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The Novel Mittag-Leffler–Galerkin Method: Application to a Riccati Differential Equation of Fractional Order

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Abstract: We present a new numerical approach to solving the fractional differential Riccati equations numerically. The approach—called the Mittag-Leffler–Galerkin method—comprises the finite Mittag-Leffler function and the Galerkin method. The error analysis of the method was studied. As a result, we present two theorems by which the error can be bounded. In addition to error analysis, the residual correction method, which allows us to estimate the error and obtain new approximate solutions, is also presented. To show how the method is applied, and the efficiency of the proposed method, some test examples were considered. When the numerical results obtained were examined, it was found that while the method achieves better results than some of the known methods in the literature, it also achieves results that are similar to those of others of the known methods.

Keywords: finite Mittag-Leffler function; fractional differential Riccati equations (FDRE); Caputo fractional derivatives; error analysis; error estimation

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1. Introduction

Fractional differential equations (FDEs) have been used to describe real-life phenomena such as continuum mechanics [1], viscoelasticity [2], finance [3], optimal control [4], variational problems [5], hydrologic modeling [6] and fluid mechanics [7], amongst others [8,9]. Due to the difficulty of obtaining exact solutions, the importance of developing effective methods for the numerical solutions of FDEs has been recognized in recent decades. The main methods used to solve FDEs include radial basis functions [10], fractional finite volume [11], Adomian decomposition [12], operational methods [13] and other numerical approaches [14–16]. In this study, we propose a new numerical solution method of solving the FDRE, defined as

$$D^{\nu}x(t) + a(t)x^2(t) + b(t)x(t) = g(t), \quad 0 < \nu \leq 1, \quad 0 \leq t \leq 1, \tag{1}$$

and the initial condition

$$x(0) = x_0. \tag{2}$$

Here, $x(t)$ is the unknown function, $a(t), b(t)$ and $g(t)$ are known functions defined in $[0,1]$ and continuous, and x_0 is a real constant.

Recently, spectral approaches have been applied to solving different types of FDEs. Esmaeili and Shamsi [17] considered a family of FDEs, and solved it by a pseudo-spectral method. Zhang et al. [18] solved the one-dimensional nonlinear space fractional Schrödinger equation, using the Crank–Nicolson–Galerkin–Legendre spectral method. Mokhtary and Ghoreishi [19] used the tau spectral method for the solutions of nonlinear fractional integrodifferential equations. Brawy et al. [20] introduced an operational approximation method, based on the spectral collocation method for the solution of fractional Schrödinger equations. Vanani and Aminataei [21] improved the algebraic formulation of fractional partial differential equations, by using matrix–vector multiplication representation, and then applied an operational approach of the tau method. Doha et al. [22] proposed a spectral method for the solution of the fractional subdiffusion equation. The approach was based on the shifted Legendre tau spectral method. Fan et al. [23] proposed a Galerkin finite element method for solving the fractional wave equation: they discretized the problem of the Crank–Nicholson scheme, and presented the stability and convergence of the numerical scheme. Saadatmandi and Dehghan [24] used a Jacobi–Gauss–Lobatto and Gauss–Radau collocation method, based on shifted Jacobi polynomials, to solve fractional Fokker–Planck equations. Kazem [25] employed an integral operational matrix, based on Jacobi polynomials, to solve FDEs.

The differential Riccati equation (DRE) is used to describe miscellaneous engineering and physical phenomena, such as the flow of rivers, the transmission line phenomenon, stochastic control, dynamic games and financial mathematics [26–28]. The FDRE, which is a generalization of the DRE, has many applications in science and engineering [29–31], so various solution strategies have been suggested. Ozturk et al. [32] used the Taylor collocation method, converting the FDRE into a system of nonlinear algebraic equations, and then solving the system. Balaji [33] applied the Legendre wavelet operational matrix method to FDRE, to obtain its approximate solution. Mokhtary and Ghoreishi [34] introduced an operational method constituted of shifted Jacobi polynomials, to solve FDREs. Kashkari and Syam [35] used the Legendre operational matrix of fractional integration, to derive a numerical solution for FDREs. According to Jafari et al., [36] adopted a modified variation iteration method for FDREs, taking into account Adomian polynomials for nonlinear terms. Bota and Caruntu [37] applied the polynomial least squares method, to find an analytical solution for FDREs. Merdan [38] applied the fractional variational iteration method, to obtain an approximate analytical solution for nonlinear FDREs. Odibat and Momani [39] applied a modification of He's homotopy perturbation method to the quadratic FDRE. The homotopy analysis transform method, based on a combination of the homotopy analysis method and the Laplace decomposition method, was employed by Saad and Al-Shomrani [40] to solve FDREs. Haq et al. [41] applied the variation of parameters method, to obtain the analytical solutions of nonlinear quadratic FDREs. Sakar et al. [42] applied an iterative reproducing kernel Hilbert space method, to obtain the solutions of FDREs. Yuzbasi [43] studied with the Bernstein collocation method, to obtain the numerical solutions of FDREs. Li et al. [44] derived the Haar wavelet operational matrix method, to solve FDREs: they simplified the calculation of the nonlinear term using the block pulse function. Raja et al. [45] introduced a new computational intelligence technique, based on artificial neural networks and sequential quadratic programming, to solve nonlinear quadratic FDREs.

In this paper, we introduce a new method of solving FDREs. We approximate the solution by an expansion in the finite Mittag-Leffler function. By applying the Galerkin method, the FDRE is reduced to a nonlinear system of algebraic equations. Solving these equations gives the approximate solution to the problem. The rest of the paper is organized as follows: in Section 2, some necessary definitions of fractional calculus and the finite Mittag-Leffler function are presented, along with some properties of the function; the proposed method is introduced in Section 4; our analysis of the error is presented in Section 5; some theorems for the error analysis of the method, with the residual correction procedure, are presented; in Section 6, we give some test examples, to illustrate the application steps of the method; we compare the results of the proposed method to the results of some other methods; finally, in Section 7, we summarize the results.

2. Fractional Calculus and Finite Mittag–Leffler Function

In this section, we first give some fundamental definitions of fractional calculus. Then, the properties of the finite Mittag-Leffler function (MLF) and its fractional derivative are introduced.

2.1. Fractional Calculus and Mittag-Leffler Function

Definition 1 ([46])**.** *The fractional integral of order $\nu > 0$ with the lower limit zero for a function x is defined as*

$$I^{\nu}x(t) = \frac{1}{\Gamma(\nu)}\int_0^t \frac{x(s)}{(t-s)^{1-\nu}}ds, \quad \nu > 0. \tag{3}$$

Here, the right-hand side is defined pointwise on $[0,\infty)$, and $\Gamma(.)$ is the Gamma function.

Definition 2 ([46])**.** *The Caputo derivative of order ν with the lower limit zero for a function x is defined as*

$$D^{\nu}x(t) = \frac{1}{\Gamma(\lceil\nu\rceil-\nu)}\int_0^t \frac{x^{(\lceil\nu\rceil)}(s)}{(t-s)^{\nu+1-\lceil\nu\rceil}}ds \tag{4}$$

$$= I^{(\lceil\nu\rceil-\nu)}x^{(\lceil\nu\rceil)}(t), \; t>0, \nu>0, \tag{5}$$

where $\lceil\nu\rceil$ is the ceiling function of ν.

The following properties [47] apply to the Caputo fractional derivative operator: we have, for constants $\xi_i, i = 1,2,...,N$,

$$D^{\nu}\sum_{i=1}^{N}\xi_i x_i(t) = \sum_{i=1}^{N}\xi_i D^{\nu}x_i(t), \tag{6}$$

as well as, from [48],

$$D^{\nu}t^{N} = \frac{\Gamma(N+1)}{\Gamma(N+1-\nu)}t^{N-\nu}, N > \nu-1. \tag{7}$$

In the case of ν as an integer, the Caputo differential operator will coincide with the usual differential operator.

2.2. Mittag-Leffler Function

The Mittag-Leffler function $E^{\xi,\eta}$ is a function that is dependent on two parameters, ξ and η. When the real component of ξ is strictly positive, it can be described by the following series:

Definition 3 ([46])**.** *The MLF of two-parameter ξ,η is defined by*

$$E^{\xi,\eta}(t) = \sum_{i=0}^{\infty}\frac{t^i}{\Gamma(\xi i+\eta)}, \quad \xi>0,\ \eta>0,\ t\in\mathbb{R},$$

where $\Gamma(\xi i+\eta)$ is the gamma function. As a special case, we have $E^{1,1}(t) = e^t$.

3. Finite Mittag-Leffler Function and Its Fractional Derivative

The novel definition of the two-parameter finite Mittag-Leffler function of any integer i is:

Definition 4. *The finite MLF of two-parameter ξ,η can be defined as*

$$E_i^{\xi,\eta}(t) = \sum_{k=0}^{i}\frac{(-1)^k t^k}{\Gamma(\xi k+\eta)}, \quad \xi>0,\ \eta>0,\ t\in\mathbb{R}, \tag{8}$$

that is:

$$E_i^{\xi,\eta}(t) = \frac{(-1)^i t^i}{\Gamma(\xi i + \eta)} + \frac{(-1)^{i-1} t^{i-1}}{\Gamma(\xi(i-1) + \eta)} + \cdots + \frac{-t}{\Gamma(\xi + \eta)} + \frac{1}{\Gamma(\eta)}.$$

Based on the above definition, we can write

$$E_i^{\xi,\eta}(t) = \frac{(-1)^i t^i}{\Gamma(\xi i + \eta)} + \text{polynomials of degree} < \text{i}. \tag{9}$$

Using (7), the fractional-order derivative of the Mittag-Leffler function (8) can be calculated as

$$\begin{aligned} D^\nu E_i^{\xi,\eta}(t) &= \sum_{k=0}^{i} \frac{D^\nu (-1)^k t^k}{\Gamma(\xi k + \eta)} \\ &= \sum_{k=0}^{i} \frac{\Gamma(k+1)(-1)^k}{\Gamma(k+1-\nu)} \frac{t^{k-\nu}}{\Gamma(\xi k + \eta)}, \;\; \xi > 0, \; \eta > 0, \; t \in \mathbb{R}. \end{aligned}$$

4. The Fundamental Concepts of the Mittag-Leffler–Galerkin Method

In this section, we apply the Galerkin method, Equation (13), which has been used to solve problems in structural mechanics, dynamics, fluid flow, hydrodynamic stability, magnetohydrodynamics, heat and mass transfer, acoustics, microwave theory, neutron transport, etc. Problems governed by ordinary differential equations, partial differential equations and integral equations have been investigated via Galerkin formulations. Steady, unsteady and eigenvalue problems have proved to be equally amenable to the Galerkin treatment. Essentially, any problem for which governing equations can be written down is a candidate for a Galerkin method [49].

For the first time, we introduce a novel numerical method, namely the Mittag-Leffler–Galerkin (MLG) method, for solving FDREs.

Let $x(t)$ be the exact solution of Equations (1) and (2). We will use the proposed new MLG method to approximate the exact solution $x(t)$, as follows:

$$x(t) \simeq x_N(t) = \sum_{i=0}^{N} c_i E_i^{\xi,\eta}(t) = \mathrm{C}\, \mathbb{E}(t). \tag{10}$$

Here, C is an unknown constant matrix of size $1 \times (N+1)$ that must be determined, and $\mathbb{E}(t)$ is a matrix of size $(N+1) \times 1$ that consists of Mittag-Leffler-basis polynomial elements, defined as

$$\mathrm{C} = [\mathrm{c}_0, \ldots, \mathrm{c}_N]_{1\times(N+1)}, \quad \mathbb{E}(t) = \left[E_0^{\xi,\eta}(t), E_1^{\xi,\eta}(t), \ldots, E_N^{\xi,\eta}(t)\right]_{(N+1)\times 1}.$$

Directly from the Mittag-Leffler-basis polynomial elements, the fractional derivative $D^\nu x(t)$ can be stated as

$$D^\nu x(t) = \mathrm{C}\, D^\nu\, \mathbb{E}(t). \tag{11}$$

To use MLG to solve Equation (1), subject to the initial condition (2), first substitute Equations (10) and (11) into Equation (1), to obtain the residual function $\mathbb{R}(t)$:

$$\Re(t) = \mathrm{C}\, D^\nu\, \mathbb{E}(t) + a(t)(\mathrm{C}\, \mathbb{E}(t))^2 + b(t)\, \mathrm{C}\, \mathbb{E}(t) - g(t). \tag{12}$$

We can obtain N nonlinear algebraic equations sets, by using the Galerkin method [49]:

$$\int_0^1 \Re_N(t) E_i^{\xi,\eta}(t)\, \mathrm{d}t = 0, \quad i = 0, \dots, N-1. \tag{13}$$

In addition, by inserting the initial condition (2) into Equation (10), we get

$$x_N(0) = \mathrm{C}\,\mathbb{E}(0) = x_0. \tag{14}$$

The resulting combination of (13) and (14) yields $N+1$ of nonlinear equations with unknown coefficients $c_0, c_1, \dots, c_N$ that can be solved by the Newton method. As a result, the MLG solution to the $x_N(t)$ problem is obtained.

5. Error Analysis

This section investigates the error $e_N(t) = x(t) - x_N(t)$, where $x(t)$ is the exact solution and $x_N(t)$ is the MLG solution. We begin by defining the error by two theorems. The approach is then configured with the residual procedure, which produces estimates of the error, and new approximate solutions.

Theorem 1. *Let us approximate $x(t) \in C^\infty[0,1]$ as $x_N(t)$, given in (10). Then, for every $t \in [0,1]$ there exists $\varpi \in [0,1]$, such that*

$$\|x(t) - x_N(t)\|_\infty \le \frac{\Gamma(\xi(N+1)+\eta)}{(N+1)!} |E_{N+1}^{\xi,\eta}(t)| \max_{\varpi \in [0,1]} |x^{(N+1)}(\varpi)|. \tag{15}$$

Proof. Let $x(t) \in C^\infty[0,1]$ be approximated by $x_N(t)$, given in (10). Let us define the function:

$$L(t) = x(t) - x_N(t) - \theta E_{N+1}^{\xi,\eta}(t).$$

Let us select the parameter θ, such that the equation $L(t) = 0$ has a solution t_0, but t_0 is not a root of $E_{N+1}^{\xi,\eta}(t)$, i.e., $E_{N+1}^{\xi,\eta}(t_0) \neq 0$. Then, if we solve the equation $L(t_0)$, with respect to θ, we get $x(t_0) - x_N(t_0) - \theta E_{N+1}^{\xi,\eta}(t_0) = 0$, so that

$$\theta = \frac{x(t_0) - x_N(t_0)}{E_{N+1}^{\xi,\eta}(t_0)}. \tag{16}$$

Given $x(t) \in C^\infty[0,1], E_N^{\xi,\eta}(t_0) \in C^N[0,\infty]$ and $E_{N+1}^{\xi,\eta}(t_0) \in C^{N+1}[0,\infty]$, then $L(t) \in C^{N+1}[0,1]$ and $L^{(N+1)}(t)$ has at least one root in the interval; that is:

$$L^{(N+1)}(\varpi) = x^{(N+1)}(\varpi) - \theta \left[E_{N+1}^{\xi,\eta}(\varpi)\right]^{(N+1)} - \left[E_N^{\xi,\eta}(\varpi)\right]^{(N)} = 0. \tag{17}$$

By using (9), the last term of (17), given $\left[E_N^{\xi,\eta}(\varpi)\right]^{(N)} = 0$ and

$$E_{N+1}^{\varepsilon,\eta}(\varpi) = \frac{\varpi^{N+1}}{\Gamma(\xi(N+1)+\eta)} + \text{lower} - \text{degree polynomials},$$

then

$$\begin{aligned}\left[E_{N+1}^{\xi,\eta}(\varpi)\right]^{(N+1)} &= \frac{(N+1)N(N-1) \times \dots \times 3 \times 2 \times 1}{\Gamma(\xi(N+1)+\eta)} \\ &= \frac{(N+1)!}{\Gamma((\xi(N+1))+\eta)}.\end{aligned}$$

Substituting $\left[E^{\xi,\eta}_{N+1}(\varpi)\right]^{(N+1)}$ into (17) yields as follows:

$$\theta = \frac{\Gamma((\xi(N+1)) + \eta)}{(N+1)!} x^{(N+1)}(\varpi). \tag{18}$$

Using Equations (16)–(18), we can write the following equation:

$$x(t_0) - x_N(t_0) = \frac{\Gamma((\xi(N+1)) + \eta)}{(N+1)!} E^{\xi,\eta}_{N+1}(t_0) x^{(N+1)}(\varpi), \tag{19}$$

and so

$$|x(t_0) - x_N(t_0)| = \frac{\Gamma((\xi(N+1)) + \eta)}{((N+1)!} |E^{\varepsilon,\eta}_{N+1}(t_0)||x^{(N+1)}(\varpi)|.$$

Finally, taking the maximum of $|x^{(N+1)}(\varpi)|$ completes the proof. □

Theorem 2. *Let $x(t) \in C^{\infty}[0,1]$ be the solution, and $x_N(t)$ be the MLG solution of (1)–(2), respectively, and let $p_N(t) = \sum_{k=0}^{N} c^p_k E^{\xi,\eta}_k(t)$. Then, for every $t \in [0,1]$, there exists $\varpi \in [0,1]$, such that the following inequality holds:*

$$e_N(t) \leq \frac{\Gamma(\xi(N+1) + \eta)}{(N+1)!} |E^{\xi,\eta}_{N+1}(t)| \max_{\varpi \in [0,1]} |p^{(N+1)}(\varpi)| + \sum_{k=0}^{N} |c^p_k - c^x_k||E^{\xi,\eta}_k(t)|.$$

Proof. As $x_N(t)$ is the MLG solution of (1)–(2), we can write the error as

$$\begin{aligned}
|x(t) - x_N(t)| &= |x(t) - p_N(t) + p_N(t) - x_N(t)| \\
&\leq |x(t) - p_N(t)| + |p_N(t) - x_N(t)| \\
&\leq \frac{\Gamma(\xi(N+1) + \eta)}{(N+1)!} |E^{\xi,\eta}_{N+1}(t)| \max_{\varpi \in [0,1]} |p^{(N+1)}(\varpi)| + |p_N(t) - x_N(t)| \\
&\leq \frac{\Gamma(\xi(N+1) + \eta)}{(N+1)!} |E^{\xi,\eta}_{N+1}(t)| \max_{\varpi \in [0,1]} |p^{(N+1)}(\varpi)| + |\sum_{k=0}^{N} (c^p_k - c^x_k) E^{\xi,\eta}_k(t)| \\
&\leq \frac{\Gamma(\xi(N+1) + \eta)}{(N+1)!} |E^{\xi,\eta}_{N+1}(t)| \max_{\varpi \in [0,1]} |p^{(N+1)}(\varpi)| + \sum_{k=0}^{N} |c^p_k - c^x_k||E^{\xi,\eta}_k(t)|.
\end{aligned}$$

□

Now, we constitute the residual correction procedure.

Theorem 3. *Let $x(t)$ be the exact solution of (1)–(2), and let $x_N(t)$ be the MLG solution, respectively. The error $e_N(t)$ satisfies the following equation:*

$$\begin{cases} D^{\nu} e_N(t) = -a(t) e^2_N(t) - 2a(t) e_N(t) x_N(t) - b(t) e_N(t) - \Re_N(t), \\ e_N(0) = x_0 - x_N(0). \end{cases} \tag{20}$$

Proof. We have

$$\Re_N(t) = D^{\nu} x_N(t) + a(t) x^2_N(t) + b(t) x_N(t) - g(t),$$

and $D^{\nu}x(t) + a(t)x^2(t) + b(t)x(t) = g(t)$. Then:

$$\begin{aligned}
D^{\nu}e_N(t) &= D^{\mu}x(t) - D^{\nu}x_N(t) \\
&= -a(t)x^2(t) - b(t)x(t) + g(t) - \Re_N(t) + a(t)x_N^2(t) + b(t)x_N(t) - g(t) \\
&= -a(t)x^2(t) - b(t)x(t) - \Re_N(t) + a(t)x_N^2(t) + b(t)x_N(t) \\
&= -a(t)x^2(t) + a(t)x_N^2(t) - b(t)x(t) + b(t)x_N(t) - \Re_N(t) \\
&= -a(t)(x^2(t) - x_N^2(t)) - b(t)(x(t) - x_N(t)) - \Re_N(t) \\
&= -a(t)e_N(t)(x(t) + x_N(t)) - b(t)e_N(t) - \Re_N(t) \\
&= -a(t)e_N(t)(e_N(t) + 2x_N(t)) - b(t)e_N(t) - \Re_N(t) \\
&= -a(t)e_N^2(t) - 2a(t)e_N(t)x_N(t) - b(t)e_N(t) - \Re_N(t).
\end{aligned}$$

□

By solving the error problem (20) by the MLG method, we get the approximation for the error, i.e.,

$$e_{N,M}(t) = \sum_{i=0}^{M} c_i^e E_i^{\xi,\eta}(t),$$

where the coefficients $c_i^e, i = 0,1,2,\ldots,M$ are unknown constants. Hence, the error can be estimated by using $e_{N,M}(t)$ in the case of $\|e_N(t) - e_{N,M}(t)\| < \varepsilon$. On the other hand, $x_N(t) + e_{N,M}(t)$ is also an approximate solution of (1)–(2). We call the solution $x_N(t) + e_{N,M}(t)$, as the corrected MLG solution. The corrected MLG solution is a better approximation than the MLG solution in any given norm, whenever

$$\|e_N(t) - e_{N,M}(t)\| < \|x(t) - x_N(t)\|.$$

6. Numerical Experiments

In this section, we present some numerical experiments to show how the method is applied, to show the efficiency of the method—by giving the precision of the results obtained using the method—and to compare the method to other methods. All the experiments were performed using Maple on a laptop with an Intel Core i3 processor and 4GB of RAM.

Example 1. *As a first example, let us apply the method to the following FDRE, whose exact solution is $x(t) = t^2$ [35,37]:*

$$D^{\frac{1}{2}}x(t) + x(t) + x^2(t) = \frac{8}{3\sqrt{\pi}}t^{\frac{3}{2}} + t^2 + t^4, \quad 0 \le t \le 1, \tag{21}$$

with the condition

$$x(0) = 0. \tag{22}$$

For $N = 2$, the MLG solution can be written as

$$\begin{aligned}
x_2(t) &= \sum_{k=0}^{2} c_k E_k^{\xi,\eta}(t) \\
&= c_0E_0^{\xi,\eta}(t) + c_1E_1^{\xi,\eta}(t) + c_2E_2^{\xi,\eta}(t).
\end{aligned}$$

From $\xi = \frac{1}{2}, \eta = 1$, Equation (12) gives Equation (A1) (see Appendix A). In addition, we have, from Equation (22),

$$c_0 + c_1 + c_2 = 0. \tag{23}$$

Finally, by solving Equations (23) and (A1) (see Appendix A), we obtain

$$c_0 = -5.07096009275 \times 10^{-13}, \quad c_1 = -1.00000000000 \text{ and } \quad c_2 = 1.00000000000.$$

Thus,

$$\begin{aligned} x_2(t) &= c_0E_0^{\frac{1}{2},1}(t) + c_1E_1^{\frac{1}{2},1}(t) + c_2E_2^{\frac{1}{2},1}(t) \\ &= c_0 + c_1(1 - \frac{2t}{\sqrt{\pi}}) + c_2(1 - \frac{2t}{\sqrt{\pi}} + t^2) \\ &= -5.07096009275 \times 10^{-13} + 1.00000000000t^2 \\ &\simeq t^2. \end{aligned}$$

From Equation (23), we construct the error equation

$$\begin{cases} D^{\frac{1}{2}}(e_2(t) + x_2(t)) + e_2(t) + x(t) + (e_2(t) + x(t))^2 = \frac{8}{3\sqrt{\pi}}t^{\frac{3}{2}} + t^2 + t^4, \\ e_2(0) = x_2(0). \end{cases} \tag{24}$$

Let us solve error Equation (24) by using the MLG method for $M = 3$. The MLG solution of the error equation is obtained as

$$e_{2,3}(t) = 4 \times 10^{-198} - \frac{9.0414 \times 10^{-99}}{\sqrt{\pi}}t + 6.6595 \times 10^{-99}t^2 - \frac{3.9160 \times 10^{-100}}{\sqrt{\pi}}t^3.$$

Adding this solution to the MLG solution yields the corrected MLG solution. We present the error, with its estimations and the corrected error, in Figure 1 for $(N, M) = (2, 3)$. We can say, from the figures, that the error estimation obtained by the procedure well fits the error.

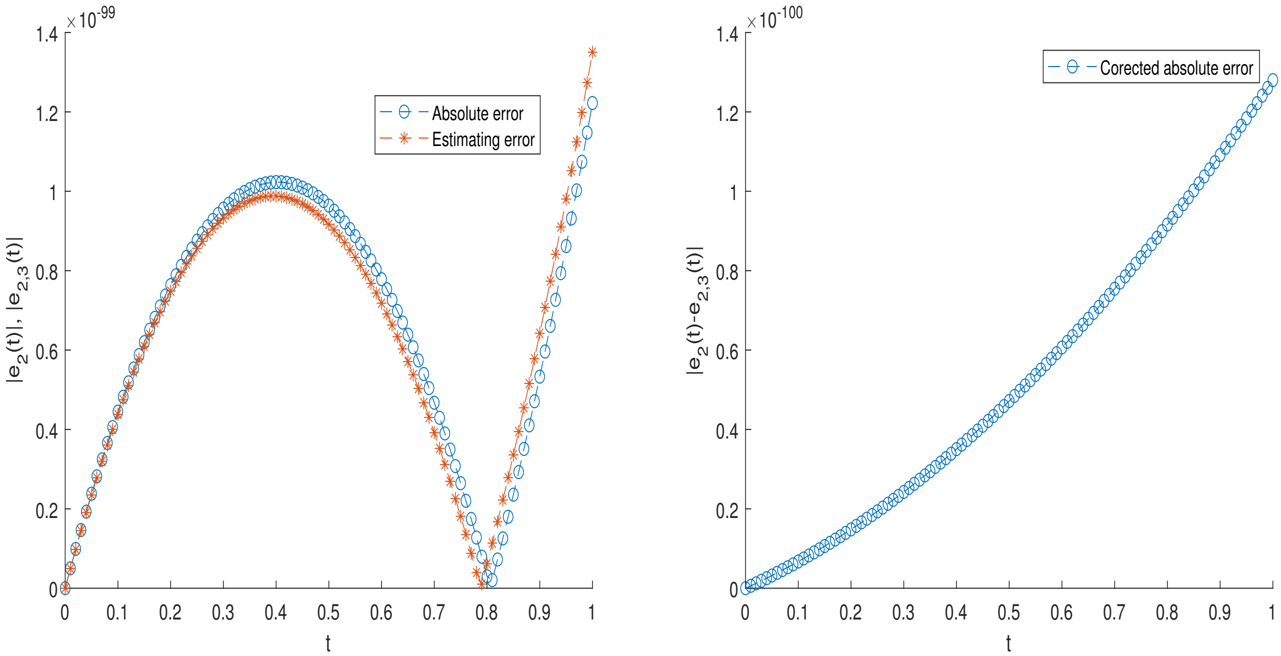

Figure 1. The error, its estimations and the corrected error, for Example 1 and $(N, M) = (2, 3)$.

The development of the basic definition, Equation (8), was realized this way, and affected the convergence of solutions; we added a comparison between the development and the original definition. Now, if we use

$$E_i^{\xi,\eta}(t) = \sum_{k=0}^{i} \frac{t^k}{\Gamma(\xi k + \eta)},$$

we have

$$c_0 = -5.10444760178, c_1 = 16.9410134078, c_2 = -11.8365658060,$$

and then

$$x_2(t) = \frac{10.2088952036t}{\sqrt{\pi}} - 11.8365658060t^2 \neq t^2.$$

It appeared to us that the original definition had a solution that was not convergent; however, in the developed definition, the solution was convergent.

Example 2. *Let us consider the following FDRE:*

$$D^{\frac{1}{2}}x(t) - tx^2(t) = \frac{16}{5}\frac{t^{\frac{5}{2}}}{\sqrt{\pi}} - t^7, \quad 0 \le t \le 1, \tag{25}$$

$$x(0) = 0. \tag{26}$$

The exact solution to the problem is

$$x(t) = t^3.$$

Let us find the MLG solution to the problem for $N = 3$, which can be written as follows:

$$x_3(t) = \sum_{k=0}^{3} c_k E_k^{\xi,\eta}(t) = c_0 E_0^{\xi,\eta}(t) + c_1 E_1^{\xi,\eta}(t) + c_2 E_2^{\xi,\eta}(t) + c_3 E_3^{\xi,\eta}(t).$$

From $\xi = \frac{1}{2}, \eta = 1$, (12) gives Equation (A2) (see Appendix A). In addition, we have, from Equation (26):

$$c_0 + c_1 + c_2 + c_3 = 0. \tag{27}$$

Finally, by solving Equations (27)–(A2), we obtain

$$c_0 = 1.1901312585 \times 10^{-9}, \quad c_1 = 3.0458221798 \times 10^{-9}, \quad c_2 = 1.3293403878,$$

and $c_3 = -1.329340392$. Thus, we obtain the MLG solution for the problem and $N = 3$ as

$$x_3(t) = -\frac{2.39 \times 10^{-9}}{\sqrt{\pi}}t + 4.24 \times 10^{-9}t^2 + \frac{1.77245385607}{\sqrt{\pi}}t^3 \simeq t^3.$$

From Equation (20), we construct the error equation

$$\begin{cases} D^{\frac{1}{2}}e_3(t) = te_3^2(t) + 2te_3(t)x_3(t) - \Re_3(t), \\ e_3(0) = -x_3(0), \end{cases} \tag{28}$$

where

$$\begin{aligned} \Re_3(t) &= D^{\frac{1}{2}}x_3(t) - tx_3^2(t) - (\frac{16}{5}\frac{t^{\frac{5}{2}}}{\sqrt{\pi}} - t^7) \\ &= 2.39449496167 \times 10^{-13}\sqrt{t}(7.53982239058 \times 10^{12}t^2 - 26640.7057024t \\ &\quad +6354.24705550) \\ &\quad -t(\frac{2.39000000000 \times 10^{-9}}{\sqrt{\pi}}t - 4.2400000000 \times 10^{-9}t^2 \\ &\quad +\frac{1.77245385607}{\sqrt{\pi}}t^3)^2 - \frac{16}{5\sqrt{\pi}}t^{\frac{5}{2}} + t^7. \end{aligned}$$

By solving the error problem (28) for $M = 4$, we obtain the estimation of the error as

$$e_{3,4}(t) = -3 \times 10^{-21} - \frac{2.40732926828 \times 10^{-9}}{\sqrt{\pi}} t + 4.30060806282 \times 10^{-9} t^2 - \frac{5.35088552742 \times 10^{-9}}{\sqrt{\pi}} t^3 + 5.4884232558 \times 10^{-11} t^4.$$

For $(N, M) = (3, 4)$, the error and the error estimation of Example 2 and the corrected error results are drawn in Figure 2.

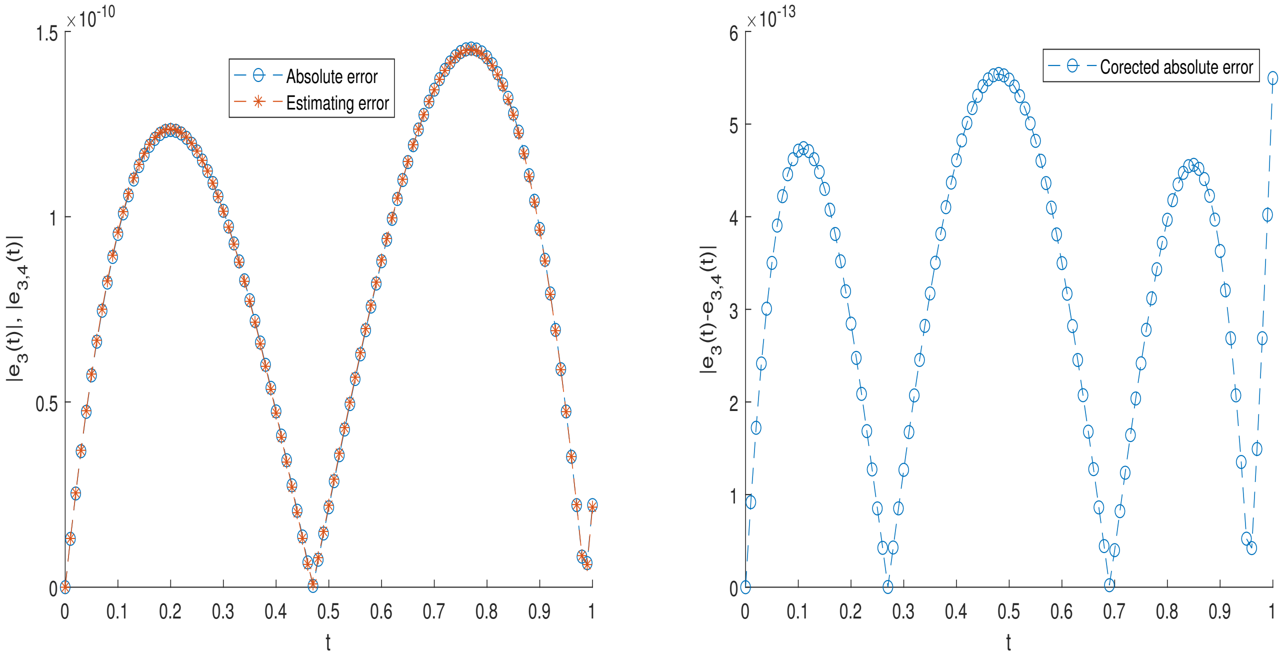

Figure 2. The error with its estimation by the procedure and the corrected error for Example 2 and $(N, M) = (3, 4)$.

In Table 1, we present the effects of the Mittag-Leffler parameters ξ and η on the error, which shows that our results are more accurate:

Table 1. The effects of the Mittag-Leffler parameters ξ and η on the error for Example 2 and $N = 3$.

ξ	η	**Absolute Error**
0.0	0.1	$5.782602642 \times 10^{-10}$
	0.2	$1.840492393 \times 10^{-9}$
	0.3	$1.900000000 \times 10^{-9}$
	0.4	$4.413182132 \times 10^{-9}$
	0.5	$1.560493121 \times 10^{-9}$
0.5	0.9	$1.853612308 \times 10^{-11}$
	1	$1.934584037 \times 10^{-11}$
	2	$1.150677936 \times 10^{-10}$

Example 3. *Consider the following FDRE [43,45]:*

$$D^{\nu}x(t) + x^2(t) - 1 = 0, \quad 0 \le t \le 1, \tag{29}$$

$$x(0) = 0. \tag{30}$$

The exact solution of the problem is $x(t) = \tanh(t)$ for $\nu = 1$.

We apply the MLG method to the problem for $\nu = 1$, $\xi = \frac{1}{2}, \eta = 1$. The MLG solutions for $N = 4$ and $N = 8$ are obtained as follows:

$$\begin{aligned} x_4(t) &= -1 \times 10^{-10} + 0.9998413015t + 0.0081494057t^2 \\ &\quad -0.3915419735t^3 + 0.1451458340t^4, \end{aligned}$$

$$\begin{aligned} x_8(x) &= 0.999794070t + 0.5516765e - 2t^2 + 0.1917058000t^4 + 0.4684949362t^6 \\ &\quad +0.0897t^8 - 0.381095428t^3 - 0.2724512057t^5 - 0.3400392225t^7 + 10^{-9}. \end{aligned}$$

By applying the method to the error equations obtained by x_4 and x_8, the estimations of e_4 and e_8 are found for $M = 7$ and $M = 9$, respectively, as follows:

$$\begin{aligned} e_{4,7}(t) &= c_0^e E_0^{\frac{1}{2},1}(t) + c_1^e E_1^{\frac{1}{2},1}(t) + c_2^e E_2^{\frac{1}{2},1}(t) + c_3^e E_3^{\frac{1}{2},1}(t) + c_4^e E_4^{\frac{1}{2},1}(t) + c_5^e E_5^{\frac{1}{2},1}(t) \\ &\quad + c_6^e E_6^{\frac{1}{2},1}(t) + c_7^e E_7^{\frac{1}{2},1}(t) \\ &= 1.8629 \times 10^{-4}t - 0.1208t^6 - 0.0090t^2 + 0.0663t^3 - 0.1822t^4 + 0.2240t^5 + 0.0214t^7, \end{aligned}$$

$$\begin{aligned} e_{8,9}(t) &= -7.3650 \times 10^{-6}t + 2.6532e - 04t^2 + 0.0171t^4 + 0.0892t^6 + 0.0480t^8 - 0.0031t^3 \\ &\quad -0.0514t^5 - 0.0894t^7 - 0.0107t^9 - 4 \times 10^{-100}, \end{aligned}$$

The results are given in Figure 3. As a result, we can say that the absolute errors e_4 and e_8 are estimated by $e_{4,7}$ and $e_{8,9}$, respectively. On the other hand, for each case, the corrected MLG solutions are better than the MLG solutions.

Table 2 presents the comparison of the approximate solution of $x(t)$ for $\nu = 1, N = 12$ to the Bernoulli wavelet operational matrix [50], the computational intelligence approach [45] and the Bernstein collocation method [43]. For $\nu = 1$, the results were compared to the operational matrices method [51], the differential squared method [52] method, the method dependent on shifted Chebyshev polynomials [53] and the hybrid functions approach [54] in Table 3. Figure 4 shows the results of the MLG method for $\nu = 1$ and different values of N. The approximate values of $x(t)$ for $N = 10$ and $\nu = \{0.75, 0.9, 0.95, 1\}$ are given in Figure 5. Although the results of the MLG method are better than the results of the hybrid functions approach [54], we can say from Tables 2 and 3 that it yields approximation results similar to the other methods for this problem. We conclude from Figure 4 that increasing $= N$ yields better approximation results for the problem.

Table 2. Comparison of the approximate solutions of $x(t)$ with the MLG method to the methods in [43,45,50] schemes, with $\nu = 1$ for Example 3.

t	Exact Solution	Method [43]	Method [45]	Method [50]	MLG Method
0.0	0.00000000000	0.000000000000	0.0000000011	0.000000000000	0.00000000000
0.2	0.19737532022	0.197375320493	0.1973918880	0.19737532017	0.19737532019
0.4	0.37994896226	0.379948962506	0.3799632287	0.379948962207	0.37994896222
0.6	0.53704956700	0.537049567214	0.5370622335	0.53704956701	0.53704956700
0.8	0.66403677027	0.664036770562	0.6640456511	0.66403677030	0.66403677029
1.0	0.76159415596	0.761594224400	0.7616019763	0.76159415595	0.76159415595

Table 3. Comparison of the errors obtained by the MLG method to the methods in [51–54] schemes, with $\nu = 1$ for Example 3.

t	Method [51]	Method [54]	Method [52]	Method [53]	MLG Method
	$N = 12$	$N = 20$	$N = 12$	$N = 12$	$N = 12$
0.1	1.11×10^{-10}	7.2701×10^{-6}	8.3141×10^{-11}	3.0881×10^{-11}	2.8244×10^{-11}
0.2	2.04×10^{-10}	1.0922×10^{-5}	9.1576×10^{-11}	4.8024×10^{-11}	3.4678×10^{-11}
0.3	2.10×10^{-12}	1.3476×10^{-5}	7.5812×10^{-11}	4.6345×10^{-11}	3.5020×10^{-11}
0.4	2.23×10^{-10}	1.4755×10^{-5}	1.1151×10^{-10}	4.8119×10^{-11}	2.7373×10^{-11}
0.5	4.03×10^{-10}	1.4778×10^{-5}	5.5890×10^{-11}	9.7303×10^{-12}	9.2649×10^{-12}
0.6	1.79×10^{-10}	1.3730×10^{-5}	7.8642×10^{-11}	1.7617×10^{-11}	1.0959×10^{-11}
0.7	8.59×10^{-11}	1.1891×10^{-5}	6.2746×10^{-11}	4.1180×10^{-11}	2.2709×10^{-11}
0.8	2.70×10^{-10}	9.5751×10^{-6}	5.3920×10^{-11}	3.2346×10^{-11}	2.3907×10^{-11}
0.9	1.89×10^{-10}	7.0732×10^{-6}	4.8389×10^{-11}	2.6919×10^{-11}	1.5571×10^{-11}

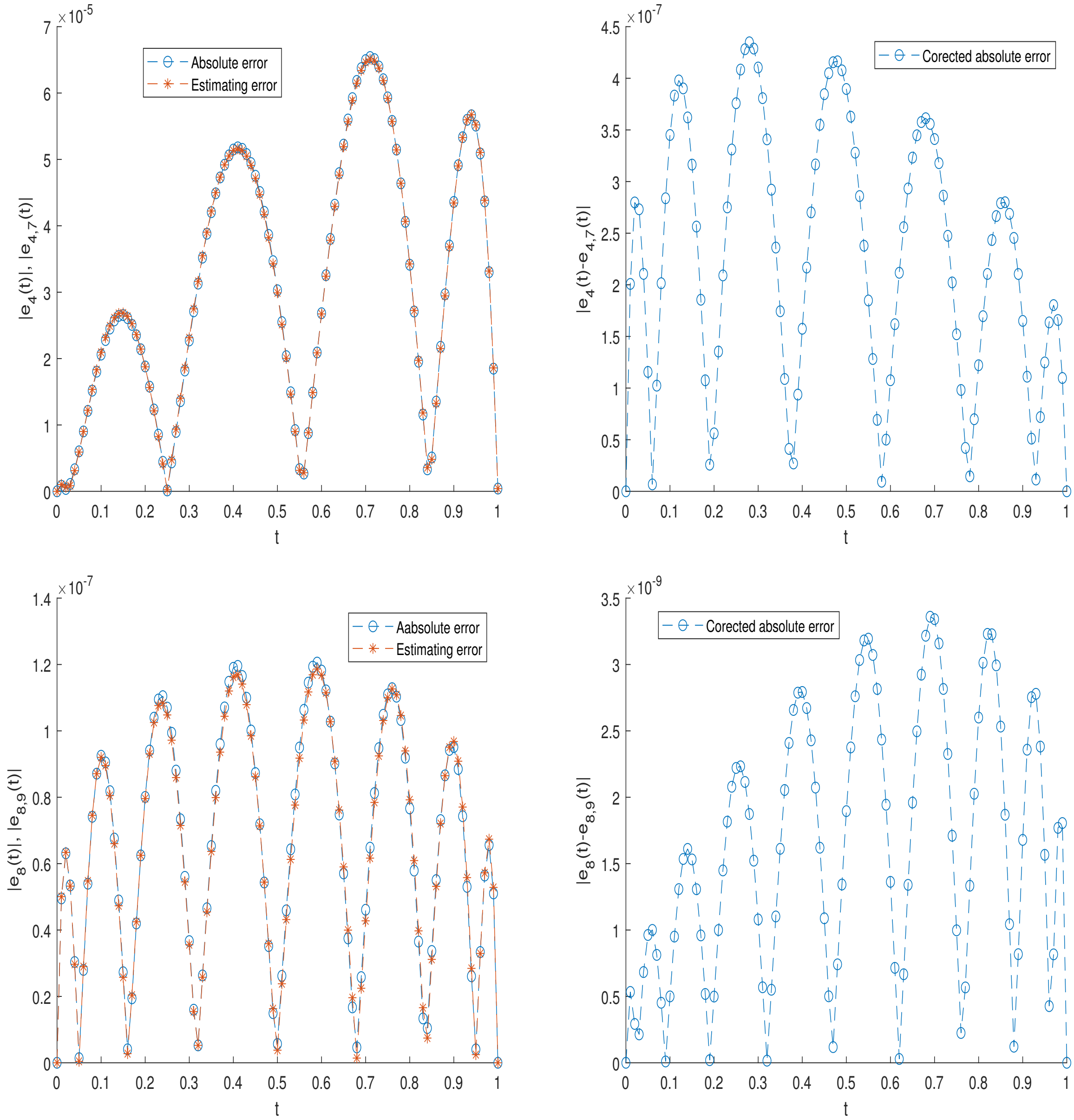

Figure 3. The absolute error e_4 with its estimation $e_{4,7}$; e_8 with its estimation $e_{8,9}$; and the corrected errors for Example 3.

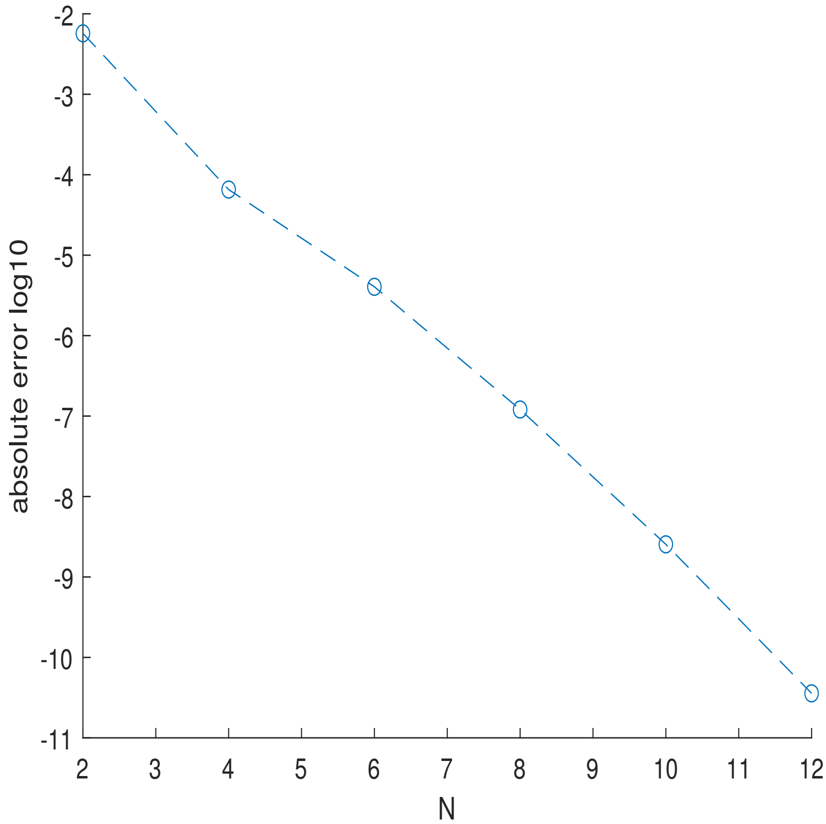

Figure 4. Log_{10} of the absolute error versus N at $\nu = 1$ for Example 3.

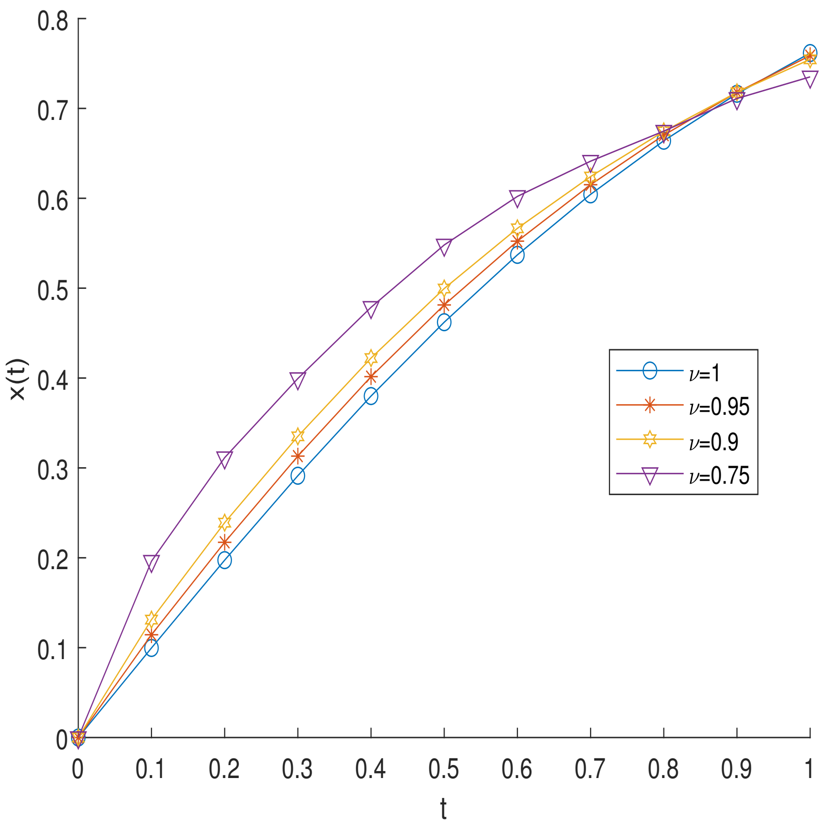

Figure 5. MLG solutions of $x(t)$ for $N = 10$ with $\nu = \{0.75, 0.9, 0.95, 1\}$ and Example 3.

Figure 6 for $N = 12$*, the absolute error for Example* 3.

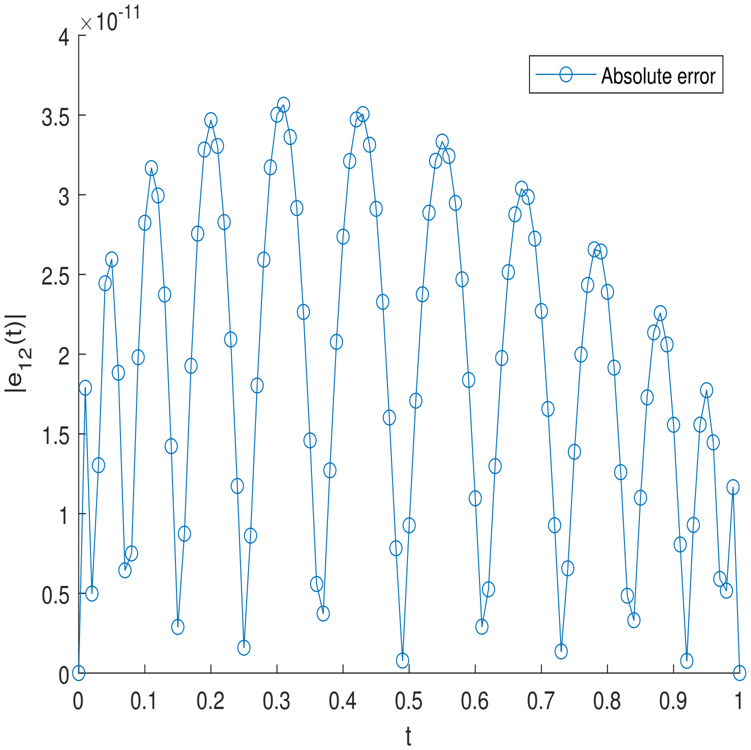

Figure 6. The absolute error for Example 3 and $N = 12$.

Example 4. *Let us consider the following FDRE [55]:*

$$D^{\nu}x(t) - x^2(t) - 1 = 0, \quad 0 \leq t \leq 1, \tag{31}$$

$$x(0) = 0. \tag{32}$$

which has the exact solution for $\nu = 1$ *is* $x(t) = \tan(t)$.

The results obtained when the method is applied for $\nu = 1$ *and* $N = \{4, 8, 12\}$ *are given below:*

$$\begin{aligned} x_4(t) &= 0.93859t + 0.53891t^2 - 0.98546t^3 + 1.06529t^4; \\ x_8(t) &= 0.99943t + 0.0192t^2 + 1.04658t^4 + 4.07934t^6 + 1.03275t^8 + 0.12481t^3 \\ &\quad -2.65049t^5 - 3.09424t^7 - 10^{-13}; \\ x_{12}(t) &= 0.99998t + 8.9376 \times 10^{-04}t^2 + 0.0986t^4 + 0.79925t^6 + 0.28047t^8 - 0.0774t^{10} \\ &\quad +0.12974t^{12} + 0.31956t^3 - 0.24372t^5 - 0.81591t^7 + 0.27357t^9 - 0.20760t^{11}. \end{aligned}$$

The solution for N = 12 and $\nu = 1$ *is given in Tables 4 and 5, by comparing the results obtained to some other numerical methods, such as the wavelet operational matrix method [44], the method dependent on shifted Chebyshev polynomials [53] and the decomposition algorithm [55]. We can conclude, from the tables, that the MLG method produces results similar to the other methods cited, except for the decomposition algorithm [55]. Figure 7 depicts the function* Log_{10} *of the absolute error for* $\nu = 1$ *and various values of N, whereas Figure 8 shows the approximate values of* $x(t)$ *for* $N = 10$ *and some values of* ν. *In conclusion, we can say that increasing N gives better approximations for this problem. The errors are given in Figures 9 and 10 for* $N = 12$ *and* $N = 10$, *respectively. In Figure 10, the estimation of absolute error and the corrected absolute error for Example 4 are given. We can say from Figures 9 and 10 that the MLG method gives more accurate results, and that the residual correction procedure estimates the error well.*

Table 4. Comparison of the approximate solution of Example 4, using the MLG method, to the solutions obtained by the methods presented in [44,53,55] for $\nu = 1$:

t	Exact Solution	Method [44]	Method [55]	Method [53]	Our Method
				$N = 14$	$N = 12$
0.0	0.00000000000		0.0000000000	0.0000000000	0.00000000000
0.1	0.10033467208	0.100342	0.1003346713	0.1003346714	0.10033465034
0.2	0.20271003550	0.202726	0.2027099297	0.2027100349	0.20271006703
0.3	0.30933624961	0.309372	0.3093343442	0.3093362509	0.30933621959
0.4	0.42279321873	0.422832	0.4227777155	0.4227932186	0.42279323380
0.5	0.54630248984	0.546363	0.5462212762	0.5463024891	0.68413677385
0.6	0.68413680834	0.684251	0.6838056920	0.6841368110	0.68413677385
0.7	0.84228838046	0.842411	0.8411449022	0.8422883779	0.84228842425
0.8	1.02963855705	1.029849	1.0261001110	1.0296385599	1.02963851386
0.9	1.26015821755	1.260573	1.2499664940	1.2601582184	1.26015825498
1.0	1.55740772465	1.557938	1.5293009690	1.5574077258	1.55740772465

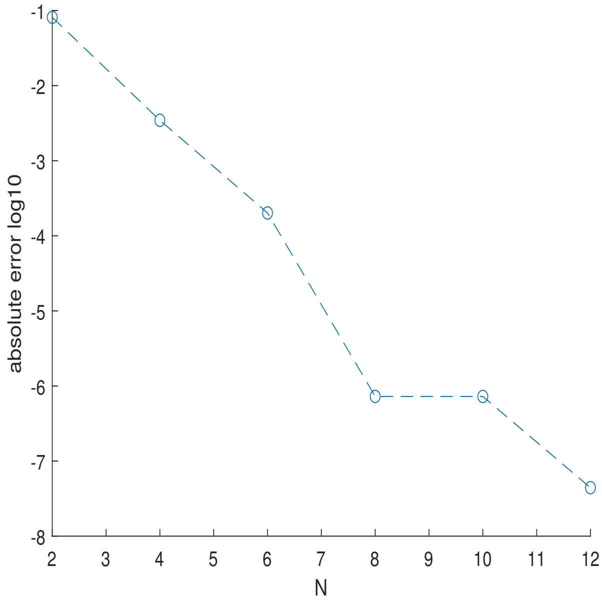

Figure 7. Log_{10} of the absolute error versus N at $\nu = 1$ for Example 4.

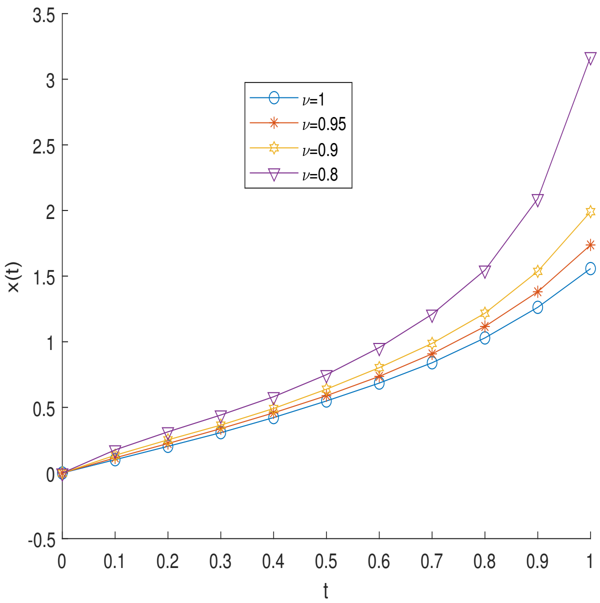

Figure 8. Approximate solutions of $x(t)$ for $N = 10$ with $\nu = \{0.8, 0.9, 0.95, 1\}$ for Example 4.

Table 5. Comparison of the absolute errors obtained using the MLG method for $\nu = 1$, in Example 4, to the absolute errors obtained using the methods in [53,55]:

t	Method [55]	Method [53]	Our Method
		$N = 12$	$N = 12$
0.0	0.00000000	0.00000000	0.00000000
0.1	8.162×10^{-10}	2.8897×10^{-8}	2.1736×10^{-8}
0.2	1.0580×10^{-7}	5.1478×10^{-8}	3.1526×10^{-8}
0.3	1.9050×10^{-6}	4.6086×10^{-8}	3.0013×10^{-8}
0.4	1.5500×10^{-5}	3.4828×10^{-8}	1.5069×10^{-8}
0.5	8.1210×10^{-5}	2.3389×10^{-8}	1.1149×10^{-8}
0.6	3.3110×10^{-4}	5.0755×10^{-8}	3.4490×10^{-8}
0.7	1.1430×10^{-3}	7.3355×10^{-8}	4.3796×10^{-8}
0.8	3.5380×10^{-3}	4.4578×10^{-8}	4.3187×10^{-8}
0.9	1.1090×10^{-3}	5.8748×10^{-8}	3.7439×10^{-8}
1.0	2.8110×10^{-3}	2.2418×10^{-8}	1.7293×10^{-15}

Figure 9 for $N = 12$, the absolute error for Example 4.

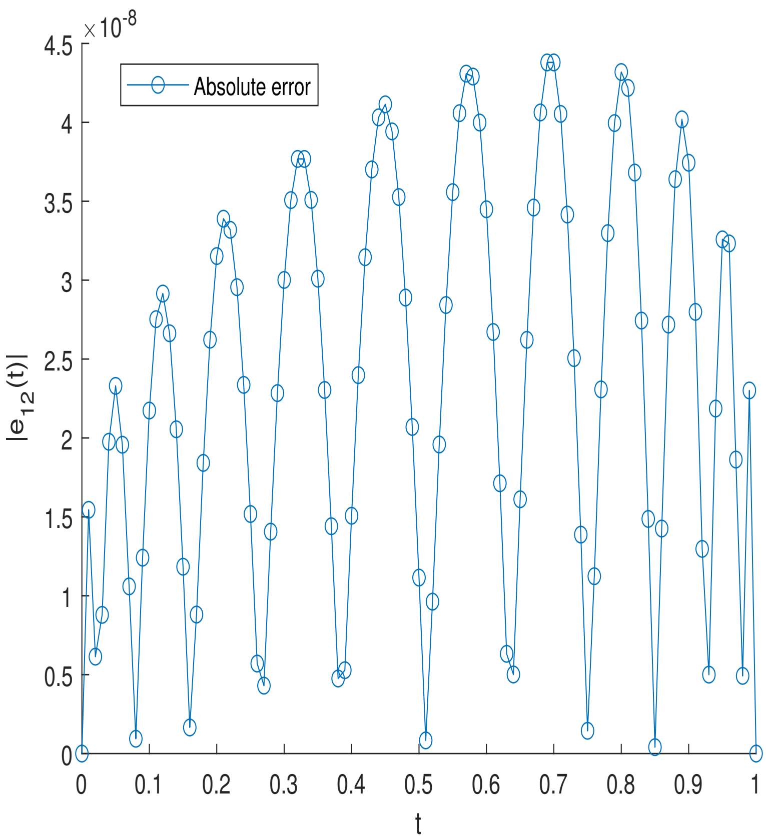

Figure 9. The graph of the absolute error for Example 4 and $N = 12$.

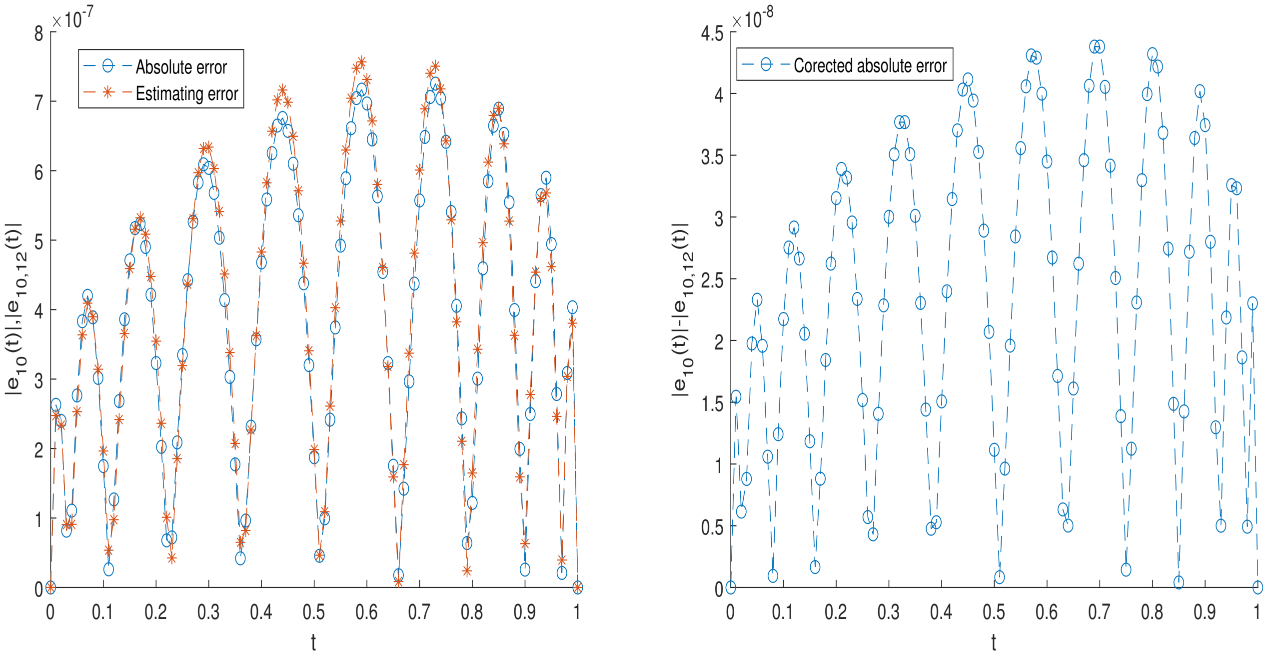

Figure 10. The absolute error, estimation of absolute error and the corrected absolute error for Example 4 for $(N, M) = (10, 12)$.

Example 5. *As a final example, let us consider the following FDRE [43,53]:*

$$D^{\nu} x(t) - 2x(t) + x^2(t) - 1 = 0, \quad 0 \leq t \leq 1, \tag{33}$$

$$x(0) = 0. \tag{34}$$

which has the exact solution for $\nu = 1$ *is* $x(t) = 1 - \sqrt{2}\frac{\sqrt{2}\tanh(\sqrt{2}t)-1}{\tanh(\sqrt{2}t)-\sqrt{2}}$.

Applying the MLG method to the problem for $\nu = 1$ *and* $N = \{4, 8, 12\}$ *yields the following MLG solutions:*

$$x_4(t) = 0.9655t + 1.26574t^2 - 0.14446t^3 - 0.39736t^4;$$

$$x_8(t) = 0.99991t + 1.00259t^2 - 0.26776t^4 - 0.58544t^6 - 0.35438t^8 + 0.31134t^3 - 0.46069t^5 + 1.04393t^7 + 4 \times 10^{-91};$$

$$x_{12}(t) = t + 0.99999t^2 - 0.33748t^4 - 0.36952t^6 - 1.60334t^8 - 2.81446t^{10} - 0.234263t^{12} + 0.33358t^3 - 0.42847t^5 + 0.93948t^7 + 2.90224t^9 + 1.30175t^{11}.$$

The results for $\nu = 1, N = 12$ *are given in Table 6 and Figure 11 with a comparison to the approximate solutions obtained by the methods in [43,53]. In addition, Figure 12 displays* Log_{10} *of absolute errors for* $\nu = 1$ *and different values of* N*. We conclude that, as in the other examples, more accurate results are obtained by increasing* N*. The approximate values of* $x(t)$ *for* $N = 10$*, and some values of* ν*, are given in Figure 13. The absolute error, with its estimation, and the corrected absolute error for Example 5 are drawn in Figure 14 for* $N = 4$*. The estimation absolute error for Example 5 are drawn in Figure 15 for* $N = 5$ *and* $N = 8$*.*

The procedure estimates the error well, and leads to a better approximate solution. We also compared the absolute errors estimated by the MLG method to the Bernstein collocation method [43] for $\nu = 0.9$*, and we give the results in Table 7.*

Table 6. Comparison of the MLG solution to the methods in [43,53] for $\nu = 1$ and Example 5.

t	Exact Solution	Method [43]	Method [53]	Our Method
			$N = 14$	$N = 12$
0.0	0.0000000000	0.000000000	0.0000000000	0.0000000000
0.2	0.2419767996	0.241977035	0.2419767995	0.24197679964
0.4	0.5678121663	0.567812472	0.5678121662	0.56781216535
0.6	0.9535662164	0.953566555	0.9535662164	0.95356621379
0.8	1.3463636554	1.346363997	1.3463636552	1.34636365269
1.0	1.6894983916	1.689510190	1.6894983916	1.68949839159

Table 7. Comparison of the estimated absolute errors for various values of N and M for $\nu = 0.9$ of Equation (33) to $\xi = 0.5, \eta = 1$.

t	$(N, M) = (5, 7)$		$(N, M) = (8, 9)$	
	Method [43]	Our Method	Method [43]	Our Method
0	0	0	0	0
0.2	2.4585×10^{-3}	1.8653×10^{-3}	5.5413×10^{-5}	9.2910×10^{-6}
0.4	2.3010×10^{-3}	1.4361×10^{-3}	6.6222×10^{-5}	1.1642×10^{-5}
0.6	2.5766×10^{-3}	2.3933×10^{-4}	6.5671×10^{-5}	6.2586×10^{-6}
0.8	1.6519×10^{-3}	4.8353×10^{-4}	5.6089×10^{-5}	9.4982×10^{-6}
1.0	6.3526×10^{-3}	1.1605×10^{-4}	6.4831×10^{-4}	4.2162×10^{-6}

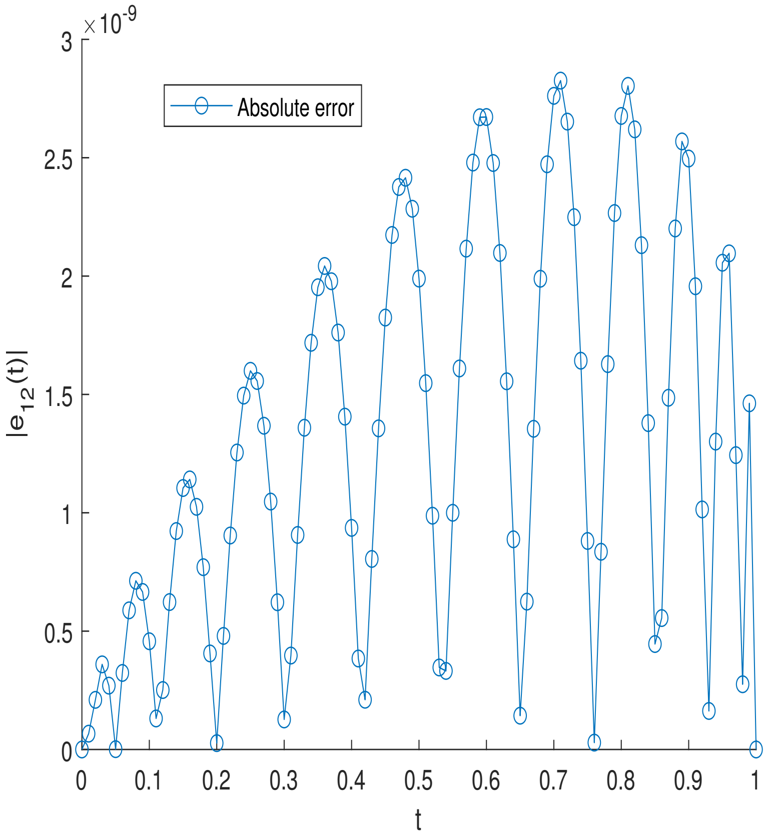

Figure 11. The absolute error for Example 5, for $N = 12$.

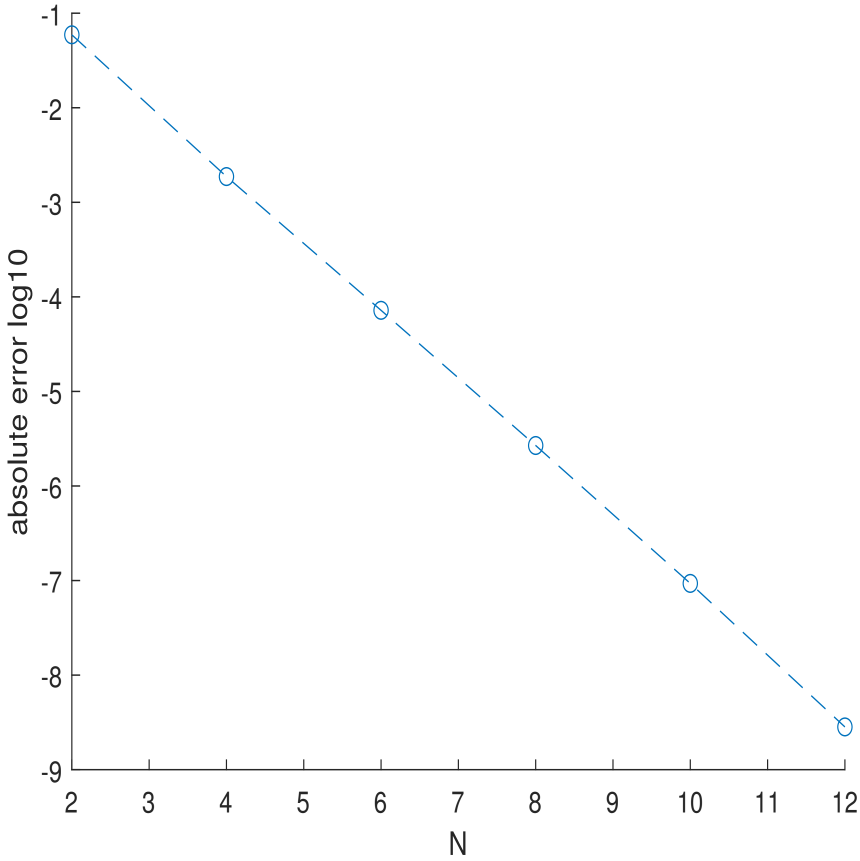

Figure 12. Log_{10} of the absolute error versus N at $\nu = 1$ for Example 5.

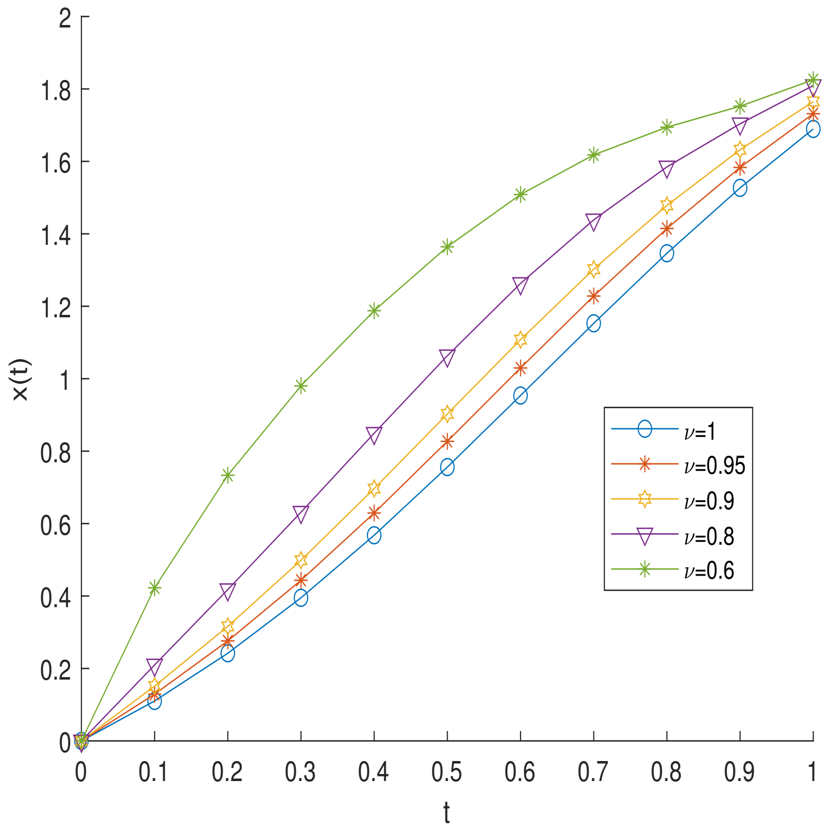

Figure 13. Approximate solutions of $x(t)$ for $N = 10$ with $\nu = \{0.6, 0.8, 0.9, 0.95, 1\}$ for Example 5.

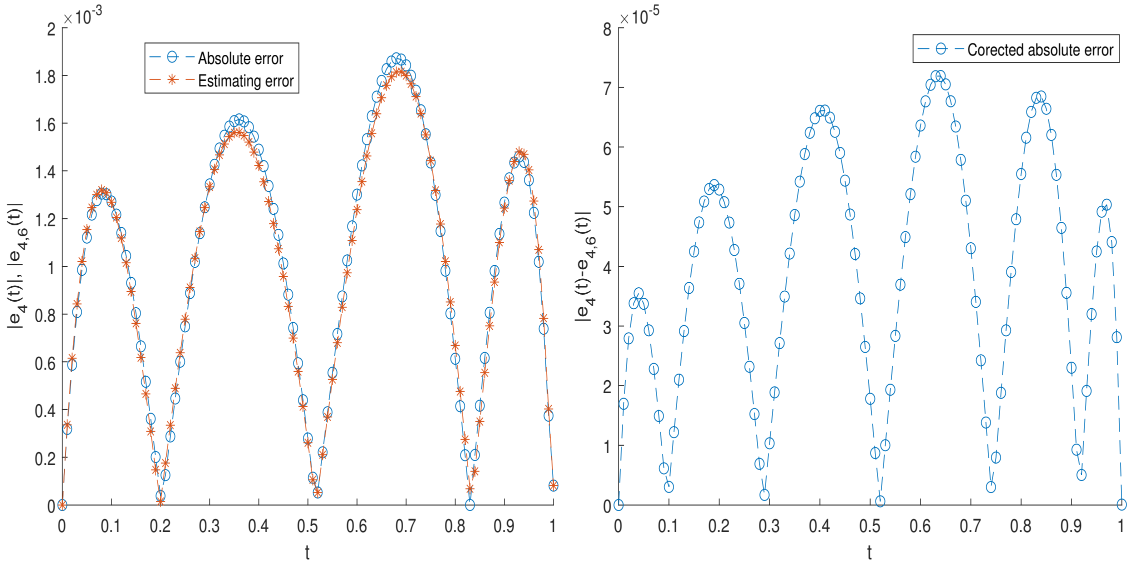

Figure 14. The absolute error, estimation of absolute error and the corrected absolute error for Example 5, for $(N, M) = (4, 6)$ with $\nu = 1$.

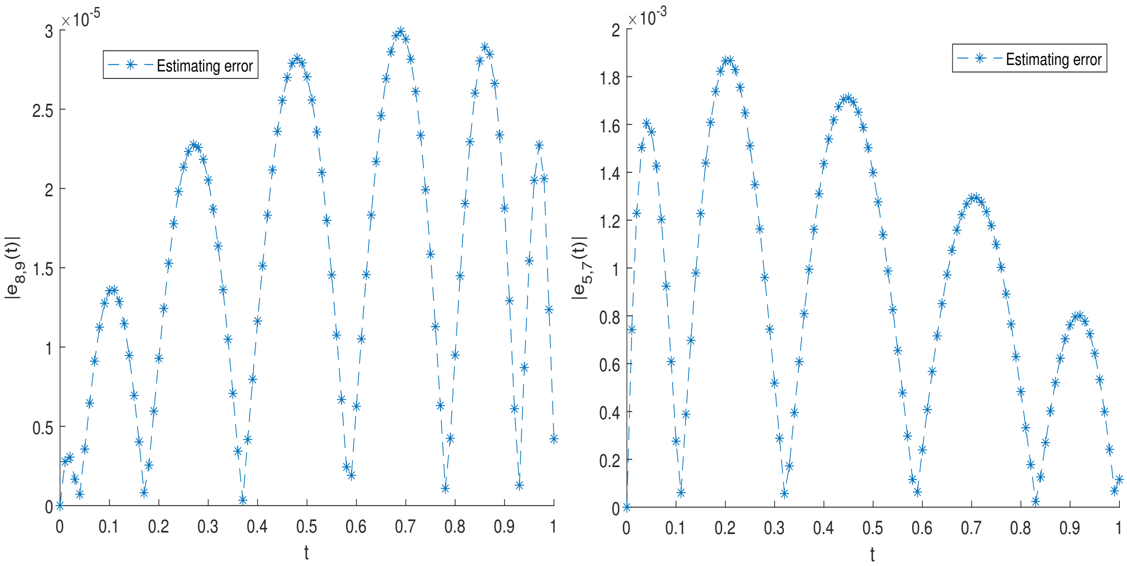

Figure 15. The estimation of absolute error for Example 5, for $(N, M) = (5, 7)$ and $(N, M) = (8, 9)$ with $\nu = 0.9$.

7. Conclusions

In this study, we introduced a new numerical method for solving fractional differential Riccati equations. The method is based on the Mittag-Leffler function and the Galerkin method. Some theorems related to the error analysis of the method were presented. The error can be bounded by these theorems. The residual correction method allows for estimating the absolute error and obtaining a new approximate solution, which was presented for the method. We applied the method to some test examples, to illustrate its effectiveness and how the method is applied. The results obtained by the method were given by comparing the results of some other previously known methods for the solutions of fractional Riccati equations. The numerical test results showed that the method gave good approximation results for the examples. In addition, while only similar results were obtained by some of the known methods, more accurate results were obtained for the problems under consideration than were obtained by others of the known methods.

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Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

$$\begin{cases} \frac{1}{15\pi^{\frac{3}{2}}}\Big(15\,c_0{}^2\pi^{3/2}+30\,c_0c_1\pi^{3/2}+40\,c_0c_2\pi^{3/2}+15\,\pi^{3/2}c_1{}^2+40\,\pi^{3/2}c_1c_2+28\,\pi^{3/2}c_2{}^2 \\ +15\,c_0\pi^{3/2}+15\,\pi^{3/2}c_1+20\,\pi^{3/2}c_2+20\,c_1{}^2\sqrt{\pi}+40\,c_1c_2\sqrt{\pi}+20\,c_2{}^2\sqrt{\pi}-30\,c_0c_1\pi \\ -30\,c_0c_2\pi-30\,c_1{}^2\pi-75\,c_1c_2\pi-45\,c_2{}^2\pi-8\,\pi^{3/2}-40\,c_1\sqrt{\pi}-40\,c_2\sqrt{\pi}-15\,c_1\pi \\ +c_2\pi-16\,\pi\Big)=0, \\ \frac{1}{210\,\pi^{\frac{5}{2}}}\Big(-420\,c_1{}^2\pi-420\,c_2{}^2\pi+672\,c_1\pi+672\,c_2\pi+560\,c_0c_1\pi^{3/2}+560\,c_0c_2\pi^{3/2} \\ -840\,c_1c_2\pi+840\,\pi^{3/2}c_1{}^2+1176\,\pi^{3/2}c_2{}^2-1050\,\pi^2c_0c_2-1680\,\pi^2c_1c_2-840\,\pi^2c_0c_1 \\ -630\,\pi^2c_1{}^2-1120\,\pi^2c_2{}^2-420\,c_1\pi^2-301\,\pi^2c_2+2016\,\pi^{3/2}c_1c_2+210\,\pi^{5/2}c_0{}^2 \\ +210\,\pi^{5/2}c_1{}^2+392\,\pi^{5/2}c_2{}^2+210\,c_0\pi^{5/2}+210\,c_1\pi^{5/2}+280\,\pi^{5/2}c_2-280\,\pi^{3/2}c_1 \\ -600\,\pi^{3/2}c_2-112\,\pi^{5/2}+560\,\pi^{5/2}c_1c_2+420\,\pi^{5/2}c_0c_1+560\,\pi^{5/2}c_0c_2-49\,\pi^2 \\ +320\,\pi^{3/2}-210\,\pi^2c_0{}^2-210\,\pi^2c_0\Big)=0. \end{cases} \tag{A1}$$

$$\begin{cases} -\frac{1}{2520\pi^{3/2}}\Big(2304\,\pi+2520\,c_0c_1\pi^{3/2}+3780\,c_0c_2\pi^{3/2}+3780\,c_3c_0\pi^{3/2}-315\,\pi^{3/2} \\ +1260\,c_0{}^2\pi^{3/2}+5040\,c_1c_2\sqrt{\pi}+7280\,c_3c_1\sqrt{\pi}+7280\,c_3c_2\sqrt{\pi}-3360\,c_0c_1\pi \\ -3360\,c_0c_2\pi-4704\,c_3c_0\pi-8736\,c_1c_2\pi-10080\,c_3c_1\pi-13056\,c_3c_2\pi+3780\,\pi^{3/2}c_1c_2 \\ +3780\,\pi^{3/2}c_1c_3+5880\,\pi^{3/2}c_2c_3+2520\,c_2{}^2\sqrt{\pi}+5320\,c_3{}^2\sqrt{\pi}-3360\,c_1{}^2\pi-5376\,c_2{}^2\pi \\ -7680\,c_3{}^2\pi+6720\,c_1\sqrt{\pi}+6720\,\sqrt{\pi}c_2+9792\,\sqrt{\pi}c_3-2688\,c_2\pi-2688\,c_3\pi \\ +2520\,c_1{}^2\sqrt{\pi}+1260\,\pi^{3/2}c_1{}^2+2940\,\pi^{3/2}c_2{}^2+2940\,\pi^{3/2}c_3{}^2\Big)=0, \\ -\frac{1}{22680\pi^{5/2}}\Big(25776\,\pi^2+60480\,\pi^{3/2}c_1+95040\,\pi^{3/2}c_2+122688\,\pi^{3/2}c_3-2835\,\pi^{5/2} \\ +45360\,c_0c_1\pi^{3/2}+45360\,c_0c_2\pi^{3/2}+65520\,c_3c_0\pi^{3/2}-72576\,c_1\pi-32256\,\pi^{3/2} \\ -72576\,c_1c_2\pi-107136\,c_3c_1\pi-107136\,c_3c_2\pi+22680\,\pi^{5/2}c_0c_1+34020\,\pi^{5/2}c_0c_2 \\ +34020\,\pi^{5/2}c_0c_3+34020\,\pi^{5/2}c_1c_2+34020\,\pi^{5/2}c_1c_3+52920\,\pi^{5/2}c_2c_3-60480\,\pi^2c_0c_1 \\ -78624\,\pi^2c_0c_2-90720\,\pi^2c_0c_3-127008\,\pi^2c_1c_2-139104\,\pi^2c_1c_3-196992\,\pi^2c_2c_3 \\ +166320\,\pi^{3/2}c_1c_2+206640\,\pi^{3/2}c_1c_3+252000\,\pi^{3/2}c_2c_3-36288\,c_1{}^2\pi-36288\,c_2{}^2\pi \\ -79808\,c_3{}^2\pi-72576\,c_2\pi-115584\,c_3\pi+11340\,\pi^{5/2}c_0{}^2+11340\,\pi^{5/2}c_1{}^2+26460\,\pi^{5/2}c_2{}^2 \\ +26460\,\pi^{5/2}c_3{}^2-45360\,\pi^2c_1{}^2-88128\,\pi^2c_2{}^2-108864\,\pi^2c_3{}^2+68040\,\pi^{3/2}c_1{}^2 \\ +98280\,\pi^{3/2}c_2{}^2+158760\,\pi^{3/2}c_3{}^2-24192\,\pi^2c_2-24192\,\pi^2c_3-15120\,c_0{}^2\pi^2\Big)=0, \\ -\frac{1}{249480\pi^{5/2}}\Big(428688\,\pi^2+950400\,\pi^{3/2}c_1+1330560\,\pi^{3/2}c_2+1828224\,\pi^{3/2}c_3-56133\,\pi^{5/2} \\ +498960\,c_0c_1\pi^{3/2}+498960\,c_0c_2\pi^{3/2}+720720\,c_3c_0\pi^{3/2}-798336\,c_1\pi-354816\,\pi^{3/2} \\ -798336\,c_1c_2\pi-1178496\,c_3c_1\pi-1178496\,c_3c_2\pi+374220\,\pi^{5/2}c_0c_1+582120\,\pi^{5/2}c_0c_2 \\ +582120\,\pi^{5/2}c_0c_3+582120\,\pi^{5/2}c_1c_2+582120\,\pi^{5/2}c_1c_3+935550\,\pi^{5/2}c_2c_3-864864\,\pi^2c_0c_1 \\ -1064448\,\pi^2c_0c_2-1292544\,\pi^2c_0c_3-1938816\,\pi^2c_1c_2-2166912\,\pi^2c_1c_3-3020160\,\pi^2c_2c_3 \\ +2162160\,\pi^{3/2}c_1c_2+2772000\,\pi^{3/2}c_1c_3+3270960\,\pi^{3/2}c_2c_3-399168\,c_1{}^2\pi-399168\,c_2{}^2\pi \\ -877888\,c_3{}^2\pi-798336\,c_2\pi-1271424\,c_3\pi+187110\,\pi^{5/2}c_0{}^2+187110\,\pi^{5/2}c_1{}^2 \\ +467775\,\pi^{5/2}c_2{}^2+467775\,\pi^{5/2}c_3{}^2-698544\,\pi^2c_1{}^2-1311552\,\pi^2c_2{}^2-1708608\,\pi^2c_3{}^2 \\ +914760\,\pi^{3/2}c_1{}^2+1247400\,\pi^{3/2}c_2{}^2+2123352\,\pi^{3/2}c_3{}^2-413952\,\pi^2c_2 \\ -413952\,\pi^2c_3-166320\,c_0{}^2\pi^2\Big)=0. \end{cases} \tag{A2}$$

References

1. Mainardi, F. Fractional calculus. In *Fractals and Fractional Calculus in Continuum Mechanics*; Springer: Vienna, Austria, 1997; pp. 291–348.
2. Koeller, R. Applications of fractional calculus to the theory of viscoelasticity. *J. Appl. Mech.* **1984**, *51*, 299–307. [CrossRef]
3. Jiang, Y.; Wang, X.; Wang, Y. On a stochastic heat equation with first order fractional noises and applications to finance. *J. Math. Anal. Appl.* **2012**, *396*, 656–669. [CrossRef]
4. Bhrawy, A.H.; Ezz-Eldien, S. A new Legendre operational technique for delay fractional optimal control problems. *Calcolo* **2016**, *53*, 521–543. [CrossRef]
5. Ezz-Eldien, S.S. New quadrature approach based on operational matrix for solving a class of fractional variational problems. *J. Comput. Phys.* **2016**, *317*, 362–381. [CrossRef]
6. Benson, D.A.; Meerschaert, M.M.; Revielle, J. Fractional calculus in hydrologic modeling: A numerical perspective. *Adv. Water Resour.* **2013**, *51*, 479–497. [CrossRef]
7. Kulish, V.V.; Lage, J.L. Application of fractional calculus to fluid mechanics. *J. Fluids Eng.* **2002**, *124*, 803–806. [CrossRef]
8. Ezz-Eldien, S.S.; El-Kalaawy, A.A. Numerical simulation and convergence analysis of fractional optimization problems with right-sided Caputo fractional derivative. *J. Comput. Nonlinear Dyn.* **2018**, *13*, 011010. [CrossRef]
9. Sadek, L. Fractional BDF Methods for Solving Fractional Differential Matrix Equations. *Int. J. Appl. Comput. Math.* **2022**, *8*, 238. [CrossRef]
10. Hosseini, V.R.; Chen, W.; Avazzadeh, Z. Numerical solution of fractional telegraph equation by using radial basis functions. *Eng. Anal. Bound. Elem.* **2014**, *38*, 31–39. [CrossRef]
11. Liu, F.; Zhuang, P.; Turner, I.; Burrage, K.; Anh, V. A new fractional finite volume method for solving the fractional diffusion equation. *Appl. Math. Model.* **2014**, *38*, 3871–3878. [CrossRef]
12. Khodabakhshi, N.; Vaezpour, S.M.; Baleanu, D. Numerical solutions of the initial value problem for fractional differential equations by modification of the Adomian decomposition method. *Fract. Calc. Appl. Anal.* **2014**, *17*, 382–400. [CrossRef]
13. Kim, M.H.; Ri, G.C.; Hyong-Chol, O. Operational method for solving multi-term fractional differential equations with the generalized fractional derivatives. *Fract. Calc. Appl. Anal.* **2014**, *17*, 79–95. [CrossRef]
14. Rabiei, K.; Razzaghi, M. Fractional-order Boubaker wavelets method for solving fractional Riccati differential equations. *Appl. Numer. Math.* **2021**, *168*, 221–234. [CrossRef]
15. Burqan, A.; Sarhan, A.; Saadeh, R. Constructing Analytical Solutions of the Fractional Riccati Differential Equations Using Laplace Residual Power Series Method. *Fractal Fract.* **2022**, *7*, 14. [CrossRef]
16. Singh, J.; Gupta, A.; Kumar, D. Computational Analysis of the Fractional Riccati Differential Equation with Prabhakar-type Memory. *Mathematics* **2023**, *11*, 644. [CrossRef]
17. Esmaeili, S.; Shamsi, M. A pseudo-spectral scheme for the approximate solution of a family of fractional differential equations. *Commun. Nonlinear Sci. Numer. Simul.* **2011**, *16*, 3646–3654. [CrossRef]
18. Zhang, H.; Jiang, X.; Wang, C.; Fan, W. Galerkin-Legendre spectral schemes for nonlinear space fractional Schrödinger equation. *Numer. Algorithms* **2018**, *79*, 337–356. [CrossRef]
19. Mokhtary, P.; Ghoreishi, F. The L2-convergence of the Legendre spectral Tau matrix formulation for nonlinear fractional integro differential equations. *Numer. Algorithms* **2011**, *58*, 475–496. [CrossRef]
20. Bhrawy, A.H.; Doha, E.H.; Ezz-Eldien, S.S.; Van Gorder, R.A. A new Jacobi spectral collocation method for solving 1+ 1 fractional Schrödinger equations and fractional coupled Schrödinger systems. *Eur. Phys. J. Plus* **2014**, *129*, 1–21. [CrossRef]
21. Vanani, S.K.; Aminataei, A. Tau approximate solution of fractional partial differential equations. *Comput. Math. Appl.* **2011**, *62*, 1075–1083. [CrossRef]
22. Doha, E.H.; Bhrawy, A.H.; Ezz-Eldien, S.S. An efficient Legendre spectral tau matrix formulation for solving fractional subdiffusion and reaction subdiffusion equations. *J. Comput. Nonlinear Dyn.* **2015**, *10*, 021019. [CrossRef]
23. Hafez, R.M.; Ezz-Eldien, S.S.; Bhrawy, A.H.; Ahmed, E.A.; Baleanu, D. A Jacobi Gauss–Lobatto and Gauss–Radau collocation algorithm for solving fractional Fokker–Planck equations. *Nonlinear Dyn.* **2015**, *82*, 1431–1440. [CrossRef]
24. Saadatmandi, A.; Dehghan, M. A new operational matrix for solving fractional-order differential equations. *Comput. Math. Appl.* **2010**, *59*, 1326–1336. [CrossRef]
25. Kazem, S. An integral operational matrix based on Jacobi polynomials for solving fractional-order differential equations. *Appl. Math. Model.* **2013**, *37*, 1126–1136. [CrossRef]
26. Bittanti, S.; Colaneri, P.; Guardabassi, G. Periodic solutions of periodic Riccati equations. *IEEE Trans. Autom. Control* **1984**, *29*, 665–667. [CrossRef]
27. Lasiecka, I.; Triggiani, R. (Eds.) *Differential and Algebraic Riccati Equations with Application to Boundary/Point Control Problems: Continuous Theory and Approximation Theory*; Springer: Berlin/Heidelberg, Germany, 1991.
28. Goldstine, H.H. *A History of the Calculus of Variations from the 17th through the 19th Century (Vol. 5)*; Springer Science & Business Media: New York, NY, USA; Heidelberg/Berlin, Germany, 2012.
29. Garrappa, R. On some explicit Adams multistep methods for fractional differential equations. *J. Comput. Appl. Math.* **2009**, *229*, 392–399. [CrossRef]
30. Feliu-Batlle, V.; Perez, R.R.; Rodriguez, L.S. Fractional robust control of main irrigation canals with variable dynamic parameters. *Control Eng. Pract.* **2007**, *15*, 673–686. [CrossRef]

31. Podlubny, I. Fractional-order systems and PI/sup/spl lambda//D/sup/spl mu//-controllers. *IEEE Trans. Autom. Control* **1999**, *44*, 208–214. [CrossRef]
32. Öztürk, Y.; Anapali, A.; Gülsu, M.; Sezer, M. A collocation method for solving fractional Riccati differential equation. *J. Appl. Math.* **2013**, *2013*, 598083. [CrossRef]
33. Balaji, S. Legendre wavelet operational matrix method for solution of fractional order Riccati differential equation. *J. Egypt. Math. Soc.* **2015**, *23*, 263–270. [CrossRef]
34. Mokhtary, P.; Ghoreishi, F. Convergence analysis of spectral Tau method for fractional Riccati differential Equations. *Bull. Iranian Math. Soc.* **2014**, *40*, 1275–1290.
35. Kashkari, B.S.; Syam, M.I. Fractional-order Legendre operational matrix of fractional integration for solving the Riccati equation with fractional order. *Appl. Math. Comput.* **2016**, *290*, 281–291. [CrossRef]
36. Jafari, H.; Tajadodi, H.; Baleanu, D. A modified variational iteration method for solving fractional Riccati differential equation by Adomian polynomials. *Fract. Calc. Appl. Anal.* **2013**, *16*, 109–122. [CrossRef]
37. Bota, C.; Caruntu, B. Analytical approximate solutions for quadratic Riccati differential equation of fractional order using the Polynomial Least Squares Method. *Chaos Solitons Fractals* **2017**, *102*, 339–345. [CrossRef]
38. Merdan, M. On the solutions fractional Riccati differential equation with modified Riemann-Liouville derivative. *Int. J. Diff. Equ.* **2012**, *2012*, 346089. [CrossRef]
39. Odibat, Z.; Momani, S. Modified homotopy perturbation method: Application to quadratic Riccati differential equation of fractional order. *Chaos Solitons Fractals* **2008**, *36*, 167–174. [CrossRef]
40. Saad, K.M.; Al-Shomrani, A.A. An application of homotopy analysis transform method for Riccati differential equation of fractional order. *J. Fract. Calc. Appl.* **2016**, *7*, 61–72.
41. Haq, E.U.; Ali, M.; Khan, A.S. On the solution of fractional Riccati differential equations with variation of parameters method. *Eng. Appl. Sci. Lett.* **2020**, *3*, 1–9.
42. Sakar, M.G.; Akgül, A.; Baleanu, D. On solutions of fractional Riccati differential equations. *Adv. Differ. Equations* **2017**, *2017*, 1–10. [CrossRef]
43. Yüzbasi, S. Numerical solutions of fractional Riccati type differential equations by means of the Bernstein polynomials. *Appl. Math. Comput.* **2013**, *219*, 6328–6343.
44. Li, Y.; Sun, N.; Zheng, B.; Wang, Q.; Zhang, Y. Wavelet operational matrix method for solving the Riccati differential equation. *Commun. Nonlinear Sci. Numer. Simul.* **2014**, *19*, 483–493. [CrossRef]
45. Raja, M.A.Z.; Manzar, M.A.; Samar, R. An efficient computational intelligence approach for solving fractional order Riccati equations using ANN and SQP. *Appl. Math. Model.* **2015**, *39*, 3075–3093. [CrossRef]
46. Oldham, K.; Spanier, J. *The Fractional Calculus Theory and Applications of Differentiation and Integration to Arbitrary Order*; Elsevier: New York, NY, USA; London, UK, 1974.
47. Kazem, S.; Abbasb, y, S.; Kumar, S. Fractional-order Legendre functions for solving fractional-order differential equations. *Appl. Math. Model.* **2013**, *37*, 5498–5510. [CrossRef]
48. Arafa, A.A.M.; Rida, S.Z. Numerical solutions for some generalized coupled nonlinear evolution equations. *Math. Comput. Model.* **2012**, *56*, 268–277. [CrossRef]
49. Fletcher, C.A.; Fletcher, C.A.J. *Computational Galerkin Methods*; Springer: Berlin/Heidelberg, Germany, 1984; pp. 72–85.
50. Keshavarz, E.; Ordokhani, Y.; Razzaghi, M. Bernoulli wavelet operational matrix of fractional order integration and its applications in solving the fractional order differential equations. *Appl. Math. Model.* **2014**, *38*, 6038–6051. [CrossRef]
51. Parand, K.; Delkhosh, M. Operational matrices to solve nonlinear Riccati differential equations of arbitrary order, Petersburg Polytech. *Univ. J. Phys. Math.* **2017**, *3*, 242–254.
52. Hou, J.; Yang, C. Numerical solution of fractional-order Riccati differential equation by differential quadrature method based on Chebyshev polynomials. *Adv. Differ. Equ.* **2017**, *2017*, 1–13. [CrossRef]
53. Ezz-Eldien, S.S.; Machado, J.A.T.; Wang, Y.; Aldraiweesh, A. A. An algorithm for the approximate solution of the fractional Riccati differential equation. *Int. J. Nonlinear Sci. Numer. Simul.* **2019**, *20*, 661–674. [CrossRef]
54. Maleknejad, K.; Torkzadeh, L. Hybrid functions approach for the fractional Riccati differential equation. *Filomat* **2016**, *30*, 2453–2463. [CrossRef]
55. Odetunde, O.S.; Taiwo, O.A. A decomposition algorithm for the solution of fractional quadratic Riccati differential equations with Caputo derivatives. *Am. J. Comput. Appl. Math.* **2014**, *4*, 83–91.