The Number of Different Distances Determined by a Set of Points in the Euclidean Plane*

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Abstract. In 1946 P. Erdös posed the problem of determining the minimum number $d(n)$ of different distances determined by a set of n points in the Euclidean plane. Erdös proved $d(n) \geq cn^{1/2}$ and conjectured that $d(n) \geq cn/\sqrt{\log n}$. If true, this inequality is best possible as is shown by the lattice points in the plane. We show $d(n) \geq n^{4/5}/(\log n)^c$.

1. Introduction

In 1946 Erdös [3] posed the problem of determining the minimum number $d(n)$ of different distances determined by a set of n points in the Euclidean plane. Erdös proved $d(n) \geq cn^{1/2}$ and conjectured that $d(n) \geq cn/\sqrt{\log n}$. If true, this inequality is best possible as is shown by the lattice points in the plane.

Despite its apparent simplicity, progress on this problem has been slow. In 1952 Moser [4] proved that $d(n) \geq cn^{2/3}$ and this result stood as the best-known lower bound on $d(n)$ for more than 30 years. In 1984 Chung [2] proved $d(n) \geq cn^{5/7}$. Subsequently Beck [1] improved this to $d(n) \geq n^{(58/81)-\varepsilon}$.

Using the methods introduced by Szemerédi and Trotter [6], [7] on incidences

between points and lines, and by Spencer, Szemerédi, and Trotter [5] on unit distances, it is relatively straightforward to show that $d(n) \geq cn^{3/4}$. However, in this article we combine techniques to show that $d(n) \geq n^{4/5}/(\log n)^c$.

Throughout this paper we consider finite sets of points in the Euclidean plane. We impose a cartesian coordinate system and choose a unit distance. When $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are points in the plane, we let $dist(x, y)$ denote the distance between x and y, i.e., $dist(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$.

When $P = \{x_1, x_2, \ldots, x_n\}$ is a set of n distinct points in the plane, we let $D(P)$ denote the set of distances determined by the points in P, i.e., $D(P) = \{dist(x_i, x_j): 1 \leq i < j \leq n\}$. We also let $d(P) = |D(P)|$ so that $d(P)$ counts the number of different distances determined by P. Using this notation, the principal goal of this paper is to prove the following theorem.

Theorem 1. *There exists an absolute constant $c_1 > 0$ so that if $n \geq 10$ and P is any set of n distinct points in the Euclidean plane, then the number $d(P)$ of different distances determined by P satisfies $d(P) > n^{4/5}/(\log n)^{c_1}$.*

Our proof for Theorem 1 depends in part on the following inequality of Szemerédi and Trotter [7].

Theorem 2. *There exists an absolute constant $c_2 > 0$ so that if $2 \leq k \leq \sqrt{n}$ and P is a set of n points in the plane, then the number of lines containing at least k points of P is less than $c_2 n^2/k^3$.*

2. Distances—The Bipartite Version

In order to establish Theorem 1, we first prove a bipartite version from which Theorem 1 follows as an immediate corollary. When P and Q are disjoint sets of points in the plane, we let $D(P, Q) = \{dist(x, y): x \in P, y \in Q\}$ and $d(P, Q) = |D(P, Q)|$.

Theorem 3. *There exists an absolute constant $c_3 > 0$ so that if $n \geq 10$, and P and Q are disjoint point sets in the Euclidean plane with $|P| = |Q| = n$, then $d(P, Q) > n^{4/5}/(\log n)^{c_3}$.*

Proof. We will show that the desired inequality holds when $c = c_3$ is sufficiently large. Our argument requires that c be suitably large in comparison with the absolute constant c_2 in Theorem 2. Although it is widely believed that Theorem 2 holds for some c_2 less than 10, the proof given in [7] yields the value $c_2 = 10^{180}$. So in the remainder of the proof, we take $c = c_3 = 10^{200}$, admittedly a generous choice.

Now suppose that the conclusion of the theorem fails for this value of c. Choose the least value of n for which this occurs and let P and Q be disjoint n-element point sets in the plane for which $d(P, Q) \leq n^{4/5}(\log n)^c$.

For each $i = 0, 1, 2, \ldots, 20$, let $M_i = (\log n)^{10i}$ and let $\lceil n/M_i \rceil$. Note that $n_0 = n$. In what follows, we identify subsets of P and Q which will satisfy certain desired properties. This is done in a series of (at most 20) steps, and at each step we select a subset of the set resulting from the previous step. The sizes of the sets behave like the sequence $n_0, n_1, \ldots, n_{20}$. The reader may note that the argument only requires that n_{i+1} be a small positive fraction of n_i, except at one critical step where we actually require that n_{i+1} be smaller than n_i by a logarithmic factor. However, the presentation of the argument is simplified by using the most restrictive requirement at each step.

Of all the discs which cover at least n_1 points of P, choose one, say D_1, so that the radius r_1 of D_1 is as small as possible. We consider points on the circular boundary of the disc to be covered by the disc. Also, choose a disc D_2 of minimum radius r_2 covering at least n_1 points of Q. We assume $r_1 \leq r_2$; else reverse the roles played by P and Q in the remainder of the argument. Let u_0 denote the center of disc D_1, and choose a subset P_1 consisting of n_1 points of P covered by D_1.

Claim. *There exists a number r_0 satisfying $r_0 \geq 10^{10} r_1$ for which there are at least $10^{20} n_1$ points of Q in the annulus $\{y \in \mathbf{R}^2 : r_0 \leq dist(u_0, y) \leq r_0 + 3r_1\}$.*

Proof. Suppose the claim is false. Set $K = \lceil n/10^{40} n_1 \rceil$, and define a sequence $s_0, s_1, \ldots, s_{2K-1}$ and pairwise disjoint subsets $S_0, S_1, S_2, \ldots, S_{2K-1}$ of Q as follows. First, set $s_0 = 10^{10} r_1$ and $S_0 = \{y \in Q : 0 \leq dist(u_0, y) \leq s_0\}$. It is easy to see that S_0 can be covered by 10^{25} circles of radius $r_1/2$, so we can safely say $|S_0| < 10^{30} n_1$.

When i is even, let s_{i+1} be the least number exceeding s_i for which the set $S_{i+1} = \{y \in Q : s_i \leq dist(u_0, y) \leq s_{i+1}\}$ contains at least $10^{20} n_1$ points. Since $d(P, Q) = d < n_1$, we can safely say $10^{20} n_1 \leq |S_{i+1}| \leq 10^{30} n_1$ when i is even. The upper bound follows from the fact that it is impossible for any circle centered at u_0 to contain $2n_1$ points of Q, for this would imply that any point of P not located at u_0 determines n_1 different distances with these points.

When i is odd, we take $s_{i+1} = s_i + 3r_1$ and

$$S_{i+1} = \{y \in Q : s_i < dist(u_0, y) < s_{i+1}\}.$$

By assumption, $|S_{i+1}| < 10^{30} n_1$. Now consider the pairwise disjoint subsets $S_1, S_3, S_5, \ldots, S_{2K-1}$. Each contains more than n_1 points so that

$$d(P_1, S_{2j-1}) > n_1^{4/5}/(\log n_1)^c$$

for each $j = 1, 2, \ldots, K$. Furthermore, if $x \in P_1$ and $y \in S_{2j-1}$, then

$$s_{2j-2} - r_1 \leq dist(x, y) \leq s_{2j-1} + r_1.$$

If $1 \leq j < k \leq K$, then $s_{2j-1} + 3r_1 \leq s_{2k-2}$, so $s_{2j-1} + r_1 \leq (s_{2k-2} - r_1) - r_1$, i.e., all distances in $D(P_1, S_{2j-1})$ are smaller than all distances in $D(P_1, S_{2k-1})$. Thus

$$d(P, Q) \geq \frac{Kn_1^{4/5}}{(\log n_1)^c}$$

$$> \frac{\lceil n/10^{40}n_1 \rceil n_1^{4/5}}{(\log n)^c}$$

$$\geq \frac{((\log n)^2/10^{40})n^{4/5}}{(\log n)^c}.$$

Since $n^{4/5}/(\log n)^c > d(P, Q)$, it follows that $\log n \leq 10^{20}$. To obtain a contradiction and complete the proof of the claim, we employ Erdős' original argument [3]. Choose distinct points x, $x' \in P$. If there exists a circle C centered at x containing a subset $Q(C)$ of at least $\sqrt{n}$ points of Q, then $d(\{x'\}, Q(C)) \geq |Q(C)|/2 \geq \sqrt{n}/2$. Thus $n^{4/5}/(\log n)^c \geq \sqrt{n}/2$, which fails when $(\log n) \leq 10^{20}$ which is forced by the size of the constant c. However, if no such circle exists, then $d(P, Q) \geq d(\{x\}, Q) \geq \sqrt{n}$, which cannot hold when $(\log n) \leq 10^{20}$. The contradiction completes the proof of the claim. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

By the pigeon-hole principle, we can choose two rays emanating from u_0 and enclosing an angle of $5°$ for which there is a subset Q_0 of $10^{10}n_1$ points of Q so that all points of Q_0 are in the annulus $\{y \in R^2 : r_0 \leq dist(u_0, y) \leq r_0 + 3r_1\}$ and are within the $5°$ angular region formed by the rays (Fig. 1).

Using rays emanating from u_0, partition the region containing Q_0 into 10^8 regions each covering at least $90n_1$ points from Q_0. Denote the subsets by $Q_1, Q_2, \ldots, Q_{10^8}$ and the angles determining them as $\theta_1, \theta_2, \ldots, \theta_{10^8}$. Let $\theta_0 = \min\{\theta_i : 1 \leq i \leq 10^8\}$. Without loss of generality, there is some i_0 with $1 \leq i_0 \leq 10^8/2$ for which $\theta_0 = \theta_{i_0}$. Relabel Q_{i_0} as Q_1' and Q_{10^8} as Q_2'. Note that the angle θ in Fig. 2 is at least $10^7\theta_0$.

Now the boundary of the region containing Q_1' is very nearly a rectangle of width $3r_1$ and height $\theta_0 r_0$. We claim $\theta_0 r_0 \geq r_1$. Suppose not. Then the region containing Q_1' has area at most $4r_1^2$ and covers $90n_1$ points of Q. Furthermore, if $r_1 > \theta_0 r_0$, this region can be covered by eight discs of radius r_1 with centers at the points indicated in Fig. 3. This implies that there exists a disc of radius less than r_1 covering n_1 points of Q. The contradiction shows $\theta_0 r_0 \geq r_1$, as claimed.

Now that we have some control on the relative sizes of the three regions and
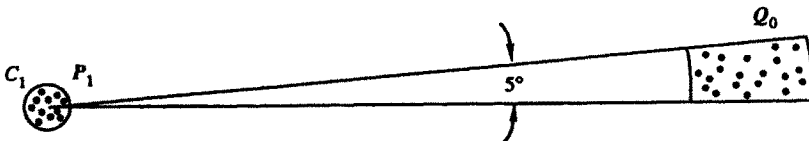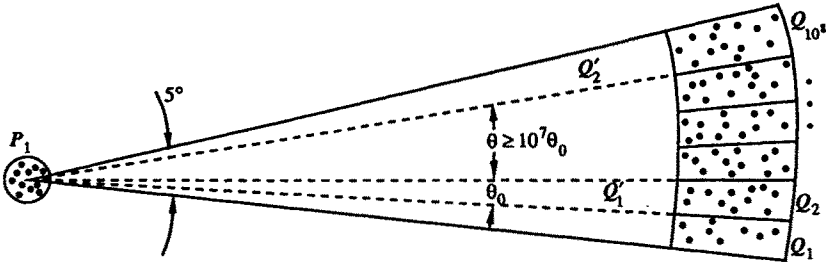

Fig. 1

Fig. 2

the separation between them, it is easy to use the pigeon-hole principle to find a number $z \leq \theta_0 r_0$ and rectangles R_0, R_1, as shown in Fig. 4, with R_0 covering a subset P_2 of n_2 points from P_1, while R_1 covers a subset Q_1'' of n_2 points from Q_1'. Recall that $n_2 = \lceil n/(\log n)^{20} \rceil < n_1$. For convenience, we change scale so that the distance (along line L) from the lower right corner of R_0 to the upper left corner of R_1 is exactly 1. Then let Q_2'' be any subset of n_2 points from Q_2'. Note that if $x \in P_2$, $y_1 \in Q_1''$, and $y_2 \in Q_2''$, then the angle formed by the segments xy_1 and xy_2 is at least $10^6 z$.

At this point we want to apply the covering lemma (see [5]–[7]). First extend the two vertical sides of rectangle R_0 upward to form a rectangle R_0' of width $z/10$ and height 0.01. Then choose a family $\mathcal{R}$ of subrectangles of R_0' so that:

(1) The sides of each rectangle $R \in \mathcal{R}$ are parallel to those of R_0'.
(2) The interiors of the rectangles in $\mathcal{R}$ are pairwise disjoint.
(3) The ratio w/h of width divided by height is $10z$ for every rectangle in $\mathcal{R}$.
(4) Each rectangle in $\mathcal{R}$ contains at least $n_2^{3/5}$ and at most $1000\, n_2^{3/5}$ points of P.
(5) There are at least $n_2^{2/5}/10^6$ rectangles in $\mathcal{R}$.

For a rectangle R, let $P_2(R)$ denote those points of P_2 covered by R. When $x \in P_2$ and $y \in Q_1'' \cup Q_2''$, we let $C(x, y)$ denote the circle centered at y and passing through x. Now let $R \in \mathcal{R}$, $x \in P_2(R)$, and $y \in Q_1''$. We claim that the circle $C(x, y)$ cannot cross both vertical sides of R (Fig. 5).

To see that this claim is valid, let R have height h and width $w = 10zh$. Now the point x is covered by R_0, so x is at most z above the line L. It is an easy exercise to show that in moving laterally an amount w, the circle $C(x, y)$ must rise at least $w/5z$ and this quantity exceeds h. This shows that $C(x, y)$ cannot cross
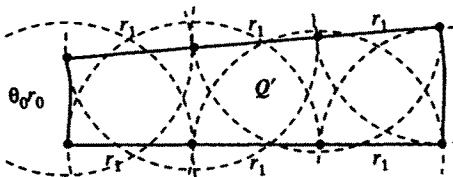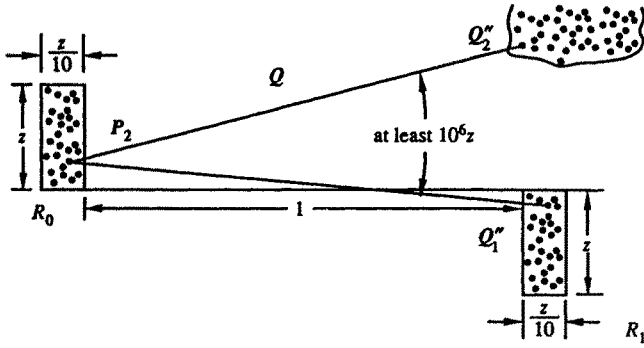

Fig. 3

Fig. 4

both vertical sides of R. However, it is indeed possible for $C(x, y)$ to cross both the top and bottom sides of R.

Now let $R \in \mathcal{R}$, $x \in P_2(R)$, $y_1 \in Q_1''$, and $y_2 \in Q_2''$. We claim that it cannot happen that $C(x, y_1)$ and $C(x, y_2)$ both cross the top and bottom sides of R (Fig. 6).

To see that this claim holds, we consider the angle formed by the line segments xy_1 and xy_2. In view of the relative separation between Q_1'' and Q_2'', this angle is at least $10^6 z$. However, if both $C(x, y_1)$ and $C(x, y_2)$ cross the top and the bottom of R, then this angle is at most $100z$. The contradiction completes the proof of the claim.

The Crossing Property.   Let $R \in \mathcal{R}$. Subdivide R into four subrectangles of equal size by taking the perpendicular bisectors of the sides of R. Let R' be any one of
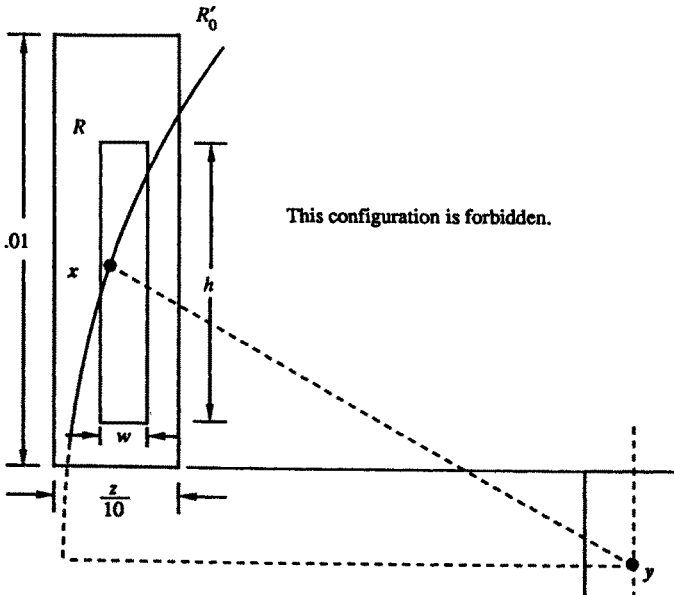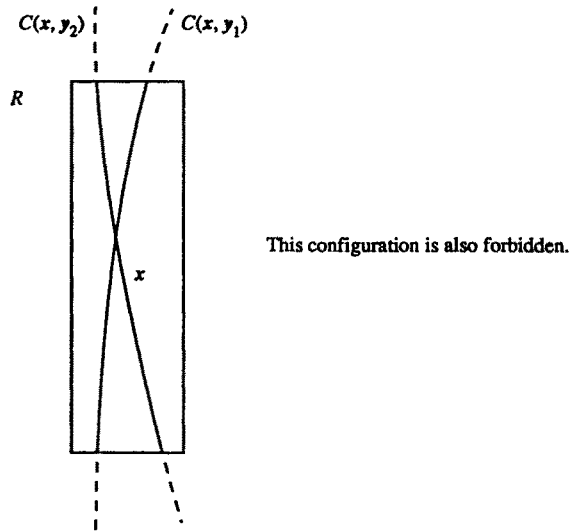

Fig. 5

Fig. 6

these subrectangles and let x, $x' \in P_2(R')$. Then, for every $y_1 \in Q_1''$ and for every $y_2 \in Q_2''$, either $C(x, y_1)$ and $C(x', y_2)$ cross at a point inside R, or $C(x', y_1)$ and $C(x, y_2)$ cross at a point inside R.

We comment that the proof of the Crossing Property is elementary and proceeds by a case-by-case analysis of the four subrectangles and the relative positions of x and x' within them. However, we do note that the desired crossing may occur inside R but not inside R' as illustrated in Fig. 7.

The reason for introducing the Crossing Property is that we will force pairs of circles with centers in Q to cross at a point in the plane not occupied by a point of P. We refer to this as a *wasted crossing*. The final contradiction is obtained by showing that the number of wasted crossings is larger than the number of pairs of circles from $\{C(x, y): x \in P, y \in Q\}$. Now set $i = 1$. We describe a procedure which
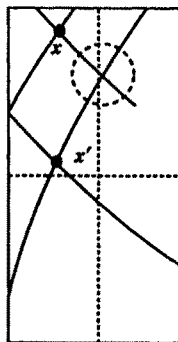

Fig. 7

identifies particular points in $P_2 \cup Q_i''$ and certain rectangles in $\mathscr{R}$. This argument will be repeated when $i = 2$. Let $R \in \mathscr{R}$, $x \in R$, and $y \in Q_i''$. We call (x, y) a *rich* pair if $C(x, y)$ contains at least two points of $P(R)$.

Lemma. *The number of rich pairs is less than n_8^2.*

Proof. Suppose the lemma is false and let S be a set of n_8^2 rich pairs of the prescribed form. Choose a number $a \geq 2$ for which there are at least n_9^2 rich pairs (x, y) for which $C(x, y)$ contains at least a and at most $2a$ points of $P(R)$ where R is the rectangle containing x. Call such pairs *uniform rich* pairs. (Note that here we actually use the logarithmic factor.)
Then let

$$\mathscr{T} = \{x, x', y\colon \text{there exists a rectangle } R \in \mathscr{R} \text{ containing both } x \text{ and } x', y \in Q_i,$$
$$x' \in C(x, y) \text{ and both } (x, y) \text{ and } (x', y) \text{ are uniform rich pairs}\}.$$

Then $|\mathscr{T}| \geq n_9^2 a$. For each $R \in \mathscr{R}$, let $\mathscr{T}(R) = \{(x, x', y) \in \mathscr{T} : x, x' \in R\}$. Since R contains at most $1000 n_2^{3/5}$ points of P and Q_i'' contains n_2 points, there are at most $1000 n_2^{8/5}$ uniform rich pairs (x, y) where $x \in R$ and $y \in Q_i''$. For each such pair, there are at most $2a$ different $x' \in R$ and for which (x, x', y) is a triple in $\mathscr{T}(R)$. It follows that $|\mathscr{T}(R)| \leq 2000 n_2^{8/5} a < n_2^{8/5} a$. We also observe that $n_2^{2/5}/10^6 \leq |\mathscr{R}| < n_1^{2/5}$. Let $\mathscr{T}_0 = \bigcup \{\mathscr{T}(R) : |\mathscr{T}(R)| < n_9^2 a/2 n_1^{2/5}\}$. Then $|\mathscr{T}_0| < n_9^2 a/2$ and $|\mathscr{T} - \mathscr{T}_0| \geq n_9^2 a/2$. So there exists a number $b \geq n_9^2 a/2 n_1^{2/5}$ and a subfamily $\mathscr{R}' \subseteq \mathscr{R}$ so that $b \leq |\mathscr{T}(R)| \leq 2b$ for every $R \in \mathscr{R}'$ and $|\bigcup \{\mathscr{T}(R) : R \in \mathscr{R}'\}| \geq n_9^2 a/10 \log n \geq n_{10}^2 a$.

Let $\mathscr{T}_1 = \bigcup \{\mathscr{T}(R) : R \in \mathscr{R}'\}$ and let $R \in \mathscr{R}'$. For each distinct pair $x, x' \in P(R)$, let $T(x, x') = \{y \in Q_1 : (x, x', y) \in \mathscr{T}_1\}$. For each $R \in \mathscr{R}'$, let

$$\mathscr{T}_2(R) = \{(x, x', y) \in \mathscr{T}(R) : |T(x, x')| < b/(2 \cdot 10^6 n_2^{6/5})\}.$$

Since $|R| \leq 10^3 n_2^{3/5}$, we see that $|\mathscr{T}_2(R)| < b/2$. Set $\mathscr{T}_3(R) = \mathscr{T}(R) - \mathscr{T}_2(R)$. Since $\mathscr{T}(R) \geq b$, we see that $|\mathscr{T}_3(R)| \geq b/2$. Then set $\mathscr{T}_3 = \bigcup \{\mathscr{T}_3(R) : R \in \mathscr{R}'\}$, and observe that $|\mathscr{T}_3| \geq |\mathscr{T}_1|/4 \geq n_{10}^2 a/4$.

When x and x' are distinct points in P, we call (x, x') a *standard* pair if there is some $R \in \mathscr{R}'$ for which $x, x' \in P(R)$ and $|T(x, x')| \geq b/(2 \cdot 10^6 n_2^{6/5})$. When (x, x') is a standard pair, we let $L(x, x')$ denote the perpendicular bisector of the line segment passing through x and x', and we let $\mathscr{L}$ denote the set of all perpendicular bisectors of standard pairs. Note that each line $L(x, x') \in \mathscr{L}$ contains at least $b/(2 \cdot 10^6 n_2^{6/5})$ points from the set Q_1'' (Fig. 8).

For each line $L \in \mathscr{L}$, let $|L|$ denote the number of points of Q_1'' which lie on L. As noted previously, $|L| \geq b/(2 \cdot 10^6 n_2^{6/5})$, but it is possible for $|L|$ to be much larger. Choose a value $e \geq b/(2 \cdot 10^6 n_2^{6/5})$ and a subset $\mathscr{L}_0 \in \mathscr{L}$ so that

(1) $e \leq |L| \leq 2e$ for every $L \in \mathscr{L}_0$;
(2) there are at least $n_{11}^2 a$ triples in $\mathscr{T}_3$ of the form (x, x', y) where (x, x') is a standard pair; and
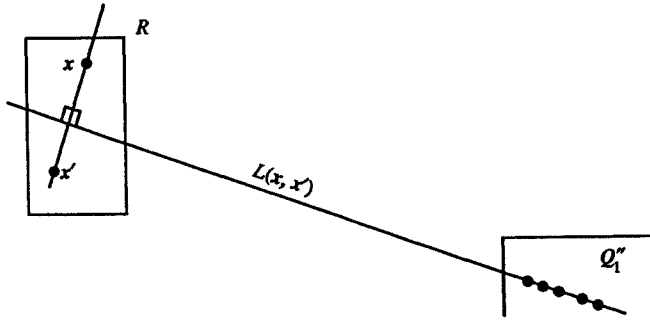(3) $L(x, x') \in \mathscr{L}_0$.

Fig. 8

Case 1: $e \geq 10\sqrt{n}$. Since $|Q_1''| < n$, a simple counting argument shows $|\mathscr{L}_0| \leq 100n/e$. Let $d = n^{4/5}/(\log n)^c$. For each line $L \in \mathscr{L}_0$, we know that L contains at most $2e$ points of Q_1. For each $y \in Q_1$ which lies on L, there are at most $2ad$ standard pairs (x, x') for which $L = L(x, x')$ and $(x, x', y) \in \mathscr{T}$. It follows that

$$n_{11}^2 a \leq \frac{100n}{e} 2e \cdot 2ad < \frac{10^3 n^{9/5} a}{(\log n)^c}$$

which is clearly false.

Case 2: $e < 10\sqrt{n}$. In this case we use the inequality in Theorem 2 to bound $|\mathscr{L}_0|$. Note that $e \geq 2$, and even if $e > \sqrt{n}$, we can certainly say $|\mathscr{L}_0| < 10^{200} n^2/e^3$. It follows that

$$n_{11}^2 a \leq \frac{10^{200} n^2}{e^3} \cdot 2e \cdot 2ad.$$

Thus $e \leq 10^{101} n^{7/5}/n_{11}(\log n)^{c/2}$. However, we already know $e \geq b/(2 \cdot 10^6 n_2^{6/5}) \geq n_9^2 a/(4 \cdot 10^6 \cdot n_2^{6/5} \cdot n_1^{2/5})$. Since $a \leq 2$, these inequalities for e can be combined to show $n_{12}^3 (\log n)^{c/2} \leq 10^{110} n^3$, which is false. The contradiction completes the proof of the lemma. □

Again set $i = 1$. We next describe a selection process which identifies particular points and rectangles. This process will also be repeated when $i = 2$. Let $R \in \mathscr{R}$, $x \in P_2(R)$, and $y \in Q_i''$. We call a pair (x, y) a *lean* pair if $C(x, y)$ contains no other point of $P(R)$ besides x. In view of the lemma, we may choose a set W of n_3^2 lean pairs.

Let $R \in \mathscr{R}$ and let $x \in P(R)$. We say that x is *normal* if there are at least n_4 points in the set $N_i(x) = \{y \in Q_i'' : (x, y) \text{ is lean}\}$. Call a rectangle $R \in \mathscr{R}$ *normal* if it contains at least $n_5^{3/5}$ normal points. So we may choose a set $\mathscr{R}_0$ of $n_6^{2/5}$ normal rectangles.

From each $R \in \mathcal{R}_0$, we may select a set $S_0(R)$ of $n_5^{3/5}$ normal points. For each $x \in S_0(R)$, we may choose a set $N_i'(x)$ of n_4 points from $N_i(x)$.

Consider the set $G = \bigcup \{S_0(R): R \in \mathcal{R}_0\}$. Then $|G| = n_6^{2/5} n_5^{3/5} > n_6$. Now we repeat the argument in our lemma when $i = 2$ but consider only the pairs of the form (x, y) where $x \in G$ and $y \in Q_2$. We conclude that there exists a set $\mathcal{R}_1$ of $n_7^{2/5}$ rectangles from $\mathcal{R}_0$ so that:

For every rectangle $R \in \mathcal{R}_1$, there is a set $S_1(R)$ of $n_7^{3/5}$ points covered by R so that:

(1) For each $x \in S_1(R)$, there are sets $N_1''(x) \subseteq Q_2''$ and $N_2''(x) \subseteq Q_2''$ so that $|N_1''(x)| = |N_2''(x)| = n_7$.

(2) For each $x \in S_1(R)$ and each $i = 1, 2$, and for every $y \in N_i''(x)$, (x, y) is a lean pair.

We now count wasted crossings. It is clear that the number of wasted crossings cannot exceed $n^2 d^2$. However, there are $n_7^{2/5}$ rectangles in $\mathcal{R}_1$. In the appropriate subrectangle, there are at least $n_8^{3/5}$ points, so at least $n_8^{6/5}$ pairs of points. For each pair, there are at least n_7^2 wasted crossings. This requires $n_7^{2/5} n_9^{6/5} n_7^2 \leq n^2 d^2$ which is false. The contradiction completes the proof of Theorem 3.   □

3. Concluding Remarks

It is probably possible to modify the arguments in this paper to show that $d(n) \geq n^{4/5}/\omega(n)$ where $\omega(n) \to \infty$ arbitrarily slowly and n is sufficiently large. In fact, perhaps it can be shown that $d(n) \geq \varepsilon n^{4/5}$ for some absolute constant $\varepsilon > 0$. We find little reason to attempt such improvements since they would still leave us far from the conjectured lower bound which we suspect is correct.

Finally, we comment that the argument presented in this paper does not show that there is some point from which there are at least $n^{4/5}/(\log n)^c$ different distances.

Acknowledgment

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