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THE NUMBER OF EQUATIONS NEEDED TO DEFINE AN ALGEBRAIC SET

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Let B be a commutative Noetherian ring, $X = \operatorname{Spec} B$ the associated affine scheme, $I \subset B$ an ideal and $V = V(I) \subset X$ the closed subset defined by I.

DEFINITION. *Elements $f_1, \ldots, f_s \in I$ define V set-theoretically (equivalently, V is defined set-theoretically by s equations $f_1 = 0$, $f_2 = 0, \ldots, f_s = 0$) if $\sqrt{(f_1, \ldots, f_s)} = \sqrt{I}$.*

Hilbert's Nullstellensatz implies that in the case when B is a finitely generated algebra over an algebraically closed field k, this definition agrees with the usual one, i.e., all $f_1, \ldots, f_s$ vanish at a k-rational point if and only if it belongs to V. In the sequel "defined" always means "defined set-theoretically".

The question we are dealing with here concerns the minimum number of equations needed to define a given $V \subset X$. A classical result that goes back to L. Kronecker **[Kr]** says that if B is n-dimensional, then $n+1$ equations would suffice for every $V \subset X$. Our first theorem describes those $V \subset X$ which can be defined by n equations.

THEOREM A. *Let k be an algebraically closed field, X a smooth affine n-dimensional variety over k with coordinate ring B, and $V = V' \cup P_1 \cup P_2 \cup \cdots \cup P_r$ an algebraic subset of $X = \operatorname{Spec} B$, where V' is the union of irreducible components of positive dimensions and $P_1, P_2, \ldots, P_r$ some isolated closed points (which do not belong to V'). Then V can be defined by n equations if and only if one of the following conditions holds.*

(i) *$r = 0$, i.e., V consists only of irreducible components of positive dimension.*

(ii) *V' is empty, i.e., V consists only of closed points and there exist positive integers $n_1, n_2, \ldots, n_r$ such that $n_1P_1 + n_2P_2 + \cdots + n_rP_r = 0$ in $A_0(X)$.*

(iii) *V' is nonempty, $r \geq 1$ and there exist positive integers $n_1, n_2, \ldots, n_r$ such that $n_1P_1 + n_2P_2 + \cdots + n_rP_r$ belongs to the image of the natural map $A_0(V') \to A_0(X)$ induced by the inclusion $V' \to X$.*

Here $A_0(\cdot)$ stands for the group of zero-cycles modulo rational equivalence **[Fu]**.

SKETCH OF PROOF. Our proof consists of three steps. In Step 1 we construct an ideal $I \subset B$ such that $\sqrt{I}$ is the defining ideal of V, and in addition I has some other special properties. In Step 2 we, in a special way, pick some ideals $Q_1, \ldots, Q_n$ such that $\sqrt{Q_i}$ is a maximal ideal containing I for each i and J/J^2 is n-generated, where $J = I \cap Q_1 \cap \cdots \cap Q_n$. In Step 3 we

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prove that the 0-dimensional Segre class of J equals 0, so by [**Mu2**, Theorem 2] and the Suslin cancellation theorem [**Su**], J is n-generated. □

If B is not assumed to be regular, the result of Theorem A need not hold. In fact, for every $n \geq 0$, we have constructed an example of a finitely generated n-dimensional algebra B over a suitable algebraically closed field, such that its singular locus consists of just one closed point and for every d between 1 and $n-1$ it contains a d-dimensional subvariety which cannot be defined by n equations.

Storch [**St**] and Eisenbud-Evans [**EE**] proved that every algebraic set in $\mathbf{A}_k^n$ can be defined by n equations, while Cowsik-Nori [**CN**] proved that every curve in $\mathbf{A}_k^n$, where $\operatorname{char} k = p > 0$, can be defined by $n-1$ equations. Our next theorem sharpens these results.

THEOREM B. *If* $\operatorname{char} k = p > 0$, *then every algebraic set* $V \subset \mathbf{A}_k^n$ *consisting only of irreducible components of positive dimensions can be defined by* $n-1$ *equations.*

SKETCH OF PROOF. The main idea of the proof is contained in the following lemma.

LEMMA. *Let* $f_1, \ldots, f_r, g_1, \ldots, g_r \in I$ *and* $a \in B$ *such that*

(i) $f_1, \ldots, f_r$ *generates* I_a *up to radical.*

(ii) $g_1, \ldots, g_r$ *generate* $(I + (a))/(a) \subset B/(a)$ *up to radical. Then there exist* $r+1$ *elements* $h_1, \ldots, h_{r+1}$ *which generate* I *up to radical.*

Now assume all irreducible components of V have dimension $d \geq 1$ and use induction on d, the case $d = 1$ being settled by [**CN**]. Let $I = I(V) \subset B = k[x_1, \ldots, x_n]$ be the defining ideal of V. By a change of variables we can assume that $k[x_n]$ has zero intersection with every minimal prime overideal of I. By induction the extension of I in $k(x_n)[x_1, \ldots, x_{n-1}]$ can be generated up to radical by $n-2$ elements. Thus there exists a square-free polynomial $a(x_n)$ such that $I_{a(x_n)}$ can be generated up to radical by $n-2$ elements. Since $k[x_n]/a(x_n)$ is a product of fields, by induction $(I+(a))/(a) \subset (k[x_n]/a(x_n))[x_1, \ldots, x_{n-1}]$ also can be generated up to radical by $n-2$ elements. Now by the lemma, I can be generated up to radical by $n-1$ elements.

If V is not equidimensional, the proof is considerably harder, but the main idea remains the same. □

The rest of this paper is devoted to a question of M. P. Murthy [**Mu1**], who asked whether every locally complete intersection (l.c.i.) subscheme $V \subset \mathbf{A}_k^n$ is a set-theoretic complete intersection. The answer is known to be positive if $\dim V = 1$ [**Fe**, **Sz**, **Bo**, **MK**].

We generalize this result as follows:

THEOREM C. *Every l.c.i. subscheme* $V \subset \mathbf{A}_k^n$ *of constant positive dimension can be defined by* $n-1$ *equations.*

A proof of this theorem is similar to that of the equidimensional case of Theorem B. □

Moreover, we have obtained the following:

PROPOSITION. *Let k be an algebraically closed field of characteristic $p > 0$, I a l.c.i. of equidimension d in $B = k[x_1, \dots, x_n]$ and $n \geq 3d$. Set $A = B/\sqrt{I}$. Then there exists a l.c.i. $J \subset I$ such that $\sqrt{J} = \sqrt{I}$ and $J/J \cdot \sqrt{I}$, the reduced conormal module of J, is isomorphic to $P \oplus A^{n-2d+1}$, where P is a projective A-module of rank $d-1$ with trivial determinant.*

SKETCH OF PROOF. By the Ferrand construction [**Fe**, p. 24; **Va**, p. 89] we are reduced to the case when the determinant of the conormal module of I is trivial. Let $c_d \in F^d(\operatorname{Spec} A)$ be the top Chern class of the reduced conormal module of I, where $F^d(\operatorname{Spec} A)$ is the dth component of the Grothendieck γ-filtration [**FL**, p. 48]. Since $F^d(\operatorname{Spec} A)$ is divisible, we can write $c_d = (1-p^d)c$. Let Q be a projective A-module of rank d such that $Q - A^d \in F^d(\operatorname{Spec} A)$ and the top Chern class of Q equals c. Set $I/I \cdot \sqrt{I} \approx M \oplus Q$, where M is projective of rank $n - 2d \geq d$. We show that there exists a l.c.i. $J_1 \subset I$ such that $\sqrt{J_1} = \sqrt{I}$ and the reduced conormal module of J_1 is $M \oplus F(Q)$, where $F(Q)$ is the Frobenius of Q. It is straightforward to compute that the top Chern class of $M \oplus F(Q)$ is zero. Taking the iterated Frobenius of J_1, if necessary, we obtain, by [**Mu2**, Theorem 5] an l.c.i. J with required properties. □

THEOREM D. *Let k be any algebraically closed field of characteristic $p > 0$ and $V \subset \mathbf{A}_k^n$ a locally complete intersection subscheme of constant dimension d, such that $2 \leq d \leq n-4$. Then V can be defined by $n-2$ equations. In particular, every locally complete intersection surface in $\mathbf{A}_k^n$, where $n \geq 6$, is a set-theoretic complete intersection.*

PROOF. By induction, using the lemma we reduce to the case $d = \dim V = 2$. By the proposition there exists l.c.i. $J \subset I$ such that $\sqrt{J} = \sqrt{I}$ and the reduced conormal module of J is free. Now by [**MK**, Theorem 5] J is $(n-2)$-generated. □

THEOREM E. *Let k be any algebraically closed field of characteristic 2 and $V \subset \mathbf{A}_k^n$ a locally complete intersection subscheme of constant dimension d, such that $3 \leq d \leq n-6$. Then V can be defined by $n-3$ equations. In particular, every locally complete intersection threefold in $\mathbf{A}_k^n$, where $n \geq 9$, is a set-theoretic complete intersection.*

SKETCH OF PROOF. By an inductive argument with the help of the lemma we are reduced to the case $d = 3$. By the proposition, we get $J \subset I$ such that $\sqrt{J} = \sqrt{I}$ and the reduced conormal module of J is $P \oplus A^{n-5}$, where P is projective of rank 2 with trivial determinant. By a special argument in characteristic 2, we then obtain a new l.c.i. ideal $J_1 \subset J$ such that $\sqrt{J_1} = \sqrt{I}$ and the reduced conormal module of J_1 is free. Now we are done by [**MK**, Theorem 5]. □

Complete proofs will appear elsewhere.

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