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THE NUMBER OF MAXIMUM INDEPENDENT SETS IN GRAPHS

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Abstract. In this paper, we study the problem of determining the largest number of maximum independent sets of a graph of order n. Solutions to this problem are given for various classes of graphs, including general graphs, trees, forests, (connected) graphs with at most one cycle, connected graphs and triangle-free graphs. Extremal graphs achieving the maximum values are also given.

1. Introduction

In a graph G, an *independent set* is a subset S of $V(G)$ such that no two vertices in S are adjacent. A *maximal independent set* is an independent set that is not a proper subset of any other independent set. A *maximum independent set* is an independent set of maximum size. Note that a maximum independent set is maximal but the converse is not always true. The set of all independent sets (respectively, maximal independent sets and maximum independent sets) of a graph G is denoted by $I(G)$ (respectively, $\mathrm{MI}(G)$ and $\mathrm{XI}(G)$) and its cardinality by $i(G)$ (respectively, $\mathrm{mi}(G)$ and $\mathrm{xi}(G)$).

Erdős and Moser raised the problem of determining the maximum value of $\mathrm{mi}(G)$ for a general graph G of order n and those graphs achieving this maximum value. This problem was solved by Erdős, and later Moon and Moser [25]. Two decades later, the problem was extensively studied for various classes of graphs, including trees, forests, (connected) graphs with at most one cycle, bipartite graphs, connected graphs, k-connected graphs, triangle-free graphs and connected triangle-free graphs; for a survey see [16].

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Although the problem of finding the size $\alpha(G)$ of a maximum independent set of a graph G has been extensively studied (see [6]), there are very few works about counting the number of maximum independent sets (see [9, 10, 14, 17, 21, 28]). The purpose of this paper is to determine the largest number of maximum independent sets for various classes of graphs, including general graphs, trees, forests, (connected) graphs with at most one cycle, connected graphs and triangle-free graphs. Extremal graphs achieving the maximum values are also given.

2. Preliminary

Denote by K_n a *complete graph* with n vertices, C_n a *cycle* with n vertices, P_n a *path* with n vertices and $K_{m,n}$ a *complete bipartite graph* whose partite sets have m and n vertices respectively. A graph is *triangle-free* if it does not contain a K_3 as an induced subgraph.

The *neighborhood* $N_G(x)$ of a vertex x is the set of vertices adjacent to x and the *closed neighborhood* $N_G[x]$ is $N_G(x) \cup \{x\}$. The *degree* of x is $\deg_G(x) = |N_G(x)|$. A vertex is *isolated* if $\deg_G(x) = 0$, and is a *leaf* if $\deg_G(x) = 1$. Two vertices x and y are *duplicated* if $N_G(x) = N_G(y)$.

For a set $A \subseteq V(G)$, the *deletion* of A from G is the graph $G - A$ obtained from G by removing all vertices in A and their incident edges. Two graphs G and H are *disjoint* if $V(G) \cap V(H) = \emptyset$. The *union* of two disjoint graphs G and H is the graph $G \cup H$ with vertex set $V(G \cup H) = V(G) \cup V(H)$ and edge set $E(G \cup H) = E(G) \cup E(H)$. nG is the short notation for the union of n copies of disjoint graphs isomorphic to G.

The *distance* $d_G(x, y)$ from a vertex x to another vertex y is the minimum number of edges in an x-y path. The *distance* $d_G(x, S)$ from a vertex x to a set S is $\min_{y \in S} d_G(x, y)$.

For a vertex x, let $\mathrm{XI}_x(G) = \{S \in \mathrm{XI}(G) : x \in S\}$ and $\mathrm{XI}_{-x}(G) = \{S \in \mathrm{XI}(G) : x \notin S\}$. The cardinalities of $\mathrm{XI}_x(G)$ and $\mathrm{XI}_{-x}(G)$ are denoted by $\mathrm{xi}_x(G)$ and $\mathrm{xi}_{-x}(G)$, respectively.

Lemma 1. *For any graph G,* $\mathrm{xi}(G) \leq \mathrm{mi}(G)$.

Proof. The lemma follows from the fact that any maximum independent set is maximal. ■

Lemma 2. *If $G = \cup_{i=1}^{k} G_i$, then* $\mathrm{xi}(G) = \Pi_{i=1}^{k} \mathrm{xi}(G_i)$.

Proof. The lemma follows from the fact that S is a maximum independent set of G if and only if $S = \cup_{i=1}^{k} S_i$, where S_i is a maximum independent set of G_i for $1 \leq i \leq k$. ■

Lemma 3. *For any vertex x in G,* $\mathrm{xi}(G) = \mathrm{xi}_x(G) + \mathrm{xi}_{-x}(G)$ *and* $\mathrm{xi}_x(G) \leq \mathrm{xi}(G - N_G[x])$.

Proof. For any vertex x, $\mathrm{xi}(G) = \mathrm{xi}_x(G) + \mathrm{xi}_{-x}(G)$ follows from the fact that $\mathrm{XI}(G)$ is the disjoint union of $\mathrm{XI}_x(G)$ and $\mathrm{XI}_{-x}(G)$. Also, $\mathrm{xi}_x(G) \leq \mathrm{xi}(G - N_G[x])$ follows from the fact that if $S \in \mathrm{XI}_x(G)$ then $S - \{x\} \in \mathrm{XI}(G - N_G[x])$. ∎

Lemma 4. *If $x_1, x_2, \cdots, x_k$ are $k \geq 2$ leaves adjacent to the same vertex y in a graph G, then* $\mathrm{xi}(G) = \mathrm{xi}(G - \{x_1, x_2, \cdots, x_k, y\})$.

Proof. Suppose there exists a set $S \in \mathrm{XI}_y(G)$. Then $S' = (S - \{y\}) \cup \{x_1, x_2, \cdots, x_k\}$ is an independent set of G with $|S'| = |S| - 1 + k > |S|$, a contradiction. So, $\mathrm{xi}_y(G) = 0$. By Lemma 3, $\mathrm{xi}(G) = \mathrm{xi}_y(G) + \mathrm{xi}_{-y}(G) = \mathrm{xi}_{-y}(G)$. The mapping $f : \mathrm{XI}_{-y} \to \mathrm{XI}(G - \{x_1, x_2, \cdots, x_k, y\})$ defined by $f(S) = S - \{x_1, x_2, \cdots, x_k\}$ is a bijection. Thus, $\mathrm{xi}_{-y}(G) = \mathrm{xi}(G - \{x_1, x_2, \cdots, x_k, y\})$ and so the lemma holds. ∎

Lemma 5. *If x is a leaf adjacent to y in a graph G, then* $\mathrm{xi}(G) \leq 2\mathrm{xi}(G - \{x, y\})$. *Moreover,* $\mathrm{xi}(G) = 2\mathrm{xi}(G - \{x, y\})$ *implies that $T \cap N_G(y) = \emptyset$ for any maximum independent set T of $G - \{x, y\}$.*

Proof. For any $S \in \mathrm{XI}_{-x}(G)$, we have $y \in S$, for otherwise $S \cup \{x\}$ is a larger independent set than S in G. Then, the mapping $f : \mathrm{XI}_{-x} \to \mathrm{XI}(G - \{x, y\})$ defined by $f(S) = S - \{y\}$ is one-to-one. Thus, $\mathrm{xi}_{-x}(G) \leq \mathrm{xi}(G - \{x, y\})$. By Lemma 3, $\mathrm{xi}(G) = \mathrm{xi}_x(G) + \mathrm{xi}_{-x}(G) \leq 2\mathrm{xi}(G - \{x, y\})$. Also, $\mathrm{xi}(G) = 2\mathrm{xi}(G - \{x, y\})$ implies $\mathrm{xi}_x(G) = \mathrm{xi}_{-x}(G) = \mathrm{xi}(G - \{x, y\})$. Hence the mapping f is onto, i.e., for any $T \in \mathrm{XI}(G - \{x, y\})$, there exists some $S \in \mathrm{XI}_{-x}(G)$ such that $y \in S$ and $T = S - \{y\}$. Hence, $T \cap N_G(y) = \emptyset$. ∎

3. Main Results

Erdős, and later Moon and Moser [25] established an upper bound of $\mathrm{mi}(G)$ for a general graph G of order n. This gave the first result of counting maximal independent sets.

Theorem 6. *If G is a graph of order $n \geq 2$, then*

$$\mathrm{mi}(G) \leq g(n) = \begin{cases} 3^s, & if\ n = 3s, \\ 4 \cdot 3^{s-1}, & if\ n = 3s + 1, \\ 2 \cdot 3^s, & if\ n = 3s + 2. \end{cases}$$

Furthermore, $\mathrm{mi}(G) = g(n)$ *if and only if*

$$G \cong G(n) = \begin{cases} sK_3, & if\ n = 3s, \\ K_4 \cup (s-1)K_3 \ or \ 2K_2 \cup (s-1)K_3, & if\ n = 3s+1, \\ K_2 \cup sK_3, & if\ n = 3s+2. \end{cases}$$

Theorem 7. *If G is a graph of order $n \geq 2$, then* $\mathrm{xi}(G) \leq g(n)$. *Furthermore*, $\mathrm{xi}(G) = g(n)$ *if and only if* $G \cong G(n)$.

Proof. The theorem follows from Lemma 1, Theorem 6, and the fact that $\mathrm{xi}(G(n)) = \mathrm{mi}(G(n)) = g(n)$. Note that we use Lemma 2 to compute $\mathrm{xi}(G(n))$. ∎

Two decades later, Wilf [27] studied the problem of counting maximal independent sets for trees. His proof was algebraic in nature. Cohen [2] provided the first graph-theoretical proof. Sagan [26] finally presented an elegant proof, in which trees attaining the upper bound were also found (as did Griggs and Grinstead [7] independently). Jou and Chang [13, 14, 15] gave alternative proofs for the same result.

Theorem 8. *If G is a tree of order $n \geq 1$, then*

$$\mathrm{mi}(G) \leq t(n) = \begin{cases} 2^{s-1} + 1, & if\ n = 2s, \\ 2^s, & if\ n = 2s+1. \end{cases}$$

Furthermore, $\mathrm{mi}(G) = t(n)$ *if and only if*

$$G \cong T(n) \in \begin{cases} B(2, s-1) \ or \ B(4, s-2), & if\ n = 2s, \\ B(1, s), & if\ n = 2s+1, \end{cases}$$

where $B(i, j)$ is the set of batons, which are the graphs obtained from a path P of $i \geq 1$ vertices by attaching $j \geq 0$ paths of length two to the endpoints of P in all possible ways (*see Figure* 1).

Unlike general graphs, trees have different upper bounds for $\mathrm{xi}(G)$ from $\mathrm{mi}(G)$ when n is odd. The following result was first established by Zito [28] by means of a structure theorem. We now give a simple proof.

Theorem 9. *If G is a tree of order $n \geq 2$, then*

$$\mathrm{xi}(G) \leq t'(n) = \begin{cases} 2^{s-1} + 1, & if\ n = 2s, \\ 2^{s-1}, & if\ n = 2s+1. \end{cases}$$

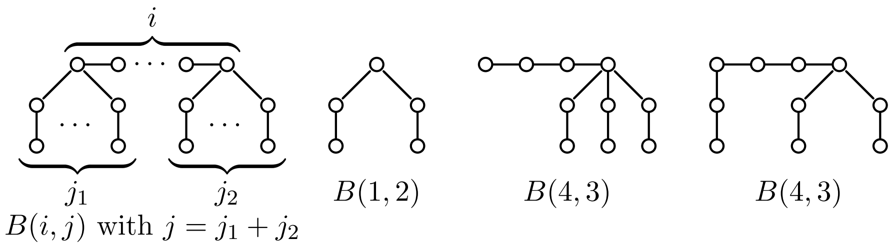

Figure 1. Batons.

Figure 2. The graph $T'(n)$.

Furthermore, $\mathrm{xi}(G) = t'(n)$ *if and only if* $G \cong T'(n)$*, where* $T'(n)$ *is as in Figure* 2*. The vertex* z *in* $T'(n)$ *is called the central vertex of* $T'(n)$*.*

Proof. It is straightforward to check that $\alpha(T'(n)) = \lceil n/2 \rceil$ and $\mathrm{xi}(T'(n)) = t'(n)$.

For $n = 2s$, since $T'(n) \in B(2, s-1)$, the theorem follows from Lemma 1, Theorem 8 and the fact that $\mathrm{xi}(T(n)) < \mathrm{mi}(T(n))$ for those $T(n)$ that are not $T'(n)$ and $\mathrm{xi}(T'(n)) = \mathrm{mi}(T(n)) = t(n) = t'(n)$.

So, we only need to prove the theorem for a tree G of order $n = 2s+1 \geq 3$. The theorem is trivial when G is a star $K_{1,n-1}$. Suppose now G is not a star. Then $n \geq 5$. Choose a vertex y that is adjacent to $k = \deg_G(y) - 1 \geq 1$ leaves $x_1, x_2, \cdots, x_k$. Such a vertex exists, as, for instance, the vertex adjacent to the end vertex of a longest path in G is as desired. Note that $G' = G - \{x_1, x_2, \cdots, x_k, y\}$ is a tree of order $n - k - 1$. For the case in which $k \geq 2$, by Lemma 4 and the induction hypothesis,

$$\begin{aligned}\mathrm{xi}(G) &= \mathrm{xi}(G') \leq t'(n-k-1) \leq \max\{t'(n-3), t'(n-4)\} \\ &= \max\{2^{s-2}+1, 2^{s-3}\} \leq t'(n).\end{aligned}$$

Also, $\mathrm{xi}(G) = t'(n)$ implies $n = 5$ and $\mathrm{xi}(G') = t'(2)$. By the result for even n, $G' \cong K_2$ and so $G \cong T'(5)$. For $k = 1$, by Lemma 5 and the induction hypothesit,

$$\mathrm{xi}(G) \leq 2\mathrm{xi}(G - N_G[x]) \leq 2t'(n-2) = t'(n).$$

Also, $\mathrm{xi}(G) = t'(n)$ implies $\mathrm{xi}(G - N_G[x]) = t'(n-2)$ and $\mathrm{xi}(G) = 2\mathrm{xi}(G - N_G[x])$. By the induction hypothesis, $G - N_G[x] \cong T'(n-2)$. By Lemma 5, y is adjacent to a vertex in $G - N_G[x]$ that is not in any maximal independent set of $G - N_G[x] \cong T'(n-2)$. The only possible vertex with this property is the central vertex of $T'(n-2)$. Thus $G \cong T'(n)$. ■

Jou and Chang [13, 14, 15] derived a result of $\mathrm{mi}(G)$ for forests (see Theorem 10 below) and then applied this result to derive an alternative proof of Theorem 8. They also used this method to get results on (connected) graphs with at most one cycle (see Theorems 12 and 14 below).

Theorem 10. *If G is a forest of order $n \geq 1$, then*

$$\mathrm{mi}(G) \leq f(n) = \begin{cases} 2^s, & if\ n = 2s, \\ 2^s, & if\ n = 2s+1. \end{cases}$$

Furthermore, $\mathrm{mi}(G) = f(n)$ *if and only if*

$$G \cong F(n) = \begin{cases} sK_2, & if\ n = 2s, \\ B(1, s-r) \cup rK_2\ for\ some\ 0 \leq r \leq s, & if\ n = 2s+1. \end{cases}$$

Theorem 11. *If G is a forest of order $n \geq 1$, then* $\mathrm{xi}(G) \leq f(n)$. *Furthermore,* $\mathrm{xi}(G) = f(n)$ *if and only if*

$$G \cong F'(n) = \begin{cases} sK_2, & if\ n = 2s, \\ K_1 \cup sK_2, & if\ n = 2s+1. \end{cases}$$

Proof. By Lemma 1 and Theorem 10, we have $\mathrm{xi}(G) \leq \mathrm{mi}(G) \leq \mathrm{mi}F(n)) = f(n)$. Moreover, $\mathrm{xi}(G) = f(n)$ if and only if $\mathrm{xi}(G) = \mathrm{mi}(G) = \mathrm{mi}(F(n)) = f(n)$. However, $\mathrm{xi}(B(1, s-r)) = 1 = 2^{s-r} = \mathrm{mi}(B(1, s-r))$ if and only if $s = r$. Therefore, $\mathrm{xi}(G) = f(n)$ if and only if $G \cong F'(n)$. ■

Theorem 12. *If G is a graph of order $n \geq 2$ with at most one cycle, then*

$$\mathrm{mi}(G) \leq c(n) = \begin{cases} 2^s, & if\ n = 2s, \\ 3 \cdot 2^{s-1}, & if\ n = 2s+1. \end{cases}$$

Furthermore, for $n \geq 6$, $\mathrm{mi}(G) = c(n)$ *if and only if*

$$G \cong C(n) = \begin{cases} sK_2, & if\ n = 2s, \\ K_3 \cup (s-1)K_2, & if\ n = 2s+1. \end{cases}$$

Theorem 13. *If* G *is a graph of order* $n \geq 2$ *with at most one cycle, then* $\mathrm{xi}(G) \leq c(n)$. *Furthermore, for* $n \geq 6$, $\mathrm{xi}(G) = c(n)$ *if and only if* $G \cong C(n)$.

Proof. This is the same as the proof of Theorem 7 except that now Theorem 12 is used. ■

Theorem 14. *If* G *is a connected graph of order* $n \geq 3$ *with at most one cycle, then*

$$\mathrm{mi}(G) \leq d(n) = \begin{cases} 3 \cdot 2^{s-2}, & if\ n = 2s, \\ 2^s + 1, & if\ n = 2s+1. \end{cases}$$

Furthermore, for $n \geq 6$, $\mathrm{mi}(G) = d(n)$ *if and only if* $G \cong D(n)$ *(see Figure 3). More precisely, for* $n = 2s$, $D(n)$ *is the graph obtained from* $B(1, s-2)$ *by adding a* K_3 *and a new edge joining a vertex of* K_3 *and the only vertex in the basic path of* $B(1, s-2)$. *For* $n = 2s+1$, $D(n)$ *is the graph obtained from* $B(1, s-1)$ *by adding a* K_3 *with one vertex identified with the only vertex in the basic path of* $B(1, s-1)$.

Theorem 15. *If* G *is a connected graph of order* $n \geq 2$ *with at most one cycle, then*

$$\mathrm{xi}(G) \leq d'(n) = \begin{cases} t'(n) = 2^{s-1} + 1, & if\ n = 2s, \\ d(n) = 2^s + 1, & if\ n = 2s+1. \end{cases}$$

Furthermore, for $n \neq 5$, $\mathrm{xi}(G) = d'(n)$ *if and only if*

$$G \cong D'(n) = \begin{cases} T'(n), & if\ n = 2s, \\ D(n), & if\ n = 2s+1. \end{cases}$$

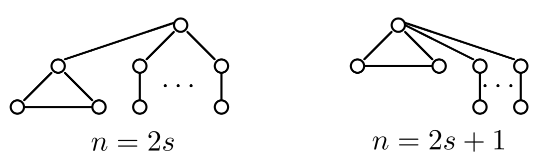

Figure 3. The graph $D(n)$.

Proof. According to Theorem 9 and a straightforward computation, we have $\mathrm{xi}(D'(n)) = d'(n)$.

The theorem is true for the case in which $n = 2s+1$ by Lemma 1, Theorem 14 and the fact that $\mathrm{xi}(D'(n)) = \mathrm{mi}(D(n)) = d(n) = d'(n)$.

We shall prove the theorem for $n = 2s$ by induction. It is trivial for $n = 2, 4$. Suppose it is true for all $n' < n$ and G is a connected graph of order $n = 2s \geq 6$. If G is a tree, then the theorem follows from Theorem 9. So, we may assume that G contains a unique cycle C. Let r be the largest distance from a vertex to C and $V_i = \{v : d_G(v, C) = i\}$ for $0 \leq i \leq r$. Note that $V_0 = C$ and all vertices in V_r are leaves when $r \geq 1$.

If $r = 0$, then $G \cong C_n$ and so $\mathrm{xi}(G) = 2 < d'(n)$. So $r \geq 1$. Choose a vertex $y \in V_{r-1}$ that is adjacent to $k \geq 1$ vertices $x_1, x_2, \cdots, x_k$ in V_r. Note that $G' = G - \{x_1, x_2, \cdots, x_k, y\}$ is a connected graph of order $n-k-1$ with at most one cycle. For $k \geq 2$, by Lemma 4 and the induction hypothesis,

$$\begin{aligned}\mathrm{xi}(G) &= \mathrm{xi}(G') \leq d'(n-k-1) \leq \max\{d'(n-3), d'(n-4)\}\\ &= \max\{2^{s-2}+1, 2^{s-3}+1\} = 2^{s-2}+1 < d'(n).\end{aligned}$$

So, $k = 1$. Suppose $G - N_G[x_1] \ncong T'(n-2)$. By Lemma 5 and the induction hypothesis,

$$\mathrm{xi}(G) \leq 2\mathrm{xi}(G - N_G[x_1]) \leq 2(d'(n-2)-1) = 2^{s-1} < d'(n).$$

So, $G - N_G[x_1] \cong T'(n-2)$, which is a tree. This is possible only when $r = 1$ and $y \in C$ is adjacent to two vertices in C. Therefore, $n = 6$ or 8 and $C = C_m$ with $s \leq m \leq s+2$. There are six such graphs. By a straightforward calculation, $\mathrm{xi}(G) < d'(n)$. ∎

Answering a question of Wilf [27], Griggs *et al.* [8] gave the maximum value of $\mathrm{mi}(G)$ for a connected graph G of order $n \geq 6$ and the extremal graphs achieving this value. Fűredi [5] presented the same result for $n \geq 50$.

Theorem 16. *If G is a connected graph of order $n \geq 6$, then*

$$\mathrm{mi}(G) \leq h(n) = \begin{cases} 2\cdot 3^{s-1} + 2^{s-1}, & \text{if } n = 3s,\\ 3^s + 2^{s-1}, & \text{if } n = 3s+1,\\ 4\cdot 3^{s-1} + 3\cdot 2^{s-2}, & \text{if } n = 3s+2.\end{cases}$$

Furthermore, $\mathrm{mi}(G) = h(n)$ *if and only if* $G \cong H(n)$, *where* $H(n)$ *is shown as in Figure* 4. *More precisely, if* $n = 3s + r$ *with* $0 \leq r \leq 2$, *then* $H(n)$ *is obtained from* $(s-r)K_3 \cup rK_4$ *by fixing a copy of* $K_{\lceil r/2\rceil+3}$ *and a vertex* x *in it, and then adding edges joining* x *to a vertex in each of the other copies of* K_3 *or* K_4.

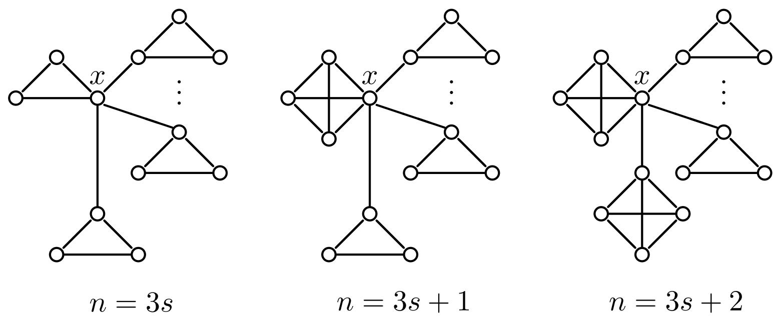

Figure 4. The graph $H(n)$ for $n \geq 6$.

Theorem 17. *If* G *is a connected graph of order* $n \geq 6$, *then* $\mathrm{xi}(G) \leq h(n)$. *Furthermore*, $\mathrm{xi}(G) = h(n)$ *if and only if* $G \cong H(n)$.

Proof. This is the same as the proof of Theorem 7 except that now Theorem 16 is used. ∎

Hujter and Tuza [11] discovered the maximum value of $\mathrm{mi}(G)$ for a triangle-free graph G of order $n \geq 4$ and found the extremal graphs achieving this value.

Theorem 18. *If* G *is a triangle-free graph of order* $n \geq 4$, *then*

$$\mathrm{mi}(G) \leq \ell(n) = \begin{cases} 2^s, & if\ n = 2s, \\ 5 \cdot 2^{s-2}, & if\ n = 2s+1. \end{cases}$$

Furthermore, $\mathrm{mi}(G) = \ell(n)$ *if and only if*

$$G \cong L(n) = \begin{cases} sK_2, & if\ n = 2s, \\ C_5 \cup (s-2)K_2, & if\ n = 2s+1. \end{cases}$$

Theorem 19. *If* G *is a triangle-free graph of order* $n \geq 4$, *then* $\mathrm{xi}(G) \leq \ell(n)$. *Furthermore*, $\mathrm{xi}(G) = \ell(n)$ *if and only if* $G \cong L(n)$.

Proof. This is the same as the proof of Theorem 7 except that now Theorem 18 is used. ∎

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