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The Number of Baxter Permutations

F. R. K. CHUNG,* R. L. GRAHAM*, V. E. HOGGATT, JR.,† AND M. KLEIMAN‡

*Bell Laboratories, Murray Hill, New Jersey, †San Jose State University San Jose California, and ‡Stuyvesant High School, New York City, New York

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BACKGROUND

Baxter permutations apparently first arose in attempts to prove the "commuting function" conjecture of Dyer (see [1]), namely, if f and g are continuous functions mapping [0, 1] into [0, 1] which commute under composition, then they have a common fixed point. Although numerous partial results were obtained for the conjecture (e.g., see [1, 3, 7, 10]), it was ultimately shown in 1967 to be false by Boyce [5] and independently, by Hunecke [8]. However, it has recently been pointed out by Boyce [6] that Baxter permutations are of more general significance in analysis than had previously been realized. This comes about as follows.

For a continuous function $h : [0, 1] \rightarrow [0, 1]$, let $[h] = \{x : h(x) = x\}$ denote the set of fixed points of h and let $[h]^* \subseteq [h]$ denote the set of crossing points of h, i.e., $\alpha \in [h^*]$ if and only if α is a limit point of both $\{x : h(x) < x\}$ and $\{x : h(x) > x\}$ (if $\alpha = 0$ then only the first condition must hold; if $\alpha = 1$ then only the second must hold). For continuous functions $f, g : [0, 1] \rightarrow [0, 1]$, if $\alpha \in [g \circ f]$ then

$$(f \circ g)(f(\alpha)) = f(g(f(\alpha))) = f(\alpha),$$

i.e., $f(\alpha) \in [f \circ g]$. In fact, when $[f \circ g]$ (and therefore, $[g \circ f]$) is finite, f is a 1–1 map of $[g \circ f]$ onto $[f \circ g]$ and therefore induces a permutation π_f of $\{1, 2,..., M\}$ onto itself as follows: If we write $[g \circ f] = \{x_1 ,..., x_M\}$, $[f \circ g] = \{y_1 \ ... \ y_M\}$ then for $i \in \{1 \ 2,..., M\}$,

$$\pi_f(i) = j,$$

where $f(x_i) = y_j$.

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It was shown by Baxter [1] that in this case, $|[g \circ f]| = 2n - 1$ for some n; the corresponding induced permutation $\pi_f{}^*$ is called a Baxter permutation on $\{1, 2, ..., 2n - 1\} \equiv I_{2n-1}$.

Baxter permutations π on I_{2n-1} have the following intrinsic characterization (see [1]):

(i) π maps odd numbers to odd numbers and even numbers to even numbers;

(ii) If $\pi(x) = i$, $\pi(y) = i + 1$, and z is between x and y then $\pi(z) \geqslant i$ if i is even and $\pi(z) \geqslant i + 1$ if i is odd (where we say that a is between b and c if $b \leqslant a \leqslant c$ or $c \leqslant a \leqslant b$).

The purpose of this note is to answer a question first raised in [4], namely, to determine the number $B(n)$ of Baxter permutations on I_{2n-1}. The answer turns out to be surprisingly nice:

THEOREM.

$$B(n) = \binom{n+1}{1}^{-1} \binom{n+1}{2}^{-1} \sum_{k=1}^{n} \binom{n+1}{k-1} \binom{n+1}{k} \binom{n+1}{k+1}. \qquad (1)$$

Proof. The proof we give is not entirely straightforward. We have no idea whether or not a purely combinatorial proof of (1) can be given, based on the special form of the sum.

THE FIRST RECURRENCE

It was already noted in [4] that a Baxter permutation is actually determined by its action on the *odd* numbers in its domain. Hence, by (i), to each Baxter permutation π on I_{2n-1} there corresponds a unique "reduced" Baxter permutation $\hat{\pi}$ on I_n, defined by:

$$\hat{\pi}(k) = \tfrac{1}{2}(1 + \pi(2k + 1)), \qquad k \in I_n.$$

The condition corresponding to (ii) becomes

$\widehat{\text{(ii)}}$ If $\pi(x) = i$, $\pi(y) = i + 1$ then for some k_i between x and y: $\pi(z) \leqslant i$ if z is between x and k_i, $\pi(z) \geqslant i + 1$ if z is between $k_i + 1$ and y.

In fact, it is somewhat surprising (but not too hard to prove) that all the values k_i in $\widehat{\text{(ii)}}$ are *distinct*, although this fact will not be used in what follows.

If we regard a permutation π on I_n as an arrangement of I_n into the sequence $(\pi(1), \pi(2), ..., \pi(n))$ then it is easy to see that $\widehat{\text{(ii)}}$ can be expressed as follows: An arrangement $A = (a_1, \ldots, a_n)$ of I_n corresponds to a reduced Baxter permutation if and only if there do not exist indices $i < j < k < l$ such that

$$a_k + 1 < a_i + 1 = a_l < a_j$$

or

$$a_j + 1 < a_l + 1 = a_i < a_k .$$

Let us call such an arrangement *admissible*. For example, (2, 6, 3, 1, 5, 4) and (5, 1, 4, 3, 7, 6, 2) are admissible while (2, 4, 1, 3) is not. The problem we face now is simply that of enumerating the admissible arrangements of I_n. We shall denote the set of admissible arrangements of I_n by A_n.

Consider an admissible arrangement $\bar{a} = (a_1, a_2, \ldots, a_n) \in A_n$. Let $i^*(\bar{a})$ denote the index i for which $a_i = n$. It is easy to see that if we delete $a_{i^*(\bar{a})} = n$ from $\bar{a}$ then the resulting $(n-1)$-tuple $\bar{a}'$ is admissible, i.e., $\bar{a}' \in A_{n-1}$. Thus we can think of generating elements of A_{n+1} by *inserting* $n+1$ at various positions in $\bar{a}$. Let us partition the $n+1$ positions into which $n+1$ may be inserted in $\bar{a}$ into two classes: the *allowed* positions in which after $n+1$ is inserted the resulting arrangement $\bar{a}^+$ is still admissible, and the *prohibited* positions where this is not the case. We will represent each allowed position by a 0 and each prohibited position by a 1, thereby generating the *insertion vector* $P(\bar{a}) = (p_0, p_1, \ldots, p_n)$. Also, it will be convenient to indicate in $P(\bar{a})$ the location of $i^*(\bar{a})$ by placing the symbol * between $p_{i^*(a)-1}$ and $p_{i^*(a)}$. For example,

$$\bar{a} = (2, 6, 3, 1, 5, 4) \Rightarrow P(\bar{a}) = (0, 0,^* 0, 1, 1, 0, 0),$$

$$\bar{a} = (5, 1, 4, 3, 7, 6, 2) \Rightarrow P(\bar{a}) = (0, 1, 1, 1, 0,^* 0, 0, 0).$$

Suppose now that $n+1$ is inserted into $\bar{a}$, say in the kth position, to form $\bar{a}^+ \in A_{n+1}$. Thus, by definition we must have $p_k = 0$. Then it is not hard to see that $P(\bar{a}^+)$ can be formed from $P(\bar{a})$ as follows:

(i) Replace all 0's between p_k and the * in $P(\bar{a})$ by 1;

(ii) Replace $p_k = 0$ by $0,^* 0$ and remove the old *.

As an example, for $\bar{a} = (2, 6, 3, 1, 5, 4)$ mentioned earlier,

$$\bar{a}^+ = (7, 2, 6, 3, 1, 5, 4) \Rightarrow P(\bar{a}^+) = (0,^* 0, 1, 0, 1, 1, 0, 0,),$$

$$\bar{a}^+ = (2, 6, 7, 3, 1, 5, 4) \Rightarrow P(\bar{a}^+) = (0, 0, 0,^* 0, 1, 1, 0, 0),$$

$$\bar{a}^+ = (2, 6, 3, 1, 5, 4, 7) \Rightarrow P(\bar{a}^+) = (0, 0, 1, 1, 1, 1, 0,^* 0).$$

Note that once a position is prohibited in $\bar{a}$, it remains prohibited no matter how many insertions are made into $\bar{a}$.

All admissible arrangements are therefore generated by starting with $\bar{a}_0 = (1)$ and recursively inserting $n + 1$ in all possible valid ways into each $\bar{a} \in A_n$. We show the beginning of this process in Fig. 1, where we also list $P(\bar{a})$ below each $\bar{a}$.

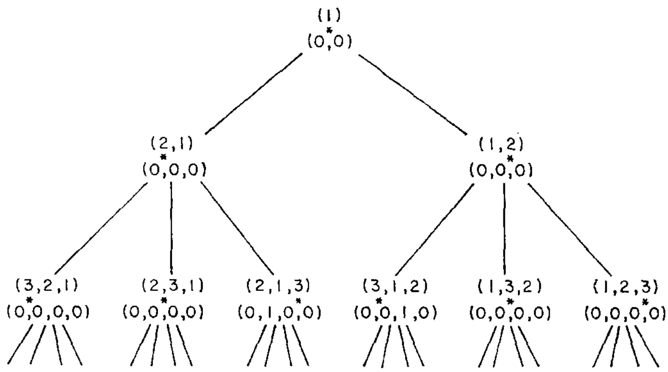

FIGURE 1

Now, the number of ways that $n + 1$ can be validly inserted into $\bar{a}$ to form an admissible $\bar{a}^+$ is just equal to the number of 0's in $P(\bar{a})$. Furthermore, we know the number of 0's in $P(\bar{a}^+)$ if we know how many 0's were changed to 1's in going from $P(\bar{a})$ to $P(\bar{a}^+)$. In fact, since the number and location of the 1's in $P(\bar{a})$ does not affect the number of "descendants" of $\bar{a}$, we can just as well delete them. We can make this precise as follows. Let $T_n(i, j)$ denote the number of ways of obtaining an $\bar{a} \in A_{n+1}$ for which $P(\bar{a})$ has $i + 1$ 0's preceding the * and $j + 1$ 0's following the *. Then $T_n(i, j)$ satisfies the following recurrence:

$$T_{n+1}(i + 1, j + 1) = \sum_{k=1}^{\infty} (T_n(i + k, j) + T_n(i, j + k)), \qquad n \geqslant 0, \quad (2)$$

where

$$T_0(i, j) = 1 \qquad \text{if } i = 0 = j,$$
$$= 0 \qquad \text{otherwise.}$$

This equation follows at once from the preceding algorithm described for generating $P(\bar{a}^+)$ from $P(\bar{a})$. Of course, the sum in (2) is a finite sum since for any fixed n, only finitely many of the $T_n(i, j)$ are nonzero. The value of $B(n)$ is obtained from the T's by

$$B(n + 1) = \sum_{i,j \geqslant 0} T_n(i, j), \qquad n \geqslant 0. \quad (3)$$

It follows from (2) and (3) that $T_n(i,j) = T_n(j,i)$ and

$$B(n) = \sum_i T_n(i,0) = T_{n+1}(1,0). \tag{4}$$

We can write the $T_n(i,j)$ in a triangular array as shown in Fig. 2, where unlisted values are zero. In Table I we list some values of $T_n(i,j)$ for small n, i, j.

$$\begin{array}{ccccccc} & & & T_n(0,0) & & & \\ & & T_n(1,0) & & T_n(0,1) & & \\ & T_n(2,0) & & T_n(1,1) & & T_n(0,2) & \\ T_n(3,0) & & T_n(2,1) & & T_n(1,2) & & T_n(0,3) \\ & & & \cdots & & & \\ T_n(n,0) & & T_n(n-1,1) & \cdots & T_n(1,n-1) & & T_n(0,n) \end{array}$$

FIGURE 2

We have reduced the problem of determining $B(n)$ to that of determining the $T_n(i,j)$. However, it is still far from clear how (2) and (3) imply (1). To remedy this situation, further transformations must be made.

TABLE I

Some Values of $T_n(i,j)$

$n = 0$	1
$n = 1$	0
	1 1
$n = 2$	0
	1 1
	1 2 1
$n = 3$	0
	2 2
	3 4 3
	1 3 3 1
$n = 4$	0
	6 6
	9 12 9
	6 11 11 6
	1 4 6 4 1
$n = 5$	0
	22 22
	33 44 33
	26 43 43 26
	10 24 30 24 10
	1 5 10 10 5 1

The Second Recurrence

From (2) we can count the number of times $T_m(i, j)$ occurs in the expansion of $B(n + m + 1)$ given by (3). Because of the form of the recurrence (2), it is easy to see that this number is independent of m; we shall denote it by $C_n(i, j)$. For example, from (3) it follows that $C_0(i\ j) = 1$ for all i and j. Similarly,

$$\begin{aligned} B(n+2) &= \sum_{i,j\geqslant 0} T_{n+1}(i,j) \\ &= \sum_{i,j\geqslant 0} \sum_{k=1}^{\infty} (T_n(i+k-1, j-1) + T(i-1, j+k-1)) \\ &= \sum_{p,q\geqslant 0} ((p+1)\, T_n(p,q) + (q+1)\, T_n(p,q)) \\ &= \sum_{i,j\geqslant 0} (i+j+2)\, T_n(i,j) \end{aligned}$$

and consequently, $C_1(i, j) = i + j + 2$.

TABLE II

Some Values of $C_n(i, j)$

$n = 0$	1
	1 1
	1 1 1
	1 1 1 1
	1 1 1 1 1
	. . . . . .
$n = 1$	2
	3 3
	4 4 4
	5 5 5 5
	6 6 6 6 6
	. . . .
$n = 2$	6
	11 11
	17 18 17
	24 26 26 24
	32 35 36 35 32
	. . . .
$n = 3$	22
	46 46
	79 86 79
	122 138 138 122
	. . . . . .

In general, this argument yields the following recurrence for $C_n(i, j)$:

$$C_{n+1}(i, j) = \sum_{0 \leqslant i' \leqslant i} C_n(i', j + 1) + \sum_{0 \leqslant j' \leqslant j} C_n(i + 1, j'), \qquad n \geqslant 0,$$

where $C_0(i, j) = 1$ for $i, j \geqslant 0$.

We list some values of $C_n(i, j)$ in Table II, using the same format that we used for $T_n(i, j)$. Since, by definition

$$B(n + m + 1) = \sum_{i,j \geqslant 0} C_m(i, j)\, T_n(i, j), \tag{6}$$

then setting $n = 0$ and using the fact that the only nonzero value of $T_0(i, j)$ is $T_0(0, 0) = 1$, we obtain (replacing m by n)

$$B(n + 1) = C_n(0, 0). \tag{7}$$

We are now ready for our next transformation.

Generating Functions

Let us introduce the following generating function:

$$f_{n+1}(x, y) = \sum_{i,j \geqslant 0} C_n(i, j)\, x^i y^j, \qquad n \geqslant 0. \tag{8}$$

Then

$$\begin{aligned} \frac{f_n(x, y) - f_n(x, 0)}{(1 - x)\, y} &= \frac{1}{(1 - x)\, y} \sum_{i,j \geqslant 0} C_{n-1}(i, j)\, x^i y^j \\ &\quad - \frac{1}{(1 - x)\, y} \sum_{i \geqslant 0} C_{n-1}(i, 0)\, x^i \\ &= \frac{1}{(1 - x)} \sum_{i,j \geqslant 0} C_{n-1}(i, j + 1)\, x^i y^j \\ &= \sum_{i,j \geqslant 0} \sum_{0 \leqslant i' \leqslant i} C_{n-1}(i', j + 1)\, x^i y^j \end{aligned}$$

and, in a similar way,

$$\frac{f_n(x, y) - f_n(0, y)}{(1 - y)\, x} = \sum_{i,j \geqslant 0} \sum_{0 \leqslant j' \leqslant j} C_{n-1}(i + 1, j')\, x^i y^j.$$

Therefore, by (5),

$$\frac{f_n(x, y) - f_n(x, 0)}{(1-x)\,y} + \frac{f_n(x, y) - f_n(0, y)}{(1-y)\,x}$$

$$= \sum_{i,j\geqslant 0} \sum_{0\leqslant i'\leqslant i} C_{n-1}(i', j+1) + \sum_{0\leqslant j'\leqslant j} C_{n-1}(i+1, j'))\, x^i y^j \qquad (9)$$

$$= \sum_{i,j\geqslant 0} C_n(i,j)\, x^i y^j = f_{n+1}(x, y), \qquad n \geqslant 1.$$

The first few $f_n(x, y)$ are as follows.

$$f_1(x, y) = \sum_{i,j\geqslant 0} C_0(i,j)\, x^i y^j = \sum_{i,j\geqslant 0} x^i y^j = \frac{1}{(1-x)(1-y)},$$

$$f_2(x, y) = \frac{1}{(1-x)^2(1-y)} + \frac{1}{(1-x)(1-y)^2},$$

$$f_3(x, y) = \frac{1}{(1-x)^3(1-y)} + \frac{4-x-y}{(1-x)^2(1-y)^2} + \frac{1}{(1-x)(1-y)^3},$$

$$f_4(x, y) = \frac{1}{(1-x)^4(1-y)} + \frac{10-5x-4y+xy+x^2}{(1-x)^3(1-y)^2}$$

$$+ \frac{10-5y-4x+xy+y^2}{(1-x)^2(1-y)^3} + \frac{1}{(1-x)(1-y)^4} \qquad (10)$$

Let us write

$$f_n(x, y) = \sum_{k=1}^{n} \frac{P_{n,k}(x, y)}{(1-x)^{n+1-k}(1-y)^k}. \qquad (11)$$

Thus,

$$f_{n+1}(x, y) = \sum_{k=1}^{n+1} \frac{P_{n+1,k}(x, y)}{(1-x)^{n+2-k}(1-y)^k}$$

$$= \frac{f_n(x, y) - f_n(x, 0)}{(1-x)\,y} + \frac{f_n(x, y) - f_n(0, y)}{(1-y)\,x}$$

$$= \sum_{k=1}^{n} \left\{ \frac{1}{(1-x)\,y} \left(\frac{P_{n,k}(x, y)}{(1-x)^{n+1-k}(1-y)^k} - \frac{P_{n,k}(x, 0)}{(1-x)^{n+1-k}} \right) \right.$$

$$\left. + \frac{1}{(1-y)\,x} \left(\frac{P_{n,k}(x, y)}{(1-x)^{n+1-k}(1-y)^k} - \frac{P_{n,k}(0, y)}{(1-y)^k} \right) \right\}$$

$$= \sum_{k=1}^{n} \left\{ \frac{P_{n,k}(x, y)}{y(1-x)^{n+2-k}(1-y)^k} - \frac{P_{n,k}(x, 0)(1-y)^k}{y(1-x)^{n+2+k}(1-y)^k} \right.$$

$$\left. + \frac{P_{n,k}(x, y)}{x(1-x)^{n+1-k}(1-y)^{k+1}} - \frac{P_{n,k}(0, y)(1-x)^{n+1-k}}{(1-x)^{n+1-k}(1-y)^{k+1}} \right\}.$$

Therefore,

$$P_{n+1,k}(x, y) = (1/y)(P_{n,k}(x, y) - P_{n,k}(x, 0)(1 - y)^k) + (1/x)(P_{n,k-1}(x, y) - P_{n,k-1}(0, y)(1 - x)^{n+2-k})$$

for $n \geqslant 1$, $1 \leqslant k \leqslant n + 1$, where

$$P_{1,1}(x, y) = 1, P_{r,s}(x, y) = 0 \qquad \text{for } s < 0 \text{ and } s > r.$$

The recurrence (11) consequently determines an array of *polynomials* $P_{n,k}(x, y)$. The value of $B(n)$ is obtained from these polynomials by:

$$B(n) = C_{n-1}(0, 0) = f_n(0, 0) = \sum_{k=1}^{n} P_{n,k}(0, 0). \tag{13}$$

Our job now becomes that of determining the $P_{n,k}(x, y)$.

The Polynomials $Q_{n,k}(x, y)$

In order to avoid complications with $\pm$ signs which could occur later, we shall define polynomials $Q_{n,k}(x, y)$ by

$$Q_{n,k}(x, y) = P_{n,k}(-x, -y).$$

Thus, the $Q_{n,k}(x, y)$ satisfy (from (12))

$$Q_{n+1,k}(x, y) = (1/y)(Q_{n,k}(x, 0)(1 + y)^k - Q_{n,k}(x, y)) + (1/x)(Q_{n,k-1}(0, y)(1 + x)^{n+2-k} - Q_{n,k-1}(x, y)) \tag{14}$$

for $n \geqslant 1$, $1 \leqslant k \leqslant n + 1$, where

$$Q_{1,1}(x, y) = 1, \qquad Q_{r,s}(x, y) = 0 \qquad \text{for } s < 0 \text{ and } s > r.$$

Let us write

$$Q_{n,k}(x, y) = \sum_{i,j \geqslant 0} D_{n,k,i,j} x^i y^j. \tag{15}$$

If we substitute the expression for $Q_{n,k}(x, y)$ in (15) into (14) then we obtain

$$D_{n+1,k,i,j} = D_{n,k,i,0} \binom{k}{j+1} - D_{n,k,i,j+1} + D_{n,k-1,0,j} \binom{n+2-k}{i+1} - D_{n,k-1,i+1,j} \,. \tag{16}$$

At this stage of the proof one would ordinarily pull out of a hat an explicit expression for $D_{n,k,i,j}$ which would then be shown by induction to satisfy (14). For this problem, however, this process is not completely trivial. To illustrate this, we give in Table III the coefficients $D_{9,k,i,j}$. Values not shown are zero. It follows from (14) that

$$D_{n,k,i,j} = D_{n,n+1-k,j,i} \tag{17}$$

so that the arrays for $k = 6, 7, 8, 9$ not listed in Table III are just transposes of arrays for $k = 4, 3, 2, 1$, respectively. The most striking aspect of the coefficients, generally, is the lack of large prime factors. For example, no $D_{9,k,i,j}$ has any prime factor exceeding 13. It was perhaps this property more

TABLE III

Values of $D_{9,k,i,j}$

$k = 1$:

$i \backslash j$	0
0	1

$k = 2$:

$i \backslash j$	0	1
0	120	84
1	210	126
2	252	126
3	210	84
4	120	36
5	45	9
6	10	1
7	1	0

$k = 3$:

$i \backslash j$	0	1	2
0	2520	3360	1176
1	6048	7392	2352
2	7560	8400	2352
3	5760	5760	1344
4	2700	2400	420
5	720	560	56
6	84	56	0

$k = 4$:

$i \backslash j$	0	1	2	3
0	14112	26460	17640	4116
1	35280	61740	38220	8232
2	40320	65520	36960	7056
3	25200	37800	18900	2940
4	8400	11550	4900	490
5	1176	1470	490	0

$k = 5$:

$i \backslash j$	0	1	2	3	4
0	24696	56448	52920	23520	4116
1	56448	120960	105840	43680	7056
2	52920	105840	85050	31500	4410
3	23520	43680	31500	9800	980
4	4116	7056	4410	980	0

than any other which convinced us that there must be a relatively simple expression for $D_{n,k,i,j}$.

Indeed, after several hours of reflection, the following expression emerged:

$$D_{n,k,i,j} = \binom{n+1}{1}^{-1} \binom{n+1}{2}^{-1} \binom{n+1}{k} \binom{n+1}{k+i+1} \binom{n+1}{k-j-1}$$
$$\times \left[\binom{k+i-2}{i} \binom{n+j-k-1}{j} - \binom{k+i-2}{i-1} \binom{n+j-k-1}{j-1} \right], \tag{18}$$

where $\binom{x}{y}$ is taken to be 0 if $y < 0$ or $y > x$. A straightforward substitution of (18) into (16) now yields (after clearing the denominators) the following equivalent equation (19), which holds if and only if the asserted value of $D_{n,k,i,j}$ in (18) satisfies (16):

$$\begin{aligned}&(k-1)\,k(i+1)(n-k-i)(n-k+j+2)(n-k+j+1)(n-k+j)\\ &\quad + (j+1)(k-j-1)(n-k+1)(n-k)(k+i+1)(k+i)(k+i-1)\\ &\quad + (n-k+1)(n-k)(i+1)(n-k-i)(k-j-1)[i(j+1)\\ &\quad - (k-1)(n-k-1)]\\ &\quad + k(k-1)(j+1)(n-k-i)(k-j-1)[j(i+1)-(k-2)(n-k)]\\ &\quad + (n-1)\,n(n+1)(i+1)(j+1)[ij-(k-1)(n-k)] \stackrel{?}{=} 0.\end{aligned} \tag{19}$$

As unlikely as it seems, (19) does indeed hold identically and, consequently, since (18) gives the correct values of $D_{n,k,i,j}$ for small values of n, then by induction, (18) gives the value of $D_{n,k,i,j}$ for all values of the parameters.

We now use (18) to deduce (1), the main result of the paper. From (18),

$$D_{n,k,0,0} = \binom{n+1}{1}^{-1} \binom{n+1}{2}^{-1} \binom{n+1}{k-1} \binom{n+1}{k} \binom{n+1}{k+1}. \tag{19}$$

So, from (13), (15), and the definition of $Q_{n,k}(x, y)$,

$$\begin{aligned} B(n) = f_n(0,0) &= \sum_{k=1}^{n} P_{n,k}(0,0)\\ &= \sum_{k=1}^{n} Q_{n,k}(0,0)\\ &= \sum_{k=1}^{n} D_{n,k,0,0}\\ &= \binom{n+1}{1}^{-1} \binom{n+1}{2}^{-1} \sum_{k=1}^{n} \binom{n+1}{k-1} \binom{n+1}{k} \binom{n+1}{k+1}, \end{aligned}$$

which is just (1). This completes the proof of the theorem.

Some Remarks

In Table IV we give some small values of $B(n)$. It was pointed out to us by A. M. Odlyzko that the first few terms in the asymptotic expansion of $B(n)$ are given by

$$B(n) = \frac{32 \cdot 2^{3n}}{n^4 \pi 3^{1/2}} \left\{1 - \frac{22}{3n} + O(n^{-2})\right\}. \tag{20}$$

This approximation is not too bad, even for relatively small n. For example, from Table IV

$$B(50) = 1.16356... \times 10^{39},$$

whereas the first two terms of the asymptotic expansion give

$$B(50) \approx 1.14598 \times 10^{39}.$$

We point out in passing that $B(n)$ satisfies the following linear recurrence (derived from (1) by Paul S. Bruckman):

$$\begin{aligned}(n+1)&(n+2)(n+3)(3n-2)\, B(n) \\ &= 2(n+1)(9n^3+3n^2-4n+4)\, B(n-1) \\ &\quad + (3n-1)(n-2)(15n^2-5n-14)\, B(n-2) \\ &\quad + 8(3n+1)(n-2)^2\,(n-3)\, B(n-3)\end{aligned} \tag{21}$$

for $n \geqslant 4$, where $B(1) = 1$, $B(2) = 2$, $B(3) = 6$.

As mentioned at the beginning, there is no proof of (1) known which enumerates classes of Baxter permutations corresponding in a natural way to the individual summands in (1). There are almost certainly other classes of

TABLE IV

n	$B(n)$
1	1
2	2
3	6
4	22
5	92
6	422
7	2074
8	10754
9	58202
10	326240
20	29949238543316
30	7101857696077190042814
40	2554987813422078288794169298972
50	1163558691573487855005674103586862832160

restricted permutations for which similar techniques can be applied although none of us has done this yet.

A few historical notes may be in order here. The recurrence for $T_n(i, j)$ in (2) was derived in 1967 by one of the authors (R. L. Graham) in response to a query of W. M. Boyce, who had already tabulated the values of $B(n)$ for small values of n (see [4]). These values subsequently appeared in the unique handbook of Sloane [11] as Sequence No. 652. In 1977, another of the authors (V. E. Hoggatt) discovered that the first 10 row sums of a certain array of generalized binomial coefficients happened to agree exactly with the values of $B(n)$ tabulated in Sloane. It had not been suspected beforehand that they might be given by such a simple expression.

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Note added in proof. C. L. Mallows of Bell Laboratories has just shown that the individual summand $D_{n,k,0,0}$ in (19) is in fact just the number of reduced Baxter permutations on I_n which have exactly k *rises*, i.e., indices i for which $\pi(i) < \pi(i + 1)$.

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