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The differential closure of a given ordinary differential field k is characterized to within (differential) k -isomorphism as a differentially closed (differential) extension field \hat{k} of k which is k -isomorphic to a subfield of any differentially closed extension field of k . It has been conjectured that, in analogy to the cases of the algebraic closure of a field and the real closure of an ordered field, the differential closure of any differential field k is minimal, that is, not k -isomorphic to a proper subfield of itself. The conjecture is here shown to be false.

Let k be a differential field (ordinary, that is with one specified derivation) of characteristic zero and let $k\{y\}$ be the differential ring of differential polynomials over k in the differential indeterminate y . Recall that the *order* of a nonzero differential polynomial in $k\{y\}$ is simply the smallest integer $r \geq -1$ such that the differential polynomial involves none of the derivatives $y^{(r+1)}, y^{(r+2)}, \dots$. According to Lenore Blum's definition, k is *differentially closed* if, for any $f, g \in k\{y\}$ with g of smaller order than f , there is a zero of f in k that is not a zero of g . For any differential field k , a *differential closure* of k is a differential extension field \hat{k} of k that is differentially closed and that can be k -embedded in any differentially closed differential extension field of k . Blum has used the methods of model theory to show the existence of \hat{k} and to derive a number of its properties [2], appreciably extending and simplifying a theory initiated by Abraham Robinson [5]. The uniqueness of \hat{k} to within differential k -isomorphism follows from a recent result of Shelah [7]. The differential closure \hat{k} of k is called *minimal* if there is no (differential) k -isomorphism of \hat{k} with a proper subfield of itself. One of the unsolved problems of the theory has been to determine whether or not \hat{k} is always minimal. Sacks has conjectured [6] that \hat{k} is minimal over k in the special case $k = \mathbb{Q}$. It is proved here, among other things, that this conjecture is false. It was learned after the completion of this paper that this result has also been proved by Kolchin [4] and announced by Shelah [8]. The author is greatly indebted to Lenore Blum for calling his attention to the problem and for numerous conversations on her work.

We begin by recalling some facts outlined in a recent paper of Ax [1]. Let $k \subset K$ be fields. There is a K -module $\Omega_{K/k}^1$, the space of differential forms of degree one of K/k , and a k -linear map $d: K \rightarrow \Omega_{K/k}^1$ such that $d(xy) = xdy + ydx$ for all $x, y \in K$ (and these can be

constructed just by insisting on universality for these properties) which is the usual dual space of the K -module of k -derivations of K , a vector space over K of dimension $\text{tr. deg. } K/k$ if the latter is finite and the field characteristic is zero. For any derivation D of K such that $Dk \subset k$, there is a map $D^1: \Omega_{K|k}^1 \rightarrow \Omega_{K|k}^1$ (most easily constructed using the universal properties of $\Omega_{K|k}^1$) which is characterized by the following properties: for all $\omega, \eta \in \Omega_{K|k}^1$ and all $f \in K$ we have $D^1(\omega + \eta) = D^1\omega + D^1\eta$, $D^1(f\omega) = (Df)\omega + f(D^1\omega)$, $D^1(df) = d(Df)$.

The following generalizes a lemma in Ax's paper [1, Lemma 3].

LEMMA 1. *Let $k \subset K$ be fields of characteristic zero, D a derivation of K such that $Dk \subset k$, C the D -constants of k , u and t elements of K that are algebraically dependent over C . Consider the k -differential of K given by udt . Then $D^1(udt) = d(uDt)$.*

For $D^1(udt) = (Du)dt + udDt$, while $d(uDt) = (Dt)du + udDt$, so we have to show that $(Du)dt = (Dt)du$. Let U, T be indeterminates over C and let $F(U, T) \in C[U, T]$ be an irreducible polynomial such that $F(u, t) = 0$. If u is transcendental over C then t is algebraic over $C(u)$ and $F(u, T)$ is irreducible over $C(u)$, so that $(\partial F/\partial T)(u, t) \neq 0$. Similarly if t is transcendental over C then $(\partial F/\partial U)(u, t) \neq 0$. The relation $(Du)dt = (Dt)du$ follows from the equations

$$\begin{aligned} \frac{\partial F}{\partial U}(u, t)du + \frac{\partial F}{\partial T}(u, t)dt &= 0, \\ \frac{\partial F}{\partial U}(u, t)Du + \frac{\partial F}{\partial T}(u, t)Dt &= 0 \end{aligned}$$

unless $(\partial F/\partial U)(u, t)$ and $(\partial F/\partial T)(u, t)$ are both zero, which can happen only if u and t are both algebraic over C , in which case both du and dt are zero.

PROPOSITION 1. *Let k be a differential field of characteristic zero, C its field of constants, x an indeterminate over C , and $f(x)$ a nonzero element of $C(x)$ such that $1/f(x)$ has the form*

$$\frac{1}{f(x)} = \sum_{i=1}^n c_i \frac{\partial u_i(x)/\partial x}{u_i(x)} + \frac{\partial v(x)}{\partial x},$$

where $c_1, \dots, c_n \in C$ and $u_1(x), \dots, u_n(x), v(x) \in C(x)$. Let x_1, x_2 be elements of a differential extension field of k whose constants are all algebraic over k , each of x_1, x_2 being a solution of the differential equation $x' = f(x)$, and suppose that x_1, x_2 are algebraically dependent over k . Then either x_1 or x_2 is algebraic over k or $(v(x_1))' = (v(x_2))'$.

The field $K = k(x_1, x_2)$ is a differential extension field of k , so for $j = 1, 2$ we may apply the Lemma to $dx_j/f(x_j) \in \Omega_{K/k}^1$ and $D = '$ to get

$$D^1\left(\frac{dx_j}{f(x_j)}\right) = d\left(\frac{Dx_j}{f(x_j)}\right) = D(1) = 0.$$

Assuming that neither x_1 nor x_2 is algebraic over k , each $dx_j/f(x_j)$ is a nonzero element of the one-dimensional K -module $\Omega_{K/k}^1$, so that we can write $dx_2/f(x_2) = cdx_1/f(x_1)$, for some nonzero $c \in K$. Hence

$$0 = D^1\left(\frac{dx_2}{f(x_2)}\right) = D^1\left(c\frac{dx_1}{f(x_1)}\right) = (Dc)\frac{dx_1}{f(x_1)} + cD^1\left(\frac{dx_1}{f(x_1)}\right) = (Dc)\frac{dx_1}{f(x_1)},$$

so that $Dc = 0$. Thus c is a constant of K , hence, by assumption, algebraic over k . Now for $j = 1, 2$,

$$\frac{dx_j}{f(x_j)} = \sum_{i=1}^n c_i \frac{\frac{\partial u_i}{\partial x}(x_j)}{u_i(x_j)} dx_j + \frac{\partial v}{\partial x}(x_j) dx_j = \sum_{i=1}^n c_i \frac{du_i(x_j)}{u_i(x_j)} + dv(x_j),$$

so that

$$\sum_{i=1}^n c_i \frac{du_i(x_2)}{u_i(x_2)} + dv(x_2) = c \left(\sum_{i=1}^n c_i \frac{du_i(x_1)}{u_i(x_1)} + dv(x_1) \right).$$

From the well-known fact that a linear combination with constant coefficients of normal differentials of third kind can be exact only if it is zero (cf. [1, Prop. 2], which generalizes the usual residue considerations) we deduce

$$\sum_{i=1}^n c_i \frac{du_i(x_2)}{u_i(x_2)} = \sum_{i=1}^n c_i \frac{du_i(x_1)}{u_i(x_1)}, \quad dv(x_2) = cdv(x_1).$$

Thus

$$\begin{aligned} (v(x_2))' &= \frac{\partial v}{\partial x}(x_2)x_2' = \frac{\partial v}{\partial x}(x_2)f(x_2) = \frac{dv(x_2)}{dx_2/f(x_2)} = \frac{cdv(x_1)}{c(dx_1/f(x_1))} \\ &= \frac{dv(x_1)}{dx_1/f(x_1)} = (v(x_1))'. \end{aligned}$$

Note that if C is algebraically closed, then any element of $C(x)$ can be written in the form prescribed for $1/f(x)$ in Proposition 1, as is seen by looking at partial fractions with respect to $C[x]$. Note also that since $(v(x_j))' = (\partial v/\partial x)(x_j)x_j' = (\partial v/\partial x)(x_j)f(x_j)$, $j = 1, 2$, the conclusion of Proposition 1 can be written

$$\left(\frac{\partial v}{\partial x}(x_1)\right)^{-1} \sum_{i=1}^n c_i \frac{\frac{\partial u_i}{\partial x}(x_1)}{u_i(x_1)} = \left(\frac{\partial v}{\partial x}(x_2)\right)^{-1} \sum_{i=1}^n c_i \frac{\frac{\partial u_i}{\partial x}(x_2)}{u_i(x_2)}.$$

REMARK. The condition in Proposition 1 that x_1 and x_2 be elements of a differential extension field of k whose constants are algebraic over k will certainly be satisfied if all the constants of $k(x_1, x_2)$ are algebraic over C , and this latter condition will automatically hold for most $f(x)$ of interest, in virtue of Lemma 2 and Proposition 2 below. For the same reason, the condition on constants in the following Corollary is superfluous. But we do not need this information for the nonminimality proof.

COROLLARY. *Let k be a differential field of characteristic zero, and suppose that x_1, x_2 are elements of a differential extension field of k whose constants are all algebraic over k , both x_1 and x_2 being solutions of the differential equation $x' = f(x)$, where $f(x)$ is either $x/(x + 1)$ or $x^3 - x^2$. Then if x_1 and x_2 are algebraically dependent over k , either x_1 or x_2 is algebraic over k , or $x_1 = x_2$.*

First note that Proposition 1 is applicable since $1/f(x)$ is of the correct form, namely either

$$\frac{x + 1}{x} = \frac{1}{x} + 1 = \frac{\partial x / \partial x}{x} + \frac{\partial x}{\partial x}$$

or

$$\frac{1}{x^3 - x^2} = \frac{1}{x - 1} - \frac{1}{x} - \frac{1}{x^2} = \frac{\frac{\partial}{\partial x} \left(\frac{x - 1}{x} \right)}{(x - 1)/x} + \frac{\partial}{\partial x} \left(\frac{1}{x} \right).$$

For $j = 1, 2$, in the case $f(x) = x/(x + 1)$ we have $(v(x_j))' = x_j' = x_j/(x_j + 1)$, while in the case $f(x) = x^3 - x^2$ we have $(v(x_j))' = (1/x_j)' = -x_j'/x_j^2 = 1 - x_j$, so the Corollary follows directly from the Proposition.

Now let C be a differential field of constants. We shall show that its differential closure \hat{C} is not minimal over C . Let x be an indeterminate over C , $f(x)$ a nonzero element of $C(x)$. For any x_1, x_2, \dots, x_n in \hat{C} , the differential equation $y' = f(y)$ has at least one solution in \hat{C} not annulling $(y - x_1)(y - x_2) \dots (y - x_n)$. Hence the differential equation $y' = f(y)$ has an infinity of solutions in \hat{C} . Since there are only a finite number of constant solutions of $y' = f(y)$, namely the zeros of $f(y)$, we can find distinct nonconstant elements x_1, x_2, \dots of \hat{C} such that $x_i' = f(x_i)$ for all $i = 1, 2, \dots$. We claim that in either of the special cases $f(x) = x/(x + 1)$ or $f(x) = x^3 - x^2$, the set $\{x_1, x_2, \dots\}$ is a set of indiscernibles over C (or, in the terminology of Sacks [4], a set of *conjugates over C*) and this fact will prove the nonminimality of \hat{C} over C [6, p. 633]. What has to be shown is that for any $n = 1, 2, \dots$ and any distinct positive integers i_1, \dots, i_n , the differential isomorphism class of $(x_{i_1}, \dots, x_{i_n})$ over C is

independent of the choice of i_1, \dots, i_n . Since $x'_i = f(x_i)$, $i = 1, 2, \dots$, it suffices to prove that the algebraic isomorphism class of $(x_{i_1}, \dots, x_{i_n})$ over C is independent of the choice of i_1, \dots, i_n , which will certainly be true if x_{i_1}, \dots, x_{i_n} are always algebraically independent over C . Hence we are reduced to proving that x_1, x_2, \dots are algebraically independent over C . As a preliminary, note that the constants of $C(x_1, x_2, \dots)$ are among the constants of \hat{C} , which are precisely the algebraic closure \bar{C} of C , an easy consequence of Blum's theory [2]. We now assume that for a certain $n = 1, 2, \dots$, the elements x_1, x_2, \dots, x_n are algebraically dependent over C , and we have to derive a contradiction. Taking n minimal and changing our notation, if necessary, we may assume that no proper subset of $\{x_1, \dots, x_n\}$ is algebraically dependent over C . If $n > 1$, then x_{n-1} and x_n are algebraically dependent over the differential field $C(x_1, \dots, x_{n-2})$ and are distinct solutions of the differential equation $x' = f(x)$, so the previous Corollary implies that either x_{n-1} or x_n is algebraic over $C(x_1, \dots, x_{n-2})$, a contradiction of the minimality of n , while if $n = 1$ we have x_1 algebraic over C , therefore a constant, again a contradiction. This proves that x_1, x_2, \dots are algebraically independent over C , and hence that \hat{C} is not minimal over C .

It is of interest to generalize somewhat the argument of the preceding paragraph. Let k be any differential field of characteristic zero and let x_1, x_2, \dots, x_n be distinct elements of a differential extension field of k , none algebraic over k , such that for each $i = 1, \dots, n$ we have $x'_i = f(x_i)$, where $f(x)$ is either $x/(x+1)$ or $x^3 - x^2$. Then x_1, \dots, x_n are algebraically independent over k and the constant subfields of $k(x_1, \dots, x_n)$ and of k are the same. To see this, we use the argument of the preceding paragraph, supplemented by Lemma 2 and Proposition 2 below. The Remark following Proposition 1 enables us to follow the above proof literally to get x_1, \dots, x_n algebraically independent over k , after which the equality of the constant subfields of $k(x_1, \dots, x_n)$ and of k is a direct consequence of Proposition 2.

LEMMA 2. *Let K be a differential field, algebraic over its differential subfield k . Then the constants of K are algebraic over the subfield of constants of k .*

For let c be a constant of K , let $n = [k(c):k]$, and pick $a_1, \dots, a_n \in k$ such that $c^n + a_1 c^{n-1} + \dots + a_n = 0$. Differentiation gives $a'_1 c^{n-1} + \dots + a'_n = 0$, from which we deduce that each $a'_i = 0$, so each a_i is a constant of k .

LEMMA 3. *Let $k \subset K$ be differential fields of characteristic zero,*

$C \subset \mathcal{E}$ their respective subfields of constants, and suppose that k is algebraically closed in K and that K is a finite field extension of k of transcendence degree one. Then if $C \neq \mathcal{E}$, C is algebraically closed in \mathcal{E} and \mathcal{E} is a finite field extension of C of transcendence degree one of genus at most that of K/k .

Start the proof by noting that since $C = k \cap \mathcal{E}$ and k is algebraically closed in K , we have C algebraically closed in \mathcal{E} . Suppose that $C \neq \mathcal{E}$ and let $t \in \mathcal{E}$, $t \notin C$. Then t is transcendental over C , and indeed over k . If also $u \in \mathcal{E}$, then t and u are algebraically dependent over k , so there exists an irreducible $f(T, U) \in k[T, U]$, T and U being indeterminates over k , such that $f(t, u) = 0$. The minimal polynomial of u over $k(t)$ is $f(t, U)$, up to a factor in $k(t)$, and $f(T, U)$ is unique, up to a factor in k , with the degree in U of $f(T, U)$ at most $[K:k(t)]$. Let $f(T, U) = \sum_{i,j} a_{ij} t^i u^j$, with each $a_{ij} \in k$, and with at least one of the a_{ij} 's equal to 1. Applying the derivation D of K , we get $\sum_{i,j} (Da_{ij}) t^i u^j = 0$. Now $\sum_{i,j} (Da_{ij}) T^i U^j$ must equal a multiple of $f(T, U)$, necessarily by an element of k , and this element of k must be 0 since one of the a_{ij} 's is 1. Thus $Da_{ij} = 0$ for all i, j , so that each $a_{ij} \in k \cap \mathcal{E} = C$. Therefore u is algebraic over $C(t)$, of degree at most $[K:k(t)]$. Therefore \mathcal{E} is algebraic over $C(t)$, with $[\mathcal{E}:C(t)] \leq [K:k(t)]$. It remains to prove the genus statement, and here we give two proofs, each relying on well-known facts about ground field extensions of algebraic function fields that may be found in [3]. First, if $\omega = fdg$ is a differential of first kind of \mathcal{E}/C , with $f, g \in \mathcal{E}$, then ω can also be considered a differential of K/k ; in fact we have a natural injection of differentials $\Omega_{\mathcal{E}/C}^1 \rightarrow \Omega_{K/k}^1$. For any k -place P of K , if f, g are finite at P then ω , considered as a differential of K/k , is also finite at P . If either f or g is not finite at P , then P induces a C -place p of \mathcal{E} , and since ω is finite at p we can write $\omega = f_1 dg_1$, with $f_1, g_1 \in \mathcal{E}$ both finite at p , so that again ω is finite at P . Therefore ω , considered as a differential of K/k , is of the first kind. Let $\omega_1, \dots, \omega_g$ be a C -basis for the space of differentials of first kind of \mathcal{E}/C ($g = \text{genus of } \mathcal{E}/C$). If $\omega_1, \dots, \omega_g$, considered as differentials of K/k , are linearly dependent over k , then there exist $a_1, \dots, a_g \in k$, not all zero, such that $a_1 \omega_1 + \dots + a_g \omega_g = 0$. Suppose that we have such a_1, \dots, a_g , with a minimal number of nonzero a_i 's, one of which is 1. Since each $\omega_i/\omega_1 \in \mathcal{E}$, applying D to $a_1(\omega_1/\omega_1) + \dots + a_g(\omega_g/\omega_1) = 0$ we get $(Da_1)(\omega_1/\omega_1) + \dots + (Da_g)(\omega_g/\omega_1) = 0$. At least one Da_i is 0, so that each $Da_i = 0$, so each $a_i \in \mathcal{E}$. Thus $a_i \in \mathcal{E} \cap k = C$, contradicting the linear independence of $\omega_1, \dots, \omega_g$ over C . Therefore $\omega_1, \dots, \omega_g$ are k -linearly independent differentials of first kind of K/k , so that the genus of K/k is at least g . For the second proof of the genus statement, consider what happens

when we extend the ground field C of the function field \mathcal{E}/C from C to k . Since C is algebraically closed in k , $\mathcal{E} \otimes_C k$ is an integral domain, isomorphic to $\mathcal{E}[k] \subset K$, and so the ground field extension, which preserves the genus of \mathcal{E}/C , gives us $\mathcal{E}(k)/k$. Since $\mathcal{E}(k)$ is a subfield of K that contains k , its genus is at most that of K/k . This completes the second proof.

PROPOSITION 2. *Let k be a differential field of characteristic zero, with derivation D and constants C . Let $k(x)$ be a pure transcendental extension field of k , let $f(x)$ be a nonzero element of $k(x)$, and make $k(x)$ a differential extension field of k by setting $Dx = f(x)$. Suppose that $1/f(x)$ is of neither of the forms*

$$(\text{element of } C) \frac{\partial u(x)/\partial x}{u(x)} \quad \text{nor} \quad \frac{\partial v(x)}{\partial x},$$

for $u(x), v(x) \in C(x)$. Then every constant of $k(x)$ is in C .

To prove this, first assume that C is algebraically closed. Suppose that not all constants of $k(x)$ are in C . By Lemma 3, the subfield of constants of $k(x)$ is an algebraic function field of one variable over C of genus zero, hence, since C is algebraically closed, of the form $C(t)$, for some $t \in k(x)$, $t \notin k$. Now consider the nonzero differentials dt and $dx/f(x)$ of $k(x)/k$. We can write $dx/f(x) = \alpha dt$, for some $\alpha \in k(x)$. Applying the operator D^1 on $\Omega_{k(x)/k}^1$, we get $D^1(dx/f(x)) = D^1(\alpha dt) = (D\alpha)dt + \alpha dD^1t = (D\alpha)dt$. By Lemma 1, $D^1(dx/f(x)) = d(Dx/f(x)) = d(1) = 0$, so $D\alpha = 0$, so that $\alpha \in C(t)$. That is, $dx/f(x) = \alpha dt$, with $\alpha \in C(t)$. Now write $dx/f(x)$ in the form

$$\frac{dx}{f(x)} = \sum_{i=1}^n c_i \frac{du_i(x)}{u_i(x)} + dv(x),$$

with $c_1, \dots, c_n \in C$ and $u_1(x), \dots, u_n(x), v(x) \in C(x)$, which can be done immediately by looking at the partial fraction expansion of $1/f(x)$ with respect to $C[x]$. Using the logarithmic derivative identities

$$\frac{d(ab)}{ab} = \frac{da}{a} + \frac{db}{b}, \quad \frac{da^\nu}{a^\nu} = \nu \frac{da}{a},$$

we can, if necessary, modify $n, c_1, \dots, c_n, u_1(x), \dots, u_n(x)$ so that c_1, \dots, c_n are linearly independent over the rational numbers \mathbb{Q} . Looking at the partial fraction decomposition of α with respect to $C[t]$, we get an expression

$$\alpha dt = \sum_{i=1}^m \gamma_i \frac{dw_i}{w_i} + dy,$$

where $\gamma_1, \dots, \gamma_m \in C$ and $w_1, \dots, w_m, y \in C(t)$. Extend c_1, \dots, c_n to a basis $c_1, \dots, c_n, c_{n+1}, c_{n+2}, \dots, c_N$ of the \mathbf{Q} -vector space $\mathbf{Q}c_1 + \dots + \mathbf{Q}c_n + \mathbf{Q}\gamma_1 + \dots + \mathbf{Q}\gamma_m$. Using the logarithmic derivative identities, we can modify $m, \gamma_1, \dots, \gamma_m, w_1, \dots, w_m$, so that the same expression for αdt holds with $m = N$, and $\gamma_1 = c_1/M, \dots, \gamma_N = c_N/M$ for some positive integer M . The above expression for $dx/f(x)$ remains true if we replace n by N , taking $u_{n+1}(x) = u_{n+2}(x) = \dots = 1$. Hence we may assume that in the displayed expressions for $dx/f(x)$ and αdt we have $m = n, c_1, \dots, c_n$ linearly independent over \mathbf{Q} , and $M\gamma_1 = c_1, \dots, M\gamma_n = c_n$, for some positive integer M . From the equation $dx/f(x) = \alpha dt$ we now infer

$$\sum_{i=1}^n c_i \frac{d((u_i(x))^M/w_i)}{(u_i(x))^M/w_i} + Md(v(x) - y) = 0.$$

At this point we again apply, in more precise form than was necessary for the proof of Proposition 1, the argument about when a linear combination of normal differential forms of third kind is exact [1, Prop. 2] to deduce that each $d((u_i(x))^M/w_i)$ and $d(v(x) - y)$ are zero. (This conclusion can be directly verified in the present case by expressing each $(u_i(x))^M/w_i$ as a power product of irreducible elements of $k[x]$ and $v(x) - y$ in terms of partial fractions.) Therefore $(u_1(x))^M/w_1, \dots, (u_n(x))^M/w_n, v(x) - y \in k$, so that also $D((u_1(x))^M/w_1), \dots, D((u_n(x))^M/w_n), D(v(x) - y) \in k$. Since w_1, \dots, w_n, y are constants, we deduce that

$$(Du_1(x))/u_1(x), \dots, (Du_n(x))/u_n(x), Dv(x) \in k.$$

But $u_1(x), \dots, u_n(x), v(x)$ are in the differential field $C(x)$, so that $(Du_1(x))/u_1(x), \dots, (Du_n(x))/u_n(x), Dv(x) \in k \cap C(x) = C$. Now for any $\phi(x) \in C(x)$ we have $D\phi(x) = (\partial\phi(x)/\partial x)Dx = (\partial\phi(x)/\partial x)f(x)$. At least one of the quantities $u_1(x), \dots, u_n(x), v(x)$ is not in k , for otherwise $dx = 0$, so at least one of

$$\frac{\partial u_1(x)/\partial x}{u_1(x)} f(x), \dots, \frac{\partial u_n(x)/\partial x}{u_n(x)} f(x), \frac{\partial v(x)}{\partial x} f(x)$$

is a nonzero element of C , implying that $1/f(x)$ is of one of the excluded forms. It remains to prove the Proposition when C is not algebraically closed. Suppose that there are constants of $k(x)$ that are not in C . The differential field structures on k and $k(x)$ extend uniquely to differential field structures on $k(\bar{C})$ and $(k(\bar{C}))(x)$, \bar{C} being the algebraic closure of C , and we get constants of $(k(\bar{C}))(x)$ that are not in the subfield of constants \bar{C} of $k(\bar{C})$, since $k(x) \cap \bar{C} = C$. Hence $1/f(x)$ is of the form $a(\partial u/\partial x)/u$ for some $a \in \bar{C}, u \in \bar{C}(x)$, or of the form $1/f(x) = \partial v/\partial x$, for some $v \in \bar{C}(x)$. Suppose first that $1/f(x) =$

$a(\partial u/\partial x)/u$, with a and u as above. Take u , as we may, to be a quotient of monic elements of $\bar{C}[x]$. We shall be done if we show that $a \in C$, $u \in C(x)$. For any $\sigma \in \text{Aut}(\bar{C}(x)/C(x)) \approx \text{Aut}(\bar{C}/C)$ we get $1/f(x) = a^\sigma(\partial u^\sigma/\partial x)/u^\sigma$, so that $a(\partial u/\partial x)/u = a^\sigma(\partial u^\sigma/\partial x)/u^\sigma$, or

$$(\partial(u^\sigma/u)/\partial x)/(u^\sigma/u) = a/a^\sigma \in \bar{C}.$$

Writing u^σ/u as a power product of distinct monic linear elements of $\bar{C}[x]$, we see that we get a nonconstant function on the left of the equation for a/a^σ unless $u^\sigma/u = 1$. Hence $u^\sigma = u$. Since this is true for all $\sigma \in \text{Aut}(\bar{C}/C)$, we get $u \in C(x)$, hence also $a \in C(x) \cap \bar{C} = C$, showing $1/f(x)$ to be of the desired form. Suppose, finally, that we have $1/f(x) = \partial v/\partial x$, for some $v \in \bar{C}(x)$. We may take v such that its partial fraction expansion with respect to $\bar{C}[x]$ has constant term zero. We wish to show $v \in C(x)$. For any $\sigma \in \text{Aut}(\bar{C}/C)$ we get $1/f(x) = (\partial v/\partial x)^\sigma = \partial v^\sigma/\partial x$, so that $\partial v^\sigma/\partial x = \partial v/\partial x$. Hence $v^\sigma = v$, and since this is true for all $\sigma \in \text{Aut}(\bar{C}/C)$ we get $v \in C(x)$, as desired.

Clearly neither of the two special values for $f(x)$ of which we have made so much use, namely $x/(x+1)$ and $x^3 - x^2$, is of the special form indicated in Proposition 2.

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