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THE NONTRIVIALITY OF THE FIRST RATIONAL HOMOLOGY GROUP OF SOME CONNECTED INVARIANT SUBSETS OF PERIODIC TRANSFORMATIONS

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ABSTRACT. This note was inspired by some results of P. A. Smith [S]. One proves that for any periodic map of a manifold M and any codimension two invariant submanifold P of M containing part of the stationary point set, connected invariant subsets of the complement of P must carry nontrivial one-dimensional rational cycles, provided that M satisfies some simple homological conditions (Theorem A). This fact has interesting consequences in transformation group theory.

0. Introduction. If $m \geqslant 2$ is a positive integer let $Z_m = \mathbf{Z}_m$ be the cyclic group of order m and if $m = \infty$ let Z_m resp. $\mathbf{Z}_m$ be the infinite cyclic group, respectively, $S^1 = \{z \in \mathbf{C} \mid |z| = 1\}$. We also denote by $G_{m,n}$, $m, n = 2, 3, 4, \ldots, \infty$, a semidirect product $\mathbf{Z}_m \ltimes_t \mathbf{Z}_n$ for $t\colon \mathbf{Z}_n \to \operatorname{Aut} \mathbf{Z}_m$. Such a semidirect product has the inclusion $\mathbf{Z}_m \xrightarrow{i} G_{m,n}$, the projection $G_{m,n} \xrightarrow{\pi} \mathbf{Z}_n$ and the section $s\colon \mathbf{Z}_n \to G_{m,n}$ ($\pi \circ s = \mathrm{id}$) as part of the data. Clearly if $n = \infty$, $G_{m,n} = \mathbf{Z}_m \times S^1$. The groups $G_{m,n}$ are regarded as compact Lie groups.

Given a compact Lie group G, $\mu\colon G \times M \to M$ a topological action, N an invariant submanifold, and $x \in M$, then the action $\mu\colon G \times (M, N) \to (M, N)$ is called locally smooth at x if there exists a smooth action $\tilde{\mu}\colon G \times V \to V$, an invariant submanifold $W \subset V$, and an invariant neighborhood $\mathcal{U}$ of $x \in M$ together with an equivariant homeomorphism $\psi(\mathcal{U}, \mathcal{U} \cap N) \to (V, W)$. The main result of this note is the following:

THEOREM A. *Let* $\mu\colon \mathbf{Z}_m \times M^n \to M^n$ *be a smooth* (*topological*) *orientation preserving action,* $P^{n-2} \subset M^n$ *an invariant smooth* (*locally flat*) *closed oriented submanifold whose orientation is also invariant. Assume that* $H_1(M)$ *is torsion prime to* m, $H_2(M; \mathbf{Q}) = 0$, *and*

(i) $M^{\mathbf{Z}_m} \neq \varnothing$[2] *and* $P \cap M^{\mathbf{Z}_m} \neq \varnothing$ *is a union of connected components of* $M^{\mathbf{Z}_m}$,

(ii) *if* μ *is a topological action then there exists* $y \in P \cap M^{\mathbf{Z}_m}$ *so that* $\mu\colon \mathbf{Z}_m \times (M, P) \to (M, P)$ *is locally smooth at* y.

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[2]For an action $\mu\colon G \times M \to M$ one denotes by M^G the fixed point set of μ.

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Then

(a) *for any connected invariant subset* $X \subseteq M \setminus P$, $H_1(X; Q) \to H_1(M \setminus P; Q)$ *and* $H_1(X; Z_m) \to H_1(M \setminus P; Z_m)$ *are nontrivial,*

(b) $P \supset M^{\mathbf{Z}_m}$.

Theorem A is applied to transformation groups through

COROLLARY A′. *Let* M, μ, P *be as in Proposition* A *and let* $K^r \subset M^n$ *be a nonempty connected invariant submanifold with* $H_1(K^r; k) = 0$ *for* $k = \mathbf{Q}$ *or* $\mathbf{Z}_m$. *Then* $K^r \cap P \neq 0$ *and* $\dim(K^r \cap P) \geqslant r - 2$.

Corollary A′ implies Theorem B and is an important tool in the proof of Theorem C.

THEOREM B (P. A. SMITH). *Suppose* $G_{p,q}$ *acts locally smoothly and effectively on the acyclic topological manifold* M^n *preserving the orientation. Let* p, q *be distinct primes or* ∞, *and* $\dim(M^{\mathbf{Z}_q}) = n - 2$. *Then* $\dim M^{G_{p,q}} \neq \varnothing$ *and* $\dim M^{G_{p,q}} = (\dim M^{\mathbf{Z}_p}) - 2$.

Theorem B is due to P. A. Smith whose paper **[S]** has inspired this note. Smith has stated it for $G_{p,q} = Z_{pq}$ and without any conclusion about the dimension $M^{G_{p,q}}$. Notice that if we drop the condition $\dim(M^{\mathbf{Z}_p}) = n - 2$ the conclusion is false (see Kister **[K2]**). A systematic class of fixed point free $G_{p,q}$-actions has been constructed by Assadi **[A]**.

A manifold M^{n+r} is called r-spheric if its universal cover has the homotopy type of S^r.

THEOREM C. (1) *Let* M^{n+2} *be a 2-spheric topological manifold and* $\mu\colon \mathbf{Z}_m \times M^{n+2} \to M^{n+2}$ *be a locally smooth effective action with m prime or* ∞. *If* $M^{\mathbf{Z}_m} \neq \varnothing$ *then*

$$H_*\left(M^{\mathbf{Z}_m}; Z_m\right) = H_*\left(S^0 \times K\left(\pi_1(M), 1\right); Z_m\right)$$

and $\dim M^{\mathbf{Z}_m} = n$; *hence, if* $m = \infty$, $\pi_1(M)$ *is a Poincaré duality group of formal dimension* n.

(2) *Let* M^{n+2} *be a 2-spheric topological manifold and* $\mu\colon G_{m,r} \times M^{n+2} \to M^{n+2}$ *an orientation preserving effective locally smooth action with* $M^{\mathbf{Z}_m} \neq \varnothing$ *and* $M^{\mathbf{Z}_r} \neq \varnothing$. *Then m and r are finite and* $M^{\mathbf{Z}_m} \cap M^{\mathbf{Z}_r}$ *is a nonempty manifold with all connected components of dimension n and* $n - 2$. *If there are connected components of dimension n then* $G_{m,r} = \mathbf{Z}_{mr}$. *If all components have dimension n then* $M^{\mathbf{Z}_m} = M^{\mathbf{Z}_r} = M^{G_{m,r}}$ *and the action is semifree. If* $m = r$ *all components have dimension n.*

The proof of Theorem A and Corollary A′ will be given in §1, and the proof of Theorems B and C in §2.

1.

PROOF OF THEOREM A. It is easy to see that (a) implies (b) because any stationary point outside P forms a connected invariant subset with trivial homology. To prove (a) we observe that one can choose a closed tubular neighborhood of P, $\tilde{\pi}\colon \tilde{P} \to P$, $\tilde{P} \subset M$ so that for at least one $y \in P \cap M^{\mathbf{Z}_m}$, $\tilde{\pi}^{-1}(y)$ is invariant. If the action is smooth, a closed invariant tubular neighborhood is the right choice for $\tilde{P}$. If the action is not smooth, let $y \in P \cap M^{\mathbf{Z}_m}$ so that $\mu\colon \mathbf{Z}_m \times (M, P) \to (M, P)$ is locally smooth at y. We choose an invariant tubular neighborhood with fibre discs for a small neighborhood of y in P. Eventually shrinking this neighborhood, we can extend this to a closed tubular neighborhood of P because, for locally flat P, this is possible by a theorem of Kirby **[K1]** which states the existence and uniqueness of the closed tubular neighborhood for closed locally flat embeddings in codimension 2. Let $E = \partial\tilde{P}$ and $\pi = \tilde{\pi}|\partial\tilde{P}$; π: $E \to P$ being the restriction of $\tilde{\pi}$ to $\partial\tilde{P} = E$ is a bundle with fibre S^1. Let us denote by $\tilde{E}$ the complement of $\mathrm{Int}(\tilde{P})$ in M so that $E = \tilde{E} \cap \tilde{P}$. The first step in our proof is to show that $H_1(M \setminus P) = H_1(\tilde{E}; Z)$ is isomorphic to $Z \oplus G$ where G is a torsion group with $G \otimes Z_m = 0$, and for any $z \in P$ the map i_z: $S^1 = \pi^{-1}(z) \to E \to M \setminus P$ represents in the homology group $H_1(M \setminus P; Z)$ an element of the form $(t, g) \in Z \oplus G$ with t prime to m.

Assuming the first step proved, the proof of Theorem A(a), for $m = \infty$, goes as follows: Let μ_x: $S^1 \to M \setminus P$ be the restriction of the action μ to $S^1 \times x$ where $x \in M \setminus P$. Since any connected invariant subset of $M \setminus P$, say X, contains $\mu_{x_0}(S^1)$ for $x_0 \in X$, it suffices to show that for each x, μ_x: $S^1 \to M \setminus P$ induces a nontrivial homomorphism $(\mu_x)^*$: $H_1(S^1; Q) \to H_1(M \setminus P; Q)$. Because $M \setminus P$ is connected it suffices to verify this for one x since if $x', x'' \in M \setminus P$, $\mu_{x'}$ is homotopic to $\mu_{x''}$. Now choose $x \in \pi^{-1}(y)$ where $y \in P \cap M^{S^1}$ is a point where $\tilde{\pi}^{-1}(y)$ is S^1-invariant; μ_x: $S^1 \to \mu_x(S^1) = \pi^{-1}(y)$ is a finite cover. Therefore the image of the generator $u \in H_1(S^1; Z)$ in $H_1(M \setminus P; Z)$ is (kt, kg) with $k \neq 0$. This proves our statement for $m = \infty$.

If $m \neq \infty$ the proof requires supplementary considerations. Let μ: $\mathbf{Z}_m \times M \to M$ be our action and let σ: $M \to M$ the homeomorphism induced by the generator. Given an invariant subset $U \subset M$, a 1-dimensional singular simplex l: $[0, 1] \to U$ is called special if $l(1) = \sigma(l(0))$. Each special simplex l defines a map μ^l: $S^1 \to U$ by taking

$$\mu^l(e^{i\theta}) = \begin{cases} l\left(\dfrac{\theta m}{2\pi}\right) & \text{if } 0 \leqslant \theta \leqslant \dfrac{2\pi}{m}, \\ \sigma \circ l\left(\dfrac{\theta m - 2\pi}{2\pi}\right) & \text{if } \dfrac{2\pi}{m} \leqslant \theta \leqslant \dfrac{4\pi}{m}, \\ \sigma^{m-1} \circ l\left(\dfrac{\theta m - 2\pi(m-1)}{2\pi}\right) & \text{if } \dfrac{2\pi}{m}(m-1) \leqslant \theta \leqslant 2\pi, \end{cases}$$

which represents the integral cycle $\Sigma l = l + \sigma l + \cdots + \sigma^{m-1} l$. If l, l': $[0, 1] \to U$ are two special simplices let γ: $[0, 1] \to U$ be a singular 1-dimensional simplex with $\gamma(0) = l(0)$, $\gamma(1) = l'(0)$. The 1-dimensional singular chain $\Delta = \gamma + l' - \sigma\gamma - l$ is obviously a cycle and $\Sigma l = \Sigma l' - \Delta - \sigma\Delta - \cdots - \sigma^{m-1}\Delta$. In the sequel, let U be $M \setminus P$. As a second step of our proof we will show that: if σ_* is the isomorphism of $H_1(M \setminus P; Z) = Z \oplus G$, resp. of $H_1(M \setminus P; Z_m) = Z_m$ induced by σ, then $\sigma_*(s, g) = (s, \theta(s, g))$ where $s \in Z$, $g, \theta(s, g) \in G$, resp. $\sigma_*(s) = s$, $s \in Z_m$. Assuming this second step proved, it suffices to show that for some special 1-simplex l, the integral homology class of $[\Sigma l]$ resp. the homology class mod m, $[\Sigma l]_m$ is of the form $(t', g') \in Z \oplus G$ with t' prime to m resp. $\mathbf{t}' \in Z_m$ where $\mathbf{t}' \neq 0$. Then we can conclude that for any other special simplex l', $[\Sigma l'] = [\Sigma l] + [\Delta] + \sigma_*[\Delta] + \cdots + \sigma_*^{m-1}[\Delta] = (t' + mr, g'') \in Z \oplus G$ where $[\Delta] = (r, g'')$ resp. $[\Sigma l']_m = [\Sigma l]_m = \mathbf{t}'$. Hence any integral homology class $[\Sigma l']$ is of infinite order and nontrivial resp. any mod m homology class $[\Sigma l'] = \mathbf{t}'$ is hence nontrivial. Now let us take l to be a special cycle lying on $\pi^{-1}(y)$, $y \in P \cap M^{\mathbf{Z}_m}$ where $\tilde{\pi}^{-1}(y)$ is invariant. We can choose l so that $[\Sigma l] = c[\pi^{-1}(y)]$ where $0 < c < m$ and c divides m. Hence $[\Sigma l] = [ct, g']$ with $ct \neq 0 \bmod m$; thus $[\Sigma l]_m \neq 0$ which verifies the claim of Theorem A(a).

To prove Step 1, consider the exact sequence

$$(*) \qquad \to H_2(\tilde{E}; k) \xrightarrow{i_2} H_2(M; k) \xrightarrow{j_2} H_2(M, \tilde{E}; k) \to H_1(\tilde{E}; k) \xrightarrow{i_1} H_1(M; k) \to \cdots$$

associated with the pair $(M, \tilde{E})$ and observe that because $H_2(M; Q) = 0$ and

$$H_i(M, \tilde{E}; k) = H_i(\tilde{P}, E; k) = \begin{cases} 0 & \text{if } i = 0, 1, \\ k & \text{if } i = 2, \end{cases}$$

j_2 is zero. Hence $(*)$ shortens to

$$0 \to k \to H_1(\tilde{E}; k) \to H_1(M; k) \quad \text{and} \quad H_2(\tilde{E}; k) \xrightarrow{i_2} H_2(M; k) \to 0.$$

By hypotheses $H_1(M; Z)$ is torsion prime to m and for $k = Z_m$, $H_1(\tilde{E}; Z_m) = Z_m$. This implies $H_1(\tilde{E}; Z) = Z \oplus G$ with G a torsion group so that $G \otimes Z_m = 0$.

The Mayer-Vietoris sequence applied to $M = \tilde{E} \cup \tilde{P}$, $E = \tilde{E} \cap \tilde{P}$ gives the exact sequence

$$(**) \qquad \to H_2(\tilde{E}; k) \oplus H_2(\tilde{P}; k) \xrightarrow{i_2 \oplus \pi_2} H_2(M; k) \xrightarrow{\delta} H_1(E; k)$$
$$\xrightarrow{i \oplus \pi_1} H_1(\tilde{E}; k) \oplus H_1(\tilde{P}; k) \to H_1(M; k).$$

Because i_2 is surjective, so is $i_2 \oplus \pi_2$, hence δ is zero. Thus $(**)$ shortens to

$$0 \to H_1(E; k) \xrightarrow{i \oplus \pi_1} H_1(\tilde{E}; k) \oplus H_1(\tilde{P}; k) \to H_1(M; k).$$

Since in the diagram below the horizontal line is the abelianization of the homotopy exact sequence of the bundle $S^1_y \xrightarrow{i_y} E \to P$

$$\begin{array}{ccccc} H_1(S^1_y; k) & \xrightarrow{(i_y)_1} & H_1(E; k) & \xrightarrow{\pi_1} & H_1(P; k) \to 0 \\ \uparrow\simeq & \nearrow & & & \\ H_2(\tilde{\pi}^{-1}(y), \pi^{-1}(y); k) & & & & \\ \downarrow\simeq & \nearrow & & & \\ H_2(\tilde{P}, E, k) & & & & \end{array}$$

and δ^E is injective, $(i_y)_1$ has to be injective. δ^E is injective because in the commutative diagram below $\Delta^{\tilde{E}}$ is injective and w is an isomorphism:

$$\begin{array}{ccc} H_2(M, \tilde{E}; k) & \xrightarrow{\delta^{\tilde{E}}} & H_1(\tilde{E}; k) \\ w \uparrow\simeq & & \uparrow \\ H_2(\tilde{P}, E; k) & \xrightarrow{\delta^E} & H_1(E; k) \end{array}$$

Consequently we have the exact sequence

$$0 \to H_1(S_y^1; k) \to H_1(E; k) \to H_1(P; k) \to 0.$$

Since for $k = Z_m$, $i_1 \oplus \pi_1$ is an isomorphism, the composition $H_1(S_y^1; Z_m) \to H_1(E; Z_m) \to H_1(\tilde{E}; Z_m)$ is an isomorphism, hence $t \equiv 1 \bmod m$ and the first step is proved.

To prove Step 2 we observe that by Step 1 the component Z of $H_1(\tilde{E}; Z) = H_1(M \setminus P; Z)$ comes from $H_2(M, \tilde{E}; Z) = H_2(M, M \setminus P; Z)$ which is invariant under the action of $\mathbf{Z}_m$ since $H_2(M, \tilde{E}; Z) \overset{\approx}{\leftarrow} H_2(\tilde{\pi}^{-1}(y), \pi^{-1}(y); Z)$ and the action preserves the orientations of M and P; so that it is trivial on $H_2(\tilde{\pi}^{-1}(y), \pi^{-1}(y); Z)$.

Proof of Corollary A$'$. If $K^r \cap P = \varnothing$ or $\dim K^r \cap P < n - 2$, then $K \setminus K \cap P$ will be connected and provides an example of an invariant set with

$$H_1(K \setminus K \cap P: Q) = 0,$$

which is impossible by Proposition A.

The following example (due to R. Stong) shows that condition (i) of Theorem A cannot be weakened to $P^{\mathbf{Z}_m} \neq \varnothing$.

Example. Consider $M^n = S^{2j+1}$ with $\mathbf{Z}_m$ action $\rho(z_0, z_1, \ldots, z_j) = (z_0, z_1, e2\pi i z_2/m, \ldots, e2\pi i z_j/m)$ and let $P^{n-2} = S^{2j-1} = \{z | z_0 = 0\}$, where $n = 2j + 1$ and $j \geqslant 2$. Clearly M and P are connected, $H_2(M, Q) = H_1(M, Z) = 0$, $P^{\mathbf{Z}_m} \neq \varnothing$ and $M^{\mathbf{Z}_m} \not\subset P$. Also every subset of $S^1 = \{z | z_i = 0 \text{ for } i > 0\}$ is invariant and need not have a nontrivial image in $H_1(M - P; \mathbf{Z}_m)$. In fact, there are invariant circles which are homologically trivial.

2.

Proof of Theorem B. Let $P = M^{Z_q}$ and $K = M^{Z_p}$, and take μ: $Z_q \times M \to M$ to be the induced action by a subgroup isomorphic to Z_q. Then K is invariant under the Z_q-action and $K - M^{G_{p,q}} \subset M - P$. By Smith theory $\overline{H}_*(K; Z_p) = 0 = \overline{H}_*(P; Z_q)$. Thus K and P are both connected. If $p = \infty$, then $\overline{H}_*(K; Q) = 0$ and the conclusion follows from Corollary A$'$. If $p < \infty$ and $\dim M^{G_{p,q}} < \dim M^{Z_p} - 2$ (in particular if $M^{G_{p,q}} = \varnothing$), then $H_1(K - M^{G_{p,q}})$ will be p-divisible. The linking number of any 1-cycle of $M - P$ with P is well defined since $H_1(M) = H_2(M) = 0$.

If we choose $\tilde{\pi}$: $\tilde{P} \to P$ and $y \in P$ as in the proof of Theorem A, it is easily seen that one can find special 1-cycles in $\tilde{\pi}^{-1}(y) - \{y\} \subset M - P$ whose linking number with P is one. However, if a special 1-cycle lies on $K \setminus M^{G_{p,q}}$, its linking number with P must vanish, being p-divisible. This contradiction establishes Theorem B.

Proof of Theorem C. Let $\tilde{M} \xrightarrow{f} M$ be the universal covering of M. Since $M^{\mathbf{Z}_m} \neq \varnothing$, the action μ can be lifted to an action $\tilde{\mu}$: $\mathbf{Z}_m \times \tilde{M} \to \tilde{M}$ with $f^{-1}(M^{\mathbf{Z}_m}) = \tilde{M}^{\mathbf{Z}_m}$; hence $\tilde{M}^{\mathbf{Z}_m} \xrightarrow{f} M^{\mathbf{Z}_m}$ is a principal covering with group $\pi_1(M) = \pi$. If m is a prime or ∞, by Smith theory $\tilde{H}_*(M^{\mathbf{Z}_m}; Z_m) = H_*(S^k; Z_m)$ where $k = 0$ or 2; if $k = 2$ Smith theory equally implies that $H_*(\tilde{M}^{\mathbf{Z}_m}; Z_m) \to H(M; Z_m)$ is an isomorphism [**B**, Chapter III] which implies (comparing the spectral sequences associated with the fibrations $\tilde{M} \to M \to K(\pi, 1)$ and $\tilde{M}^{\mathbf{Z}_m} \to M^{\mathbf{Z}_m} \to K(\pi, 1)$) that M and $M^{\mathbf{Z}_m}$ have the same homology with coefficients in Z_m; this is impossible, hence $k = 0$ and $f^{-1}(M^{\mathbf{Z}_m})$ consists of two connected components $\tilde{P}_1$ and $\tilde{P}_2$ each of them with trivial homology mod m and being the total space of principal coverings with group π. By spectral sequence arguments applied to the fibrations $\tilde{P}_i \to K(\pi, 1)$ we have $H_*(P_i; Z_m) = H_*(K(\pi, 1); Z_m)$. We also claim that dim $P_i = n$ because the composition $P_i \to M \to K(\pi, 1)$ induces an isomorphism for homology with coefficients in Z_m which combined with the Gysin sequence of the fibration $\tilde{M} \to M \to K(\pi, 1)$ implies $H_n(K(\pi, 1); Z_m) = H_n(P_i; Z_m) = Z_m$. This proves Theorem C(1).

Before we start the proof of (2) we notice that $\tilde{M} \setminus \tilde{P}_i$ satisfies $H_1(\tilde{M} \setminus \tilde{P}_i; Z_m) = 0$ and $H_2(\tilde{M} \setminus \tilde{P}_i; Q) = 0$ for m prime and finite. To see this, one considers the exact sequence of the pair $(\tilde{M}, \tilde{M} \setminus \tilde{P}_i)$,

$$\to H_3(\tilde{M}, \tilde{M} \setminus \tilde{P}_i; Z_m) \to H_2(\tilde{M} \setminus \tilde{P}_i; Z_m) \to H_2(\tilde{M}; Z_m)$$

$$\overset{\psi}{\to} H_2(\tilde{M}, \tilde{M} \setminus \tilde{P}_i; Z_m) \to H_1(\tilde{M} \setminus P_i; Z_m) \to 0$$

which, after applying the isomorphism $H_3(\tilde{M}, \tilde{M} \setminus \tilde{P}_i; Z_m) = H_1(\tilde{P}_i; Z_m)$ and Smith theory, $H_1(\tilde{P}_i; Z_m) = 0$ becomes

$$0 \to H_2(\tilde{M} \setminus \tilde{P}_i; Z_m) \to Z_m \overset{\psi}{\to} Z_m \to H_1(\tilde{M} \setminus \tilde{P}_i; Z_m) \to 0.$$

If m is prime (and finite) ψ is not zero, hence it is an isomorphism. To see this, assume $\psi = 0$; hence $Z_m = H_2(\tilde{M}, \tilde{M} \setminus \tilde{P}_i; Z_m) \to H_1(\tilde{M} \setminus \tilde{P}_i; Z_m)$ is an isomorphism. If $\tilde{\tilde{P}}_i$ is a closed invariant tubular neighborhood of $\tilde{P}_i$ and π: $\hat{P}_i = \partial \tilde{\tilde{P}}_i \to \tilde{P}_i$ the equivariant map which is a bundle with fibre $S^1_x = \pi^{-1}(x)$, $x \in \tilde{P}_i$, then $H_1(S^1_x; Z_m) \to H_1(\hat{P}_i; Z_m) \to H_1(\tilde{M} \setminus \tilde{P}_i; Z_m)$ is an isomorphism for each x and it is exactly the isomorphism $H_2(\tilde{M}; \tilde{M} \setminus \tilde{P}_i; Z_m) \to H_1(\tilde{M} \setminus \tilde{P}_i; Z_m)$. But since $\tilde{P}_i$ consists of fixed points, S^1_x is a cycle of type Σl for a special simplex l, therefore its homology class mod m is the same as that of any other cycle $\Sigma l'$ with l' special (as in the proof of Theorem A). If we choose l' to be the degenerate special simplex which lies in $\tilde{P}_2$ if $i = 1$ or $\tilde{P}_1$ if $i = 2$, clearly the homology class (mod m) $[\Sigma l']$ is zero which is in contradiction with our assumption that $H_2(\tilde{M}, \tilde{M} \setminus \tilde{P}_i; Z_m) = Z_m \to H_1(\tilde{M} \setminus \tilde{P}_i; Z_m)$ is an isomorphism. Consequently ψ is an isomorphism and therefore $H_1(\tilde{M} \setminus \tilde{P}_i; Z_m) = 0$ and $H_2(\tilde{M} \setminus \tilde{P}_i; Z_m) = 0$.

To prove (2) let μ_ε, $\varepsilon = m$ or r, be the restriction of the action of μ to $\mathbf{Z}_\varepsilon$. Since μ_ε has fixed points, let $\tilde{\mu}_\varepsilon$ be liftings of μ_ε to the actions $\tilde{\mu}_\varepsilon$: $\mathbf{Z}_\varepsilon \to \tilde{M} \to \tilde{M}$ with $f^{-1}(M^{\mathbf{Z}_\varepsilon}) = \tilde{M}^{\mathbf{Z}_\varepsilon}$. Notice that $\tilde{M}^{\mathbf{Z}_m}$ is invariant for $\tilde{\mu}_r$ because $M^{\mathbf{Z}_m}$ is invariant for μ_r. If $G_{m,r} = \mathbf{Z}_m \ltimes {}_t\mathbf{Z}_r$, then $\tilde{M}^{\mathbf{Z}_r}$ is invariant for $\tilde{\mu}_m$. Assume $m = \infty$ and let $\tilde{P}_1$ and $\tilde{P}_2$ resp. $\tilde{R}_1$ and $\tilde{R}_2$ be the connected components of $\tilde{M}^{\mathbf{Z}_m}$ resp. $\tilde{M}^{\mathbf{Z}_r}$. By Smith theory we know that $\tilde{P}_i$ have trivial homology for any coefficients; we also know that $\tilde{P}_i$ is invariant to $\tilde{\mu}_r$ and the $(\tilde{R}_1 \cup \tilde{R}_2) \cap \tilde{P}_i = \tilde{P}_i^{\mathbf{Z}_n}$. Since $H_*(\tilde{P}_i^{\mathbf{Z}_n}; Z_n) = H_*(M; Z_n)$, $\tilde{P}_i^{\mathbf{Z}_m}$ is nonempty so $\tilde{P}_i$ intersects at least one of the components $\tilde{R}_1$ or $\tilde{R}_2$, say $\tilde{R}_1$. Moreover if $\tilde{R}_1 \cap \tilde{P}_i \neq \tilde{P}_i$, $\tilde{R}_1 \cap \tilde{P}_i$ has to be of dimension at most $n - 2$. Since $\tilde{R}_1 \cap \tilde{P}_i \to \tilde{R}_1 \cap \tilde{P}_i$ is a principal covering with group π, one concludes that $H_n(K(\pi, 1); Z_n) = 0$. But as we have seen in the proof of (1) this is not possible, hence $\tilde{R}_1 \cap \tilde{P}_i = \tilde{P}_i$. Since $\tilde{R}_1 \supset \tilde{P}_i$, and both are closed connected submanifolds of dimension n, we have $R_1 = P_i$, which shows that $\tilde{M}^{\mathbf{Z}_m} = \tilde{M}^{\mathbf{Z}_n}$. Of course this is not possible because this will imply that $G_{m,r}$ acts without fixed points on the fibre S^1_x of the boundary of a tubular neighborhood of $M^{G_{m,r}}$. If $r = \infty$ then $G_{m,r} = \mathbf{Z}_m \times S^1$

which is a group of the type $G_{\infty,m}$ previously considered. Now assume m and r are finite. We apply Corollary A′ to $\tilde{M}\setminus\tilde{R}_i$ equipped with the action μ_r ($\tilde{M}\setminus\tilde{R}$ as M and $\tilde{R}_j$ as P, $j=2$ if $i=1$ and $j=1$ if $i=2$ and $\tilde{P}_j$ as R); we conclude that $\tilde{P}_i \cap \tilde{R}_j \neq \varnothing$, and the dimension of each connected component is $\geqslant n-2$. Since $\tilde{P}_i^{\mathbf{Z}_r} = \tilde{P}_i \cap (\tilde{R}_1 \cup \tilde{R}_2)$, then either $\tilde{R}_j = \tilde{P}_i$ or $\dim(\tilde{P}_i \cap \tilde{R}_j) \leqslant n-2$; hence the dimension of each component of $\tilde{P}_i \cap \tilde{R}_j$ is either n or $n-2$. In the first case $\tilde{R}_j = \tilde{P}_i$ which implies that $G_{m,r}$ acts freely on S^1 which is possible only if $G_{m,r}$ is a cyclic group, hence $G_{m,r} = Z_{mr}$. The same conclusion holds if $M^{\mathbf{Z}_m} = M^{\mathbf{Z}_n}$. If $m=r$, $M^{\mathbf{Z}_m} \cap M^{\mathbf{Z}_r}$ cannot have components of dimension $n-2$ because if they have, then $\tilde{M}^{\mathbf{Z}_m} \cap \tilde{M}^{\mathbf{Z}_r}$ has components whose homology with Z_m coefficients is trivial by Smith theory and consequently, $H_n(K(\pi,1); Z_m) = 0$; clearly this is not the case because of (1).

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