The Annals of Probability
2017, Vol. 45, No. 6A, 4071–4100
DOI: 10.1214/16-AOP1158

THE NUMBER OF OPEN PATHS IN ORIENTED PERCOLATION

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We study the number N_n of open paths of length n in supercritical oriented percolation on $\mathbb{Z}^d \times \mathbb{N}$, with $d \geq 1$, and we prove the existence of the connective constant for the supercritical oriented percolation cluster: on the percolation event $\{\inf N_n > 0\}$, $N_n^{1/n}$ almost surely converges to a positive deterministic constant.

The proof relies on the introduction of adapted sequences of regenerating times, on subadditive arguments and on the properties of the coupled zone in supercritical oriented percolation. This global convergence result can be deepened to give directional limits and can be extended to more general random linear recursion equations known as linear stochastic evolutions.

1. Introduction and main results. Consider supercritical oriented percolation on $\mathbb{Z}^d \times \mathbb{N}$. Let $N(a,b)$ denote the number of open paths from a to b. By concatenation of paths, we get $N(a,c) \geq N(a,b)N(b,c)$. In other words, the following superadditivity property holds:

$$\log N(a,c) \geq \log N(a,b) + \log N(b,c).$$

In the case of deterministic lattices, such a subadditive inequality immediately gives the existence of the so-called connective constant: if N_n denotes the number of self-avoiding paths starting from the origin, the connective constant is the limit of $N_n^{1/n}$. Coming back to percolation on a lattice, subadditive ergodic theorems suggest that, on the percolation event "the cluster of the origin is infinite", the number N_n of self-avoiding open paths with length n starting from the origin should grow exponentially fast in n. However, the possibility for edges to be closed implies that $\log N(\cdot,\cdot)$ may be infinite and, therefore, not integrable. This prevents from using subadditive techniques, at least in their simplest form.

In terms of directed polymers in random environment, the limit, when it exists, of $\log N_n/n$ corresponds to the quenched free energy of the model. There has been recently much activity around these subjects: Fukushima and Yoshida proved in [6] that $\underline{\lim} \log N_n/n$ is almost surely strictly positive on the percolation event, and Lacoin in [10] studied the discrepancy between the quenched free energy $\underline{\lim} \log N_n/n$ and the annealed free energy $\lim \log \mathbb{E} N_n/n$. See also the survey by Comets, Shiga and Yoshida [3] on directed polymers in random environment.

Received November 2015; revised September 2016.
MSC2010 subject classifications. Primary 60K35; secondary 82B43.
Key words and phrases. Subadditive ergodic theorem, oriented percolation.

In spite of these studies, to our knowledge, there was no proof of the convergence of $N_n^{1/n}$ in full generality in the literature. Note, however, that such a convergence has been obtained for a relaxed kind of percolation called ρ-percolation. Let $\rho \in (0, 1)$ and let $N_n(\rho)$ denotes the number of paths with length n using at least ρn open edges. The existence of the limit $N_n(\rho)^{1/n}$ has been proved in Comets–Popov–Vachkovskaia [2] and in Kesten–Sidoravicius [9] by different methods.

The present paper aims to prove that in supercritical oriented percolation, $N_n^{1/n}$ has an almost sure limit on the percolation event. The proof relies on essential hitting times which have been introduced in Garet–Marchand [7] in order to establish a shape theorem for the contact process in random environment. We then extend this result to obtain a similar result for the number of paths with a prescribed slope, and for a general model of random linear recursion equations introduced by Yoshida [12] and called Linear Stochastic Evolutions (LSE). As a special case, we obtain the existence of the free energy for the directed polymer in random environment model with potentials taking their values in $\mathbb{R} \cup \{-\infty\}$.

The existence of a quenched connective constant is also believed to hold in the nonoriented supercritical percolation cluster on $\mathbb{Z}^d$, and Lacoin [11] proved the noncoincidence of this quenched connective constant and the annealed connective constants on the supercritical planar percolation cluster. But here again, there is to our knowledge no proof for the existence of a limit for $N_n^{1/n}$. The result we present here strongly relies on the oriented structure of the graph, and we do not know how to adapt it to the nonoriented context.

Before stating precisely our results, let us first define the oriented percolation setting we work with.

1.1. *Oriented percolation in dimension $d+1$.* Let $d \geq 1$ be fixed, and let $\|\cdot\|_1$ be the ℓ_1-norm on $\mathbb{R}^d$. We consider the oriented graph whose set of sites is $\mathbb{Z}^d \times \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$, and we put an oriented edge from (z_1, n_1) to (z_2, n_2) if and only if

$$n_2 = n_1 + 1 \quad \text{and} \quad \|z_2 - z_1\|_1 \leq 1;$$

the set of these edges is denoted by $\overrightarrow{\mathbb{E}}^{d+1}_{\text{alt}}$.

We say that $\gamma = (\gamma_i, i)_{m \leq i \leq n} \in (\mathbb{Z}^d \times \mathbb{N})^{n-m+1}$ is a *path* if and only if

$$\forall i \in \{m, \ldots, n-1\} \qquad \|\gamma_{i+1} - \gamma_i\|_1 \leq 1.$$

Fix now a parameter $p \in [0, 1]$, and open independently each edge with probability p. More formally, consider the probability space $\Omega = \{0, 1\}^{\overrightarrow{\mathbb{E}}^{d+1}_{\text{alt}}}$, endowed with its Borel σ-algebra and the probability

$$\mathbb{P}_p = \bigl(\mathrm{Ber}(p)\bigr)^{\otimes \overrightarrow{\mathbb{E}}^{d+1}_{\text{alt}}},$$

where $\mathrm{Ber}(p)$ stands for the Bernoulli law of parameter p. For a configuration $\omega = (\omega_e)_{e \in \overrightarrow{\mathbb{E}}^{d+1}_{\mathrm{alt}}} \in \Omega$, say that the edge $e \in \overrightarrow{\mathbb{E}}^{d+1}_{\mathrm{alt}}$ is open if $\omega_e = 1$ and closed otherwise. A path is said *open* in the configuration ω if all its edges are open in ω. For two sites (v, m), (w, n) in $\mathbb{Z}^d \times \mathbb{N}$, we denote by $\{(v, m) \to (w, n)\}$ the existence of an open path from (v, m) to (w, n). By extension, we denote by $\{(v, m) \to +\infty\}$ the event that there exists an infinite open path starting from (v, m). There exists a critical probability $\overrightarrow{p_c}^{\mathrm{alt}}(d + 1) \in (0, 1)$ such that

$$\mathbb{P}_p\big((0, 0) \to +\infty\big) > 0 \quad \Longleftrightarrow \quad p > \overrightarrow{p_c}^{\mathrm{alt}}(d + 1).$$

In the following, we assume $p > \overrightarrow{p_c}^{\mathrm{alt}}(d + 1)$, and we will mainly work under the following conditional probability:

$$\overline{\mathbb{P}}_p(\cdot) = \mathbb{P}_p\big(\cdot|(0, 0) \to +\infty\big).$$

1.2. *Global convergence result and previous results.* Denote by N_n the number of open paths of length n emanating from $(0, 0)$. Note first that $\mathbb{E}_p(N_n) = ((2d + 1)p)^n$. As noticed by Darling [4], the sequence $(N_n((2d + 1)p)^{-n})_{n\ge 0}$ is a nonnegative martingale, so there exists a nonnegative random variable W such that

$$\mathbb{P}_p\text{-a.s.} \qquad \frac{N_n}{(2d + 1)^n p^n} \longrightarrow W \quad \text{and} \quad \mathbb{E}_p[W] \le 1.$$

Therefore, it is easy to see that

$$\frac{1}{n}\log N_n \to \log\big((2d + 1)p\big) \qquad \text{on the event } \{W > 0\}.$$

So when $W > 0$, N_n has the same growth rate as its expectation. In his paper [4], Darling was seeking for conditions implying that $W > 0$. Actually, it is not always the case that $W > 0$. Let us summarize some known results:

- Yoshida [12] showed that $W = 0$ a.s. if $d = 1$ or $d = 2$.
- There exist $\overrightarrow{p_{c,2}}^{\mathrm{alt}}(d + 1)$ and $\overrightarrow{p_{c,3}}^{\mathrm{alt}}(d + 1)$ in $[\overrightarrow{p_c}^{\mathrm{alt}}(d + 1), 1]$, with $\overrightarrow{p_{c,2}}^{\mathrm{alt}}(d + 1) < \overrightarrow{p_{c,3}}^{\mathrm{alt}}(d + 1)$ such that (see Lacoin [10], Sections 2.2 and 2.3):
  - $\overline{\mathbb{P}}_p(W > 0) = 1$ when $p > \overrightarrow{p_{c,3}}^{\mathrm{alt}}(d + 1)$ and $\overline{\mathbb{P}}_p(W > 0) = 0$ when $p < \overrightarrow{p_{c,3}}^{\mathrm{alt}}(d + 1)$.
  - $\overline{\lim} \frac{1}{n} \log N_n = \log(p(2d + 1))$ $\overline{\mathbb{P}}_p$-a.s. when $p > \overrightarrow{p_{c,2}}^{\mathrm{alt}}(d + 1)$ and $\overline{\lim} \frac{1}{n} \log N_n < \log(p(2d + 1))$ $\overline{\mathbb{P}}_p$-a.s. when $p < \overrightarrow{p_{c,2}}^{\mathrm{alt}}(d + 1)$.
  - $\overrightarrow{p_{c,3}}^{\mathrm{alt}}(d + 1) < 1$ if $d \ge 3$.
- It is believed that $\overrightarrow{p_{c,2}}^{\mathrm{alt}}(d + 1) > \overrightarrow{p_c}^{\mathrm{alt}}(d + 1)$, and thus $\overrightarrow{p_{c,3}}^{\mathrm{alt}}(d + 1) > \overrightarrow{p_c}^{\mathrm{alt}}(d + 1)$ when $d \ge 2$. Lacoin [10] proved that the inequality is indeed strict for L-spread-out percolation for $d \ge 5$ and L large.

In any case, it is clear that we need a proof of the existence of a limit for $\frac{1}{n}\log N_n$ that would not require $W > 0$, and this is our main result.

THEOREM 1.1. *Let $p > \overrightarrow{p_c}^{\mathrm{alt}}(d+1)$. There exists a strictly positive constant $\tilde{\alpha}_p(0)$ such that, $\overline{\mathbb{P}}_p$-almost surely and in $L^1(\overline{\mathbb{P}}_p)$,*

$$\lim_{n\to+\infty} \frac{1}{n} \log N_n = \tilde{\alpha}_p(0).$$

Our next result focuses on open paths with a prescribed slope.

1.3. *Directional convergence results.* We first need to give a few more notations and results. Oriented percolation is known as the analogue in discrete time for the contact process. Usually, results are proved for one model, and it is commonly admitted that the proofs could easily be adapted to the other one. For the results concerning supercritical oriented percolation we use in this work, we will thus sometimes give the reference for the property concerning the contact process without any further explanation.

We define

$$\xi_n = \{y \in \mathbb{Z}^d : (0,0) \to (y,n)\} \quad \text{and} \quad H_n = \bigcup_{0\le k\le n} \xi_k.$$

As for the contact process, the growth of the sets $(H_n)_{n\ge 0}$ is governed by a shape theorem when conditioned to survive: for every $p > \overrightarrow{p_c}^{\mathrm{alt}}(d+1)$, there exists a norm μ_p on $\mathbb{R}^d$ such that for every $\varepsilon > 0$, $\overline{\mathbb{P}}_p$ almost surely,

$$\text{(1)} \qquad \exists N\ \forall n \ge N \qquad B_{\mu_p}\big(0, (1-\varepsilon)n\big) \subset H_n + [0,1]^d \subset B_{\mu_p}\big(0, (1+\varepsilon)n\big),$$

where $B_{\mu_p}(x,r) = \{y \in \mathbb{R}^d : \mu_p(y-x) \le r\}$. See, for the supercritical contact process, Durrett [5] or Garet–Marchand [7].

For every set $A \subset B_{\mu_p}(0,1)$, we denote by $N_{nA,n}$ the number of open paths starting from $(0,0)$, with length n and whose extremity lies in $nA \cap \mathbb{Z}^d$.

THEOREM 1.2. *Fix $p > \overrightarrow{p_c}^{\mathrm{alt}}(d+1)$. There exists a concave function*

$$\tilde{\alpha}_p : \mathring{B}_{\mu_p}(0,1) \longrightarrow \big(0, \log\big(p(2d+1)\big)\big],$$

with the same symmetries as the grid $\mathbb{Z}^d$, such that, for every set A such that $\overline{\mathring{A}} = \overline{A} \subset \mathring{B}_{\mu_p}(0,1)$, $\overline{\mathbb{P}}_p$-almost surely,

$$\text{(2)} \qquad \lim_{n\to+\infty} \frac{1}{n} \log N_{nA,n} = \sup_{x\in A} \tilde{\alpha}_p(x).$$

It will appear in the proofs that the limit in Theorem 1.1 is the maximum of the function $\tilde{\alpha}_p$. Since $\tilde{\alpha}_p$ is even and concave, this constant is $\tilde{\alpha}_p(0)$.

By considering, in Theorem 1.2, the set $A = B_{\mu_p}(x,\varepsilon)$ for $x \in \mathring{B}_{\mu_p}(0,1)$ and for a small ε, we see that $\tilde{\alpha}_p(x)$ characterises the growth of the number of open paths with length n and prescribed slope x. Using the very same technics of proof, one could for instance prove the following directional convergence result. If $x \in \mathbb{Z}^d$, denote by $N_{x,n}$ the number of open paths from $(0,0)$ to (x,n).

REMARK 1.3. Fix $p > \overrightarrow{p_c}^{\mathrm{alt}}(d+1)$ and $(y,h) \in \mathbb{Z}^d \times \mathbb{N}^*$ such that $\mu_p(y) < h$. Extract from the sequence (ny, nh) the (random) subsequence, denoted $\psi : \mathbb{N} \to \mathbb{N}$, of indices k such that $(0,0) \to k \cdot (y,h)$. Then $\overline{\mathbb{P}}_p$ almost surely,

$$\lim_{n \to +\infty} \frac{1}{\psi(n)h} \log N_{\psi(n)\cdot(y,h)} = \tilde{\alpha}_p(y/h).$$

Take now as a random environment a realization of oriented percolation on $\mathbb{Z}^d \times \mathbb{N}$ with parameter p such that 0 percolates. Once this random setting is fixed, choose a random open path with length n, uniformly among all open paths with length n, and ask for the behavior of the extremity of this random path. More precisely, for every set A with $\overline{\mathring{A}} = \overline{A} \subset \mathring{B}_{\mu_p}(0,1)$, the probability that the extremity of the random path stands in nA is $N_{nA,n}/N_n$. Then Theorem 1.2 can be rephrased as a quenched large deviations principle for the extremity of this random open path (or directed polymer on a oriented-percolation cluster).

REMARK 1.4. Fix $p > \overrightarrow{p_c}^{\mathrm{alt}}(d+1)$.
For every set A such that $\overline{\mathring{A}} = \overline{A} \subset \mathring{B}_{\mu_p}(0,1)$, $\overline{\mathbb{P}}_p$-almost surely,

$$\lim_{n \to +\infty} \frac{1}{n} \log \frac{N_{nA,n}}{N_n} = -\inf_{x \in A} \bigl(\tilde{\alpha}_p(0) - \tilde{\alpha}_p(x)\bigr).$$

Remarks and open questions.

- Is the following statement true?

$$\forall x \in \mathring{B}_{\mu_p}(0,1) \setminus \{0_{\mathbb{Z}^d}\} \qquad \tilde{\alpha}_p(x) < \tilde{\alpha}_p(0).$$

  If the statement held, then the extremity of a random open path with length n, uniformly chosen among open paths with length n, would concentrate near $0_{\mathbb{Z}^d}$.
- Is $\tilde{\alpha}_p$ strictly concave? This would imply the previous statement.
- Is $\tilde{\alpha}_p$ continuous in p?
- The function $\tilde{\alpha}_p$ probably does not vanish when x tends to the boundary of $\mathring{B}_{\mu_p}(0,1)$. Here is a very schematic version of an argument in dimension $1+1$, due to Ryoki Fukushima. Along the rightmost path γ_n to level n, we are looking for "left-turns;" a left-turn is the succession of an edge oriented in the North-East direction and of an edge oriented in the North–West direction. We can find, uniformly in $p \le 1-\varepsilon$, a number $\Theta(n)$ of left turns along the rightmost path. The two edges of a left turn are the right half of a square: each time the two edges of the left half of this square are open, we double the number of open paths going to the extremity of γ_n. As we are considering the rightmost path, anything on the left of it is independent, and hence these open left half of squares occur independently with probability p^2. This shows that the number of open paths along the rightmost path already grows exponentially at a uniformly positive rate. This argument also shows that the growth rate does not vanish as p tends to the critical probability.

1.4. *Extension to linear stochastic evolutions.* We extend here the study to (an independent subcase of) the LSE introduced by Yoshida [12].

We define a set of oriented edges $\overrightarrow{\mathbb{E}}^d$ of $\mathbb{Z}^d$ in the following way: in $(\mathbb{Z}^d, \overrightarrow{\mathbb{E}}^d)$, there is an oriented edge between two points z_1 and z_2 in $\mathbb{Z}^d$ if and only if $\|z_1 - z_2\|_1 \le 1$. The oriented edge in $\overrightarrow{\mathbb{E}}^{d+1}_{\text{alt}}$ from (z_1, n_1) to (z_2, n_2) can be identified with the couple $((z_1, z_2), n_2) \in \overrightarrow{\mathbb{E}}^d \times \mathbb{N}^*$. Thus, we identify $\overrightarrow{\mathbb{E}}^{d+1}_{\text{alt}}$ and $\overrightarrow{\mathbb{E}}^d \times \mathbb{N}^*$.

We consider the oriented graph $(\mathbb{Z}^d \times \mathbb{N}, \overrightarrow{\mathbb{E}}^d \times \mathbb{N}^*)$, and a collection $(A_{e,n})_{(e,n)\in\overrightarrow{\mathbb{E}}^d\times\mathbb{N}^*}$ of independent and identically distributed nonnegative random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We are interested in the asymptotic behavior of the solution $(N_{x,n})_{(x,n)\in\mathbb{Z}^d\times\mathbb{N}}$ of the following (random) recurrence relations:

$$\text{(LSE)} \qquad \begin{cases} N_{0,0} = 1 \quad \text{and} \quad \forall x \in \mathbb{Z}^d \setminus \{0\} \qquad N_{x,0} = 0, \\ \forall n \in \mathbb{N}\ \forall x \in \mathbb{Z}^d \qquad N_{x,n+1} = \displaystyle\sum_{y:\|y-x\|_1\le 1} A_{(y,x),n+1} N_{y,n}, \end{cases}$$

and especially on the growth rate of the partition function $N_n = \sum_{x\in\mathbb{Z}^d} N_{x,n}$.

We say that an edge e is open if and only if $A_e > 0$: the states of the edges, open or closed, induce an oriented percolation on $\mathbb{Z}^d \times \mathbb{N}$ with parameter $p = \mathbb{P}(A > 0)$, and $\overline{\mathbb{P}}$ is, as before, the probability $\mathbb{P}$ conditionally to the event $\{(0,0) \to +\infty\}$ for this oriented percolation. Our last result is the following.

THEOREM 1.5. *Assume that*

$$p \stackrel{\text{def}}{=} \mathbb{P}(A > 0) > \overrightarrow{p_c}^{\text{alt}}(d+1), \tag{3}$$

$$\exists \gamma > 0 \qquad \mathbb{E}\big(A^\gamma + A^{-\gamma} | A > 0\big) < +\infty. \tag{4}$$

There exists a constant $\tilde{\alpha}(0)$ *such that*, $\overline{\mathbb{P}}$-*almost surely and in* $L^1(\overline{\mathbb{P}})$,

$$\lim_{n\to+\infty} \frac{1}{n} \log N_n = \tilde{\alpha}(0).$$

Assumption (3) is optimal: if $\mathbb{P}(A > 0) \le \overrightarrow{p_c}^{\text{alt}}(d+1)$, the cluster of open edges starting from the origin is almost surely finite, and thus, almost surely, $N_n = 0$ for every n large enough. On the contrary, Assumption (4) seems relatively soft and simple to us, and we did not try to optimize it.

Taking for $(A_{e,n})_{(e,n)\in\overrightarrow{\mathbb{E}}^d\times\mathbb{N}^*}$ i.i.d. Bernoulli random variables with parameter p, we recover the number of open paths with length n in oriented percolation with parameter p as a particular case of LSE.

Directed polymer in a random environment also falls in the class of LSE. In this model, instead of considering a path uniformly chosen among open paths with length n, we first weight paths accordingly to the potential of their edges.

We thus consider a family $(B_{e,n})_{(e,n)\in\overrightarrow{\mathbb{E}}^d\times\mathbb{N}^*}$ of independent and identically distributed random variables, modelling the random environment. Then we associate to a path $\gamma=(\gamma_i,i)_{0\le i\le n}\in(\mathbb{Z}^d\times\mathbb{N})^{n+1}$ starting from the origin a Hamiltonian:

$$H_n(\gamma)=\sum_{(e,n)\in\gamma}B_{e,n},$$

and we build a probability measure on paths with length n starting from the origin:

$$\mu_n(\gamma)=\frac{1}{Z_n}\exp\bigl(H_n(\gamma)\bigr)\qquad\text{where } Z_n=\sum_{\gamma:(0,0)\to\mathbb{Z}^d\times\{n\}}\exp\bigl(H_n(\gamma)\bigr)$$

is the partition function. Thus, paths using edges with high potential are favorized. Setting

$$Z_{x,n}=\sum_{\gamma:(0,0)\to(x,n)}\exp\bigl(H_n(\gamma)\bigr),$$

we see that the family $(Z_{x,n})_{(x,n)\in\mathbb{Z}^d\times\mathbb{N}}$ satisfies the random recursion relations (LSE), with $A_{e,n}=\exp(B_{e,n})$. In our setting, as we allow $A_{e,n}$ to be 0, we can consider potential taking the value $-\infty$ with positive probability. This amounts to study the directed polymer on a supercritical oriented percolation cluster, or in other words to forbid each edge of $\mathbb{Z}^d\times\mathbb{N}$ independently, with the same probability $1-p=\mathbb{P}(B_{e,n}=-\infty)$. Theorem 1.5 gives the existence of the (quenched) free energy of the directed polymer in this setting. The analogues of Theorem 1.2 and of Remarks 1.3 and 1.4 can also be established under the assumption of Theorem 1.5 with the very same technics.

Very recently, Comets, Fukushima, Nakajima and Yoshida studied in [1] the directed polymer with unbounded jumps in random environment. In particular, they also prove the existence of the free energy for inverse temperature $\beta=-\infty$, that is, when potentials are allowed to take the value $-\infty$, without using any subadditivity, but rather by proving a continuity property. Note, however, that their model is quite different from ours, since the existence of unbounded jumps rules out the percolation transition.

Organization of the paper. First, in Section 2, we recall results for supercritical oriented percolation, and we build the essential hitting times.

Then, in Section 3, we fix a vector $(y,h)\in\mathbb{Z}^d\times\mathbb{N}^*$ and we build an associated sequence of regenerating times $(S_n(y,h))_n$ [see Definition (7)]. These random times satisfy $(0,0)\to(ny,S_n(y,h))\to+\infty$ and have good invariance and integrability properties with respect to $\overline{\mathbb{P}}_p$. We can thus apply Kingman's subadditive ergodic theorem to obtain, in Lemma 3.2, the existence of the following limit:

$$\frac{1}{S_n(y,h)}\log(N_{ny,S_n(y,h)})\to\alpha_p(y,h).$$

Section 4 is devoted to the proof of Theorem 1.1. The asymptotic behavior of $\log(N_n)/n$ should come from the "direction" (y,h) in which open paths are more abundant, that is, in the "direction" (y,h) that maximizes $\alpha_p(y,h)$. The key step to recover a full limit from the limit of a random subsequence is the continuity Lemma 4.2: using the coupled zone, we prove in essence that two points close in $\mathbb{Z}^d \times \mathbb{N}^*$ and reached from $(0,0)$ by open paths should have similar number of open paths arriving to them.

In Sections 5 and 6, the same ideas are respectively used to prove Theorem 1.2 and Theorem 1.5. The arguments are however more intricate. That is why we chose to present an independent proof of Theorem 1.1 where to our opinion, each type of argument—regenerating time, coupling—appears in a simpler form.

Notation. For $n \geq 1$, $x \in \mathbb{Z}^d$ and any set $A \subset \mathbb{R}^d$, we denote by:

- N_n the number of open paths from $(0,0)$ to $\mathbb{Z}^d \times \{n\}$,
- $\overline{N}_n$ the number of open paths from $(0,0)$ to $\mathbb{Z}^d \times \{n\}$ that are the beginning of an infinite open path,
- $N_{x,n}$ the number of open paths from $(0,0)$ to (x,n),
- $N_{A,n}$ the number of open paths from $(0,0)$ to $(A \cap \mathbb{Z}^d) \times \{n\}$.

2. Preliminary results.

2.1. *Exponential estimates for supercritical oriented percolation.* We work with the oriented percolation model in dimension $d+1$, as defined in the Introduction. We set, for $n \in \mathbb{N}$ and $x \in \mathbb{Z}^d$,

$$\xi_n^x = \{y \in \mathbb{Z}^d : (x,0) \to (y,n)\}, \qquad H_n^x = \bigcup_{0 \leq k \leq n} \xi_k^x,$$

$$\xi_n^{\mathbb{Z}^d} = \bigcup_{x \in \mathbb{Z}^d} \xi_n^x, \qquad K_n'^x = \bigcap_{k \geq n} (\xi_k^x \Delta \xi_k^{\mathbb{Z}^d})^c,$$

$$\tau^x = \min\{n \in \mathbb{N} : \xi_n^x = \varnothing\}.$$

To simplify, we often write ξ_n, τ, H_n, K_n' instead of ξ_n^0, τ^0, H_n^0, $K_n'^0$.

For instance, τ is the length of the longest open path starting from the origin, and the percolation event is equal to $\{\tau = +\infty\}$. First, finite open paths cannot be too long (see Durrett [5]):

$$\forall p > \overrightarrow{p_c}^{\mathrm{alt}}(d+1)\ \exists A, B > 0\ \forall n \in \mathbb{N} \qquad \mathbb{P}_p(n \leq \tau < +\infty) \leq Ae^{-Bn}. \tag{5}$$

The set $K_n' \cap H_n$ is called the coupled zone, and will play a central role in our proofs, by allowing to compare numbers of open paths with close extremities. We will particularly use the situation in Figure 1.

As for the contact process, the growth of the sets $(H_n)_{n\geq 0}$ and the coupled zones $(K_n' \cap H_n)_{n \geq 0}$ is governed by a shape theorem and related large deviations inequalities.

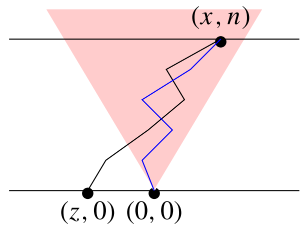

FIG. 1. *Coupled zone. If x is in the shaded coupled zone K'_n, and is reached by an open path starting from some $(z,0)\in\mathbb{Z}^d\times\{0\}$, then $(0,0)\to(x,n)$ (in blue).*

PROPOSITION 2.1 (Large deviations inequalities, Garet–Marchand [8]). *Fix $p>\overrightarrow{p_c}^{\mathrm{alt}}(d+1)$. For every $\varepsilon>0$, there exist $A,B>0$ such that*

$$\forall n\geq 1\qquad \overline{\mathbb{P}}_p\left(\begin{array}{c}B_{\mu_p}(0,(1-\varepsilon)n)\subset (K'_n\cap H_n)+[0,1]^d\\ \subset H_n+[0,1]^d\subset B_{\mu_p}(0,(1+\varepsilon)n)\end{array}\right)\geq 1-Ae^{-Bn}.$$

2.2. *Essential hitting times and associated translations*. We now introduce the analogues, in the discrete setting of oriented percolation, of the essential hitting times used by Garet–Marchand to study the supercritical contact process conditioned to survive in [7] and [8]; we give their main properties in Proposition 2.2.

For a given $x\in\mathbb{Z}^d$, the essential hitting time will be a random time $\sigma(x)$ such that:

- $\overline{\mathbb{P}}_p$ almost surely, $(0,0)\to(x,\sigma(x))\to\infty$,
- the associated random translation of vector $(x,\sigma(x))$ leaves $\overline{\mathbb{P}}_p$ invariant.

Thus, $\sigma(x)$ will be interpreted as a regenerating time of the oriented percolation conditioned to percolate.

Remember that $\overrightarrow{\mathbb{E}}^d$ has been defined in Section 1.4, and that we identify $\overrightarrow{\mathbb{E}}^{d+1}_{\mathrm{alt}}$ and $\overrightarrow{\mathbb{E}}^d\times\mathbb{N}^*$. We also define, for $(y,h)\in\mathbb{Z}^d\times\mathbb{N}$, the translation $\theta_{(y,h)}$ on Ω by

$$\theta_{(y,h)}\big((\omega_{(e,k)})_{e\in\overrightarrow{\mathbb{E}}^d,k\geq 1}\big)=(\omega_{(e+y,k+h)})_{e\in\overrightarrow{\mathbb{E}}^d,k\geq 1}.$$

At some point, we will also need to look backwards in time. So, as set of sites, we replace $\mathbb{Z}^d\times\mathbb{N}$ by $\mathbb{Z}^d\times\mathbb{Z}$, and we introduce the following reversed time translation defined on $\{0,1\}^{\mathbb{Z}^d\times\mathbb{Z}}$ by

$$\theta^{\downarrow}_{(y,h)}\big((\omega_{(e,k)})_{e\in\overrightarrow{\mathbb{E}}^d,k\in\mathbb{Z}}\big)=(\omega_{(e+y,h-k)})_{e\in\overrightarrow{\mathbb{E}}^d,k\in\mathbb{Z}}.\tag{6}$$

Fix $p>\overrightarrow{p_c}^{\mathrm{alt}}(d+1)$.

We now recall the construction of the essential hitting times and the associated translations introduced in [7] (see Figure 2). Fix $x\in\mathbb{Z}^d$. The essential hitting time $\sigma(x)$ is defined through a family of stopping times as follows: we set $u_0=v_0=0$ and we define recursively two increasing sequences of stopping times $(u_n)_{n\geq 0}$ and $(v_n)_{n\geq 0}$ with $u_0=v_0<u_1<v_1<u_2\cdots$ as follows:

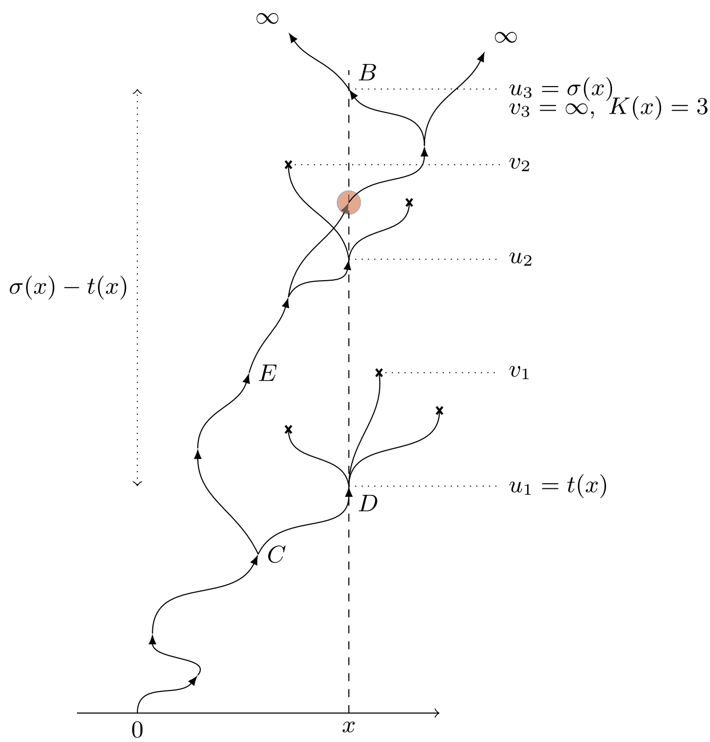

FIG. 2. *The essential hitting time. The shaded point is skipped by the process, as it occurs during the life time of* (x, u_2).

- Assume that v_k is defined. We set $u_{k+1} = \inf\{t > v_k : x \in \xi_t^0\}$.
  If $v_k < +\infty$, then u_{k+1} is the first time after v_k where x is once again infected; otherwise, $u_{k+1} = +\infty$.
- Assume that u_k is defined, with $k \geq 1$. We set $v_k = u_k + \tau \circ \theta_{(x,u_k)}$.
  If $u_k < +\infty$, the time $\tau \circ \theta_{(x,u_k)}$ is the length of the oriented percolation cluster starting from (x, u_k); otherwise, $v_k = +\infty$.

We then set

$$K(x) = \min\{n \geq 0 : v_n = +\infty \text{ or } u_{n+1} = +\infty\}.$$

This quantity represents the number of steps before the success of this process: either we stop because we have just found an infinite v_n, which corresponds to a time u_n when x is occupied and has infinite progeny, or we stop because we have just found an infinite u_{n+1}, which says that after v_n, site 0 is never infected anymore. It is not difficult to see that

$$\mathbb{P}_p\big(K(x) > n\big) \leq \mathbb{P}_p(\tau < +\infty)^n,$$

and thus $K(x)$ is $\mathbb{P}_p$ almost surely finite. We define the essential hitting time $\sigma(x)$ by setting

$$\sigma(x) = u_{K(x)} \in \mathbb{N} \cup \{+\infty\}.$$

By construction $(0,0) \to (x, \sigma(x)) \to +\infty$ on the event $\{\tau = +\infty\}$. Note however that $\sigma(x)$ is not necessarily the first positive time when x is occupied and has infinite progeny: for instance, such an event can occur between u_1 and v_1, being ignored by the recursive construction. It can be checked that conditionally to the event $\{\tau = \infty\}$, the process necessarily stops because of an infinite v_n, and thus $\sigma(x) < +\infty$. At the same time, we define the operator $\tilde{\theta}$ on Ω, which is a random translation by

$$\tilde{\theta}_x(\omega) = \begin{cases} \theta_{(x,\sigma(x))}\omega & \text{if } \sigma(x) < +\infty, \\ \omega & \text{otherwise.} \end{cases}$$

If $(x_1, \dots, x_m)$ is a sequence of points in $\mathbb{Z}^d$, we also introduce the shortened notation:

$$\tilde{\theta}_{x_1,\dots,x_m} = \tilde{\theta}_{x_m} \circ \tilde{\theta}_{x_m - 1} \circ \cdots \circ \tilde{\theta}_{x_1}.$$

For each integer $n \geq 1$, we denote by $\mathcal{F}_n$ the σ-field generated by the maps $(\omega \mapsto \omega_{(e,k)})_{e \in \overrightarrow{\mathbb{E}}^d, 1 \leq k \leq n}$. We denote by $\mathcal{F}$ the σ-field generated by the maps $(\omega \mapsto \omega_{(e,k)})_{e \in \overrightarrow{\mathbb{E}}^d, k \geq 1}$.

PROPOSITION 2.2. *Fix $p > \overrightarrow{p_c}^{\mathrm{alt}}(d+1)$ and $x_1, \dots, x_m \in \mathbb{Z}^d$.*

(a) *Suppose $A \in \mathcal{B}(\mathbb{R})$, $B \in \mathcal{F}$. Then for each $x \in \mathbb{Z}^d$,*

$$\overline{\mathbb{P}}_p(\sigma(x) \in A, \tilde{\theta}_x^{-1}(B)) = \overline{\mathbb{P}}_p(\sigma(x) \in A)\overline{\mathbb{P}}_p(B).$$

(b) *The probability measure $\overline{\mathbb{P}}_p$ is invariant under $\tilde{\theta}_{x_1,\dots,x_m}$.*

(c) *The random variables*

$$\sigma(x_1), \qquad \sigma(x_2) \circ \tilde{\theta}_{x_1}, \qquad \sigma(x_3) \circ \tilde{\theta}_{x_1,x_2}, \qquad \dots, \qquad \sigma(x_m) \circ \tilde{\theta}_{x_1,\dots,x_{m-1}}$$

are independent under $\overline{\mathbb{P}}_p$.

(d) *Suppose $t \leq m$, $A \in \mathcal{F}_t$, $B \in \mathcal{F}$*

$$\overline{\mathbb{P}}_p(A, \tilde{\theta}_{x_1,\dots,x_m}^{-1}(B)) = \overline{\mathbb{P}}_p(A)\overline{\mathbb{P}}_p(B).$$

(e) *For every $x \in \mathbb{Z}^d$, $\mu_p(x) = \lim_{n \to +\infty} \frac{\overline{\mathbb{E}}_p(\sigma(nx))}{n} = \inf_{n \geq 1} \frac{\overline{\mathbb{E}}_p(\sigma(nx))}{n}$.*

(f) *There exists $\alpha, \beta > 0$ such that*

$$\forall x \in \mathbb{Z}^d \qquad \overline{\mathbb{E}}_p(\exp(\alpha\sigma(x))) \leq \exp(\beta(\|x\|_1 \vee 1)).$$

PROOF. To prove (a)–(d), it is sufficient to mimic the proofs of Lemma 8 and Corollary 9 in [7]. The convergence has been proved for the contact process in [7], Theorem 22. The existence of exponential moments for σ has been proved for the contact process in [8], Theorem 2. $\square$

3. Directional limits along subsequences of regenerating times. The essential hitting times have good regenerating properties, but by construction [see Proposition 2.2(e)], the vector $(x, \sigma(x))$ lies close to the border of the percolation cone $\{(y, \mu_p(y)) : y \in \mathbb{R}^d\}$. We now need to build new regenerating points such that the set of directions of these points is dense inside the percolation cone.

We define, for $(y, h) \in \mathbb{Z}^d \times \mathbb{N}^*$, a new regenerating time $s(y, h)$ by setting

$$s(y, h) = \sigma(y) + \sum_{i=1}^{h} \sigma(0) \circ \tilde{\theta}^{i-1}(0) \circ \tilde{\theta}(y),$$

and the associated translation:

$$\hat{\theta}_{(y,h)}(\omega) = \begin{cases} \theta_{(y,s(y,h))}\omega & \text{if } s(y, h) < +\infty, \\ \omega & \text{otherwise.} \end{cases}$$

Note that on $\{\tau = +\infty\}$, $(0, 0) \to (y, s(y, h)) \to +\infty$ and $\hat{\theta}_{(y,h)} = \tilde{\theta}_{y,0,\dots,0}$ (with h zeros). We can easily deduce from Proposition 2.2 the following properties of the time $s(y, h)$ under $\overline{\mathbb{P}}_p$.

LEMMA 3.1. *Fix* $p > \overrightarrow{p_c}^{\text{alt}}(d + 1)$, *and* $(y, h) \in \mathbb{Z}^d \times \mathbb{N}^*$:

(a) *The probability measure* $\overline{\mathbb{P}}_p$ *is invariant under the translation* $\hat{\theta}_{(y,h)}$.

(b) *The random variables* $(s(y, h) \circ (\hat{\theta}_{(y,h)})^j)_{j\geq 0}$ *are independent and identically distributed under* $\overline{\mathbb{P}}_p$.

(c) *The measure-preserving dynamical system* $(\Omega, \mathcal{F}, \overline{\mathbb{P}}_p, \hat{\theta}_{(y,h)})$ *is mixing.*

(d) *There exists* $\alpha, \beta > 0$ *such that*

$$\forall y \in \mathbb{Z}^d\ \forall h \in \mathbb{N}^* \qquad \overline{\mathbb{E}}_p\big(\exp(\alpha s(y, h))\big) \leq \exp\big(\beta\big((\|y\|_1 \vee 1) + h\big)\big).$$

We fix $(y, h) \in \mathbb{Z}^d \times \mathbb{N}^*$. We work under $\overline{\mathbb{P}}_p$, and we set, for every $n \geq 1$,

$$S_n = S_n(y, h) = \sum_{k=0}^{n-1} s(y, h) \circ \hat{\theta}^k_{(y,h)}. \tag{7}$$

The points $(ny, S_n(y, h))_{n\geq 1}$ are the sequence of regenerating points associated to (y, h) along which we are going to look for subadditivity properties. As, under $\overline{\mathbb{P}}_p$, the random variables $(s(y, h) \circ \hat{\theta}^j_{(y,h)})_{j\geq 0}$ are independent and identically distributed with finite first moment (see Lemma 3.1), the strong law of large numbers ensures that $\overline{\mathbb{P}}_p$-almost surely

$$\lim_{n\to+\infty} \frac{S_n(y, h)}{n} = \overline{\mathbb{E}}_p\big(s(y, h)\big) = \overline{\mathbb{E}}_p\big(\sigma(y)\big) + h\overline{\mathbb{E}}_p\big(\sigma(0)\big). \tag{8}$$

For large n, the point $(ny, S_n(y, h))$ is not far from the line $\mathbb{R}(y, \overline{\mathbb{E}}_p(s(y, h)))$.

To obtain directional limits along subsequences, we first apply Kingman's subadditive ergodic theorem to $f_n = -\log N_{(ny,S_n(y,h))}$ for a fixed (y, h) in $\mathbb{Z}^d \times \mathbb{N}^*$.

LEMMA 3.2. *Fix* $p > \overrightarrow{p_c}^{\text{alt}}(d+1)$ *and* $(y,h) \in \mathbb{Z}^d \times \mathbb{N}^*$. *There exists* $\alpha_p(y,h) \in (0, \log(2d+1)]$ *such that* $\overline{\mathbb{P}}_p$*-almost surely and in* $L^1(\overline{\mathbb{P}}_p)$,

$$\lim_{n\to+\infty} \frac{1}{S_n(y,h)} \log N_{(ny, S_n(y,h))} = \alpha_p(y,h).$$

PROOF. Fix $(y,h) \in \mathbb{Z}^d \times \mathbb{N}^*$. To avoid heavy notation, we omit all the dependence in (y,h). For instance $S_n = S_n(y,h)$ and $\hat{\theta} = \hat{\theta}_{(y,h)}$. Note that by definition, $\overline{\mathbb{P}}_p$-almost surely, for every $n \geq 1$, $(0,0) \to (ny, S_n) \to +\infty$ and consequently, $N_{(ny,S_n)} \geq 1$. For $n \geq 1$, we set

$$f_n = -\log N_{(ny,S_n)}.$$

Let $n, m \geq 1$. Note that $S_n + S_m \circ \hat{\theta}^n_{(y,h)} = S_{n+m}$. As $N_{(my,S_m)} \circ \hat{\theta}^n$ counts the number of open paths from (ny, S_n) to $((n+m)y, S_n + S_m \circ \hat{\theta}^n)$, concatenation of paths ensures that $N_{(ny,S_n)} \times N_{(my,S_m)} \circ \hat{\theta}^n \leq N_{((n+m)y, S_{n+m})}$ which implies that

$$\forall n, m \geq 1 \qquad f_{n+m} \leq f_n + f_m \circ \hat{\theta}^n.$$

As $1 \leq N_{(ny,S_n)} \leq (2d+1)^{S_n}$,

$$-S_n \log(2d+1) \leq f_n \leq 0.$$

The integrability of s thus implies the integrability of every f_n. So we can apply Kingman's subadditive ergodic theorem. By property (c) in Lemma 3.1, the dynamical system $(\Omega, \mathcal{F}, \overline{\mathbb{P}}, \hat{\theta})$ is mixing. Particularly, it is ergodic, so the limit is deterministic: if we define

$$-\alpha'_p(y,h) = \inf_{n\geq 1} \frac{\overline{\mathbb{E}}_p(f_n)}{n},$$

we have $\overline{\mathbb{P}}_p$-almost surely and in $L^1(\overline{\mathbb{P}}_p)$: $\lim_{n\to+\infty} \frac{f_n}{n} = -\alpha'_p(y,h)$.

The limit of the lemma follows then directly from (8) by setting

$$\alpha_p(y,h) = \frac{\alpha'_p(y,h)}{\overline{\mathbb{E}}_p s(y,h)}.$$

Finally, $\alpha'_p(y,h) \geq \overline{\mathbb{E}}_p(-f_1) = \overline{\mathbb{E}}_p(\log N_{(y,S_1)})$. Since $N_{(y,S_1)} \geq 1$ $\overline{\mathbb{P}}_p$-a.s. and $N_{(y,S_1)} \geq 2$ with positive probability, it follows that $\alpha'_p(y,h) > 0$, and consequently $\alpha_p(y,h) > 0$.

As $N_{(ny,S_n)} \leq (2d+1)^{S_n}$, we see that $\alpha_p(y,h) \leq \log(2d+1)$ and that the convergence also holds in $L^1(\overline{\mathbb{P}}_p)$. $\square$

We can now introduce a natural candidate for the limit in Theorem 1.1:

$$\alpha_p = \sup\{\alpha_p(y,h) : (y,h) \in \mathbb{Z}^d \times \mathbb{N}^*\} < +\infty. \tag{9}$$

Indeed, at the logarithmic scale we are working with, we can expect that the dominant contribution to the number N_n of open paths to level n will be due to the number $N_{nz,n}$ of open paths to level n in the direction $(z, 1)$ that optimizes the previous limit. Note, however, that in our construction, (y, h) has no real geometrical signification, but it is just a useful encoding: as said before, the asymptotic direction of the regenerating point $(ny, S_n(y, h))$ in $\mathbb{Z}^d \times \mathbb{N}$ is

$$\left(\frac{y}{\overline{\mathbb{E}}_p(s(y,h))}, 1\right).$$

To skip from the subsequences to the full limit, we approximate $B_{\mu_p}(0, 1)$ with a denumerable set of points: let

$$D_p = \left\{\frac{y}{\overline{\mathbb{E}}_p(s(y,h))} : y \in \mathbb{Z}^d, h \in \mathbb{N}^*\right\}. \tag{10}$$

LEMMA 3.3. *For every $p > \overrightarrow{p_c}^{\mathrm{alt}}(d+1)$, $B_{\mu_p}(0, 1) \subset \overline{D_p}$.*

PROOF. Note that the set $\{z/l : (z, l) \in \mathbb{Z}^d \times \mathbb{N}^* \text{ and } \mu_p(z) < l\}$ is dense in $B_{\mu_p}(0, 1)$. Thus, fix $(z, l) \in \mathbb{Z}^d \times \mathbb{N}^*$ such that $\mu_p(z) < l$ and consider

$$(y_n, h_n) = \left(nz, \left\lceil \frac{n(l - \mu_p(z))}{\overline{\mathbb{E}}_p(\sigma(0))} \right\rceil\right) \in \mathbb{Z}^d \times \mathbb{N}^*.$$

Then

$$\frac{y_n}{\overline{\mathbb{E}}_p(s(y_n, h_n))} = \frac{nz}{\overline{\mathbb{E}}_p(\sigma(y_n)) + h_n \overline{\mathbb{E}}_p(\sigma(0))} \to \frac{z}{l}$$

as n goes to $+\infty$. $\square$

Finally, for $(y, h) \in \mathbb{Z}^d \times \mathbb{N}^*$, we denote by

$$\forall n \in \mathbb{N} \qquad \varphi(n) = \varphi_{(y,h)}(n) = \inf\{k \in \mathbb{N} : S_k(y, h) \geq n\}. \tag{11}$$

Thus, for large n, $(\varphi(n) \cdot y, S_{\varphi(n)})$ is the first point among the sequence of regenerating points associated to (y, h) to be above level n. By the renewal theory, $\overline{\mathbb{P}}_p$ almost surely,

$$\lim_{n\to+\infty} \frac{\varphi_{(y,h)}(n)}{n} = \frac{1}{\overline{\mathbb{E}}_p(s(y,h))} \quad \text{and} \quad \lim_{n\to+\infty} \frac{S_{\varphi_{(y,h)}(n)}(y, h)}{n} = 1. \tag{12}$$

It is also not too far above level n.

LEMMA 3.4. *For every $(y, h) \in \mathbb{Z}^d \times \mathbb{N}^*$, there exist positive constants A, B such that*

$$\forall n \in \mathbb{N} \qquad \overline{\mathbb{P}}(S_{\varphi_{(y,h)}(n)} - n \geq n) \leq A \exp(-Bn).$$

PROOF. As we work in discrete time, $\varphi(n) \le n$. So

$$\overline{\mathbb{P}}_p(S_{\varphi(n)} - n \ge n) \le \overline{\mathbb{P}}_p\big(\exists k \le n : s(y,h) \circ \hat{\theta}^k_{(y,h)} \ge n\big) \le n\overline{\mathbb{P}}_p\big(s(y,h) \ge n\big).$$

As $s(y,h)$ admits exponential moments thanks to Lemma 3.1, we can conclude with the Markov inequality. □

4. Proof of Theorem 1.1. Fix $p > \overrightarrow{p_c}^{\mathrm{alt}}(d+1)$. The proof of the almost sure convergence in Theorem 1.1 is a direct consequence of the forthcoming Lemmas 4.1, 4.2 and 4.3. The L^1 convergence follows from the remark that $\frac{1}{n}\log N_n \le \log(2d+1)$. Remember that α_p is defined in (9).

LEMMA 4.1. *$\overline{\mathbb{P}}_p$-almost surely, $\underline{\lim}_{n\to+\infty} \frac{1}{n}\log \overline{N}_n \ge \alpha_p$.*

PROOF. Take $(y,h) \in \mathbb{Z}^d \times \mathbb{N}^*$. Note that $(\overline{N}_n)_{n\ge1}$ is nondecreasing, and considering the increasing sequence $S_k = S_k(y,h)$, we see that, $\overline{\mathbb{P}}_p$ almost surely, for every integer n such that $S_k \le n \le S_{k+1}$,

$$\frac{1}{n}\log \overline{N}_n \ge \frac{1}{S_{k+1}}\log \overline{N}_{S_k} \ge \frac{S_k}{S_{k+1}} \frac{\log \overline{N}_{(ky,S_k)}}{S_k}.$$

With (8) and Lemma 3.2, we deduce that $\overline{\mathbb{P}}_p$ almost surely,

$$\underline{\lim}_{n\to+\infty} \frac{1}{n}\log \overline{N}_n \ge \alpha_p(y,h),$$

which completes the proof. □

LEMMA 4.2. *$\overline{\mathbb{P}}_p$-almost surely, $\overline{\lim}_{n\to+\infty} \frac{1}{n}\log \overline{N}_n \le \alpha_p$.*

PROOF. Fix $\varepsilon > 0$ and $\eta \in (0,1)$. We first approximate $B_{\mu_p}(0,1)$ with a finite number of points: with Lemma 3.3, we can find a finite set $F \subset \mathbb{Z}^d \times \mathbb{N}^*$ such that

$$B_{\mu_p}(0,1+\varepsilon) \subset \bigcup_{(y,h)\in F} B_{\mu_p}\left(\frac{(1+\varepsilon)y}{\overline{\mathbb{E}}_p(s(y,h))}, (1-\eta)\varepsilon/2\right).$$

Then, for n large, we will control the number $\overline{N}_n$ using directional convergence along these directions. We define $M_n(y,h)$ as the first point in the sequence $(ky, S_{(y,h)}(k))_{k\ge1}$ of regenerating points associated to (y,h) to be above level $n(1+\varepsilon)$. Using the notation introduced in (11), we set

$$\begin{aligned}
\forall (y,h) \in F \qquad k_n &= k_n(y,h) = \varphi_{(y,h)}\big(n(1+\varepsilon)\big),\\
Z_n &= Z_n(y,h) = k_n \cdot y \in \mathbb{Z}^d,\\
V_n &= V_n(y,h) = S_{k_n}(y,h) \in \mathbb{N},\\
M_n &= M_n(y,h) = (Z_n, V_n).
\end{aligned}$$

For a given $(y,h)\in F$, the law of large numbers (12) says that

$$k_n(y,h)\sim\frac{n(1+\varepsilon)}{\overline{\mathbb{E}}_p(s(y,h))}\quad\text{and}\quad V_n(y,h)\sim n(1+\varepsilon). \tag{13}$$

So $\overline{\mathbb{P}}_p$ almost surely, for all n large enough,

$$\forall(y,h)\in F\qquad B_{\mu_p}\left(\frac{(1+\varepsilon)ny}{\overline{\mathbb{E}}_p(s(y,h))},(1-\eta)\varepsilon n/2\right)\subset B_{\mu_p}\big(Z_n(y,h),(1-\eta)\varepsilon n\big).$$

It follows then from the shape theorem (1) that, $\overline{\mathbb{P}}_p$ almost surely, for all n large enough,

$$\xi_n\subset B_{\mu_p}\big(0,(1+\varepsilon)n\big)\subset\bigcup_{(y,h)\in F}B_{\mu_p}\big(Z_n(y,h),(1-\eta)\varepsilon n\big). \tag{14}$$

The strategy is to prove that for n large enough, for each $x\in B_{\mu_p}(0,n(1+\varepsilon))$, the n first steps of an open path that goes from $(0,0)$ to (x,n) and then to infinity are also the n first steps of an open path which contributes to $N_{M_n(y,h)}$ for any $(y,h)\in F$ such that $x\in B_{\mu_p}(Z_n(y,h),(1-\eta)\varepsilon n)$. Concretely, we will prove that on a good event G_n with large probability,

$$\overline{N}_n\le\sum_{(y,h)\in F}\overline{N}_{M_n(y,h)}. \tag{15}$$

To do so, we will use the coupled zones, backwards in time, issued from the $M_n(y,h)$'s for $(y,h)\in F$. Define the good events

$$G_n=\bigcap_{M\in\{-2n,\dots,2n\}^d\times\{0,\dots,2n\}}\left\{\begin{array}{c}\tau<n(1+\varepsilon)\\ \text{or } K'_{n\varepsilon}\supset B_{\mu_p}\big(0,(1-\eta)\varepsilon n\big)\cap\mathbb{Z}^d\end{array}\right\}\circ\theta_M^{\downarrow}.$$

We recall that $\theta_M^{\downarrow}$ was introduced in (6) and corresponds to looking at the process backwards in time. Since $\theta_M^{\downarrow}$ preserves $\mathbb{P}_p$, we easily deduce from (5), Proposition 2.1 and a Borel–Cantelli argument that $\overline{\mathbb{P}}_p$ almost surely, G_n holds for every n large enough.

Now take n large enough such that (14) holds, G_n holds, together with $V_n(y,h)\le 2n$ for each $(y,h)\in F$, which is possible thanks to (13). Thus, ξ_n is contained in the union of the "reversed" coupled zones issued from the $M_n(y,h)$'s. See Figure 3.

Fix now $x\in\xi_n$ such that $(x,n)\to\infty$. As (14) holds, choose $(y,h)\in F$ such that $x\in B_{\mu_p}(Z_n(y,h),(1-\eta)\varepsilon n)$. Let us prove that there exists an open path from (x,n) to $M_n(y,h)$. See Figure 4. Since $(0,0)\to M_n$ and $V_n\ge n(1+\varepsilon)$, we know that $\tau\circ\theta_{M_n}^{\downarrow}\ge n(1+\varepsilon)$. Since $M_n\in\{-2n,\dots,2n\}^d\times\{0,\dots,2n\}$, $\mu_p(x-Z_n)\le(1-\eta)\varepsilon n$ and G_n holds, we have $x-Z_n\in K'_{n\varepsilon}\circ\theta_{M_n}^{\downarrow}$. Note that $V_n(y,h)\ge n(1+\varepsilon)$, so $V_n(y,h)-n\ge\varepsilon n$. Note also that $(x,n)\to\infty$ implies that $x-Z_n\in\xi^{\mathbb{Z}^d}_{V_n(y,h)-n}\circ\theta_{M_n}^{\downarrow}$. By definition of the coupled zone, we have

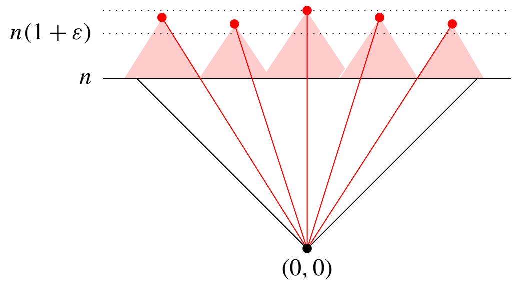

FIG. 3. *The 5 inner lines give the directions (y,h) in the finite set F; their final points are the associated $M_n(y,h)$. The set ξ_n is contained in the union of the "reversed" coupled zones issued from the $M_n(y,h)$'s.*

$x - Z_n \in \xi^0_{V_n(y,h)-n} \circ \theta^{\downarrow}_{M_n}$. Going back to the initial orientation, it means that $(x,n) \to M_n$. So, if γ is a path from $(0,0)$ to (x,n), it is clear that γ is the restriction of a path that goes from $(0,0)$ to M_n, and then to infinity. This proves (15).

Finally, we use the directional limits given by Lemma 3.2: $\overline{\mathbb{P}}_p$ almost surely,

$$\forall (y,h) \in F \qquad \lim_{n\to+\infty} \frac{1}{V_n(y,h)} \log \overline{N}_{M_n(y,h)} = \alpha_p(y,h).$$

As $V_n(y,h) \sim n(1+\varepsilon)$, it is a consequence of the shape theorem (1) that $\overline{\mathbb{P}}_p$-a.s., for all n large enough,

$$\forall (y,h) \in F \qquad \frac{1}{n(1+\varepsilon)} \log \overline{N}_{M_n(y,h)} \le \alpha_p(y,h) + \varepsilon \le \alpha_p + \varepsilon.$$

Consequently, for n large enough, we have $\overline{\mathbb{P}}_p$ almost surely,

$$\overline{N}_n \le \sum_{(y,h)\in F} \overline{N}_{M_n(y,h)} \le |F| \exp\big((\alpha_p+\varepsilon)n(1+\varepsilon)\big),$$

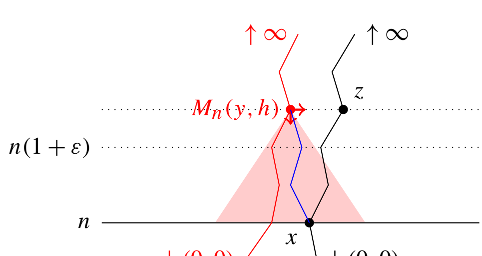

FIG. 4. *At the right, $(x,n) \to +\infty$; but (x,n) is in the "reversed" coupled zone (shaded) issued from $M_n(y,h)$: thus $(x,n) \to M_n(y,h)$ (in the middle).*

and so

$$\overline{\lim_{n\to+\infty}} \frac{1}{n} \log(\overline{N}_n) \le (1+\varepsilon)(\alpha_p+\varepsilon).$$

We complete the proof by letting ε go to 0. $\square$

Finally, we prove that working with open paths or with open paths that are the beginning of an infinite open path is essentially the same.

LEMMA 4.3. *$\overline{\mathbb{P}}_p$-almost surely,*

$$\overline{\lim_{n\to+\infty}} \frac{\log N_n}{n} = \overline{\lim_{n\to+\infty}} \frac{\log \overline{N}_n}{n} \quad \textit{and} \quad \underline{\lim_{n\to+\infty}} \frac{\log N_n}{n} = \underline{\lim_{n\to+\infty}} \frac{\log \overline{N}_n}{n}.$$

PROOF. Fix $0<\varepsilon<1$ and define, for $n \ge 1$, the following event:

$$E_n = \bigcap_{\|z\|_1 \le n} \{\tau < \varepsilon n \text{ or } \tau = +\infty\} \circ \theta_{(z,\lfloor n(1-\varepsilon)\rfloor)}.$$

Assume that E_n occurs.

Consider a path $\gamma = (\gamma_i, i)_{0\le i\le n}$ from $(0,0)$ to $\mathbb{Z}^d \times \{n\}$ and set $z = \gamma_{\lfloor n(1-\varepsilon)\rfloor}$: as $\tau \circ \theta_{(z,\lfloor n(1-\varepsilon)\rfloor)} \ge \varepsilon n$, the event E_n implies that $\tau \circ \theta_{(z,\lfloor n(1-\varepsilon)\rfloor)} = +\infty$. So $(\gamma_i, i)_{0\le i\le \lfloor n(1-\varepsilon)\rfloor}$ contributes to $\overline{N}_{\lfloor n(1-\varepsilon)\rfloor}$ and thus, on E_n,

$$N_n \le (2d+1)^{\varepsilon n+1} \overline{N}_{\lfloor n(1-\varepsilon)\rfloor},$$

so

$$\begin{aligned}\frac{1}{n}\log N_n &\le \left(\varepsilon + \frac{1}{n}\right)\log(2d+1) + \frac{1}{n}\log \overline{N}_{\lfloor n(1-\varepsilon)\rfloor} \\ &\le \left(\varepsilon + \frac{1}{n}\right)\log(2d+1) + \frac{1}{\lfloor n(1-\varepsilon)\rfloor}\log \overline{N}_{\lfloor n(1-\varepsilon)\rfloor}.\end{aligned}$$

The exponential estimate (5) ensures that

$$\forall n \ge 1 \qquad \mathbb{P}_p(E_n^c) \le C_d A n^d \exp(-B\varepsilon n) \le A' \exp(-B'n).$$

With the Borel–Cantelli lemma, this leads to

$$\overline{\lim_{n\to+\infty}} \frac{1}{n}\log N_n \le \varepsilon \log(2d+1) + \overline{\lim_{n\to+\infty}} \frac{1}{n} \log \overline{N}_n.$$

By taking ε to 0, we obtain

$$\overline{\lim_{n\to+\infty}} \frac{\log N_n}{n} \le \overline{\lim_{n\to+\infty}} \frac{\log \overline{N}_n}{n}.$$

The proof for the inequality with $\underline{\lim}$ instead of $\overline{\lim}$ is identical. Since $\overline{N}_n \le N_n$, the reversed inequalities are obvious. $\square$

5. Proof of Theorem 1.2.

5.1. *Construction and continuity of $\tilde{\alpha}_p$.* Recall that D_p was defined in (10). Our strategy is to prove that the identity

$$\tilde{\alpha}_p\left(\frac{y}{\overline{\mathbb{E}}_p(s(y,h))}\right) = \alpha_p(y,h),$$

defines a map on D_p that is uniformly continuous on every compact subset of $D_p \cap \mathring{B}_\mu(0,1)$. We first refine the argument of Lemma 4.2 using the coupled zone.

LEMMA 5.1. *Let $\beta \in (0,1)$. There exists $\alpha > 0$ such that the following holds. For every $\varepsilon > 0$, for every $\hat{x}_1, \hat{x}_2 \in B_{\mu_p}(0, 1-\beta)$, if*

$$\mu_p(\hat{x}_1 - \hat{x}_2) \le \alpha\varepsilon,$$

then for any sequences of points $(M_n^1 = (Z_n^1, V_n^1))_n$ and $(M_n^2 = (Z_n^2, V_n^2))_n$ in $\mathbb{Z}^d \times \mathbb{N}^*$, for any $C > 0$ such that, $\overline{\mathbb{P}}_p$ almost surely,

$$\frac{Z_n^1}{V_n^1} \to \hat{x}_1 \quad \textit{and} \quad \frac{V_n^1}{n} \to C(1+\varepsilon),$$

$$\frac{Z_n^2}{V_n^2} \to \hat{x}_2 \quad \textit{and} \quad \frac{V_n^2}{n} \to C,$$

we have the following property: $\overline{\mathbb{P}}_p$ almost surely, for every n large enough, if $(0,0) \to (Z_n^1, V_n^1)$ and $(0,0) \to (Z_n^2, V_n^2) \to \infty$, then $\overline{N}_{(Z_n^2, V_n^2)} \le N_{(Z_n^1, V_n^1)}$.

PROOF. Fix small $\alpha, \eta > 0$ and a large integer $K \ge 3$ such that

$$\alpha + (1-\beta) < \frac{K-2}{K}(1-\eta).$$

Fix $\varepsilon > 0$. Set $\varepsilon' = \varepsilon/K$.

$$\begin{aligned}\mu_p\left(\frac{Z_n^2}{Cn} - \frac{Z_n^1}{Cn}\right) \le{}& \mu_p\left(\frac{Z_n^2}{V_n^2}\right)\left|\frac{V_n^2}{Cn} - 1\right| + \mu_p\left(\frac{Z_n^2}{V_n^2} - \hat{x}_2\right) + \mu_p(\hat{x}_2 - \hat{x}_1) \\ &+ \mu_p\left(\hat{x}_1 - \frac{Z_n^1}{V_n^1}\right) + \mu_p\left(\frac{Z_n^1}{V_n^1}\right)\left|\frac{V_n^1}{Cn} - 1\right|.\end{aligned}$$

So $\overline{\mathbb{P}}$ almost surely,

$$\overline{\lim_{n\to+\infty}} \mu_p\left(\frac{Z_n^2}{Cn} - \frac{Z_n^1}{Cn}\right) \le (\alpha + 1 - \beta)\varepsilon < \frac{K-2}{K}(1-\eta)\varepsilon,$$

so $\overline{\mathbb{P}}$ almost surely, for every n large enough,

$$\mu_p\left(\frac{Z_n^2}{Cn} - \frac{Z_n^1}{Cn}\right) \le \frac{K-2}{K}(1-\eta)\varepsilon = (K-2)(1-\eta)\varepsilon'. \tag{16}$$

By the convergences for the V_n^i/n, we know that $\overline{\mathbb{P}}_p$ almost surely, for every n large enough,

$$|V_n^1 - Cn(1+\varepsilon)| \le Cn\varepsilon' \quad \text{and} \quad |V_n^2 - Cn| \le Cn\varepsilon'. \tag{17}$$

Define

$$G_n = \left\{ \begin{array}{l} \forall x \in [-Cn(1+2\varepsilon), Cn(1+2\varepsilon)]^d \\ \forall k \in [Cn(1+\varepsilon-\varepsilon'), Cn(1+\varepsilon+\varepsilon')] \\ (\tau \circ \theta^{\downarrow}_{(x,k)} \ge \varepsilon' Cn) \\ \Rightarrow \forall m \ge \varepsilon' Cn B_{\mu_p}(0, x, (1-\eta)m) \subset \tilde{K}'_m \circ \theta^{\downarrow}_{(x,k)} \end{array} \right\}.$$

With the large deviations for the coupled zone given in Proposition 2.1, there exist $A, B > 0$ such that

$$\forall n \text{ large enough} \qquad \overline{\mathbb{P}}_p(G_n^c) \le A \exp(-Bn).$$

Thus, the Borel–Cantelli lemma ensures that $\overline{\mathbb{P}}_p(\underline{\lim}\, G_n) = 1$.

Assume then that $\tau = +\infty$. $\overline{\mathbb{P}}_p$ almost surely, for every n large enough, we know that (16), (17) and G_n occur. Assume that, for one of these large enough n, $(0,0) \to (Z_n^1, V_n^1)$ and $(0,0) \to (Z_n^2, V_n^2) \to \infty$. Note that

$$V_n^1 - V_n^2 \ge Cn(1+\varepsilon-\varepsilon') - Cn(1+\varepsilon') \ge Cn(K-2)\varepsilon'.$$

So, on the event G_n, as $(0,0) \to (Z_n^1, V_n^1)$, we see that $\tau \circ \theta^{\downarrow}_{M_n^1} \ge \varepsilon' Cn$, so

$$K_{V_n^1 - V_n^2} \circ \theta^{\downarrow}_{M_n^1} \supset B_{\mu_p}(Z_n^1, (1-\eta)C(K-2)n\varepsilon').$$

So, with (16), we see that $Z_n^2 \in K_{V_n^1 - V_n^2} \circ \theta^{\downarrow}_{M_n^1}$. As $(Z_n^2, V_n^2) \to \infty$, then $(Z_n^2, V_n^2) \to (Z_n^1, V_n^1)$, which gives an injection from the set of open paths from $(0,0)$ to (Z_n^2, V_n^2) into the set of open paths from $(0,0)$ to (Z_n^1, V_n^1). $\square$

For $(y,h) \in \mathbb{Z}^d \times \mathbb{N}^*$, we define $M_n(y,h)$ as the first point in the sequence $(ky, S_{(y,h)}(k))$ of regenerating points associated to (y,h) to be above level n [see Definition (11)]:

$$\begin{aligned} k_n &= k_n(y,h) = \varphi_{(y,h)}(n), \\ Z_n &= Z_n(y,h) = k_n \cdot y \in \mathbb{Z}^d \quad \text{and} \quad V_n = V_n(y,h) = S_{k_n}(y,h) \in \mathbb{N}, \\ M_n &= M_n(y,h) = (Z_n, V_n). \end{aligned}$$

The law of large numbers (12) says that $\overline{\mathbb{P}}_p$ almost surely,

$$\frac{Z_n(y,h)}{n} = \frac{k_n(y,h) \cdot y}{n} \sim \frac{y}{\overline{\mathbb{E}}_p(s(y,h))} \quad \text{and} \quad \frac{V_n}{n} \sim 1.$$

The next lemma is a first step toward continuity.

LEMMA 5.2. *Let $\beta \in (0,1)$. There exists $\alpha > 0$ such that, for every $\varepsilon > 0$, for every $(y_1, h_1), (y_2, h_2) \in \mathbb{Z}^d \times \mathbb{N}^*$ such that*

$$\mu_p\left(\frac{y_1}{\overline{\mathbb{E}}_p(s(y_1,h_1))}\right) \le 1-\beta \quad \textit{and} \quad \mu_p\left(\frac{y_2}{\overline{\mathbb{E}}_p(s(y_2,h_2))}\right) \le 1-\beta,$$

if $\mu_p(\frac{y_1}{\overline{\mathbb{E}}_p(s(y_1,h_1))} - \frac{y_2}{\overline{\mathbb{E}}_p(s(y_2,h_2))}) \le \alpha\varepsilon$, then $|\alpha_p(y_1,h_1) - \alpha_p(y_2,h_2)| \le \varepsilon$.

PROOF. For $n \ge 1$, take $(Z_n^1, V_n^1) = M_{n(1+\varepsilon)}(y_1, h_1)$ and $(Z_n^2, V_n^2) = M_n(y_2, h_2)$. With the previous lemma, we obtain

$$\overline{N}_{M_n(y_2,h_2)} \le \overline{N}_{M_{n(1+\varepsilon)}(y_1,h_1)},$$

$$\frac{1}{V_n(y_2,h_2)} \log \overline{N}_{M_n(y_2,h_2)} \le \frac{H_{n(1+\varepsilon)}(y_1,h_1)}{V_n(y_2,h_2)} \frac{\log \overline{N}_{M_{n(1+\varepsilon)}(y_1,h_1)}}{H_{n(1+\varepsilon)}(y_1,h_1)},$$

$$\alpha_p(y_2,h_2) \le (1+\varepsilon)\alpha_p(y_1,h_1).$$

By symmetry, we obtain $|\alpha_p(y_2,h_2) - \alpha_p(y_1,h_1)| \le \varepsilon \log(2d+1)$. $\square$

Definition of $\tilde{\alpha}_p$. We define the following equivalence relation of the points in $\mathbb{Z}^d \times \mathbb{N}^*$:

$$(y_1,h_1) \sim (y_2,h_2) \quad \Leftrightarrow \quad \frac{y_1}{\overline{\mathbb{E}}_p(s(y_1,h_1))} = \frac{y_2}{\overline{\mathbb{E}}_p(s(y_2,h_2))}.$$

Lemma 5.2 ensures that if $(y_1,h_1) \sim (y_2,h_2)$, then $\alpha_p(y_1,h_1) = \alpha_p(y_2,h_2)$. We can thus define on the quotient set of directions D_p, defined in (10), the following directional limit:

$$\tilde{\alpha}_p\left(\frac{y}{\overline{\mathbb{E}}_p(s(y,h))}\right) = \alpha_p(y,h).$$

Lemma 5.2 ensures that the application $\tilde{\alpha}_p$ is uniformly continuous on each $D_p \cap B_{\mu_p}(0,(1-\beta))$. Note that the α given by Lemma 5.2 gives an upper bound for its modulus of continuity. By Lemma 3.3, $D_p \cap B_{\mu_p}(0,(1-\beta))$ is dense in the compact set $B_{\mu_p}(0,(1-\beta))$, so we can extend $\tilde{\alpha}_p$ to any $B_{\mu_p}(0,(1-\beta))$, and then to $\mathring{B}_{\mu_p}(0,1)$.

5.2. *Inequalities for the directional convergence.* We now prove refined versions of Lemmas 4.1 and 4.2.

LEMMA 5.3. *For every subset A of $\mathring{B}_{\mu_p}(0,1)$ such that $\mathring{A} \neq \varnothing$, $\overline{\mathbb{P}}_p$-almost surely,*

$$\varliminf_{n\to+\infty} \frac{1}{n} \log \overline{N}_{nA,n} \ge \sup_{x\in\mathring{A}} \tilde{\alpha}_p(x).$$

PROOF. Let $L \in \mathbb{R}$ with $L < \sup_{x \in \mathring{A}} \tilde{\alpha}_p(x)$ and take $x \in \mathring{A}$ with $\tilde{\alpha}_p(x) > L$. Fix $\varepsilon \in (0,1)$ such that $B(x, 8\varepsilon) \subset A$. By the continuity of $\tilde{\alpha}_p$, if we take ε small enough, we can also ensure that $\tilde{\alpha}_p > L$ on $B(x, 8\varepsilon)$. With Lemma 3.3, we can find $(y,h) \in \mathbb{Z}^d \times \mathbb{N}$ such that $\hat{y} = \frac{y}{\overline{\mathbb{E}}_p(s(y,h))} \in B(x, 4\varepsilon)$. We define $M_n(y,h)$ as the first point in the sequence $(ky, S_{(y,h)}(k))_{k \geq 1}$ of regenerating points associated to (y,h) to be above level $n(1-\varepsilon)$. Using the notation introduced in (11), we set

$$\begin{aligned} \forall (y,h) \in F \qquad k_n &= k_n(y,h) = \varphi_{(y,h)}\bigl(n(1-\varepsilon)\bigr),\\ Z_n &= Z_n(y,h) = k_n \cdot y \in \mathbb{Z}^d,\\ V_n &= V_n(y,h) = S_{k_n}(y,h) \in \mathbb{N},\\ M_n &= M_n(y,h) = (Z_n, V_n). \end{aligned}$$

The law of large numbers (12) says that

$$Z_n(y,h) \sim n(1-\varepsilon)\hat{y} \quad \text{and} \quad V_n(y,h) \sim n(1-\varepsilon). \tag{18}$$

Note

$$G_n = \bigcap_{\substack{M \in B_{\mu_p}(n(1-\varepsilon)\hat{y}, \varepsilon n) \times [n(1-\varepsilon) \cdots n(1-\varepsilon/2)],\\ k \geq \varepsilon n/2}} \bigl\{\xi_k^0 \subset B_{\mu_p}\bigl(0, (1+\varepsilon)k\bigr)\bigr\} \circ \theta_M.$$

Since θ_M preserves $\mathbb{P}_p$, we easily deduce from (5), Proposition 2.1 and a Borel–Cantelli argument that $\overline{\mathbb{P}}_p$ almost surely, G_n holds for n large enough.

Now take n large enough such that G_n holds and, with (18),

$$Z_n \in B_{\mu_p}\bigl(n(1-\varepsilon)\hat{y}, \varepsilon n\bigr) \quad \text{and} \quad (1-\varepsilon)n \leq V_n \leq (1-\varepsilon/2)n,$$

so that $\varepsilon n/2 \leq n - V_n \leq \varepsilon n$. Then G_n ensures that ($\varepsilon < 1$)

$$\xi_{n-V_n}^{Z_n} \subset B_{\mu_p}\bigl(Z_n, (1+\varepsilon)\varepsilon n\bigr) \subset B_{\mu_p}\bigl(n(1-\varepsilon)\hat{y}, 3\varepsilon n\bigr) \subset B_{\mu_p}(n\hat{y}, 4\varepsilon n) \subset n\mathring{A}.$$

So $\overline{N}_{M_n} \leq \overline{N}_{nA,n}$, and then

$$\frac{1}{n} \log \overline{N}_{nA,n} \geq \frac{V_n}{n} \frac{1}{V_n} \log \overline{N}_{M_n}.$$

With (18) and Lemma 3.2, we deduce that $\overline{\mathbb{P}}_p$ almost surely,

$$\varliminf_{n \to +\infty} \frac{1}{n} \log \overline{N}_{nA,n} \geq \frac{1}{1-\varepsilon} \alpha_p(\hat{y}) \geq \frac{1}{1-\varepsilon} L.$$

Letting ε going to 0 completes the proof. $\square$

LEMMA 5.4. *For every nonempty set A such that $\overline{A} \subset \mathring{B}_{\mu_p}(0,1)$, $\overline{\mathbb{P}}_p$-almost surely*,

$$\varlimsup_{n \to +\infty} \frac{1}{n} \log \overline{N}_{nA,n} \leq \sup_{x \in A} \tilde{\alpha}_p(x).$$

PROOF. The proof is a refinement of that of Lemma 4.2. Let $\delta > 0$. Since $\overline{A}$ is a compact subset of $\mathring{B}_\mu(0,1)$ and $z \mapsto \tilde{\alpha}_p(z)$ is continuous on $\mathring{B}_\mu(0,1)$, one can find $\varepsilon \in (0,1)$ such that

$$\sup_{A+B_{\mu_p}(0,2\varepsilon)} \tilde{\alpha}_p \le \delta + \sup_A \tilde{\alpha}_p.$$

Now take $\eta > 0$ and F as defined in the proof of Lemma 4.2 and note

$$F_B = \left\{(y,h) \in F : \frac{y}{\overline{\mathbb{E}}_p(s(y,h))} \in B\right\}.$$

Now consider $x \in nA$. Since $nA \subset B_{\mu_p}(0, n(1+\varepsilon))$, for n large enough, we can find $(y,h) \in F$ such that $x/n \in B_{\mu_p}(\frac{(1+\varepsilon)y}{\overline{\mathbb{E}}_p(s(y,h))}, (1-\eta)\varepsilon/2)$. We have

$$\begin{aligned}
\mu_p\left(\frac{y}{\overline{\mathbb{E}}_p(s(y,h))} - \frac{x}{n}\right) &\le \mu_p\left((1+\varepsilon)\frac{y}{\overline{\mathbb{E}}_p(s(y,h))} - (1+\varepsilon)\frac{x}{n}\right) \\
&\le \mu_p\left(\frac{(1+\varepsilon)y}{\overline{\mathbb{E}}_p(s(y,h))} - \frac{x}{n}\right) + \varepsilon\mu_p(x/n) \\
&\le (1-\eta)\varepsilon/2 + \varepsilon\mu_p(x/n) \le 2\varepsilon.
\end{aligned}$$

Since $x/n \in A$, we get $(y,h) \in F_{A+B_{\mu_p}(0,2\varepsilon)}$. Now, following the proof of Lemma 4.2, for n large enough, for each $x \in nA$, the n first steps of an open path that goes from $(0,0)$ to (x,n) and then to infinity are also the n first steps of an open path which contributes to $N_{M_n(y,h)}$ for any $(y,h) \in F_{A+B_{\mu_p}(0,2\varepsilon)}$, which gives

$$\overline{N}_{nA,n} \le \sum_{(y,h)\in F_{A+B_{\mu_p}(0,2\varepsilon)}} \overline{N}_{M_n(y,h)}. \tag{19}$$

As previously, we get

$$\overline{\lim_{n\to+\infty}} \frac{1}{n} \log(\overline{N}_{nA,n}) \le \sup_{F_{A+B_{\mu_p}(0,2\varepsilon)}} \alpha_p \le \sup_{A+B_{\mu_p}(0,2\varepsilon)} \tilde{\alpha}_p \le \delta + \sup_A \tilde{\alpha}_p.$$

We complete the proof by letting δ go to 0. $\square$

5.3. *Proof of equation* (2) *in Theorem* 1.2. It remains to skip from $\overline{N}_{nA,n}$ to $N_{nA,n}$. Fix $0 < \varepsilon < 1$ and define, for $n \ge 1$, the following event:

$$\begin{aligned}
G_n = &\bigcap_{\|z\|_1 \le n} \{\tau < \varepsilon n \text{ or } \tau = +\infty\} \circ \theta_{(z, \lfloor n(1-\varepsilon)\rfloor)} \\
&\cap \bigcap_{\|z\|_1 \le n} \{K'_{\varepsilon n} \subset B_{\mu_p}(0, 2\varepsilon n)\} \circ \theta^{\downarrow}_{(z,n)}.
\end{aligned}$$

As before, a Borel–Cantelli argument ensures that $\overline{\mathbb{P}}_p$-almost surely, G_n occurs for every large enough n.

Assume that G_n occurs. Consider a path $\gamma = (\gamma_i, i)_{0 \le i \le n}$ from $(0,0)$ to $nA \times \{n\}$ and set $z = \gamma_{\lfloor n(1-\varepsilon)\rfloor}$: as $\tau \circ \theta_{(z, \lfloor n(1-\varepsilon)\rfloor)} \ge \varepsilon n$, the event G_n implies that $\tau \circ \theta_{(z,\lfloor n(1-\varepsilon)\rfloor)} = +\infty$. Looking backwards in time, we see that all these z are in $nA + B_{\mu_p}(0, 2\varepsilon n)$. So $(\gamma_i, i)_{0\le i \le \lfloor n(1-\varepsilon)\rfloor}$ contributes to $\overline{N}_{nA+B_{\mu_p}(0,2\varepsilon n), \lfloor (1-\varepsilon)n\rfloor}$, and thus, on G_n,

$$N_{nA,n} \le (2d+1)^{\varepsilon n+1} \overline{N}_{nA+B_{\mu_p}(0,2\varepsilon n),\lfloor(1-\varepsilon)n\rfloor},$$

so

$$\frac{1}{n}\log N_{nA,n} \le \left(\varepsilon + \frac{1}{n}\right)\log(2d+1) + \frac{1}{n}\log \overline{N}_{nA+B_{\mu_p}(0,2\varepsilon n),\lfloor(1-\varepsilon)n\rfloor}.$$

Now, we first use Lemma 5.4 and take the $\overline{\lim}$, and then we use the continuity of $\tilde{\alpha}_p$ and let ε go to 0:

$$\overline{\lim_{n\to+\infty}} \frac{\log N_{nA,n}}{n} \le \varepsilon \log(2d+1) + \sup_{x\in A+B_{\mu_p}(0,2\varepsilon)} \tilde{\alpha}_p(x),$$

so

$$\overline{\lim_{n\to+\infty}} \frac{\log N_{nA,n}}{n} \le \sup_{x\in A} \tilde{\alpha}_p(x).$$

As $\overline{N}_{nA,n} \le N_{nA,n}$, with Lemma 5.3 we obtain

$$\lim_{n\to+\infty} \frac{\log N_{nA,n}}{n} = \sup_{x\in A} \tilde{\alpha}_p(x).$$

This completes the proof.

5.4. *Proof of the concavity of* $\tilde{\alpha}_p$. As $\tilde{\alpha}_p$ is continuous, it is sufficient to prove

$$\forall x, y \in \mathring{B}_{\mu_p}(0,1) \qquad \tilde{\alpha}_p\left(\frac{x+y}{2}\right) \ge \frac{\tilde{\alpha}_p(x)+\tilde{\alpha}_p(y)}{2}. \tag{20}$$

Write $z = (x+y)/2$. Let $0 < \varepsilon < 1/2$. Write $B_x = B_{\mu_p}(x,\varepsilon)$, $B_y = B_{\mu_p}(y,\varepsilon)$ and $B_z = B_{\mu_p}(z,\varepsilon)$. Assume that ε is small enough to ensure that B_x, B_y and B_z are included in $\mathring{B}_{\mu_p}(0,1)$. By equation (2), we have the following $\overline{\mathbb{P}}_p$ almost sure behavior:

$$\lim_{n\to+\infty} \frac{1}{n}\log N_{nB_x,n} = \sup_{u\in B_x} \tilde{\alpha}_p(u) > 0.$$

Therefore, for n large enough, we have

$$\mathbb{P}_p\Big(N_{nB_x,n} \ge \exp\Big(n(1-\varepsilon)\sup_{u\in B_x}\tilde{\alpha}_p(u)\Big)\Big) \ge \frac{1}{2}\rho,$$

where $\rho = \mathbb{P}_p((0,0) \to \infty) > 0$. Let X_n be some point of $\mathbb{Z}^d \cap nB_x$ which maximizes $\{N_{w,n} : w \in \mathbb{Z}^d \cap nB_x\}$—to ensure measurability, the tie is broken by a deterministic rule. As the cardinality of $\mathbb{Z}^d \cap nB_x$ is of order n^d, we get, for n large enough,

$$\mathbb{P}_p(A_n^x) \geq \frac{1}{2}\rho \qquad \text{where } A_n^x = \Big\{N_{X_n,n} \geq \exp\Big(n(1-2\varepsilon) \sup_{u \in B_x} \tilde{\alpha}_p(u)\Big)\Big\}.$$

Note that X_n and A_n^x are measurable with respect to $\mathcal{F}_n$. We also have, for n large enough,

$$\mathbb{P}_p(A_n^y) \geq \frac{1}{2}\rho \qquad \text{where } A_n^y = \Big\{N_{nB_y,n} \circ \theta_{X_n,n} \geq \exp\Big(n(1-\varepsilon) \sup_{u \in B_y} \tilde{\alpha}_p(u)\Big)\Big\}.$$

By independence, we thus get, for n large enough,

$$\mathbb{P}_p(A_n^x \cap A_n^y) \geq \frac{1}{4}\rho^2. \tag{21}$$

But on $A_n^x \cap A_n^y$ we have

$$N_{2nB_z,2n} \geq \exp\Big(n(1-2\varepsilon)\Big(\sup_{u \in B_x} \tilde{\alpha}_p(u) + \sup_{u \in B_y} \tilde{\alpha}_p(u)\Big)\Big). \tag{22}$$

On the other hand, by equation (2), we have the following almost sure behavior:

$$\lim_{n \to +\infty} \frac{1}{2n} \log N_{2nB_z,2n} = \sup_{u \in B_z} \tilde{\alpha}_p(u).$$

Therefore, for n large enough, we have

$$\mathbb{P}_p\Big(N_{2nB_z,2n} \leq \exp\Big(2n(1+\varepsilon) \sup_{u \in B_z} \tilde{\alpha}_p(u)\Big)\Big) \geq \Big(1 - \frac{1}{8}\rho^2\Big). \tag{23}$$

Combining (21), (22) and (23), we get

$$\frac{1-2\varepsilon}{2}\Big(\sup_{u \in B_x} \tilde{\alpha}_p(u) + \sup_{u \in B_y} \tilde{\alpha}_p(u)\Big) \leq (1+\varepsilon) \sup_{u \in B_z} \tilde{\alpha}_p(u).$$

We now let ε tend to 0 and we get (20), which completes the proof of Theorem 1.2.

6. Extension to linear stochastic evolutions.

This section is devoted to the proof of Theorem 1.5. As it is similar to the proof of Theorem 1.1, we do not provide a complete proof but rather emphasize on points that are different. Instead of comparing number of paths, we now compare weights of family of paths. The integrability assumption (4) is a simple and quite soft assumption allowing to control such weights. The following lemma follows easily from this assumption, from the polynomial growth of the number of paths with length n, and from the exponential Markov inequality:

LEMMA 6.1. *There exist $A, B, C > 0$ such that for every $n \in \mathbb{N}$, for every $t > 0$,*

$$\mathbb{P}\Bigg(\sup_{\gamma:(0,0)\to\mathbb{Z}^d\times\{n\}} \sum_{e\in\gamma} \log(A_e \wedge 1) \geq Cn + t\Bigg) \leq A\exp(-Bt),$$

$$\mathbb{P}\Bigg(\inf_{open\ \gamma:(0,0)\to\mathbb{Z}^d\times\{n\}} \sum_{e\in\gamma} \log(A_e \vee 1) \leq -Cn - t\Bigg) \leq A\exp(-Bt).$$

LEMMA 6.2. *Fix $(y, h) \in \mathbb{Z}^d \times \mathbb{N}^*$. There exists $\alpha(y, h)$ such that $\overline{\mathbb{P}}$-almost surely and in $L^1(\overline{\mathbb{P}})$,*

$$\lim_{n\to+\infty} \frac{1}{S_n(y, h)} \log N_{(ny, S_n(y,h))} = \alpha_p(y, h).$$

PROOF. Fix $(y, h) \in \mathbb{Z}^d \times \mathbb{N}^*$. Note that by definition, $\overline{\mathbb{P}}$-almost surely, for every $n \geq 1$, $(0, 0) \to (ny, S_n) \to +\infty$. For $n \geq 1$, we set

$$f_n = -\log N_{(ny, S_n)}.$$

Let us first prove that f_n is integrable. In the following equations, we consider optimums on the set of open paths γ from $(0, 0)$ to $\mathbb{Z}^d \times \{S_n\}$:

$$\inf_{\gamma} \prod_{e\in\gamma} (A_e \vee 1) \leq N_{(ny, S_n)} \leq (2d + 1)^{S_n} \Bigg(\sup_{\gamma} \prod_{e\in\gamma} (A_e \wedge 1)\Bigg),$$

$$\inf_{\gamma} \sum_{e\in\gamma} \log(A_e \vee 1) \leq \log N_{(ny, S_n)} \leq S_n \log(2d + 1) + \sup_{\gamma} \sum_{e\in\gamma} \log(A_e \wedge 1).$$

With the previous lemma, there exist $C, C' > 0$ such that

$$\overline{\mathbb{E}}\Bigg(\sup_{\gamma:(0,0)\to\mathbb{Z}^d\times\{S_n\}} \sum_{e\in\gamma} \log\max(A_e, 1) \Big| S_n\Bigg) \leq CS_n + C',$$

$$\overline{\mathbb{E}}\Bigg(\inf_{\text{open } \gamma:(0,0)\to\mathbb{Z}^d\times\{S_n\}} \sum_{e\in\gamma} \log\big(\min(A_e, 1)\big) \Big| S_n\Bigg) \geq -CS_n - C'.$$

With the integrability of S_n, we see that f_n is integrable (and in particular almost surely finite).

As before, $f_{n+p} \leq f_n + f_p \circ \hat{\theta}^n$ and we can apply Kingman's subadditive ergodic theorem:

$$-\alpha'(y, h) = \inf_{n\geq 1} \frac{\overline{\mathbb{E}}(f_n)}{n} \in \mathbb{R}, \quad \text{and} \quad \overline{\mathbb{P}}\text{-a.s.} \quad \lim_{n\to+\infty} \frac{f_n}{n} = -\alpha'(y, h).$$

The lemma follows then from (8) by setting $\alpha(y, h) = \frac{\alpha'(y,h)}{\overline{\mathbb{E}}s(y,h)}$. $\square$

We can now introduce a natural candidate for the limit in Theorem 1.5:

$$\alpha = \sup\{\alpha(y, h) : (y, h) \in \mathbb{Z}^d \times \mathbb{N}^*\} < +\infty. \tag{24}$$

LEMMA 6.3. *$\overline{\mathbb{P}}$-almost surely*, $\underline{\lim}_{n\to+\infty} \frac{1}{n}\log \overline{N}_n \geq \alpha$.

PROOF. Take $(y,h) \in \mathbb{Z}^d \times \mathbb{N}^*$, and consider the increasing sequence of integers: $(S_k = S_k(y,h))_k$.

By construction, $\overline{\mathbb{P}}$-almost surely, $(0,0) \to (ky, S_k) \to ((k+1)y, S_{k+1})$: denote by γ_k the rightmost open path from (ky, S_k) to $((k+1)y, S_{k+1})$. We see that, $\overline{\mathbb{P}}$ almost surely, for every integer n such that $S_k \leq n \leq S_{k+1}$,

$$\begin{aligned}\frac{1}{n}\log \overline{N}_n &\geq \frac{1}{S_{k+1}}\log\Bigl(\overline{N}_{(ky,S_k)}\prod_{e\in\gamma_k}(A_e \wedge 1)\Bigr)\\ &\geq \frac{S_k}{S_{k+1}}\frac{1}{S_k}\log \overline{N}_{(ky,S_k)} + \frac{1}{S_{k+1}}\sum_{e\in\gamma_k}\log(A_e\wedge 1).\end{aligned}\tag{25}$$

With (8) and Lemma 6.2, we see that $\overline{\mathbb{P}}_p$ almost surely,

$$\underline{\lim}_{k\to+\infty} \frac{S_k}{S_{k+1}}\frac{1}{S_k}\log \overline{N}_{(ky,S_k)} \geq \alpha_p(y,h).\tag{26}$$

Now take $\varepsilon > 0$. Let $\tilde{\mathcal{F}}$ be the σ-algebra generated by the field $\{\mathbf{1}_{A_e} : e \in \overrightarrow{\mathbb{E}}^{d+1}_{\mathrm{alt}}\}$. Note that γ_k is $\tilde{\mathcal{F}}$-measurable, with length $S_{k+1} - S_k$, that $S_k \geq k$, and that $\sigma^2 = \mathrm{Var}(\log(A \wedge 1))$ and $m = E(\log(A\wedge 1))$ are well defined thanks to assumption (4). With Chebyshev's inequality, we obtain

$$\begin{aligned}\overline{\mathbb{P}}\Bigl(\frac{1}{S_{k+1}}\sum_{e\in\gamma_k}\log(A_e\wedge 1) \leq -\varepsilon\Big|\tilde{\mathcal{F}}\Bigr) &\leq \frac{\overline{\mathbb{E}}((\sum_{e\in\gamma_k}\log(A_e\wedge 1))^2|\tilde{\mathcal{F}})}{\varepsilon^2 S_{k+1}^2}\\ &\leq \frac{(S_{k+1}-S_k)\sigma^2 + (S_{k+1}-S_k)^2 m^2}{\varepsilon^2 k^2},\\ \overline{\mathbb{P}}\Bigl(\frac{1}{S_{k+1}}\sum_{e\in\gamma_k}\log(A_e\wedge 1) \leq -\varepsilon\Bigr) &\leq \frac{\overline{\mathbb{E}}(s(y,h))\sigma^2 + \overline{\mathbb{E}}(s(y,h)^2)m^2}{\varepsilon^2 k^2}.\end{aligned}$$

With the Borel–Cantelli lemma, we obtain

$$\underline{\lim}_{k\to+\infty} \frac{1}{S_{k+1}}\sum_{e\in\gamma_k}\log(A_e \vee 1) \geq -\varepsilon.\tag{27}$$

Putting together (25), (26) and (27), we complete the proof. $\square$

LEMMA 6.4. *$\overline{\mathbb{P}}_p$-almost surely*, $\overline{\lim}_{n\to+\infty} \frac{1}{n}\log \overline{N}_n \leq \alpha_p$.

PROOF. Fix $\varepsilon > 0$ and $\eta \in (0,1)$. Proceeding as in Lemma 4.2, we approximate $B_\mu(0,1)$ with a finite set $F \subset \mathbb{Z}^d \times \mathbb{N}^*$ and obtain (13) and (14). The event

G_n is now:

$$G_n = \bigcap_{M \in \{-2n,\dots,2n\}^d \times \{0,\dots,2n\}} \left\{ \begin{array}{c} \tau < n(1+\varepsilon) \\ \text{or } K'_{n\varepsilon} \supset B_{\mu_p}\big(0, (1-\eta)\varepsilon n\big) \cap \mathbb{Z}^d \end{array} \right\} \circ \theta_M^{\downarrow}$$
$$\cap \bigcap_{M \in \{-2n,\dots,2n\}^d \times \{0,\dots,2n\}} \left\{ \inf_{\substack{\text{open } \gamma: \\ (0,0) \to \mathbb{Z}^d \times \{2\varepsilon n\}}} \sum_{e \in \gamma} \log(A_e \wedge 1) \geq -3C\varepsilon n \right\} \circ \theta_M^{\downarrow}.$$

As before, and using moreover Lemma 6.1, the Borel–Cantelli lemma, $\overline{\mathbb{P}}$ almost surely, G_n holds for every n large enough.

Now take n large enough such that (14) holds, G_n holds, together with $V_n(y,h) \leq n(1+2\varepsilon)$ for each $(y,h) \in F$, which is possible thanks to (13).

Fix $x \in \mathbb{Z}^d$ such that $(0,0) \to (x,n) \to \infty$. As before, choose $(y,h) \in F$ such that $x \in B_{\mu_p}(Z_n(y,h), (1-\eta)\varepsilon n)$ and conclude that $(x,n) \to M_n(y,h)$. Thus, with G_n and as $V_n(y,h) \leq n(1+2\varepsilon)$

$$\begin{aligned} \overline{N}_{M_n(y,h)} &\geq \sum_{x \in B_{\mu_p}(Z_n(y,h),(1-\eta)\varepsilon n)} \left(\overline{N}_{x,n} \sup_{\gamma:(x,n) \to M_n(y,h)} \prod_{e \in \gamma} A_e \right) \\ &\geq \sum_{x \in B_{\mu_p}(Z_n(y,h),(1-\eta)\varepsilon n)} \overline{N}_{x,n} \left(\inf_{\substack{\gamma \text{ open, from } M_n(y,h) \\ \text{to } \mathbb{Z}^d \times \{V_n(y,h) - 2\varepsilon n\}}} \prod_{e \in \gamma} (A_e \wedge 1) \right) \\ &\geq \exp(-3C\varepsilon n) \sum_{x \in B_{\mu_p}(Z_n(y,h),(1-\eta)\varepsilon n)} \overline{N}_{x,n}. \end{aligned}$$

Thus,

$$\overline{N}_n \leq \exp(3C\varepsilon n) \sum_{(y,h) \in F} \overline{N}_{M_n(y,h)}.$$

The end of the proof is as before. $\square$

LEMMA 6.5. *$\overline{\mathbb{P}}_p$-almost surely,*

$$\overline{\lim_{n \to +\infty}} \frac{\log N_n}{n} = \overline{\lim_{n \to +\infty}} \frac{\log \overline{N}_n}{n} \quad \textit{and}$$
$$\underline{\lim_{n \to +\infty}} \frac{\log N_n}{n} = \underline{\lim_{n \to +\infty}} \frac{\log \overline{N}_n}{n}.$$

PROOF. Fix $0 < \varepsilon < 1$ and define, for $n \geq 1$, the following event:

$$E_n = \bigcap_{\|z\|_1 \leq n} \{\tau < \varepsilon n \text{ or } \tau = +\infty\} \circ \theta_{(z, \lfloor n(1-\varepsilon) \rfloor)}$$
$$\cap \bigcap_{\|z\|_1 \leq n} \left\{ \sup_{\gamma:(0,0) \to \mathbb{Z}^d \times \{2\varepsilon n\}} \sum_{e \in \gamma} \log(A_e \vee 1) \leq 3C\varepsilon n \right\} \circ \theta_{(z, \lfloor n(1-\varepsilon) \rfloor)}.$$

Assume that E_n occurs.

Consider a path $\gamma = (\gamma_i, i)_{0\le i\le n}$ from $(0,0)$ to $\mathbb{Z}^d \times \{n\}$ and set $z = \gamma_{\lfloor n(1-\varepsilon)\rfloor}$: as $\tau \circ \theta_{(z,\lfloor n(1-\varepsilon)\rfloor)} \ge \varepsilon n$, the event E_n implies that $\tau \circ \theta_{(z,\lfloor n(1-\varepsilon)\rfloor)} = +\infty$. So $(\gamma_i, i)_{0\le i\le \lfloor n(1-\varepsilon)\rfloor}$ contributes to $\overline{N}_{\lfloor n(1-\varepsilon)\rfloor}$, and thus, on E_n,

$$\begin{aligned} N_n &\le (2d+1)^{\varepsilon n+1}\overline{N}_{\lfloor n(1-\varepsilon)\rfloor} \sup_{\|z\|_1\le n} \sup_{\gamma:(0,0)\to\mathbb{Z}^d\times\{2\varepsilon n\}} \sum_{e\in\gamma} \log(A_e \vee 1) \\ &\le (2d+1)^{\varepsilon n+1}\exp(3C\varepsilon n)\overline{N}_{\lfloor n(1-\varepsilon)\rfloor}. \end{aligned}$$

The end of the proof is as before. □

Acknowledgements. Olivier Garet and Régine Marchand would like to warmly thank Matthias Birkner and Sun Rongfeng for pointing out an error in a previous version of the paper. The picture in Figure 2 is courtesy of Aurelia Deshayes.

REFERENCES

[1] Comets, F., Fukushima, R., Nakajima, S. and Yoshida, N. (2015). Limiting results for the free energy of directed polymers in random environment with unbounded jumps. *J. Stat. Phys.* **161** 577–597. MR3406700

[2] Comets, F., Popov, S. and Vachkovskaia, M. (2008). The number of open paths in an oriented ρ-percolation model. *J. Stat. Phys.* **131** 357–379. MR2386584

[3] Comets, F., Shiga, T. and Yoshida, N. (2004). Probabilistic analysis of directed polymers in a random environment: A review. In *Stochastic Analysis on Large Scale Interacting Systems. Adv. Stud. Pure Math.* **39** 115–142. Math. Soc. Japan, Tokyo. MR2073332

[4] Darling, R. W. R. (1991). The Lyapunov exponent for products of infinite-dimensional random matrices. In *Lyapunov Exponents* (*Oberwolfach*, 1990). *Lecture Notes in Math.* **1486** 206–215. Springer, Berlin. MR1178959

[5] Durrett, R. (1991). The contact process, 1974–1989. In *Mathematics of Random Media* (*Blacksburg*, *VA*, 1989). *Lectures in Applied Mathematics* **27** 1–18. Amer. Math. Soc., Providence, RI. MR1117232

[6] Fukushima, R. and Yoshida, N. (2012). On exponential growth for a certain class of linear systems. *ALEA Lat. Am. J. Probab. Math. Stat.* **9** 323–336. MR3069368

[7] Garet, O. and Marchand, R. (2012). Asymptotic shape for the contact process in random environment. *Ann. Appl. Probab.* **22** 1362–1410. MR2985164

[8] Garet, O. and Marchand, R. (2014). Large deviations for the contact process in random environment. *Ann. Probab.* **42** 1438–1479. MR3262483

[9] Kesten, H. and Sidoravicius, V. (2010). A problem in last-passage percolation. *Braz. J. Probab. Stat.* **24** 300–320. MR2643568

[10] Lacoin, H. (2012). Existence of an intermediate phase for oriented percolation. *Electron. J. Probab.* **17** no. 41, 17. MR2928724

[11] Lacoin, H. (2014). Non-coincidence of quenched and annealed connective constants on the supercritical planar percolation cluster. *Probab. Theory Related Fields* **159** 777–808. MR3230008

[12] YOSHIDA, N. (2008). Phase transitions for the growth rate of linear stochastic evolutions. *J. Stat. Phys.* **133** 1033–1058. MR2462010

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