagonal part Q_D and the off-diagonal part Q_F of a matrix Q are defined to be, respectively, $Q_D = \mathrm{diag}(Q_{11}, Q_{22})$ and $Q_F = Q - Q_D$, the recursion relations (2·7)~(2·9) can be more conveniently separated into their diagonal and off-diagonal components

$$V_{jD,x} = [P, V_{j+1,F}], \qquad (j = 0, 1, \cdots, n+s) \tag{2·10}$$

$$V_{-1,F} = 0 \,, \tag{2·11}$$

$$V_{j+1,F} = \frac{1}{2} i\sigma_3 V_{jF,x} - \frac{1}{2} i\sigma_3 [P, V_{j+1,D}] - \frac{1}{2} i\sigma_3 P_t \delta_{nj} \,, \quad (j = -1, 0, 1, \cdots, n+s) \tag{2·12}$$

$$V_{n+s+1,F}=0\,, \tag{2·13}$$

$$V_{n+s+1,D}=0\,. \tag{2·14}$$

By inserting $V_{j+1,F}$ from (2·12) and (2·13) into (2·10), we obtain

$$V_{jD,x}=\frac{1}{2}i[P,\ \sigma_3 V_{jF,x}]+\frac{1}{2}i\sigma_3(P^2)_t\delta_{nj}\,,\quad (j=0,1,\cdots,n+s-1) \tag{2·15}$$

$$V_{n+s,Dx}=0\,. \tag{2·16}$$

By inserting once more V_{jF} from (2·12) we obtain

$$V_{jD,x}=-\frac{1}{4}[P,\ V_{j-1,Fxx}]+\frac{1}{4}[P,[P,\ V_{jD}]_x]+\frac{1}{2}i\sigma_3(P^2)_t\delta_{nj}$$

$$+\frac{1}{4}[P,\ P_{tx}]\delta_{n,j-1}\,. \qquad (j=0,1,\cdots,n+s-1) \tag{2·17}$$

Let us define

$$\rho_j=\mathrm{Tr}(V_{jD}) \tag{2·18}$$

and

$$\tau_j=\mathrm{Tr}(\sigma_3 V_{jD}), \tag{2·19}$$

where Tr indicates the trace of the matrix considered. The matrix V_{jD} can be rewritten as

$$V_{jD}=\frac{1}{2}(\rho_j+\sigma_3\tau_j) \tag{2·20}$$

and from (2·16), (2·17) it is easy to deduce the differential equations

$$\rho_{jx}=0\,, \qquad (j=0,1,\cdots,n+s) \tag{2·21}$$

$$\tau_{jx}=\frac{1}{2}\,\mathrm{Tr}(P\sigma_3 V_{j-1,Fxx})+\frac{1}{2}(P^2)_x\tau_j+P^2\tau_{jx}+i(P^2)_t\delta_{nj}$$

$$+\frac{1}{2}\,\mathrm{Tr}(\sigma_3 PP_{tx})\delta_{n,j-1}\,, \qquad (j=0,1,\cdots,n+s-1) \tag{2·22}$$

$$\tau_{n+s,x}=0\,. \tag{2·23}$$

They can be explicitly integrated to

$$\rho_j=\alpha_j(t), \qquad (j=0,1,\cdots,n+s) \tag{2·24}$$

$$(1-P^2)^{1/2}\tau_j=\beta_j(t)+\frac{1}{2}I\Big[\mathrm{Tr}(K\sigma_3 V_{j-1,Fxx})\Big]+iI[\mathrm{Tr}(KP_t)]\delta_{nj}$$

$$+\frac{1}{2}I[\mathrm{Tr}(\sigma_3 KP_{tx})]\delta_{n,j-1}\,,\quad (j=0,1,\cdots,n+s-1) \tag{2·25}$$

$$\tau_{n+s}=\beta_{n+s}(t), \tag{2·26}$$

where

$$I[\ \ ]=\int_{-\infty}^{x}[\ \ ]dx', \tag{2·27}$$

$$K=P/(1-P^2)^{1/2} \tag{2·28}$$

and $\alpha_j(t)$, $\beta_j(t)$ are arbitrary functions of t.

By inserting (2·20) into (2·12) we obtain the following recursion formula for the V_{jF}'s:

$$V_{-1,F}=0\ , \tag{2·29}$$

$$V_{j+1,F}=L\,V_{jF}+\frac{1}{2}i\beta_{j+1}K+M\delta_{n,j+1}+N\delta_{nj}\ ,$$

$$(j=-1,0,\cdots,n+s-1) \tag{2·30}$$

and the evolution equation

$$\sigma_3 V_{n+s-1,Fxx}+\beta_{n+s}P_x-\sigma_3 P_{tx}\delta_{n,n+s-1}+2iP_t\delta_{n,n+s}=0\ . \tag{2·31}$$

L is the matrix integro-differential operator

$$LQ=\frac{1}{2}i\sigma_3 Q_x+\frac{1}{4}iKI[\mathrm{Tr}(K\sigma_3 Q_{xx})] \tag{2·32}$$

and

$$M=-\frac{1}{2}KI[\mathrm{Tr}(KP_t)], \tag{2·33}$$

$$N=-\frac{1}{2}i\sigma_3 P_t+\frac{1}{4}iKI[\mathrm{Tr}(\sigma_3 KP_{tx})]. \tag{2·34}$$

From the recursion formulae (2·29), (2·30) one easily gets the explicit form of the off-diagonal part of V operator of the auxiliary spectral problem and, therefore, from (2·31) a NLEE in P.

The explicit form of V is obtained by means of Eqs. (2·20), (2·24)~(2·26). We consider, separately, the cases $s=0$, $s=1$ and $s\geq 2$.

In the first case ($s=0$) we get the NLEEs

$$P_t=-\frac{1}{4}\sigma_3 D^2\sum_{k=0}^{n-1}\beta_k(t)L^{n-1-k}K+\frac{1}{2}i\beta_n(t)P_x\ ,\quad (n=1,2,\cdots) \tag{2·35}$$

where $D=\partial/\partial x$.

The term $\frac{1}{2}i\beta_n P_x$ in the right-hand side can be crossed out by means of an elementary change of variable.

According to the idea exposed in the Introduction we choose as representative of the obtained hierarchy of NLEEs the following basic NLEEs:

$$P_t = c_k(t)\sigma_3 D^2 L^k K \qquad (k=0, 1, \cdots) \tag{2·36}$$

with $c_k(t)$ arbitrary functions of t.

In the following section we show that, in spite of the integro-differential character of the operator L, these equations are pure differential equations and, at $c_k=1$ reduce to the equations found by Wadati, Konno and Ichikawa.

In the case $s=1$ we get the NLEEs

$$P_{tx} + \frac{1}{2}D^2 KI[\mathrm{Tr}(KP_t)] = \frac{1}{2}iD^2 \sum_{k=0}^{n} L^{n-k}\beta_k(t)K + \beta_{n+1}(t)\sigma_3 P_x\,, \tag{2·37}$$

$$(n=0, 1, \cdots)$$

which can be once integrated to

$$P_t + \frac{1}{2}DKI[\mathrm{Tr}(KP_t)] = \frac{1}{2}iD \sum_{k=0}^{n} L^{n-k}\beta_k(t)K + \beta_{n+1}\sigma_3 P_x + \phi_1(t)\sigma_1 + \phi_2(t)\sigma_2 \tag{2·38}$$

with $\phi_1(t)$ and $\phi_2(t)$ arbitrary functions of t.

We choose as representative of the hierarchy (2·38) the following basic NLEEs

$$P_t + \frac{1}{2}DKI[\mathrm{Tr}(KP_t)] = \phi_1(t)\sigma_1 + \phi_2(t)\sigma_2 + i\phi_3(t)\sigma_3 P\,, \tag{2·39}$$

$$P_t + \frac{1}{2}DKI[\mathrm{Tr}(KP_t)] = c_k(t)DL^k K\,, \quad (k=0, 1, \cdots) \tag{2·40}$$

with ϕ_i and c_k arbitrary functions of t.

It is convenient to introduce the new functions

$$R = I[(1-P^2)^{1/2} - 1] + x\,, \tag{2·41}$$

$$S^2 = q/r\,, \tag{2·42}$$

from which we have $q = S/(1-R_x{}^2)^{1/2}$, $r = (1-R_x{}^2)^{1/2}/S$. Consequently P, K can be represented in terms of R and S.

In the following section we shall show that all the basic NLEEs (2·39), (2·40) written in terms of the new functions R and S are pure partial differential equations. Their explicit form can be more easily obtained by multiplying any basic equation, successively, by K and $\sigma_3 K$ and by taking the trace of the two obtained equations.

We write explicitly the first few of them.

From (2·39) we get the evolution equations

$$R_tR_{xx}-R_{tx}R_x=\phi_1R_x^2(1-R_x^2)^{1/2}(1+S^2)/2S$$
$$-i\phi_2R_x^2(1-R_x^2)^{1/2}(1-S^2)/2S\,, \tag{2·43}$$

$$S_tR_x-R_tS_x=\phi_1R_x(1-S^2)/2(1-R_x^2)^{1/2}-i\phi_2R_x(1+S^2)/2(1-R_x^2)^{1/2}$$
$$+i\phi_3R_xS\,. \tag{2·44}$$

From (2·40) we obtain at $k=0$,

$$R_tR_{xx}-R_{tx}R_x=-c_0R_{xx}\,, \tag{2·45}$$

$$S_tR_x-R_tS_x=c_0S_x\,, \tag{2·46}$$

that can be once integrated to

$$R_t=-c_0+d_0R_x\,, \tag{2·47}$$

$$S_t=d_0S_x \tag{2·48}$$

with d_0 an arbitrary function of t. At $k=1$, one gets the equations

$$R_tR_{xx}-R_{tx}R_x=-\frac{1}{2}ic_1R_{xx}\frac{S_x(1-R_x^2)}{R_x^2S}+\frac{1}{2}ic_1R_x\left(\frac{S_x(1-R_x^2)}{R_x^2S}\right)_x, \tag{2·49}$$

$$S_tR_x-R_tS_x=\frac{1}{2}ic_1\frac{S_x^2}{R_x^2S}-\frac{1}{2}ic_1\frac{SR_{xxx}}{R_x(1-R_x^2)}-\frac{1}{2}ic_1\frac{SR_{xx}^2}{(1-R_x^2)^2R_x^2}(3R_x^2-2), \tag{2·50}$$

that can be once integrated to

$$R_t=-\frac{1}{2}ic_1\frac{S_x(1-R_x^2)}{R_x^2S}+d_1R_x\,,$$

$$S_t=\frac{1}{2}ic_1\frac{S_x^2}{R_xS}-\frac{1}{2}ic_1\frac{SR_{xxx}}{R_x^2(1-R_x^2)}-\frac{1}{2}ic_1\frac{SR_{xx}^2}{(1-R_x^2)^2R_x^3}(3R_x^2-2)+d_1S_x \tag{2·51}$$

with d_1 an arbitrary function of t.

Equation (2·40) for general k can be written

$$R_tR_{xx}-R_{tx}R_x=c\frac{R_x^3}{2}\,\mathrm{Tr}(KDL^kK), \tag{2·52}$$

$$S_tR_x-R_tS_x=-c_k\frac{R_x^2S}{2(1-R_x^2)}\,\mathrm{Tr}(\sigma_3KDL^kK), \tag{2·53}$$

which can be integrated once to

$$R_t=-\frac{1}{2}c_kR_xI[\mathrm{Tr}(PDL^kK)]+d_kR_x\,, \tag{2·54}$$

$$S_t=-\frac{1}{2}c_k\Big[S_xI[\mathrm{Tr}(PDL^kK)]+\frac{R_xS}{(1-R_x{}^2)}\mathrm{Tr}(\sigma_3KDL^kK)\Big]+d_kS_x\,. \tag{2·55}$$

It is easy to verify that

$$I[\mathrm{Tr}(PDK)]=\mathrm{Tr}(PK)+2(1-P^2)^{1/2}+d_k'\,,$$

$$I[\mathrm{Tr}(PDL^{k+1}K)]=-\frac{1}{2}i[2i\;\mathrm{Tr}(PL^{k+1}K)+\mathrm{Tr}(P\sigma_3DL^kK)-(1-P^2)^{1/2}H_k]+d_k''\,,$$

where $H_k\equiv I[\mathrm{Tr}(K\sigma_3D^2L^kK)]$, and d_k, d_k' and d_k'' are arbitrary functions of t.

In the next section we shall prove that all L^kK and H_k, $(k=0, 1, \cdots)$ are pure differential expression. Therefore the evolution equations (2·54) and (2·55) are local partial differential equations.

If the functions ϕ_k, d_k and c_k are chosen to be real all the equations in the hierarchy are compatible with the request that R is real and S has unit modulus.

By choosing in Eqs. (2·43), (2·44), $\phi_2\equiv\phi_3\equiv0$ and $S=1$ and in Eqs. (2·45), (2·46) $S=1$ one gets the following two reduced equations:

$$R_tR_{xx}-R_{tx}R_x=\phi_1(t)R_x{}^2(1-R_x{}^2)^{1/2}\,, \tag{2·56}$$

$$R_tR_{xx}-R_{tx}R_x=-c_0(t)R_{xx}\,. \tag{2·57}$$

In the case $s\geq2$ we get the NLEEs

$$\frac{1}{2}i\sigma_3D^2\sum_{k=0}^{n+s-1}L^{n+s-1-k}\beta_k(t)K+\sigma_3D^2L^{s-2}(N+LM)+\beta_{n+s}(t)P_x=0\,, \tag{2·58}$$

which can be once integrated.

We choose as basic equations

$$DL^{s-2}(N+LM)=\phi_1(t)\sigma_1+\phi_2(t)\sigma_2+i\phi_3(t)\sigma_3P\,, \tag{2·59}$$

$$DL^{s-2}(N+LM)=c_k(t)DL^kK\,,\qquad (k=0, 1, \cdots) \tag{2·60}$$

with $\phi_i(t)$ and $c_k(t)$ arbitrary functions of t.

These equations correspond to a operator V in the auxiliary spectral equation with negative powers of λ and are integro-differential equations.

In all three cases ($s=0$, $s=1$, $s\geq2$) the more general NLEE in the hierarchy can be obtained by taking a linear combination of the basic equations.

§ 3. Properties of the operator L generating the hierarchies of NLEEs

Let us consider the recursion relation

$$K_{k+1}=LK_k\,,\qquad (k=0,1,\cdots) \tag{3·1}$$

where $K_0=K$ and L is the integro-differential operator defined by Eq. (2·32).

If we introduce the formal series

$$\mathcal{K}=\sum_{k=0}^{\infty}K_k\eta^{-k} \tag{3·2}$$

the recursion relation (3·1) can be rewritten as

$$\eta\mathcal{K}-\eta K=L\mathcal{K} \tag{3·3}$$

or, taking into account the explicit form of L,

$$\eta\sigma_3\mathcal{K}=\frac{1}{2}i\mathcal{K}_x-\frac{1}{2}iK\sigma_3\mathcal{H}\,, \tag{3·4}$$

where

$$\mathcal{H}=\frac{1}{2}I[\mathrm{Tr}(K\sigma_3\mathcal{K}_{xx})]-2i\eta\,. \tag{3·5}$$

Multiplying both sides of (3·4), successively, on the left and on the right by $\mathcal{K}_{xx}$, one gets two equations, whose sum is the following equation:

$$\eta\sigma_3(\mathcal{K}\mathcal{K}_x-\mathcal{K}_x\mathcal{K})_x=\frac{1}{2}i(\mathcal{K}_x{}^2)_x-\frac{1}{2}i(\mathcal{H}^2)_x\,. \tag{3·6}$$

This equation can be integrated to

$$\eta\sigma_3(\mathcal{K}\mathcal{K}_x-\mathcal{K}_x\mathcal{K})=\frac{1}{2}i\mathcal{K}_x{}^2-\frac{1}{2}i\mathcal{H}^2-2i\eta^2\,. \tag{3·7}$$

The constant of integration has been fixed recalling that, since $P(x,t)$ and it derivatives are supposed to vanish at $x=-\infty$, $\mathcal{K}\to 0$ and $\mathcal{H}\to -2i\eta$ as $x\to-\infty$.

By inserting $\mathcal{H}$ from (3·4) into (3·7) we get

$$\eta\sigma_3(\mathcal{K}\mathcal{K}_x-\mathcal{K}_x\mathcal{K})=\frac{1}{2}i\mathcal{K}_x{}^2+\frac{1}{2}iK^{-4}[4\eta^2(K\mathcal{K})^2-(K\mathcal{K}_x)^2$$
$$+4i\eta\sigma_3K\mathcal{K}K\mathcal{K}_x]-2i\eta^2\,. \tag{3·8}$$

By equating to zero the coefficients of the powers of η we deduce the pure differential recursion formula for K_k

$$K_{k+1}=\frac{1}{4}i\sigma_3 K\sum_{l=0}^{k}(K_l K_{k-l,x}-K_{k-l,x}K_l)+\frac{1}{2}i\sigma_3 K^{-2}\sum_{l=0}^{k}K_l K K_{k-l,x}$$

$$-\frac{1}{8}K\theta(k-1)\sum_{l=0}^{k-1}K_{l,x}K_{k-1-l,x}-\frac{1}{2}K^{-2}\theta(k-1)\sum_{l=0}^{k-1}K_{l+1}KK_{k-l}$$

$$+\frac{1}{8}K^{-2}\theta(k-1)\sum_{l=0}^{k-1}K_{l,x}KK_{k-1-l,x}\,, \qquad (k=0,1,\cdots) \tag{3·9}$$

where $\theta(k)$ is the step function defined as

$$\theta(k)=\begin{cases}1 & \text{for } k\geq 0\\ 0 & \text{for } k<0\,.\end{cases} \tag{3·10}$$

By induction one can prove that the terms $L^k K$ appearing in the considered hierarchies at $s=0$, $s=1$ and $s\geq 2$ are pure differential terms in P.

The first basic equation ($k=0$) in the WKI hierarchy can be written in a generalized Hamiltonian form as follows:

$$P_t=J\frac{\delta H}{\delta P}\,, \tag{3·11}$$

where

$$\delta H/\delta P=\mathrm{diag}(\delta H/\delta q,\,\delta H/\delta r), \tag{3·12}$$

$$H(P)=2(1-\sqrt{1-P^2}) \tag{3·13}$$

and

$$J=i\sigma_2 D^2 \tag{3·14}$$

is a symplectic 2×2 matrix operator.

WKI have proved that the Hamiltonian $H(P)$ is a common conserved density of all the basic equations in the WKI hierarchy.[6]

Let us connect this fact with the existence of the recursion relation (3·9).

From the equation

$$H_t=\mathrm{Tr}(KP_t) \tag{3·15}$$

and from the basic equation (2·36) we get

$$H_t=c_k\,\mathrm{Tr}(K\sigma_3 K_{k,xx}) \tag{3·16}$$

or, equivalently,

$$H_t=2c_k H_{k,x}\,, \tag{3·17}$$

where H_k is the coefficient of η^{-k} in the formal series (3·5) which defines $\mathcal{H}$.

Since from (3·4) we obtain

$$\mathcal{H}=K^{-2}(-2i\eta K\mathcal{K}+\sigma_3 K\mathcal{K}_x), \tag{3·18}$$

where $\mathcal{K}$ and $\mathcal{K}_x$ are formal power series in the matrix K and its derivatives, and we see that H is a conserved density for all the equations in the WKI hierarchy.

Let us, now, introduce the integro-differential operator

$$\tilde{L}=\sigma_1 L\sigma_1 \tag{3·19}$$

and rewrite the term $L^k K$ as

$$L^k K=\sigma_1\tilde{L}^k\sigma_1 K\,. \tag{3·20}$$

The operator $\tilde{L}$ can be represented in the following form

$$\tilde{L}Q=WQ+ZI[H'(P)(JQ)], \tag{3·21}$$

where W and Z are the differential operators

$$W=-\frac{1}{2}i\sigma_3 D\,, \tag{3·22}$$

$$Z=\frac{1}{4}i\sigma_1 K\,, \tag{3·23}$$

and $H'(P)$ is the Gateaux (or directional) derivative of H

$$H'(P)Q=\frac{\partial}{\partial\varepsilon}H(P+\varepsilon Q)\Big|_{\varepsilon=0}. \tag{3·24}$$

The pure differential character of the term (3·20) is related to the fact that the integral term in $\tilde{L}$ can be expressed by means of H, which is a common conserved density of all the equations (2·36).

In fact, from the recursion relation (3·1) and the definition (3·5) of $\mathcal{H}$ it follows that

$$\begin{aligned}\tilde{L}^k\sigma_1 K&=(W+ZIH'(P)J)\tilde{L}^{k-1}\sigma_1 K\\&=W\tilde{L}^{k-1}\sigma_1 K+ZI[\mathrm{Tr}(K\sigma_3 K_{k-1,xx})]\\&=W\tilde{L}^{k-1}\sigma_1 K+2ZH_{k-1}\,.\end{aligned} \tag{3·25}$$

Since H_{k-1} can be expressed in terms of K and its derivatives, by induction the term $\tilde{L}^k\sigma_1 K$ is pure differential.

The representation (3·21) for the operator $\tilde{L}$ generating the hierarchies of NLEEs related to the WKI spectral equation is not specific of this problem.

In fact, analogous representations can be written for the operators $\tilde{L}$ generating other hierarchies of NLEEs.

This can be verified, for instance, for the hierarchy related to the AKNS spectral problem in the $N\times N$ case and in the Kaup-Newell spectral problem in the 2×2 case.

We deserve this result and a generalization of the representation (3·21) in other cases to a future work.

§ 4. The Bäcklund transformation

Let us consider a generalized gauge transformation of the eigenmatrix Ψ in the spectral equation (1·1), i.e., the linear transformation

$$\bar{\Psi}=B\Psi \tag{4·1}$$

with B a non-singular 2×2 matrix.

We define B as a self-Bäcklund gauge transformation if it does not change the spectral form of U and V.[8),9)]

Then $\bar{\Psi}$ satisfies the two spectral equations

$$\bar{\Psi}_x=\bar{U}\bar{\Psi}\ , \tag{4·2}$$

$$\bar{\Psi}_t=\bar{V}\bar{\Psi}\ , \tag{4·3}$$

where $\bar{U}$ and $\bar{V}$ are the isospectral deformations of U and V obtained by substituting $P(x,t)$ with $\bar{P}(x,t)=\begin{pmatrix}0 & \bar{q}\\ \bar{r} & 0\end{pmatrix}$.

The gauge B must satisfy the two matrix differential equations

$$B_x=\bar{U}B-BU\ , \tag{4·4}$$

$$B_t=\bar{V}B-BV\ . \tag{4·5}$$

By cross differentiating (4·4) and (4·5) we get

$$B_{xt}-B_{tx}=(\bar{U}_t-\bar{V}_x+[\bar{U},\bar{V}])B-B(U_t-V_x+[U,V]). \tag{4·6}$$

Therefore, if B satisfies the system of Eqs. (4·4) and (4·5), $\bar{P}$ satisfies the same NLEE as P and the two differential equations (4·4) and (4·5) can be considered as the x-derivative and the t-derivative components, respectively, of a self-Bäcklund transformation.

According to the procedure outlined in Refs. 8) and 9) we first find the general solution B of Eq. (4·4). Afterwards we fix the time evolution of the constant of integrations of the found B by imposing it to satisfy (4·5) at $x=-\infty$.

Let us choose B to have a pole of order m at $\lambda=\infty$

$$B(\bar{P},P;\lambda)=\sum_{j=0}^{m}\lambda^j B_j(\bar{P},P). \tag{4·7}$$

Equating to zero the coefficients of the powers of λ in (4·4), introducing for convenience B_{-1} and B_{m+1} and separating the diagonal and the off-diagonal components, we get the recursion relations

$$B_{-1,D}=B_{-1,F}=0\,, \tag{4·8}$$

$$B_{j+1,Dx}=(\bar{P}B_{jF}-B_{jF}P), \qquad (j=-1,0,\cdots,m) \tag{4·9}$$

$$B_{j+1,Fx}=-2i\sigma_3B_{jF}+(\bar{P}B_{jD}-B_{jD}P), \qquad (j=-1,0,\cdots,m) \tag{4·10}$$

$$B_{m+1,D}=B_{m+1,F}=0\,. \tag{4·11}$$

Inserting B_{jF} from (4·10) into (4·9), we obtain

$$B_{j+1,Dx}=-\frac{1}{2}i\sigma_3(\bar{P}B_{j+1,Fx}+B_{j+1,Fx}P)+\frac{1}{2}i\sigma_3(\bar{P}^2-P^2)B_{jD}\,. \quad (j=-1,0,\cdots,m) \tag{4·12}$$

If $\bar{P}^2\neq P^2$, it is easy to verify that the two recursion relations (4·10) and (4·12) together with the condition $B_{m+1}=0$ imply that the gauge B is identically zero.

Therefore we impose the additional condition

$$\bar{P}^2=P^2\,. \tag{4·13}$$

Equation (4·12) can be integrated to

$$B_{jD}=-\frac{1}{2}i\sigma_3I(\bar{P}B_{jF,x}+B_{jF,x}P)+\gamma_j(t), \tag{4·14}$$

where $\gamma_j(t)$ are x-independent arbitrary diagonal matrices.

Inserting B_{jD} from (4·14) into (4·10), we obtain

$$B_{0F}=0\,, \tag{4·15}$$

$$B_{j+1,F}=I\Lambda B_{jF}+I\Pi_j\,, \quad (j=0,1,\cdots,m) \tag{4·16}$$

$$B_{m+1,F}=0\,, \tag{4·17}$$

where Λ is the integro-differential operator

$$\Lambda Q=-2i\sigma_3Q+\frac{1}{2}i\sigma_3(\bar{P}I(\bar{P}Q_x+Q_xP)+I(\bar{P}Q_x+Q_xP)P) \tag{4·18}$$

and

$$\Pi_j=(\bar{P}\gamma_j-\gamma_jP). \tag{4·19}$$

The constant of integration in (4·16) has been chosen to be zero in order to guarantee a good behaviour of the gauge B at $x=\pm\infty$.

The recursion relations (4·15)~(4·17) allow us to determine the off-diagonal part of the gauge B up to arbitrary diagonal matrices. Consequently, Eq. (4·14) furnishes the diagonal part of B.

Moreover, from Eq. (4·17) it follows that P and $\bar{P}$ are related by the integro-differential equation

$$\sum_{k=0}^{m}(\Lambda I)^{m-k}\Pi_k=0\,. \tag{4·20}$$

It has been shown in Refs. 8) and 9) that the asymptotic behaviour at $x=-\infty$ of the matrix differential equation (4·5) determines uniquely the self-Bäcklund gauge B.

Since the value $B^{(-)}$ of the gauge B at $x=-\infty$ is a diagonal matrix and, for all V in the class considered, $V^{(-)}$ and $\bar{V}^{(-)}$ are equal and diagonal, we obtain

$$B_t^{(-)}=0 \tag{4·21}$$

or B is a self-Bäcklund gauge if and only if the γ_i matrices are constants.

We conclude that the x-component of the self-BT we are looking for is the integro-differential equation (4·20) and that its t-component can be obtained by inserting into (4·6) the explicit formulae we found for V and B.

However, in contrast with the usual BTs of the other solvable NLEEs, to the two integro-differential equations defining the BT one must add the constraint

$$\bar{P}^2=P^2\,. \tag{4·22}$$

For $m=0$ we obtain the trivial BT

$$\bar{P}=\gamma_0 P\gamma_0^{-1} \tag{4·23}$$

and for $m=1$ the x-component of the BT results to be

$$-2i\sigma_3(\bar{P}\gamma_0-\gamma_0 P)+\bar{P}_x\gamma_1-\gamma_1 P_x=0\,. \tag{4·24}$$

If one considers a NLEE obtained by a reduction procedure from a NLEE in the hierarchy (2·36) as, for instance, the Harry-Dym equation or the nonlinear Schrödinger type or KdV type equation considered in Refs. 6) and 7), the constraint (4·27) reduces the BT to the identity transformation.*)

If Tr $\sigma_3\gamma_0\gamma_1^{-1}=0$ one can easily verify that the BT (4·24) reduces to the trivial BT: $\bar{P}=\gamma_1 P\gamma_1^{-1}$.

If Tr $\sigma_3\gamma_0\gamma_1^{-1}\neq 0$, we multiply (4·24) by $\bar{P}\gamma_1$ and compute the trace of the obtained matrix equation. As the result we obtain

$$A\bar{q}+B\bar{r}+C=0\,, \tag{4·25}$$

where the coefficients

*) The same result for the Harry-Dym equation has been obtained by using the method of prolongation structure of Estabrook and Wahlquist by M. Leo, R. A. Leo, L. Solombrino and G. Soliani (private communication).

$$A=\beta_2(\alpha_2 r-\beta_2 r_x), \tag{4·26}$$

$$B=\beta_1(\alpha_1 q-\beta_1 q_x), \tag{4·27}$$

$$C=\beta_1\beta_2(qr)_x+(\beta_1\alpha_2+\beta_2\alpha_1)qr \tag{4·28}$$

are computed taking into account the constraint $\bar{r}\bar{q}=rq$ and where the constants α_i, β_i are defined as follows:

$$2i\sigma_3\gamma_0=\mathrm{diag}(\alpha_1, \alpha_2), \tag{4·29}$$

$$\gamma_1=\mathrm{diag}(\beta_1, \beta_2). \tag{4·30}$$

From (4·25) by multiplying, successively, by $\bar{q}$ and $\bar{r}$ we get two quadratic equations in $\bar{q}$ and $\bar{r}$, which furnish the following explicit formulae for $\bar{q}$ and $\bar{r}$:

$$\bar{q}=\frac{1}{2}A^{-1}(-C+\Delta), \tag{4·31}$$

$$\bar{r}=\frac{1}{2}B^{-1}(-C-\Delta) \tag{4·32}$$

with $\Delta^2=(C^2-4ABqr)$.

The found equations for $\bar{q}$ and $\bar{r}$ satisfy the constraint $\bar{q}\bar{r}=qr$ but, in general, they *do not satisfy identically* the differential equation (4·24).

Therefore, in general, the constraint $\bar{P}^2=P^2$ is not compatible with the system of differential equations defining the BT of the NLEEs related to the WKI spectral problem.

We conclude that the considered NLEEs on account of their highly non linear character, either do not have self-BTs at all or have self-BTs that must be obtained with a more involved procedure than that one successfully used for the NLEEs related to the $N\times N$ AKNS[8] and 2×2 Kaup-Newell spectral problems[9] and to other spectral problems.[16)~18)]

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