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The Number of Ramified Coverings of the Sphere by the Double Torus, and a General Form for Higher Genera

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An explicit expression is obtained for the generating series for the number of ramified coverings of the sphere by the double torus, with elementary branch points and prescribed ramification type over infinity. Thus we are able to determine various linear recurrence equations for the numbers of these coverings with no ramification over infinity; one of these recurrence equations has previously been conjectured by Graber and Pandharipande. The general form of this series is conjectured for the number of these coverings by a surface of arbitrary genus that is at least two.

1. INTRODUCTION AND BACKGROUND

Let $\mu_m^{(g)}(\alpha)$ be the number of almost simple ramified coverings of $\mathbb{S}^2$ by X with ramification type α where X is a compact connected Riemann surface of genus g, and α is a partition with m parts. The problem of determining an (explicit) expression for $\mu_m^{(g)}(\alpha)$ is called the *Hurwitz Enumeration Problem*, a brief account of which is given in [2]. The terminology and notation used here will be consistent with the latter paper.

There appears to be a natural topological distinction between the low genera instances of the problem, namely for $g \leqslant 1$, on the one hand, and the higher genera instances, namely for $g \geqslant 2$, on the other hand. In this paper we address the higher genera case of the Hurwitz Enumeration Problem. The distinction between the low genera and the high genera cases manifests itself in this paper in the fact that a general form can be given for the higher genera case, but that does not specialize to the low genera case. In this paper we prove an explicit result for $g = 2$, the double torus. Thus we are able to determine various linear recurrence equations for $\mu_m^{(2)}(1^m)$, corresponding to the case of no ramification over infinity. One of these

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recurrence equations has previously been conjectured by Graber and Pandharipande [9]. Moreover, we conjecture an explicit result for $g=3$. We also give a conjecture for the general form of the generating series for $\mu_m^{(g)}(\alpha)$ for arbitrary $g \geqslant 2$.

Let $\mathscr{C}_\alpha$ be the conjugacy class of the symmetric group $\mathfrak{S}_n$ on n symbols indexed by the partition α of n. Let $\vartheta(\alpha)=\prod_{i\geqslant 1} i^{m_i} m_i!$, where α has m_i parts equal to i where $i \geqslant 1$. We write $\alpha \vdash n$ to indicate that $\alpha = (\alpha_1, ..., \alpha_r)$ is a partition of n and $\alpha \models n$ to indicate that α is a partition of n with no part equal to one. The length, r, of α is denoted by $l(\alpha)$.

Let the generating series for $\mu_m^{(g)}(\alpha)$ be defined by

$$F_g(x, \mathbf{p}) = \sum_{m,\, n \geqslant 1} \sum_{\substack{\alpha \vdash n \\ l(\alpha)=m}} \frac{\mu_m^{(g)}(\alpha)}{(n+m+2g-2)!} p_\alpha x^n, \tag{1}$$

where p_i, for $i \geqslant 1$, are indeterminates and $p_\alpha = p_{\alpha_1} p_{\alpha_2} \cdots$. Let

$$\psi_i(x, \mathbf{p}) = \sum_{n \geqslant 1} n^{i-1} a_n p_n x^n, \tag{2}$$

where $a_n = n^n/(n-1)!$ for $n \geqslant 1$. Then, as we have shown in previous work [3, 2] F_0 and F_1 can be expressed succinctly in terms of $\psi_i \equiv \psi_i(s, \mathbf{p})$ where $s \equiv s(x, \mathbf{p})$ is the unique solution of the functional equation

$$s = x e^{\psi_0(s, \mathbf{p})}. \tag{3}$$

From [3, Proposition 3.1; 2, Theorem 4.2], the expressions for F_0 and F_1 in terms of the ψ_i are given by

$$\left(x \frac{\partial}{\partial x}\right)^2 F_0(x, \mathbf{p}) = \psi_0, \tag{4}$$

$$F_1(x, \mathbf{p}) = \frac{1}{24} \left(\log(1-\psi_1)^{-1} - \psi_0\right). \tag{5}$$

In this paper we prove the following explicit expression for F_2.

THEOREM 1.1.

$$F_2(x, \mathbf{p}) = \frac{1}{5760} \left(\frac{Q_3}{(1-\psi_1)^3} + \frac{Q_4}{(1-\psi_1)^4} + \frac{Q_5}{(1-\psi_1)^5} \right), \tag{6}$$

where $Q_3 = 5\psi_4 - 12\psi_3 + 7\psi_2$, $Q_4 = 29\psi_2\psi_3 - 25\psi_2^2$, *and* $Q_5 = 28\psi_2^3$.

The form of this theorem was arrived at through a careful examination and further analysis of the expressions for $\mu_m^{(2)}(\alpha)$ for $m \leqslant 3$ that appear in the Appendix of [5]. In addition, we make the following conjecture for the general form of F_g.

Conjecture 1.2. For $g \geqslant 2$,

$$F_g(x, \mathbf{p}) = \sum_{d=2g-1}^{5g-5} \frac{1}{(1-\psi_1)^d} \sum_{n=d-1}^{d+g-1} \sum_{\substack{\theta \vdash n \\ l(\theta) = d-2(g-1)}} K_\theta{}^{(g)} \psi_\theta,$$

where $K_\theta{}^{(g)} \in \mathbb{Q}$, and $\psi_\theta = \psi_{\theta_1} \psi_{\theta_2} \cdots$.

This expression for $F_g(x, \mathbf{p})$ is a sum of rational functions of the ψ_i's with particularly simple denominators and with numerators of prescribed form. For $g = 3$ we have determined the $K_\alpha^{(g)}$ explicitly, based on this form, with the aid of Maple. The resulting expression for $F_3(x, \mathbf{p})$ is displayed in Appendix A.

For $l(\theta) = 1$ and for all g, $K_\theta^{(g)}$ may be obtained quite readily as follows. From [7],

$$\frac{n\mu_1^{(g)}(n)}{(n+2g-1)!} = a_n \frac{n^{2g-2}}{2^{2g}} [x^{2g}] \left(\frac{\sinh(x)}{x}\right)^{n-1},$$

where $[x^{2g}]$ denotes the operator giving the coefficient of x^{2g} in a formal power series. But, as a polynomial in n,

$$[x^{2g}] \left(\frac{\sinh(x)}{x}\right)^{n-1} = \frac{1}{6^g g!} n^g + \cdots + \frac{2(1-2^{2g-1}) B_{2g}}{(2g)!} n^0,$$

where B_{2g} is a Bernoulli number. Thus, from the conjectured form, we have

$$F_g = \frac{(1/24^g g!) \psi_{3g-2} + \cdots + ((1-2^{2g-1}) B_{2g}/2^{2g-1}(2g)!) \psi_{2g-2}}{(1-\psi)^{2g-1}}$$
$$+ \cdots + \frac{A_g \psi_2^{3g-3}}{(1-\psi)^{5g-5}}.$$

where A_g is a rational number.

2. PROOF OF THE MAIN RESULT

We use the approach that has been developed in [3] and extended in [2] that makes use of Hurwitz's encoding [6] of the problem as a transitive ordered factorization of a permutation with prescribed cycle type into transpositions. It has already been shown [5] by a cut-and-join analysis of the action of a transposition on the cycle structure of an arbitrary permutation in $\mathfrak{S}_n$, that if

$$F=\tilde{F}_0+z\tilde{F}_1+z^2\tilde{F}_2+\cdots,$$

where $\tilde{F}_g$ is obtained from F_g by multiplying the summand by $u^{n+m+2g-2}$, then F satisfies the partial differential equation

$$\frac{\partial F}{\partial u}=\frac{1}{2}\sum_{i,\,j\geqslant 1}\left(ijp_{i+j}z\frac{\partial^2 F}{\partial p_i\partial p_j}+ijp_{i+j}\frac{\partial F}{\partial p_i}\frac{\partial F}{\partial p_j}+(i+j)\,p_ip_j\frac{\partial F}{\partial p_{i+j}}\right). \quad (7)$$

The techniques developed in [3, 2] enable us to confirm whether a series satisfies the partial differential equation induced from this by considering only terms of given degree in z, thus grading by genus, but we do not yet possess a method for constructing the solution of such an equation in closed form. The next result gives the linear first order partial differential equation for F_2 that is induced by restricting (7) in this way to terms of degree exactly two in z.

LEMMA 2.1. *The series $f=F_2$ satisfies the partial differential equation*

$$T_0 f-T_1=0, \quad (8)$$

where

$$T_0=x\frac{\partial}{\partial x}+\sum_{i\geqslant 1}p_i\frac{\partial}{\partial p_i}+2-\sum_{i,\,j\geqslant 1}ijp_{i+j}\frac{\partial F_0}{\partial p_i}\frac{\partial}{\partial p_j}-\frac{1}{2}\sum_{i,\,j\geqslant 1}(i+j)\,p_ip_j\frac{\partial}{\partial p_{i+j}},$$

$$T_1=\frac{1}{2}\sum_{i,\,j\geqslant 1}ijp_{i+j}\frac{\partial F_1}{\partial p_i}\frac{\partial F_1}{\partial p_j}+\frac{1}{2}\sum_{i,\,j\geqslant 1}ijp_{i+j}\frac{\partial^2 F_1}{\partial p_i\partial p_j}.$$

Proof. Clearly,

$$u\frac{\partial}{\partial u}[z^2]\,F=\left(x\frac{\partial}{\partial x}+\sum_{i\geqslant 1}p_i\frac{\partial}{\partial p_i}+2\right)[z^2]\,F,$$

and the result follows by applying $[z^2]$ to (7). ∎

Lemma 2 gives a linear partial differential equation for F_2 with coefficients that involve the known series F_0 and F_1 (see (4) and (5)). The proof of Theorem 1.1 consists of showing that the expression for F_2 given in (6) does indeed satisfy this linear partial differential equation. To establish this, extensive use will be made of a number of results in [2] that were used in the determination of the generating series F_1, the case of the torus. These enable us to reduce $T_0F_2 - T_1$ to a polynomial in a new set of variables, and this is shown to be identically zero, with the aid of Maple to carry out the substantial amount of routine simplification. The details of this part of the proof are suppressed, but enough information is retained to permit it to be reproduced.

In particular we require the following results from [2], which are included for completeness. They follow from (2), (3), (4), and (5), with the aid of Lagrange's Implicit Function Theorem.

$$x\frac{\partial s}{\partial x} = \frac{s}{1-\psi_1}, \qquad \frac{\partial s}{\partial p_k} = \frac{1}{k}\frac{a_k s^{k+1}}{1-\psi_1}, \tag{9}$$

$$x\frac{\partial \psi_i}{\partial x} = \frac{\psi_{i+1}}{1-\psi_1}, \qquad \frac{\partial \psi_i}{\partial p_k} = k^{i-1}a_k s^k + \frac{a_k}{k}\frac{\psi_{i+1}s^k}{1-\psi_1}, \tag{10}$$

and

$$\frac{\partial F_0}{\partial p_k} = \frac{a_k}{k^3}s^k - \frac{a_k}{k^2}\sum_{r \geqslant 1} a_r p_r \frac{s^{k+r}}{k+r}, \tag{11}$$

$$\frac{\partial F_1}{\partial p_k} = \frac{1}{24}\frac{a_k}{k}s^k\left(\frac{1-k}{1-\psi_1} + \frac{\psi_2}{(1-\psi_1)^2}\right). \tag{12}$$

It follows immediately from (12) that

$$\frac{\partial^2 F_1}{\partial p_i \partial p_j} = \frac{1}{24}\frac{a_i s^i}{i}\frac{a_j s^j}{j}\left(\frac{i^2+j^2+ij-i-j}{(1-\psi_1)^2}\right.$$
$$\left.+\frac{2(i+j)\,\psi_2 - \psi_2 + \psi_3}{(1-\psi_1)^3} + \frac{2\psi_2^2}{(1-\psi_1)^4}\right). \tag{13}$$

These account for all of the terms in the partial differential equation (8) that do not involve F_2.

Proof of Theorem 1.1. Let G_2 denote the series written explicitly on the right hand side of (6). To prove Theorem 1.1 it is sufficient to show that G_2 is a solution of (8), since G_2, with constant term equal to zero, clearly satisfies the initial condition for F_2. The requisite partial derivatives of G_2 are obtained indirectly from (10) by differentiating with respect to the ψ_i's, giving

$$x\frac{\partial G_2}{\partial x} = \sum_{m \geqslant 1}\left(\frac{\partial G_2}{\partial \psi_m}\right)\left(x\frac{\partial \psi_m}{\partial x}\right) = \sum_{m \geqslant 1}\frac{\psi_{m+1}}{1-\psi_1}\frac{\partial G_2}{\partial \psi_m}, \tag{14}$$

$$\frac{\partial G_2}{\partial p_k} = \sum_{m \geqslant 1}\frac{a_k s^k}{k}\left(k^m + \frac{\psi_{m+1}}{1-\psi_1}\right)\frac{\partial G_2}{\partial \psi_m}. \tag{15}$$

By inspection, each summand of $T_0 G_2 - T_1$ may be expressed through (11), (12), (13), (14), and (15) entirely in terms of q_i's where $q_i = s^i p_i$ for $i \geqslant 1$. The q_i's are algebraically independent, and the dependency on s is perfectly subsumed. Now, in terms of these q_i's, let

$$M_{k,l} = \sum_{i,j \geqslant 1} q_{i+j} a_i a_j i^k j^l, \tag{16}$$

so $M_{k,l} = M_{l,k}$, and $M_{k,l}$ is homogeneous of degree one in the q_i's. Let

$$N_k = \frac{1}{2} \sum_{i,j \geqslant 1} \left(q_i q_j a_{i+j} (i+j)^k - 2 q_{i+j} a_j j^k \frac{a_i}{i} \sum_{r \geqslant 1} q_r \frac{a_r}{i+r} \right), \tag{17}$$

so N_k is homogeneous of degree two in the q_i's. Then, from (12) and (13)

$$T_1 = \frac{1}{5760} \sum_{i=2}^{4} \frac{S_i}{(1-\psi_1)^i}, \tag{18}$$

where

$$S_2 = 240 M_{2,0} + 125 M_{1,1} - 250 M_{1,0} + 5 M_{0,0},$$

$$S_3 = \psi_2 (490 M_{1,0} - 130 M_{0,0}) + 120 \psi_3 M_{0,0},$$

$$S_4 = 245 M_{0,0},$$

and, from (11), (14), (15), (16), and (17),

$$T_0 G_2 = 2 G_2 + \sum_{m \geqslant 1} \left(\frac{\psi_{m+1}}{1-\psi_1} (1 + \psi_0 - M_{-2,0} - N_0) + \psi_m - M_{-2,m} - N_m \right) \frac{\partial G_2}{\partial \psi_m}. \tag{19}$$

But

$$\frac{\partial G_2}{\partial \psi_m} = \frac{1}{5760} \sum_l \frac{R_{l,m}}{(1-\psi_1)^l},$$

where $R_{l,1} = (l-1)\, Q_{l-1}$ for $l = 4, 5, 6$, $R_{3,2} = 7$, $R_{4,2} = 29\psi_3 - 50\psi_2$, $R_{5,2} = 84\psi_2^2$, $R_{3,3} = -12$, $R_{4,3} = 29\psi_2$, $R_{3,4} = 5$, and $R_{l,m}$ is zero otherwise. Substituting these into (19), and combining with (18), we obtain the following explicit expression for the left hand side of the partial differential equation (8)

$$\begin{aligned}
&5760(T_0 G_2 - T_1)\\
&\quad = \sum_{i=2}^{4} \frac{S_i}{(1-\psi_1)^i} + \sum_{i=3}^{5} \frac{2Q_i}{(1-\psi_1)^i}\\
&\qquad + \sum_{i=4}^{7} \frac{1}{(1-\psi_1)^i} \sum_m \psi_{m+1}(1+\psi_0 - M_{-2,0} - N_0)\, R_{i-1,m}\\
&\qquad + \sum_{i=3}^{6} \frac{1}{(1-\psi_1)^i} \sum_m (\psi_m - M_{-2,m} - N_m)\, R_{i,m}.
\end{aligned} \tag{20}$$

Then $5760(T_0 G_2 - T_1)(1-\psi_1)^7$ is a polynomial of degree 6 in the q_i's, with constant term equal to zero. Let C_i be the contribution from terms of (homogeneous) total degree i in the q_i's for $i = 1, ..., 6$. The explicit expressions for these are given in Appendix B.

To complete the proof of Theorem 1.1 we show that each of these contributions is identically zero. It is convenient to introduce the symmetrization operator $\varpi_{1, ..., i}$ on the ring $\mathscr{H}_i[q_1, q_2, ...]$ of homogeneous polynomials of total degree i in $q_1, q_2, ...$, defined by

$$\varpi_{1, ..., i}(q_{\alpha_1} \cdots q_{\alpha_i}) = \sum_{\pi \in \mathfrak{S}_i} x_{\pi(1)}^{\alpha_1} \cdots x_{\pi(i)}^{\alpha_i}$$

and extended linearly to the whole of the ring. Since $\varpi_{1, ..., i} f = 0$ implies that $f = 0$ for $f \in \mathscr{H}_i[q_1, q_2, ...]$, it is sufficient to show that $\varpi_{1, ..., i} C_i = 0$ for $i = 1, ..., 6$.

Now let $w \equiv w(x)$ be the unique solution of the functional equation

$$w = xe^w, \tag{21}$$

and let $w_i \equiv w(x_i)$, where x_i is an indeterminate for $i \geqslant 1$. Let $w_i^{(j)} = x_i (\partial/\partial x_i)^j w_i$. Since $x\partial/\partial x = (w/(1-w))\, \partial/\partial w$, it is a straightforward matter to express $w_i^{(k)}$ as a rational function of w_i. For example,

$$w_i^{(1)} = \frac{w_i}{1-w_i}, \qquad w_i^{(2)} = \frac{w_i}{(1-w_i)^3}, \quad w_i^{(3)} = \frac{w_i + 2w_i^2}{(1-w_i)^5}.$$

Moreover, $w_1, w_2, ...$ are algebraically independent. Now from [2]

$$\sum_{i, j, r \geqslant 1} \frac{a_i a_j a_r j^k}{i(i+r)} x_1^{i+j} x_2^r = w_1^{(k+2)} \left(\frac{x_2}{x_1 - x_2} - \frac{w_2^{(1)}}{w_1 - w_2} - w_2^{(1)} \right),$$

so

$$\varpi_{1,2} N_k = \frac{w_1^{(k+2)} w_2^{(1)} - w_2^{(k+2)} w_1^{(1)}}{w_1 - w_2} + w_1^{(k+2)} w_2^{(1)} + w_2^{(k+2)} w_1^{(1)},$$
$$\varpi_1 M_{i,j} = w_1^{(i+2)} w_1^{(j+2)},$$
$$\varpi_1 \psi_k = w_1^{(k+1)}.$$

Each of these is a polynomial in W_1 and W_2 where $W_i = 1/(1-w_i)$, so $\varpi_{1,\ldots,i} C_i$ is a polynomial in $W_1, \ldots, W_i$ alone. Then $\varpi_{1,\ldots,i} C_i$ may be obtained from the constituents $\varpi_1 \psi_i$, $\varpi_1 M_{l,m}$ and $\varpi_{1,2} N_k$ by distributing the indeterminates $x_1, \ldots, x_i$ as disjoint subsets of arguments for these constituents in all possible ways. We have used Maple to carry out this routine but laborious task, and have thus established that $\varpi_{1,\ldots,i} C_i$ is identically zero as a polynomial in $W_1, \ldots, W_i$ for $i = 1, \ldots, 6$. This completes the proof. ∎

3. AN EXPLICIT EXPRESSION FOR THE NUMBER OF RAMIFIED COVERINGS OF THE SPHERE BY THE DOUBLE TORUS

In Theorem 1.1 we have determined the generating series F_2 for the ramification numbers $\mu_m^{(2)}(\alpha)$. In this section we expand this series and thus give an explicit expression for $\mu_m^{(2)}(\alpha)$. The following result is needed, in which $a_\alpha = a_{\alpha_1} a_{\alpha_2} \cdots$ and m_ν is the monomial symmetric function with exponents specified by the parts of the partition ν.

LEMMA 3.1. *For $\alpha \vdash n$, $n \geqslant 1$,*

$$[x^n p_\alpha] \frac{1}{1-\psi_1} \prod_{i \geqslant 1} \frac{\psi_i^{j_i}}{j_i!} = \frac{a_\alpha}{\vartheta(\alpha)} n^{l(\alpha) - l(\nu)} m_\nu(\alpha),$$

where $\nu = (1^{j_1} 2^{j_2} \ldots)$.

Proof. Let

$$\Lambda = [x^n] \frac{1}{1-\psi_1} \prod_{i \geqslant 1} \frac{\psi_i^{j_i}}{j_i!}.$$

Then by Lagrange's Implicit Function Theorem

$$\Lambda = [t^n] \left(\prod_{i \geqslant 1} \frac{\psi_i(t, \mathbf{p})^{j_i}}{j_i!} \right) e^{n\psi_0(t, \mathbf{p})}$$
$$= [t^n] \sum \prod_{i,k \geqslant 1} \frac{(k^{i-1} a_k p_k t^k)^{j_{i,k}}}{j_{i,k}!} \prod_{k \geqslant 1} \frac{(nk^{-1} a_k p_k t^k)^{d_k}}{d_k!},$$

where the summation is over $j_{i,k} \geqslant 0$, for $i, k \geqslant 1$, and $d_k \geqslant 0$ for $k \geqslant 1$, restricted by $\sum_{k \geqslant 1} j_{i,k} = j_i$, for $i \geqslant 1$. Thus, if $\alpha = (1^{b_1} 2^{b_2} \ldots)$, where $\alpha \vdash n$, then

$$[p_\alpha] \Lambda = \frac{a_\alpha}{\vartheta(\alpha)} \sum n^{d_1 + d_2 + \cdots} \prod_{k \geqslant 1} \frac{b_k!}{d_k!} \prod_{i \geqslant 1} \frac{k^{ij_{i,k}}}{j_{i,k}!},$$

where the summation is now further restricted by $\sum_{i \geqslant 1} j_{i,k} = b_k - d_k$, for $k \geqslant 1$. Then

$$[p_\alpha] \Lambda = \frac{a_\alpha}{\vartheta(\alpha)} n^{b_1 + b_2 + \cdots - j_1 - j_2 - \cdots} [y_1^{j_1} y_2^{j_2} \cdots] \prod_{k \geqslant 1} \left(1 + \sum_{i \geqslant 1} k^i y_i\right)^{b_k}$$

and the result follows. ∎

Applying Lemma 3.1 to the generating series F_2 given in terms of the ψ_i's in Theorem 1.1, we immediately obtain the following result which gives an explicit expression for $\mu_m^{(2)}(\alpha)$. This expression is a symmetric function in the parts of α, a linear combination of monomial symmetric functions. The explicit expression for $\mu_m^{(1)}(\alpha)$ obtained in [2] is a symmetric function expressed in terms of the elementary symmetric functions $e_k(\alpha)$, where $k \geqslant 1$. These forms are closely related since $e_k = m_{(1^k)}$.

COROLLARY 3.2.

$$\mu_m^{(2)}(\alpha) = (n+m+2)! \frac{a_\alpha}{\vartheta(\alpha)} \frac{n^m}{5760} \sum_{k \geqslant 0} \left(\frac{(k+1)!}{n^{k+1}} (5m_{(4\,1^k)} - 12m_{(3\,1^k)} + 7m_{(2\,1^k)}) \right.$$
$$\left. + \frac{(k+2)!}{n^{k+2}} \left(\frac{29}{2} m_{(3\,2\,1^k)} - 25m_{(2^2\,1^k)} \right) + \frac{(k+3)!}{n^{k+3}} 28m_{(2^3\,1^k)} \right).$$

Proof. This is direct from Theorem 1.1 and Lemma 3.1. ∎

4. A PROOF OF THE GRABER-PANDHARIPANDE RECURRENCE EQUATION

We conclude with an examination of the case $\alpha = (1^n)$, corresponding to no ramification over ∞. It will be convenient to denote $\mu_n^{(g)}(1^n)$ by $\mu_n^{(g)}$ for brevity. For $g \geqslant 0$, let f_g be the specialization of F_g with $p_1 = 1$, and $p_i = 0$ for $i > 1$. Then

$$f_g = \sum_{n \geqslant 1} x^n \frac{\mu_n^{(g)}}{(2n+2g-2)!},$$

and under these specializations of the p_i's we have $s=w$ where w is the unique solution of the functional equation (21) and $\psi_i=w$ for all i. Thus from (4) and (5) we have

$$D^2 f_0 = w,$$
$$f_1 = \tfrac{1}{24}(\log(1-w)^{-1} - w),$$

and from Theorem 1.1

$$f_2 = \frac{1}{5760}\left(\frac{4w^2}{(1-w)^4} + \frac{28w^3}{(1-w)^5}\right).$$

An explicit expression can in fact be obtained for $\mu_n^{(2)}$. The expression is

COROLLARY 4.1.

$$\mu_n^{(2)} = \frac{(2n+2)!}{1440n}(12A_4 + 21A_3 + 2A_2),$$

where

$$A_k = \sum_{i=0}^{n-k} \binom{i+5}{5} \frac{n^{n-i-k}}{(n-i-k)!}.$$

Proof. The results follows by applying Lagrange's Implicit Function Theorem to the above expression for f_2. ▮

Recurrence equations can be obtained for this number, and our interest in these, or rather, the corresponding differential equations for f_2, is that they may cast light on a more direct way of obtaining the $\mu_n^{(2)}$. It is convenient to introduce the operator

$$D = x\frac{d}{dx},$$

and to change variable to $W = 1/(1-w)$. Then $D = W^2(W-1)\,d/dW$. Now $Df_0 = 1/2 - W^{-2}/2$, $D^2 f_0 = 1 - 1/W$, and $D^r f_0$ is a polynomial in W for $r \geqslant 2$. Moreover, $D^r f_1$ is a polynomial in W for $r \geqslant 1$ and $D^r f_2$ is a polynomial in W for $r \geqslant 0$. Then these derivatives are algebraically dependent, so f_2 satisfies a differential equation. Clearly, this equation is not unique.

To decide upon the form that such a differential equation may take we suppose there exists a (combinatorial or geometric) construction acting on selected sheets that decomposes a covering into two connected coverings whose genera sum to the genus of the original covering. The combinatorial effect of D is to select a single sheet in all possible ways. We therefore seek a formal linear differential equation for f_2 that involves terms of the form $(D^p f_i)(D^q f_j)$ where $i+j=2$, with the additional condition that it suffices to select at most four sheets, so $p+q\leqslant 4$ and $p, q\geqslant 1$, together with terms of the form $D^r f_1$ where $2\leqslant r\leqslant 3$. Such a differential equation has the form

$$\begin{aligned}(b_1 D^2+b_2 D+b_3)\, f_2=(b_4 D^3+b_5 D^2)\, f_1&+b_6(D^2 f_0)(D^2 f_2)\\ &+b_7(D^2 f_1)^2+b_8(Df_1)(D^3 f_1)\\ &+b_9(D^2 f_2)(Df_0)+b_{10}(D^2 f_0)(Df_2).\end{aligned}$$

It follows by substituting the computed derivatives into the differential equation, equating coefficients of powers of W to obtain a system of homogeneous linear equations and solving this system, that the solution space is 4 dimensional and that

$$\begin{aligned}b_3&=-4b_1-2b_2+240b_4+120b_5,\\ b_6&=-\tfrac{11}{2}b_1-\tfrac{3}{2}b_2-72b_4-70b_5,\\ b_7&=\tfrac{47}{4}b_1+\tfrac{23}{4}b_2-1236b_4-875b_5,\\ b_8&=-\tfrac{293}{4}b_1-\tfrac{85}{4}b_2-264b_4-420b_5,\\ b_9&=13b_1+3b_2+144b_4+140b_5,\\ b_{10}&=\tfrac{35}{2}b_1+\tfrac{7}{2}b_2+336b_4+280b_5,\end{aligned}$$

where b_1, b_2, b_4, b_5 are arbitrary. The system therefore has nontrivial solutions.

COROLLARY 4.2.

$$\begin{aligned}\mu_n^{(2)}=n^2\left(\frac{97}{136}n-\frac{20}{17}\right)\mu_n^{(1)}&+\sum_{j=1}^{n-1}\binom{2n}{2j-2}\mu_j^{(0)}\mu_{n-j}^{(2)}j(n-j)\left(8n-\frac{115}{17}j\right)\\ &+\sum_{j=1}^{n-1}\binom{2n}{2j}\mu_j^{(1)}\mu_{n-j}^{(1)}j(n-j)\left(\frac{11697}{34}j(n-j)-\frac{3899}{68}n^2\right).\end{aligned}$$

Proof. By setting $b_1=4$, $b_2=6$, $b_4=97/136$, $b_5=-20/17$ we have

$$\begin{aligned}(4D^2+6D+2)\, f_2=(\tfrac{97}{136}D^3-\tfrac{20}{17}D^2)\, f_1&+8(D^2 f_2)(Df_0)\\ &+\tfrac{21}{17}(D^2 f_0)(Df_2)+\tfrac{3899}{17}(D^2 f_1)^2\\ &-\tfrac{3899}{34}(Df_1)(D^3 f_1),\end{aligned}$$

and the result follows immediately. ∎

This establishes the recurrence equation for $\mu_n^{(2)}(1^n)$, corresponding to the case of simple ramification, conjectured by Graber and Pandharipande [9].

5. A "SIMPLER" RELATIONSHIP FOR THE DOUBLE TORUS

In view of the combinatorial interpretation of the differential operator D, a differential equation such as the one given in the proof of Corollary 4.2 (or, equivalently, a recurrence equation) may have a more direct combinatorial explanation, which may in turn suggest a geometrical explanation. For this purpose it is therefore prudent to look for a differential equation with fewer terms, and whose coefficients are, at the very least, potentially more susceptible to combinatorial explanation. Since there are four independent parameters b_1, b_2, b_4, b_5, we may impose three further conditions to lessen the number of terms in the differential equation, and then divide out the remaining parameter. The obvious conditions to apply are those that remove terms from the differential equation. The next two corollaries give instances where these criteria are met.

The first instance is a second order linear differential equation for f_2 with simple coefficients that has contributions from the sphere, torus and the double torus on the right hand side.

COROLLARY 5.1.

$$(2D^2-6D+2)\,f_2=(\tfrac{1}{24}\,D^3-\tfrac{1}{10}\,D^2)\,f_1+2(D^2f_0)(D^2f_2)$$
$$+25(D^2f_1)^2+12(Df_1)(D^3f_1).$$

Proof. Set $b_9=b_{10}=0$, and $b_1=2$. ∎

The second instance is obtained by imposing conditions to eliminate the presence of contributions from the double torus (and therefore the sphere) on the right hand side of the differential equation. This gives a first order linear differential equation for f_2 with simple coefficients that has contributions only from the torus on the right hand side.

COROLLARY 5.2.

$$(2D+3)\,f_2=\tfrac{1}{4}(\tfrac{5}{12}\,D^3-\tfrac{3}{5}\,D^2)\,f_1+14(D^2f_1)^2-7(Df_1)(D^3f_1).$$

Proof. Set $b_6=b_9=b_{10}=0$. ∎

It remains to determine whether a differential equation can be found with fewer terms and with equally simple coefficients. To assist in the search, let Δf, for a polynomial f in W, be defined to be (k, l), where k and l are, respectively, the lowest and highest degrees of the terms of f in W. We will refer to this as the *degree span* of f. Then it is readily seen that $\Delta D^r f_0 = (r-2, 2r-5)$ for $r \geqslant 4$, $\Delta D^3 f_0 = (0, 1)$, $\Delta D^2 f_0 = (-1, 0)$ and $\Delta D f_0 = (-2, 0)$. Also $\Delta D f_1 = (0, 2)$, $\Delta D^r f_1 = (r, 2r)$ for $r \geqslant 2$, and $\Delta D^r f_2 = (r+2, 2r+5)$ for $r \geqslant 0$.

Consider $(b_1 D + b_2) f_2$ where b_1 and b_2 are generic real numbers. Then, from the above spans, $\Delta(b_1 D + b_2) f_2 = (2, 7)$. We now construct another expression from f_1 and f_0 that has the same span. From the above expressions for spans, $\Delta(D^2 f_1)(D f_1) = (2, 6)$, $\Delta D^2 f_1 = (2, 4)$, $\Delta D^3 f_1 = (3, 6)$, and $\Delta D^6 f_0 = (4, 7)$. Then, for generic $b_3, ..., b_6$, we have $\Delta(b_3 D^6 f_0 + b_4 (D^2 f_1)(D f_1) + b_5 D^2 f_1 + b_6 D^3 f_1) = (2, 7)$. Therefore we consider the differential equation

$$(b_1 D + b_2) f_2 = b_3 D^6 f_0 + b_4 (D^2 f_1)(D f_1) + b_5 D^2 f_1 + b_6 D^3 f_1.$$

COROLLARY 5.3.

$$f_2 = \frac{1}{6!} (7D^3 - 8D^2) f_1 - \frac{14}{15} (D^2 f_1)(D f_1).$$

Moreover,

$$\mu_n^{(2)} = 2 \binom{2n+2}{2} \left(\frac{1}{6!} (7n^3 - 8n^2) \mu_n^{(1)} - \frac{7n}{15} \sum_{j=1}^{n-1} j(n-j) \binom{2n}{2j} \mu_j^{(1)} \mu_{n-j}^{(1)} \right).$$

Proof. By the argument of the previous section we obtain a solution space of dimension 1, for the system of linear equations, and this gives a unique differential equation up to a normalizing factor. The recurrence equation is obtained by comparing coefficients of x^n on each side of the equation. ∎

The above corollary gives an equation for f_2 that certainly has fewer terms than the differential equations of the earlier corollaries. Morover, $7 = 2^3 - 1$, $8 = 2^3$ and $15 = 3!!$, (where $n!! = (2n)!/2^n n!$, the number of perfect matchings on a $2n$-set) each of which is a number with a known combinatorial interpretation.

Implicit in the above discussion is the assumption that there is no linear recurrence equation for $\mu_n^{(2)}$ that does not involve $\mu_n^{(1)}$ and $\mu_n^{(0)}$. This can be verified easily through an algebraic argument that appeals to the fact that e^w is a transcendental series in w.

APPENDIX

A. *Ramified Coverings of the Sphere by the Triple Torus*

$$
\begin{aligned}
F_3(x,\mathbf{p}) = {} & \frac{1}{2^3 9!}\,\frac{35\psi_7 - 147\psi_6 + 205\psi_5 - 93\psi_4}{(1-\psi_1)^5} \\
& + \frac{1}{2^3 9!}(-930\psi_2\psi_3 + 607\psi_4{}^2 + 1501\psi_3{}^2 \\
& + 2329\psi_2\psi_4 + 539\psi_2\psi_6 + 1006\psi_3\psi_5 \\
& - 3078\psi_3\psi_4 - 1938\psi_2\psi_5)/(1-\psi_1)^6 \\
& + \frac{1}{2^3 9!}(13452\psi_2\psi_3\psi_4 + 2915\psi_3{}^3 - 16821\psi_2\psi_3{}^2 - 12984\psi_2{}^2\psi_4 \\
& + 12885\psi_2{}^2\psi_3 + 4284\psi_2{}^2\psi_5 - 1395\psi_2{}^3)/(1-\psi_1)^7 \\
& + \frac{1}{2^3 9!}\,\frac{22260\psi_2{}^3\psi_4 + 43050\psi_2{}^2\psi_3{}^2 - 55300\psi_2{}^3\psi_3 + 10710\psi_2{}^4}{(1-\psi_1)^8} \\
& + \frac{1}{2^3 9!}\,\frac{81060\psi_2{}^4\psi_3 - 31220\psi_2{}^5}{(1-\psi_1)^9} + \frac{245}{20736}\,\frac{\psi_2{}^6}{(1-\psi_1)^{10}}
\end{aligned}
$$

B. *The Expression for* $5760(T_0G_2 - T_1)(1-\psi_1)^7$

Listed below for $i = 1, \ldots, 6$ is C_i, the contribution to $5760(T_0G_2 - T_1)(1-\psi_1)^7$ from terms of total degree i in the q_j's. The $M_{l,m}$, N_k and ψ_i are series in the q_j's.

$$
\begin{aligned}
C_1 = {} & -240M_{2,0} - 125M_{1,1} + 250M_{1,0} - 5M_{0,0} - 7M_{-2,2} + 3\psi_4 \\
& - 29\psi_3 + 21\psi_2 + 12M_{-2,3} - 5M_{-2,4} + 5\psi_5
\end{aligned}
$$

$$
\begin{aligned}
C_2 = {} & 20\psi_1 M_{-2,4} + 29\psi_3{}^2 - 9\psi_1\psi_4 + 87\psi_1\psi_3 - 63\psi_1\psi_2 \\
& - 48\psi_1 M_{-2,3} - 490\psi_2 M_{1,0} + 130\psi_2 M_{0,0} + 28\psi_1 M_{-2,2} \\
& + 7\psi_3\psi_0 - 7\psi_3 M_{-2,0} - 29\psi_2 M_{-2,3} + 5\psi_5\psi_0 - 5\psi_5 M_{-2,0} \\
& - 12\psi_4\psi_0 + 12\psi_4 M_{-2,0} - 15\psi_1\psi_5 - 29M_{-2,2}\psi_3 \\
& + 50M_{-2,2}\psi_2 - 15M_{-2,1}\psi_4 + 36M_{-2,1}\psi_3 - 21M_{-2,1}\psi_2 \\
& + 625\psi_1 M_{1,1} - 1250\psi_1 M_{1,0} + 25\psi_1 M_{0,0} - 5N_4 + 12N_3 \\
& - 7N_2 - 120\psi_3 M_{0,0} + 30\psi_2\psi_3 - 79\psi_2{}^2 + 44\psi_4\psi_2 + 1200\psi_1 M_{2,0}
\end{aligned}
$$

$$
\begin{aligned}
C_3 = {} & 15\psi_1{}^2\psi_5 + 36\psi_1\psi_4\psi_0 - 36\psi_1\psi_4 M_{-2,0} - 15\psi_1\psi_5\psi_0 + 15\psi_1\psi_5 M_{-2,0} \\
& + 45\psi_1 M_{-2,1}\psi_4 - 108\psi_1 M_{-2,1}\psi_3 + 63\psi_1 M_{-2,1}\psi_2 + 87\psi_1 M_{-2,2}\psi_3 \\
& + 20\psi_1 N_4 - 42\psi_1{}^2 M_{-2,2} + 9\psi_1{}^2\psi_4 - 87\psi_1{}^2\psi_3 + 63\psi_1{}^2\psi_2 \\
& + 72\psi_1{}^2 M_{-2,3} - 30\psi_1{}^2 M_{-2,4} - 29N_2\psi_3 + 50N_2\psi_2 - 15N_1\psi_4 \\
& + 36N_1\psi_3 - 21N_1\psi_2 + 1960\psi_1\psi_2 M_{1,0} - 48\psi_1 N_3 + 28\psi_1 N_2 \\
& - 520\psi_1\psi_2 M_{0,0} + 480\psi_1\psi_3 M_{0,0} - 88\psi_1\psi_4\psi_2 - 29\psi_3{}^2 M_{-2,0} \\
& - 58\psi_1\psi_3{}^2 - 116M_{-2,1}\psi_2\psi_3 + 86\psi_2\psi_3 M_{-2,0} - 60\psi_1\psi_2\psi_3 \\
& + 44\psi_2\psi_4\psi_0 - 86\psi_2\psi_3\psi_0 - 44\psi_2\psi_4 M_{-2,0} + 21\psi_0\psi_2{}^2 \\
& - 21M_{-2,0}\psi_2{}^2 + 158\psi_1\psi_2{}^2 + 100M_{-2,1}\psi_2{}^2 - 84\psi_2{}^2 M_{-2,2} \\
& + 29\psi_3{}^2\psi_0 - 1250\psi_1{}^2 M_{1,1} + 2500\psi_1{}^2 M_{1,0} - 50\psi_1{}^2 M_{0,0} \\
& - 2400\psi_1{}^2 M_{2,0} - 21\psi_1\psi_3\psi_0 - 150\psi_1 M_{-2,2}\psi_2 + 21\psi_1\psi_3 M_{-2,0} \\
& + 87\psi_1\psi_2 M_{-2,3} + 40\psi_2{}^3 - 29N_3\psi_2 - 245\psi_2{}^2 M_{0,0} - 5\psi_5 N_0 \\
& - 7\psi_3 N_0 + 12\psi_4 N_0 + 200\psi_3\psi_2{}^2
\end{aligned}
$$

$$
\begin{aligned}
C_4 = {} & 200\psi_2{}^2\psi_3\psi_0 - 200\psi_2{}^2\psi_3 M_{-2,0} - 100\psi_0\psi_2{}^3 - 3\psi_1{}^3\psi_4 + 29\psi_1{}^3\psi_3 \\
& - 21\psi_1{}^3\psi_2 - 48\psi_1{}^3 M_{-2,3} + 20\psi_1{}^3 M_{-2,4} - 140\psi_2{}^3 M_{-2,1} \\
& + 100M_{-2,0}\psi_2{}^3 + 44\psi_1{}^2\psi_4\psi_2 - 2940\psi_1{}^2\psi_2 M_{1,0} + 780\psi_1{}^2\psi_2 M_{0,0} \\
& - 720\psi_1{}^2\psi_3 M_{0,0} + 28\psi_1{}^3 M_{-2,2} + 172\psi_1\psi_2\psi_3\psi_0 + 88\psi_1\psi_2\psi_4 M_{-2,0} \\
& - 172\psi_1\psi_2\psi_3 M_{-2,0} + 232\psi_1 M_{-2,1}\psi_2\psi_3 - 200\psi_1 M_{-2,1}\psi_2{}^2 \\
& + 29\psi_1{}^2\psi_3{}^2 + 72\psi_1{}^2 N_3 - 42\psi_1{}^2 N_2 - 30\psi_1{}^2 N_4 - 88\psi_1\psi_2\psi_4\psi_0 \\
& - 58\psi_1\psi_3{}^2\psi_0 + 58\psi_1\psi_3{}^2 M_{-2,0} + 30\psi_1{}^2\psi_2\psi_3 - 42\psi_1\psi_0\psi_2{}^2 \\
& + 42\psi_1 M_{-2,0}\psi_2{}^2 + 86\psi_3 N_0\psi_2 - 79\psi_1{}^2\psi_2{}^2 + 100N_1\psi_2{}^2 - 29\psi_3{}^2 N_0 \\
& - 21\psi_2{}^2 N_0 - 40\psi_1\psi_2{}^3 - 116N_1\psi_2\psi_3 - 108\psi_1 N_1\psi_3 + 63\psi_1 N_1\psi_2 \\
& + 735\psi_1\psi_2{}^2 M_{0,0} - 36\psi_1\psi_4 N_0 + 21\psi_1\psi_3 N_0 + 87\psi_1 N_3\psi_2 \\
& + 15\psi_1\psi_5 N_0 - 45\psi_1{}^2 M_{-2,1}\psi_4 + 87\psi_1 N_2\psi_3 - 150\psi_1 N_2\psi_2 \\
& + 45\psi_1 N_1\psi_4 - 200\psi_1\psi_3\psi_2{}^2 - 21\psi_1{}^2\psi_3 M_{-2,0} - 87\psi_1{}^2\psi_2 M_{-2,3} \\
& + 15\psi_1{}^2\psi_5\psi_0 - 15\psi_1{}^2\psi_5 M_{-2,0} - 36\psi_1{}^2\psi_4\psi_0 + 36\psi_1{}^2\psi_4 M_{-2,0} \\
& + 108\psi_1{}^2 M_{-2,1}\psi_3 - 87\psi_1{}^2 M_{-2,2}\psi_3 + 150\psi_1{}^2 M_{-2,2}\psi_2 \\
& + 21\psi_1{}^2\psi_3\psi_0 - 63\psi_1{}^2 M_{-2,1}\psi_2 - 5\psi_1{}^3\psi_5 + 168\psi_1\psi_2{}^2 M_{-2,2}
\end{aligned}
$$

$$+2400\psi_1{}^3M_{2,0}+1250\psi_1{}^3M_{1,1}-2500\psi_1{}^3M_{1,0}+50\psi_1{}^3M_{0,0}$$
$$-84N_2\psi_2{}^2-44\psi_4N_0\psi_2+140\psi_2{}^4$$

$$\begin{aligned}C_5 = {} & 100\psi_2{}^3N_0+200\psi_1\psi_2{}^2\psi_3M_{-2,0}+1960\psi_1{}^3\psi_2M_{1,0}+140\psi_1\psi_2{}^3M_{-2,1} \\
& -200\psi_1\psi_2{}^2\psi_3\psi_0+140\psi_2{}^4\psi_0-140\psi_2{}^4M_{-2,0}-48\psi_1{}^3N_3 \\
& +28\psi_1{}^3N_2+20\psi_1{}^3N_4-7\psi_1{}^4M_{-2,2}+12\psi_1{}^4M_{-2,3}-5\psi_1{}^4M_{-2,4} \\
& -12\psi_1{}^3\psi_4M_{-2,0}+100\psi_1\psi_0\psi_2{}^3-100\psi_1M_{-2,0}\psi_2{}^3 \\
& -520\psi_1{}^3\psi_2M_{0,0}+480\psi_1{}^3\psi_3M_{0,0}-36\psi_1{}^3M_{-2,1}\psi_3 \\
& +21\psi_1{}^3M_{-2,1}\psi_2+29\psi_1{}^3M_{-2,2}\psi_3-50\psi_1{}^3M_{-2,2}\psi_2 \\
& -7\psi_1{}^3\psi_3\psi_0+7\psi_1{}^3\psi_3M_{-2,0}+29\psi_1{}^3\psi_2M_{-2,3}-5\psi_1{}^3\psi_5\psi_0 \\
& +5\psi_1{}^3\psi_5M_{-2,0}+12\psi_1{}^3\psi_4\psi_0-735\psi_1{}^2\psi_2{}^2M_{0,0} \\
& -21\psi_1{}^2\psi_3N_0-87\psi_1{}^2N_3\psi_2-15\psi_1{}^2\psi_5N_0+15\psi_1{}^3M_{-2,1}\psi_4 \\
& +36\psi_1{}^2\psi_4N_0-21\psi_1{}^2M_{-2,0}\psi_2{}^2+100\psi_1{}^2M_{-2,1}\psi_2{}^2 \\
& -84\psi_1{}^2\psi_2{}^2M_{-2,2}+29\psi_1{}^2\psi_3{}^2\psi_0-29\psi_1{}^2\psi_3{}^2M_{-2,0} \\
& +86\psi_1{}^2\psi_2\psi_3M_{-2,0}-116\psi_1{}^2M_{-2,1}\psi_2\psi_3-87\psi_1{}^2N_2\psi_3 \\
& +150\psi_1{}^2N_2\psi_2-45\psi_1{}^2N_1\psi_4+108\psi_1{}^2N_1\psi_3-63\psi_1{}^2N_1\psi_2 \\
& -86\psi_1{}^2\psi_2\psi_3\psi_0-44\psi_1{}^2\psi_2\psi_4M_{-2,0}-172\psi_1\psi_3N_0\psi_2 \\
& +88\psi_1\psi_4N_0\psi_2+44\psi_1{}^2\psi_2\psi_4\psi_0+232\psi_1N_1\psi_2\psi_3-200\psi_1N_1\psi_2{}^2 \\
& +58\psi_1\psi_3{}^2N_0+42\psi_1\psi_2{}^2N_0+168\psi_1N_2\psi_2{}^2+21\psi_1{}^2\psi_0\psi_2{}^2 \\
& -1200\psi_1{}^4M_{2,0}-625\psi_1{}^4M_{1,1}+1250\psi_1{}^4M_{1,0}-25\psi_1{}^4M_{0,0} \\
& -140N_1\psi_2{}^3-200\psi_3N_0\psi_2{}^2\end{aligned}$$

$$\begin{aligned}C_6 = {} & 5\psi_1{}^3\psi_5N_0-12\psi_1{}^3\psi_4N_0+7\psi_1{}^3\psi_3N_0+29\psi_1{}^3N_3\psi_2 \\
& -116\psi_1{}^2N_1\psi_2\psi_3+86\psi_1{}^2\psi_3N_0\psi_2-44\psi_1{}^2\psi_4N_0\psi_2+240\psi_1{}^5M_{2,0} \\
& +125\psi_1{}^5M_{1,1}-250\psi_1{}^5M_{1,0}+5\psi_1{}^5M_{0,0}+100\psi_1{}^2N_1\psi_2{}^2 \\
& -29\psi_1{}^2\psi_3{}^2N_0-21\psi_1{}^2\psi_2{}^2N_0-84\psi_1{}^2N_2\psi_2{}^2 \\
& +29\psi_1{}^3N_2\psi_3-50\psi_1{}^3N_2\psi_2+15\psi_1{}^3N_1\psi_4-36\psi_1{}^3N_1\psi_3 \\
& +21\psi_1{}^3N_1\psi_2+245\psi_1{}^3\psi_2{}^2M_{0,0}+12\psi_1{}^4N_3-7\psi_1{}^4N_2 \\
& -5\psi_1{}^4N_4+200\psi_1\psi_3N_0\psi_2{}^2-100\psi_1\psi_2{}^3N_0+140\psi_1N_1\psi_2{}^3 \\
& -490\psi_1{}^4\psi_2M_{1,0}+130\psi_1{}^4\psi_2M_{0,0}-120\psi_1{}^4\psi_3M_{0,0}-140\psi_2{}^4N_0\end{aligned}$$

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REFERENCES

1. I. P. Goulden, A differential operator for symmetric functions and the combinatorics of multiplying transpositions, *Trans. Amer. Math. Soc.* **344** (1994), 421–440.
2. I. P. Goulden and D. M. Jackson, A proof of a conjecture for the number of ramified coverings of the sphere by the torus, *J. Combin. Theory Ser. A*, to appear, math. AG/9902009.
3. I. P. Goulden and D. M. Jackson, Transitive factorizations into transpositions and holomorphic mappings on the sphere, *Proc. Amer. Math. Soc.* **125** (1997), 51–60.
4. I. P. Goulden and D. M. Jackson, "Combinatorial Enumeration," Wiley, New York, 1983.
5. I. P. Goulden, D. M. Jackson, and A. Vainshtein, The number of ramified coverings of the sphere by the torus and surfaces of higher genera, *Ann. of Combin.*, to appear, math.AG/9902125.
6. A. Hurwitz, Ueber Riemann'sche Flächen mit gegebenen Verzweigungspunkten, *Math. Ann.* **39** (1891), 1–60.
7. D. M. Jackson, Some combinatorial problems associated with products of conjugacy classes of the symmetric group, *J. Combin. Theory Ser. A* **49** (1988), 363–369.
8. V. Strehl, Minimal transitive products of transpositions—The reconstruction of a proof by A. Hurwitz, *Sém. Lothar. Combin.* **37** (1996), Art. S37c.
9. R. Vakil, Enumerative geometry of plane curves of low genus, math.AG/9803007.
10. R. Vakil, Recursions, formulas, and graph-theoretic interpretations of ramified coverings of the sphere by surfaces of genus 0 and 1, math.CO/9812105.