The Number of Edge Colorings with no Monochromatic Triangle

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Abstract

Let $F(n,k)$ denote the maximum number of two edge colorings of a graph on n vertices that admit no monochromatic K_k, (a complete graph on k vertices). The following results are proved: $F(n,3) = 2^{\lfloor n^2/4 \rfloor}$ for all $n \geq 6$. $F(n,k) = 2^{(\frac{k-2}{2k-2}+o(1))n^2}$. In particular, the first result solves a conjecture of Erdös and Rothschild.

1 Introduction

All graphs considered here are finite, undirected and simple. Given a graph $G = (V,E)$, denote by $F(G,r,k)$ the number of distinct edge colorings of G with r colors having no monochromatic K_k, (the complete graph on k vertices). Let $F(n,r,k) = \max\{F(G,r,k) \mid G \text{ is a graph on } n \text{ vertices }\}$. It is easy to see that $F(n,r,k) \leq r^{t_{q-1}(n)}$ where $t_{q-1}(n)$ is the maximum size of a graph that does not contain a K_q (the Turán graph) where q is the Ramsey number that guarantees the existence of a monochromatic K_k in any r edge coloring of K_q (Cf. [1] for definitions). It is also trivial that $F(n,r,k) \geq r^{t_{k-1}(n)}$, since every r edge coloring is acceptable for the corresponding Turán graph. It seems likely that the lower bound is closer to the truth, at least for $r = 2$. Indeed, Erdös and Rothschild conjectured some ten year ago [3] that $F(n,2,3) = 2^{\lfloor n^2/4 \rfloor}$ for all sufficiently large n. In this paper we prove this conjecture. Since we are mainly concerned with the case $r = 2$ and $k = 3$ we put $F(n) = F(n,2,3)$, $F(G) = F(G,2,3)$, $F(n,k) = F(n,2,k)$ and $F(G,k) = F(G,2,k)$. Hence, we prove the following theorem:

Theorem 1.1 *$F(1) = 1$. $F(2) = 2$. $F(3) = 6$. $F(4) = 18$. $F(5) = 82$. $F(n) = 2^{\lfloor n^2/4 \rfloor}$ for all $n \geq 6$.*

We conjecture that the lower bound for $F(n,k)$ is the correct one, provided n is sufficiently large.

Conjecture 1.2 *For all $k \geq 3$ there is a constant $N = N(k)$ such that for all $n \geq N$, $F(n,k) = 2^{t_{k-1}(n)}$.*

Although we are unable to prove this conjecture at the moment, we can prove an asymptotic version of it.

Theorem 1.3 $F(n,k) = 2^{t_{k-1}(n)+o(n^2)} = 2^{(\frac{k-2}{2k-2}+o(1))n^2}$.

The rest of this paper is organized as follows: In section 2 we prove the necessary lemmas, and also establish the values of $F(n)$ for $n \leq 6$. In section 3 we prove Theorem 1.1. Section 4 is devoted to the proof of Theorem 1.3.

2 The lemmas

In the following two sections, the term *coloring* refers to a red-blue edge coloring that contains no monochromatic triangles. Let G be a graph, and G' a subgraph of G. A coloring $\hat{f}$ of G is called an *extension* of a coloring f of G' to G if the image of $\hat{f}$ on G' equals f. Given a coloring f of G', denote by $E_G(f)$ the set of all extensions of f to G. We denote by $\delta(G)$ the minimum degree of a vertex of G. Given a vertex x of G, denote by $N(x)$ the set of vertices adjacent to x in G (the neighbors of x). Put $d(x) = |N(x)|$ (the degree of x). The following lemma establishes an upper bound on $|E_G(f)|$ whenever $G' = G \setminus \{x\}$ is the subgraph induced by all vertices of G, but x.

Lemma 2.1 *Let x be any vertex of a graph G. Let P_i, $i = 1, \ldots, k$ be a set of vertex disjoint paths in the subgraph induced by $N(x)$. Let T_j, $j = 1, \ldots, l$ be a set of vertex disjoint triangles in the subgraph induced by $N(x)$ that are pairwise disjoint with the paths. Let z_i be the number of vertices on P_i. Then for every coloring f of $G' = G \setminus \{x\}$,*

$$|E_G(f)| \leq (2^{d(x)-3l-\sum_{i=1}^{k} z_i}) \cdot (3^l) \cdot (\prod_{i=1}^{k} a_{z_i+2})$$

where a_t denotes the t'th Fibonacci number. In particular,

$$F(G) \leq F(G')(2^{d(x)-3l-\sum_{i=1}^{k} z_i}) \cdot (3^l) \cdot (\prod_{i=1}^{k} a_{z_i+2})$$

Proof Let H be the subgraph of G induced by $N(x) \cup \{x\}$, and let H' be the subgraph of H induced by $N(x)$. Note that H' is also a subgraph of G'. Let f' be the restriction of f to H'. Every extension of f to G uniquely determines an extension of f' to H, and if $\hat{f}_1, \hat{f}_2 \in E_G(f)$ are distinct extensions then the extensions of f' to H that they determine are also distinct. Hence it suffices to show that $|E_H(f')| \le (2^{d(x)-3l-\sum_{i=1}^k z_i}) \cdot (3^l) \cdot (\prod_{i=1}^k a_{z_i+2})$ where f' is an arbitrary coloring of H'. Recall the definition of the Fibonacci sequence: $a_1 = 1$, $a_2 = 1$, $a_{t+2} = a_t + a_{t+1}$. Let $P_i = (v_1, \dots, v_{z_i})$. Let H_t, $t = 0, \dots, z_i$ be obtained from H by deleting all edges adjacent to x except $(x, v_1), \dots, (x, v_t)$. We prove by induction on t that $|E_{H_t}(f')| \le a_{t+2}$. For $t = 0$ this follows from $H_0 = H'$ and $a_2 = 1$. For $t = 1$ this follows from $a_3 = 2$ and the fact that $H_1 = H' \cup \{(x, v_1)\}$ and we may color the edge (x, v_1) in two colors. Assuming it is true for $t-2$ and $t-1$, we show it is true for t. Let r_s (b_s) be the number of extensions in $E_{H_s}(f')$ in which the edge (x, v_s) is colored red (blue). By the induction hypothesis, $r_{t-1} + b_{t-1} \le a_{t+1}$. Clearly, $r_{t-1} \le |E_{H_{t-2}}(f')| \le a_t$ and similarly, $b_{t-1} \le a_t$. Now suppose first that the edge (v_{t-1}, v_t) is blue. Then $b_t \le r_{t-1}$ and $r_t \le r_{t-1} + b_{t-1}$. Thus we obtain

$$r_t + b_t = 2r_{t-1} + b_{t-1} \le a_{t+1} + r_{t-1} \le a_{t+1} + a_t = a_{t+2}$$

Similar arguments apply when (v_{t-1}, v_t) is red. We have shown that $|E_{H_{z_i}}(f')| \le a_{z_i+2}$.

Now consider a triangle T_j. Given the colors that f' induces on T_j, (clearly, it induces two blue edges and one red edge or two red edges and one blue edge) there are exactly 3 ways to color the 3 edges connecting x to the vertices of T_j without introducing a monochromatic triangle.

Since the paths P_i are vertex disjoint, and the triangles T_j are vertex disjoint, and the triangles are pairwise disjoint from the paths, and since the edges connecting x to vertices that do not belong to any path or triangle can be colored with at most two colors, it follows that $|E_H(f')| \le (2^{d(x)-3l-\sum_{i=1}^k z_i}) \cdot (3^l) \cdot (\prod_{i=1}^k a_{z_i+2})$. □

The bound in Lemma 2.1 cannot be improved since if the edges of each path P_i are all colored with the same color, and $N(x)$ contains no other edges but the edges of these paths, the bound is achievable.

The next lemma establishes a bound for $F(G)$ whenever G has a special structure.

Lemma 2.2 *Let G be a graph on $n \ge 7$ vertices, having a vertex c with $d(c) = n-1$, such that $G \setminus \{c\}$ is a complete bipartite graph with vertex classes of sizes k and $l = n-1-k$, where $l \ge 3$. Then*

$$F(G) \le (2^l - 2)(6 \cdot 2^{l-3})^k + 2(2^l + 1)^k$$

Proof Let $x_1, \dots, x_k$ be the vertices of one side of the bipartite graph $G \setminus \{c\}$ and let H be their common set of neighbors. Note that H is a star with root c and l leaves. Denote by H_i the subgraph induced by $H \cup \{x_i\}$. It suffices to prove that for any non-monochromatic coloring f of H, $|E_{H_i}(f)| \le 6 \cdot 2^{l-3}$ and for every monochromatic coloring b (e.g. the all blue coloring) of H, $|E_{H_i}(b)| = 2^l + 1$.

Let f be a non-monochromatic coloring of H. W.l.o.g there is a substar H' with three leaves with two blue edges and one red edge. There are exactly six ways to color the four edges joining x_i to this substar without creating a monochromatic triangle in $H' \cup \{x\}$. The remaining $l-3$ edges of x_i can be colored in at most 2^{l-3} ways.

Now let b be the all blue coloring of H. There is only one extension of b that colors the edge (x_i, c) blue, since all other edges of x_i must be colored red. All 2^l extensions coloring the edge (x_i, c) red are possible. □

The next lemma asserts that every graph G contains a K_5-free spanning subgraph G' with $F(G) \leq F(G')$.

Lemma 2.3 *If e is an edge belonging to a K_5 of G, then $F(G) \leq F(G \setminus \{e\})$.*

Proof Any coloring of a K_5 contains exactly five blue and five red edges. Therefore, if f' is any coloring of $G \setminus \{e\}$, then, assuming it is extendible to G, the color of e is uniquely determined in the extension. Hence $|E_G(f')| \leq 1$. □

We need the following simple but useful lemma in our proof of Theorem 1.1.

Lemma 2.4 *Let G be a graph on n vertices, $\delta = \delta(G)$. The following four claims hold:*

1. *There is a path containing $\delta + 1$ vertices in G.*
2. *If $n \geq 2\delta + 2$ then there are two vertex disjoint paths P_1 and P_2 in G, P_i containing z_i vertices $i = 1, 2$, $z_1 + z_2 = 2\delta + 2$. Furthermore, if $\delta \geq 3$ then $z_1 \geq 4$ and $z_2 \geq 4$.*
3. *If $n < 2\delta + 2$ then G contains a Hamiltonian path.*
4. *If $n < 2\delta$ and G does not contain a K_4, then the vertices of G can be partitioned into t triangles and a path containing $n - 3t$ vertices, for $t = 1, \ldots, 2\delta - n$.*

Proof The first and third claims above are well known (cf. [2]).

For the second claim, let P be a longest path in G. If P has $2\delta + 2$ vertices, we are done. If P has at most 2δ vertices, it is well known that there is also a cycle C with the same set of vertices, and by the maximality of P, the subgraph induced by this set of vertices is a connected component of G. Since this is not the whole graph, it follows from the first claim that there is another path P_2 of length $\delta + 1$ outside of this connected component. If $\delta \geq 3$ both paths have length at least 4.

The remaining case is when P contains exactly $2\delta + 1$ vertices. Put $P = (a_1, \ldots, a_{2\delta+1})$. Note that all edges of a_1 and of $a_{2\delta+1}$ connect them to vertices inside P. If the set of vertices of P forms a connected component then, as before, we can obtain another path of length $\delta + 1$ outside of this connected component. otherwise, there is a vertex $b \notin P$ such that (b, a_j) is an edge for some $2 \leq j \leq 2\delta$. W.l.o.g. $j \leq \delta+1$. Define $P_1 = (a_1, \ldots, a_j, b)$, $P_2 = (a_{j+1}, \ldots, a_{2\delta+1})$. If $\delta \geq 3$ and $j > 2$ then both paths have length at least 4. If $j = 2$, note that b is connected to at least two more vertices. Therefore there is a vertex $c \neq a_{2\delta}$ such that (b, c) is an edge. If $c \in P$ we could have chosen c instead of a_2. If $c \notin P$ we can redefine $P_1 = (a_1, a_2, b, c)$.

For the fourth claim, take a partition of the vertices of G into t triangles and a path of length $n - 3t$, where t is maximal. Such a partition certainly exists by the third claim. Let H be the graph induced by the $N = n - 3t$ vertices not on triangles. Since H does not contain a K_4, we have $\delta(H) \geq \delta - 2t$. On the other hand, by the maximality of t we have $\delta(H) \leq N/2$. To see this, note that if $\delta(H) > N/2$ then H contains a triangle, so we have $t+1$ triangles, and in the graph H' induced by the remaining $N-3$ vertices $\delta(H') \geq \delta(H) - 2 > (N-2)/2 - 1$ and therefore H' contains a Hamiltonian path by the third claim. This contradicts the maximality of t. We have shown that

$$\delta - 2t \leq \delta(H) \leq \frac{N}{2} = \frac{n-3t}{2}$$

therefore $t \geq 2\delta - n$. $\square$

We conclude this section with two somewhat technical lemmas. The first lemma establishes the values of $F(n)$ for $n \leq 6$. However, we need this lemma not only for the sake of completeness (if this were the only reason, we might not have bothered proving it, since it is a finite problem). Our proof of Theorem 1.1 is by induction on n. Therefore, we must show that $F(6) = 512$ to establish 6 as a basis to our induction. The second lemma handles four exceptional cases that are encountered in the proof of Theorem 1.1. These cases are, again, finitely checkable. However, we give analytic proofs for them.

Lemma 2.5 $F(1) = 1$, $F(2) = 2$, $F(3) = 6$, $F(4) = 18$, $F(5) = 82$, $F(6) = 512$.

Proof The values of $F(1)$, $F(2)$ and $F(3)$ are obvious. To determine $F(4)$ we only need to check graphs with more than four edges. There are only two such graphs: K_4 and K_4^- (K_k^- denotes the complete graph missing one edge). It is easy to check that both $F(K_4) = 18$ and $F(K_4^-) = 18$.

To determine $F(5)$ we only need to consider graphs G with $\delta(G) \geq 3$, since by Lemma 2.1 and by the fact that $F(4) = 18$ any graph G on 5 vertices with a vertex of degree ≤ 2 has $F(G) \leq 72$. There are only three graphs on 5 vertices with $\delta(G) \geq 3$. These are K_5, K_5^- and K_5^{--} (the graph obtained from K_5 by removing two independent edges). It is easy to check that $F(K_5) = 12$ (in fact, any coloring of K_5 is a partition into two Hamiltonian cycles), $F(K_5^-) = 72$ and $F(K_5^{--}) = 82$.

To determine $F(6)$ we, again, only need to consider graphs G with $\delta(G) \geq 3$. Assume first that $\delta(G) = 3$. If there is a vertex x of degree 3, such that $N(x)$ is not independent, then by Lemma 2.1 we get $F(G) \leq 6F(5) \leq 492$. If for every vertex x of degree 3, $N(x)$ is independent, then G

must be the complete bipartite graph $K_{3,3}$, for which $F(K_{3,3}) = 512$. We may now assume that $\delta(G) \geq 4$. There are only four possible graphs to consider: K_6, K_6^-, K_6^{--} and $K_{2,2,2}$ (the complete 3-partite graph with equal vertex classes). It is well known that $F(K_6) = 0$. It is also not difficult to see that $F(K_{2,2,2}) = 450$ and $F(K_6^{--}) = 194$. Since K_6^- contains a K_5 we can use Lemma 2.3 to obtain $F(K_6^-) \leq 194$. □

Lemma 2.6 *Let x be a vertex of a graph G. Let H be the induced subgraph on $N(x)$. The following holds for a coloring f of $G \setminus \{x\}$.*

1. *If H is a graph on 7 vertices and $\delta(H) \geq 3$ then $|E_G(f)| \leq 32$.*
2. *If H is a graph on 6 vertices and $\delta(H) \geq 3$ then $|E_G(f)| \leq 16$.*
3. *If H is a graph on 10 vertices and $\delta(H) \geq 5$ then $|E_G(f)| \leq 128$.*
4. *If H is a graph on 5 vertices and $\delta(H) \geq 3$ then $|E_G(f) \leq 8$.*

Proof

1. G contains a Hamiltonian path. However, a path of length 7 only implies $|E_G(f)| \leq a_9 = 34$. To remedy this, take any edge not on the path, and consider the subgraph of H that consists of this edge and the Hamiltonian path. It is easy to verify that there are at most 32 ways to extend a given coloring of this subgraph.

2. If H is a $K_{3,3}$ (a complete bipartite graph), one may easily check that $|E_G(f)| \leq 16$. Otherwise, there must exist a partition of H into a triangle and a path on three vertices. By Lemma 2.1 $|E_G(f)| \leq 3a_5 = 15$.

3. If H is a $K_{5,5}$ one may check directly that $|E_G(f)| \leq 128$. (It is enough to take a Hamiltonian cycle with a chord between two antipodal vertices, and verify that any coloring of it cannot be extended in more than 128 ways). Otherwise, there is a partition of H into a triangle and a path on 7 vertices. By Lemma 2.1 $|E_G(f)| \leq 3a_9 = 102$.

4. H must contain two triangles with one vertex in common. Let these triangles be $\{a, b, c\}$ and $\{a, d, e\}$. Given f, there are 3 ways to color the edges $(x, a), (x, b), (x, c)$ without creating a monochromatic cycle in $\{a, b, c, x\}$. Similarly, there are 3 ways to color the edges $(x, a), (x, d), (x, e)$. The only way all the 9 pairs of extensions are possible, is when the color of (x, a) is completely determined. In this case, take any path of length 4 whose vertices are (b, c, d, e) (such a path must exist). There are at most $a_6 = 8$ ways to color the edges $(x, b), (x, c), (x, d), (x, e)$. Since (x, a) is completely determined, then $E_G(f) \leq 8$ also in this case. □

3 Proof of Theorem 1.1

We prove Theorem 1.1 by induction on n, for all $n \geq 6$. The value $F(6) = 512$ was computed in Lemma 2.5, and indeed $512 = 2^{6^2/4}$. By Lemma 2.3 it suffices to prove the theorem for graphs not containing a K_5. We therefore fix a graph G of order n not containing a K_5. Assuming that the theorem is true for all graphs of order $n-1$, we must show that $F(G) \leq 2^{\lfloor n^2/4 \rfloor}$.

Let x be a vertex of minimal degree in G. Put $G' = G \setminus \{x\}$. Our objective is to show that for any coloring f of G', $|E_G(f)| \leq 2^{\lfloor n/2 \rfloor}$ (however, in some cases we show that $F(G) \leq 2^{\lfloor n^2/4 \rfloor}$ directly). This suffices, since by the induction hypothesis we obtain

$$F(G) \leq F(G')2^{\lfloor n/2 \rfloor} \leq 2^{\lfloor (n-1)^2/4 \rfloor + \lfloor n/2 \rfloor} \leq 2^{\lfloor n^2/4 \rfloor}$$

where the last inequality is true for all n.

Fix a coloring f of G'. If $d(x) \leq \lfloor n/2 \rfloor$ then, clearly, $|E_G(f)| \leq 2^{\lfloor n/2 \rfloor}$. We may therefore assume that $d(x) = \lfloor n/2 \rfloor + t$ for some $t \geq 1$. Let H be the subgraph induced by $N(x)$. Thus

$$\delta(H) \geq 2t - 1 \;\; if \;\; n \;\; is \;\; odd \tag{1}$$

$$\delta(H) \geq 2t \;\; if \;\; n \;\; is \;\; even \tag{2}$$

Let P be a longest path in H. P has z vertices and by the first case of Lemma 2.4, $z \geq 2t$.

It is convenient to distinguish between a few cases. Some cases are reduced to other cases, but always to cases appearing above them in the text. Therefore, all possible cases are covered.

Case 1: $t = 1$, $z \geq 4$.
By Lemma 2.1 $|E_G(f)| \leq 2^{\lfloor n/2 \rfloor + 1 - z} a_{z+2}$. For all $z \geq 4$ we have $2^{1-z} a_{z+2} \leq 1$, and therefore $|E_G(f)| \leq 2^{\lfloor n/2 \rfloor}$.

Case 2: $t = 1$, $z = 2$ and $n \geq 8$.
In this case, H is simply a perfect matching, and since $n \geq 8$, $\lfloor n/2 \rfloor + 1 \geq 5$ so this matching contains at least three edges (in fact $n \geq 10$ must hold). Therefore by Lemma 2.1 $|E_G(f)| \leq 2^{\lfloor n/2 \rfloor + 1 - 6} a_4^3 < 2^{\lfloor n/2 \rfloor}$.

Case 3: $t = 1$, $z = 2$ and $n = 7$.
As in the previous case, H must be a perfect matching, say $(a, b), (c, d)$ are its edges. Thus a, b, c, d are also of degree 4 and $N(a) = \{b, x, u, v\}$, $N(b) = \{a, x, u, v\}$, $N(c) = \{d, x, u, v\}$, $N(d) = \{c, x, u, v\}$. $N(a)$ is not a perfect matching, since (b, u), (b, v) and (b, x) are all edges. Similarly, $N(b)$, $N(c)$ and $N(d)$ are not perfect matchings. If (u, v) is an edge, then we can reduce this case to case 1 ($t = 1$, $z = 4$) above by letting a play the role of x. Otherwise, (u, v) is not an edge, and the graph G is completely determined. It is a complete bipartite graph to which two

edges have been added. One vertex class is $\{x, u, v\}$ the other is $\{a, b, c, d\}$ and the additional edges are $(a, b), (c, d)$. It is easy to compute (manually) that $F(G) = 4 \cdot 9^3 < 2^{12}$.

Case 4: $t = 1$, $z = 3$ and H is not a star.

Since $n \geq 7$ there is an edge e in H whose endpoints are not on P. By Lemma 2.1 $|E_G(f)| \leq 2^{\lfloor n/2 \rfloor + 1 - 5} a_5 a_4 < 2^{\lfloor n/2 \rfloor}$.

Case 5: $t = 1$, $z = 3$ and H is a star.

Note that n must be odd, since by (2) for even n the minimal degree in H is at least 2. We may assume that for every vertex of minimal degree of G, the set of neighbors is a star (otherwise, x could have been chosen such that one of the cases above would apply). This implies that G must be the unique graph on n vertices having a vertex a of degree $n-1$ such that $G \setminus \{a\}$ in a complete bipartite graph with $(n-1)/2$ vertices in each vertex class. By lemma 2.2 we have

$$F(G) \leq (2^{(n-1)/2} - 2)(6 \cdot 2^{(n-7)/2})^{(n-1)/2} + 2(2^{(n-1)/2} + 1)^{(n-1)/2} \leq 2^{(n^2-1)/4} = 2^{\lfloor n^2/4 \rfloor}.$$

Case 6: $t > 1$, $z \geq 4t$.

We require the following inequality for the Fibonacci numbers:

$$a_{j+2} \leq 2^{0.75j}, \quad for \ \ j \notin \{1, 2, 3\} \tag{3}$$

By Lemma 2.1 we have $|E_G(f)| \leq 2^{\lfloor n/2 \rfloor + t - z} a_{z+2}$. We must show that $2^{t-z} a_{z+2} \leq 1$. Indeed, by (3)

$$2^{t-z} a_{z+2} \leq 2^{-0.75z} a_{z+2} \leq 2^{-0.75z} 2^{0.75z} = 1.$$

Case 7: $t > 1$, $z < 4t$, and $\lfloor n/2 \rfloor \geq 3t$.

In this case the conditions of Lemma 2.4 (the second case) are satisfied. The minimal degree of H is at least 3, and $\lfloor n/2 \rfloor + t \geq 2(2t-1) + 2$. Therefore, according to the lemma, H contains two vertex disjoint paths P_1 and P_2. The number of vertices of P_i is z_i, $i = 1, 2$. $z_1 + z_2 = 2(2t-1) + 2 = 4t$ and $z_1 \geq 4$, $z_2 \geq 4$. By lemma 2.1 and by (3) we have:

$$|E_G(f)| \leq 2^{\lfloor n/2 \rfloor + t - 4t} a_{z_1+2} a_{z_2+2} \leq 2^{\lfloor n/2 \rfloor - 3t} 2^{0.75 \cdot 4t} \leq 2^{\lfloor n/2 \rfloor}.$$

Case 8: $t > 1$ $z < 4t$ and $\lfloor n/2 \rfloor < 3t$, n is even.

The conditions of case 4 of Lemma 2.4 are satisfied for the graph H, since H does not contain a K_4 (since G does not contain a K_5), $\delta(H) \geq 2t$ by (2) and $2\delta(H) \geq 4t > n/2 + t = |H|$. Therefore, we can partition the vertices of H into $4t - (n/2 + t) = 3t - n/2$ triangles, and a path containing $2n - 8t$ vertices. By Lemma 2.1 we have

$$|E_G(f)| \leq 3^{3t - n/2} a_{2n-8t+2}.$$

We must show that $3^{3t-n/2}a_{2n-8t+2} \leq 2^{n/2}$. Note that $2n-8t \notin \{1,2,3\}$. Therefore by (3) we must show

$$3^{3t-n/2}2^{0.75(2n-8t)} \leq 2^{n/2}.$$

Taking Logarithmic factors and rearranging the terms, the last inequality is equivalent to

$$t(3\log 3 - 6) \leq n(\frac{\log 3}{2} - 1)$$

Which is equivalent to $t \geq n/6$ which holds by our assumption.

Case 9: $t > 1$ $z < 4t$ and $\lfloor n/2 \rfloor \in \{3t-1, 3t-2\}$, n is odd.

Note that $\lfloor n/2 \rfloor = (n-1)/2$ and that H contains a Hamiltonian path since the conditions of Lemma 2.4 (case 3) are satisfied for H $(2(2t-1)+2 > (n-1)/2+t)$. By Lemma 2.1

$$E_G(f) \leq a_{(n-1)/2+t+2}.$$

We must show that $a_{(n-1)/2+t+2} \leq 2^{(n-1)/2}$. Equivalently, since $t = (n+1)/6$ or $t = (n+3)/6$ in our case, we must show that $a_{(2n-1)/3+2} \leq 2^{(n-1)/2}$ or $a_{2n/3+2} \leq 2^{(n-1)/2}$. The first inequality is true for all $n \geq 17$. Note that $n = -1 \bmod 6$ in this case, so the only possible value of n for which this does not hold is $n = 11$. In this case $t = 2$ and H is a graph on 7 vertices and minimal degree at least 3. Therefore, we can use Lemma 2.6 to obtain $E_G(f) \leq 32$, as required. The second inequality is true for all $n \geq 21$. Note that $n = 3 \bmod 6$ in this case, so the only possible values of n for which this does not hold are $n = 15$ and $n = 9$. For $n = 15$, $t = 3$ and H is a graph on 10 vertices with minimal degree at least 5. For $n = 9$, $t = 2$ and H is a graph on 6 vertices and minimal degree at least 3. In both cases we can use Lemma 2.6 to obtain $E_G(f) \leq 2^{(n-1)/2}$, as required.

Case 10: $t > 1$ $z < 4t$ and $\lfloor n/2 \rfloor \leq 3t-3$, n is odd.

As in case 8, the conditions of case 4 of Lemma 2.4 are satisfied for the graph H. Therefore, we can partition the vertices of H into $2(2t-1)-((n-1)/2+t) = 3t-n/2-3/2$ triangles, and a path containing $2n-8t+4$ vertices. By Lemma 2.1 we have

$$|E_G(f)| \leq 3^{3t-n/2-3/2}a_{2n-8t+6}.$$

We must show that $3^{3t-n/2-3/2}a_{2n-8t+6} \leq 2^{(n-1)/2}$. Assume first that $2n-8t+4 \geq 4$ and $3t-3 > (n-1)/2$. By (3) it suffices to show

$$3^{3t-n/2-3/2}2^{0.75(2n-8t+4)} \leq 2^{(n-1)/2}.$$

Taking Logarithmic factors and rearranging the terms, the last inequality is equivalent to

$$t \geq n/6 + 0.5 + 1/(12 - 6\log 3)$$

but according to our assumption, $3t-3 > (n-1)/2$ so $t \geq n/6 + 7/6 \geq n/6 + 0.5 + 1/(12 - 6\log 3)$.

Now assume that $2n - 8t + 4 \geq 4$ and $3t - 3 = (n-1)/2$. Note that this implies $n \geq 13$. In this case there is only one triangle, and the path has $(2n-8)/3$ vertices. Therefore, we must show $3a_{(2n-2)/3} \leq 2^{(n-1)/2}$. Which holds for $n \geq 13$.

Finally, assume $2n - 8t + 4 < 4$. Since n is odd, we have $2n - 8t + 4 = 2$. In this case, the path is simply an edge, and there are $n/4 - 3/4$ triangles. We must show that $3^{(n+1)/4} \leq 2^{(n-1)/2}$. This holds for $n \geq 11$. Since $n = 3 \bmod 4$ in this case, the only value of n for which this does not hold is $n = 7$. In this case H is a graph on 5 vertices and minimum degree at least 3 and by Lemma 2.6 $E_G(f) \leq 8$. $\square$

4 The asymptotic behavior of $F(n,k)$

In this section we prove Theorem 1.3. The proof is based on the Regularity Lemma of Szemerédi [4] together with some additional ideas. In order to state this lemma we introduce a few definitions, most of which follow [4]. If $G = (V, E)$ is a graph, and A, B are two disjoint subsets of V, let $e(A,B) = e_G(A,B)$ denote the number of edges of G with an endpoint in A and an endpoint in B. If A and B are non-empty, define the *density of edges* between A and B by $d(A,B) = \frac{e(A,B)}{|A||B|}$. For $\epsilon > 0$, the pair (A,B) is called *ϵ-regular* if for every $X \subset A$ and $Y \subset B$ satisfying $|X| \geq \epsilon|A|$ and $|Y| \geq \epsilon|B|$, the inequality

$$|d(A,B) - d(X,Y)| < \epsilon$$

holds.

An *equitable partition* of a set V is a partition of V into pairwise disjoint classes $C_0, C_1, \ldots, C_p$, in which all the classes C_i for $1 \leq i \leq p$ have the same cardinality. The class C_0 is called the *exceptional class* and may be empty. An equitable partition of the set of vertices V of G into the classes $C_0, C_1, \ldots, C_p$, with C_0 being the exceptional class, is called *ϵ-regular* if $|C_0| \leq \epsilon|V|$, and all but at most ϵp^2 of the pairs (C_i, C_j) for $1 \leq i < j \leq p$ are ϵ-regular. The following lemma is proved in [4].

Lemma 4.1 (The Regularity Lemma [4]) *For every $\epsilon > 0$ and every positive integer t there are integers $T = T(\epsilon, t)$ and $N = N(\epsilon, t)$ such that every graph with $n \geq N$ vertices has an ϵ-regular partition into $p+1$ classes, where $t \leq p \leq T$.* $\square$

A variant of the following lemma often appears in conjunction with the Regularity Lemma. We prove it here in a form that suits our purpose.

Lemma 4.2 *Let $C_1, \ldots, C_k$ be pairwise disjoint equal sized vertex classes, $|C_i| = m$ for all i. Suppose that (C_i, C_j) are ϵ-regular and $d(C_i, C_j) \geq \delta$ for $1 \leq i < j \leq k$. If $(k-1)\epsilon < (\delta/2)^{k-1}$ then there are vertices $(v_1, \ldots, v_k)$, $v_i \in C_i$ such that $(v_i, v_j) \in E(C_i, C_j)$ for $1 \leq i < j \leq k$.*

Proof Note that the lemma is trivial for $k = 1$ so we assume $k \geq 2$. We claim that for every p, $1 \leq p \leq k$, there are vertices $v_i \in C_i$ for all $i < p$, and subsets $B_i \subset C_i$, $|B_i| \geq (\delta/2)^{p-1}m$ for all $p \leq i \leq k$ with the following property: Each v_i, $i < p$ is adjacent to all vertices in

$$\{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{p-1}\} \cup_{j=p}^{k} B_j.$$

The assertion of the lemma follows from the above claim for $p = k$. The proof of the claim is by induction on p. For $p = 1$ simply take $B_i = C_i$ for all i. Assume the claim is true for p, we must show it is true for $p + 1$. Consider the set B_p. By the assumption, the cardinality of each B_j is at least ϵm. For each j, $p < j \leq k$, let D_j denote the set of vertices in B_p that have less than $(\delta - \epsilon)|B_j|$ neighbors in B_j. We claim that $|D_j| < \epsilon m$ for each j. This is because otherwise the two sets $X = D_j$ and $Y = B_j$ would contradict the ϵ-regularity of the pair (C_p, C_j), since $d(D_j, B_j) < \delta - \epsilon$ whereas $d(C_p, C_j) \geq \delta$. Therefore, the cardinality of the set $B_p \setminus \cup_{j=p+1}^{k} D_j$ is at least

$$|B_p| - (k-p)\epsilon m \geq (\frac{\delta}{2})^{p-1} m - (k-1)\epsilon m \geq ((\frac{\delta}{2})^{k-1} - (k-1)\epsilon) m > 0.$$

We can now choose arbitrarily a vertex v_p in $B_p \setminus \cup_{j=p+1}^{k} D_j$ and replace each B_j for $p < j \leq k$ by the set of neighbors of v_p in B_j. Since $\delta - \epsilon > \delta/2$ this will not decrease the cardinality of each B_j by more than a factor of $\delta/2$, and therefore the claim holds for $p + 1$. $\Box$

We are now ready to prove Theorem 1.3. Let $\epsilon > 0$ be given. we show that there is an $n_0 = n_0(\epsilon)$ such that for every graph G on $n \geq n_0$ vertices,

$$F(G, k) \leq 2^{(\frac{k-2}{2k-2} + \epsilon)n^2}. \tag{4}$$

Given a parameter γ, put $\delta = 2(\gamma(k-1))^{1/(k-1)}$. Now let $\gamma = \gamma(\epsilon)$ satisfy

$$2.5\gamma < \epsilon/4. \tag{5}$$

and

$$H(\delta) < \epsilon/2 \tag{6}$$

where H denotes, as usual, the entropy function. Let $t = \lceil 1/\gamma \rceil$ and let $T = T(\gamma, t)$, $N = N(\gamma, t)$ be as in Lemma 4.1. Finally, let $n_0 \geq N$ satisfy

$$T^2 + \log((n_0 + 1)!) \leq \frac{\epsilon}{8} n_0^2. \tag{7}$$

Note that $n_0 = n_0(\epsilon)$ and that every value greater than n_0 also respects (7).

Let $G = (V, E)$ be a fixed graph on $n = |V| \geq n_0$ vertices. Denote by $\mathcal{F}$ the set of all red-blue colorings of G containing no monochromatic K_k, where the number of blue edges is not less than the number of red edges. Clearly,

$$2|\mathcal{F}| \geq F(G, k). \tag{8}$$

Every coloring $f \in \mathcal{F}$ determines a unique spanning subgraph $G_f = (V, E_f)$ of G (the subgraph induced by the blue edges). We apply lemma 4.1 on G_f to obtain a γ-regular partition $C_0, \dots C_p$ of V where $t \leq p \leq T$. We define the graph D_f (the *dense pairs graph*) as follows: The vertex set of D_f is $\{1, \dots, p\}$. The edge set of D_f is:

$$E(D_f) = \{(i,j) \mid d(C_i, C_j) > \delta \;\; (C_i, C_j) \;\; is \;\; \gamma - regular\}.$$

Claim: D_f contains at most $\frac{k-2}{2k-2}p^2$ edges.

Proof By Turán's Theorem, it suffices to show that D_f does not contain a K_k. Assume, for contradiction, that it does. W.l.o.g. $\{1, \dots, k\}$ are the vertices of a K_k. The conditions of Lemma 4.2 are satisfied for $C_1, \dots, C_k$. This is because $d(C_i, C_j) > \delta = 2((k-1)\gamma)^{1/(k-1)}$ and hence $(k-1)\gamma < (\delta/2)^{k-1}$. The lemma implies the existence of a K_k in G_f, which is impossible. $\Box$

With a coloring $f \in \mathcal{F}$ and a regular partition of G_f we associate a *configuration* which is the ordered collection $(C_0, C_1, \dots, C_p, D_f)$. We call p the *index* of the configuration. Let $\mathcal{C}$ denote the set of all configurations. We claim that

$$|\mathcal{C}| \leq 2^{(\epsilon/8)n^2}. \tag{9}$$

To see this, note that given p, there are at most $\gamma(n+1)$ possible sizes for the exceptional class, and that the size of the exceptional class determines the size of all other classes. Therefore there are at most $\gamma(n+1)n!$ possible γ-regular partitions with $p+1$ classes. Given such a partition, there are at most $2^{\binom{p}{2}}$ possible graphs on p vertices, and hence the total number of configurations with index p is $\gamma(n+1)!2^{\binom{p}{2}}$. we use the fact that $p \leq T$ and inequality (7) to obtain

$$|\mathcal{C}| \leq T\gamma(n+1)!2^{\binom{T}{2}} \leq 2^{T^2 + \log(n+1)!} \leq 2^{(\epsilon/8)n^2}.$$

Next, we give an upper bound on the number of colorings $f \in \mathcal{F}$ that are associated with the same configuration. For a given configuration $C = (C_0, C_1 \dots, C_p, D)$, let $s(C)$ be the number of colorings in $\mathcal{F}$ whose configuration is C. We claim that

$$s(C) \leq 2^{(3\epsilon/4 + (k-2)/(2k-2))n^2}. \tag{10}$$

To see this, we must consider all possible arrangements of edges of a G_f that may lead to the configuration C. This is done as follows:

- There are at most γn^2 edges adjacent to vertices of the exceptional class C_0. Therefore, the total number of arrangements of these edges is at most

$$2^{\gamma n^2}. \tag{11}$$

- There are at most $\binom{C_1}{2}p \leq (n^2/2p) \leq (n^2/2)\gamma$ edges with both endpoints in the same (non exceptional) class. Therefore, the total number of arrangements of these edges is at most

$$2^{(\gamma/2)n^2}. \tag{12}$$

- There are at most γp^2 non γ-regular pairs. There are (significantly) less than $2^{\binom{p}{2}}$ possible arrangements of these sets of pairs. Given the non γ-regular pairs, there are at most $|C_1|^2\gamma p^2 \leq n^2\gamma$ edges with endpoints in non-regular pairs. Therefore, the total number of arrangements of these edges is at most

$$2^{p^2+\gamma n^2}. \tag{13}$$

- In every γ-regular pair (C_i, C_j) such that $(i,j) \notin E(D)$ there are at most $|C_1|^2\delta$ edges. There are at most $2^{\binom{p}{2}}$ possible arrangements of these sets of pairs (Note that given C, we know which are the non edges of D, but some of them may correspond to non γ-regular pairs, while others may correspond to γ-regular pairs with density at most δ). For every such pair, there are at most

$$\sum_{j=0}^{\lfloor \delta |C_1|^2 \rfloor} \binom{|C_1|^2}{j} \leq 2^{H(\delta)|C_1|^2} \leq 2^{(\epsilon/2)n^2/p^2}$$

possible arrangements of the edges within the pair (here we implicitly used Stirling's formula). It follows that for all pairs in a given set of pairs there are at most $2^{\binom{p}{2}(\epsilon/2)n^2/p^2} \leq 2^{(\epsilon/4)n^2}$ possible arrangements of the edges. Therefore, taking all possible arrangements into consideration, the total number of arrangements of these edges is at most

$$2^{(\binom{p}{2}+\epsilon/4)n^2}. \tag{14}$$

- In every pair (C_i, C_j) for which $(i,j) \in E(D)$ there are at most $|C_1|^2 \leq n^2/p^2$ edges. However, we know which pairs these are, since D is given, and we know by the above claim that there are at most $\frac{k-2}{2k-2}p^2$ edges in D. Therefore, the total number of arrangements of these edges is at most

$$2^{\frac{k-2}{2k-2}n^2}. \tag{15}$$

Multiplying (11),(12),(13),(14) and (15) we have that

$$s(C) \leq 2^{(\gamma+\gamma/2+\gamma+\epsilon/4+\frac{k-2}{2k-2})n^2+p^2+\binom{p}{2}} \leq 2^{(2.5\gamma+\epsilon/4+\epsilon/4+\frac{k-2}{2k-2})n^2} \leq 2^{(3\epsilon/4+\frac{k-2}{2k-2})n^2}.$$

Theorem 1.3 now follows from (9) (10) and (8) by observing that $\mathcal{F} = \sum_{C \in \mathcal{C}} s(C)$. □

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