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THE NUMBER OF SEMIGROUPS OF ORDER n

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ABSTRACT. The number of semigroups on n elements is counted asymptotically for large n. It is shown that "almost all" semigroups on n elements have the following property: The n elements are split into sets A, B and there is an $e \in B$ so that whenever $x, y \in A$, $xy \in B$, but if x or y is in B, $xy = e$.

1. **The problem.** Fix a labelled n-element set $[n] = \{1, \dots, n\}$. A *semigroup* SG on $[n]$ is an associative binary operation (denoted by concatenation). Let $S(n)$ denote the number of semigroups on $[n]$. We find an asymptotic approximation to $S(n)$. Let

$$f(t) = \binom{n}{t} t^{1+(n-t)^2}. \tag{1.1}$$

THEOREM 1.

$$S(n) = \left[\sum_{t=1}^{n} f(t)\right](1 + o(1)). \tag{1.2}$$

Define $t_0 = t_0(n)$ as that t which maximizes $f(t)$. One can show

$$t_0 \sim n/(2 \ln n) \tag{1.3}$$

and $f(t)$ has a sharp peak at t_0. Equation (1.2) simplifies to

$$S(n) = f(t_0)(1 + o(1)) \tag{1.4}$$

except in those "rare" instances when $f(t_0 + 1)$ or $f(t_0 - 1)$ are "near" $f(t_0)$.

For the remainder of the paper we adopt the convention that any inequality about functions of n is meant to be true only for all n sufficiently large, where how large depends on the statement.

2. **Construction of the semigroups.** Let $A \subseteq [n]$, $|A| = n - t$, $t \geqq 1$. Set $B = A^0$. We construct a family $\mathbb{S}(A)$ of semigroups on $[n]$ as follows:

(i) Select $e \in B$.

(ii) For $x, y \in A$ define xy to be an arbitrary member of B.

(iii) For $x \in B$ or $y \in B$ define $xy = e$.

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Then for all $x, y, z \in [n]$, $xy \in B$ so $(xy)z = e$. Similarly, $x(yz) = e$. This yields

$$|\mathfrak{S}(A)| = t^{1+(n-t)^2} \tag{2.1}$$

semigroups. Call a semigroup SG of type T (for trivial) if it is of the above form for some A, e.

Let $\mathfrak{S}_1(A)$ denote those semigroups $SG \in \mathfrak{S}(A)$ such that for no $a \in A$ is $ax = xa = e$ for all $x \in [n]$. Then

$$|\mathfrak{S}_1(A)| = (1 + o(1))|\mathfrak{S}(A)| \tag{2.2}$$

and the $\mathfrak{S}_1(A)$ are disjoint. Hence, there are $(\sum_{t=0}^{n} f(t))(1 + o(1))$ semigroups of type T. This implies

$$S(n) \geqq \left[\sum_{t=0}^{n} f(t)\right](1 + o(1)). \tag{2.3}$$

3. **An upper-bound.** Let $\mathfrak{T}(A)$ denote the family of semigroups for which A is a minimal (in cardinality) set of generators. (I.e. all $x \in [n]$ may be expressed as $x = a_1 \cdots a_k$ for some $k \geqq 1, a_1, \ldots, a_k \in A$.) Set $N(A) = |\mathfrak{T}(A)|$. Then

$$S(n) \leqq \sum_{A \subseteq [n]} N(A). \tag{3.1}$$

If $a_1, a_2 \in A$ then

$$a_1 a_2 \in \{a_1, a_2\} \cup B, \tag{3.2}$$

as otherwise $A - \{a_1 a_2\}$ would be a smaller set of generators. We bound $N(A)$ by constructing all $SG \in \mathfrak{T}(A)$ as follows:

(i) For $a_1, a_2 \in A$ choose $a_1 a_2 \in \{a_1, a_2\} \cup B$. (This may be done in $(t + 2)^{(n-t)^2}$ ways.)

(ii) Let A' be a minimal (in cardinality) subset of A so that $A'A \supseteq AA \cap B$. For each $b \in AA \cap B$ there exist a_b', a_b so that $b = a_b' a_b$. The set $X = \{a_b' \colon b \in B\}$ satisfies $XA \supseteq AA \cap B$, so that

$$|A'| \leqq |X| \leqq |B| = t. \tag{3.3}$$

Since $A' \subseteq A$, $|A'| \leqq \min(t, n - t)$.

Now for all $a \in A'$, $b \in B$ choose ab arbitrarily. (This may be done in at most $n^{|A'||B|} \leqq n^{t \min(t,n-t)}$ ways.)

Claim 1. SG is determined by $AA \cup A'B$.

PROOF. As all $x, y \in [n]$ are finite products of A's, it suffices to show that all $a_1 a_2 \cdots a_k$ are determined. We show this by induction, it being trivial for $k \leqq 2$. For $k > 2$ set $a_1 a_2 = b$. If $b \in A$, $a_1 a_2 \cdots a_k = b a_3 \cdots a_k$ is determined by induction. If $b \notin A$, $b \in B \cap AA$ so $b = cd$ for some $c \in A'$, $d \in A$. Then $a_1 \cdots a_k = c(d a_3 \cdots a_k)$. Now $x = d a_3 \cdots a_k$ is determined by induction and cx is determined since $A'[n]$ is determined.

We have shown

$$N(A) \leqq (t+2)^{(n-t)^2} n^{t\min(t,n-t)}. \tag{3.4}$$

Thus there are at most $\binom{n}{t}(t+2)^{(n-t)^2} n^{t\min(t,n-t)}$ semigroups having a minimal set of generators of cardinality t.

From (3.1), (3.4),

$$S(n) \leqq \sum_{t=1}^{n} \binom{n}{t}(t+2)^{(n-t)^2} n^{t\min(t,n-t)} \tag{3.5}$$

so that

$$S(n) = [n/(2e+o(1))\ln n]^{n^2}. \tag{3.6}$$

Let $\varepsilon > 0$ be fixed and small ($\varepsilon = 10^{-6}$ will do). By elementary calculations and (2.3),

$$\sum{}' \binom{n}{t}(t+2)^{(n-t)^2} n^{t\min(t,n-t)} = o(S(n))$$

where $\sum'$ runs over t, $|t-t_0| > \varepsilon n/\ln n$. Thus: *Almost all semigroups are in* $\mathfrak{T}(A)$ *for some* A, $|A| = n-t$, $|t-t_0| \leqq \varepsilon n/\ln n$. *We restrict our attention to* t *in this range for the duration of this paper.*

4. **The easy case.** Our ultimate objective is to show

$$N(A) = t^{1+(n-t)^2}(1+o(1)) \tag{4.1}$$

for $|A| = n-t$, $|t-t_0| < \varepsilon n/\ln n$. In this section we show the corresponding result for a restricted class of semigroups.

We say that a semigroup in $\mathfrak{T}(A)$ has property E (for easy) if $a_1, a_2 \in A$ imply $a_1a_2 \neq a_1$ and $a_1a_2 \neq a_2$. Let $\mathfrak{T}^*(A)$ denote the subclass of $\mathfrak{T}(A)$ satisfying property E and $N^*(A) = |\mathfrak{T}^*(A)|$. We shall show

$$N^*(A) = t^{1+(n-t)^2}(1+o(1)) \tag{4.2}$$

for $|A| = n-t$, $|t-t_0| < \varepsilon n/\ln n$.

For $SG \in \mathfrak{T}^*(A)$ set

$$G_{ax} = \{b \in B\colon ab = x\} \qquad \text{for } a \in A,\ x \in [n].$$

Set

$$F(a) = \max_{x\in[n]} |G_{ax}|$$

and

$$\delta = t - \min_{a\in A} F(a). \tag{4.3}$$

The number δ defined above may be thought of as follows. Consider the rows aB, $a \in A$. Each row has a most frequent entry. δ gives a uniform upper bound (over $a \in A$) for the number of entries in aB not equal to the most frequent one. Trivial semigroups have $\delta = 0$. Let $\mathfrak{T}^*_\delta(A)$ be those semigroups in $\mathfrak{T}^*(A)$ with given δ and $N^*_\delta(A) = |\mathfrak{T}^*_\delta(A)|$.

We now bound $N_\delta^*(A)$. The semigroups in $\mathfrak{T}_\delta^*(A)$ may be constructed by the following procedure.

(i) Pick $g \in A$ so that $t - \delta = F(g)$. (This may be done in $n - t \leqq n$ ways.)

(ii) Set A' equal a minimal subset of A containing g so that $A'A \supseteq AA \cap B$. fix A'. (Since $|A'| \leqq t$, this may be done in at most n^t ways.)

(iii) Determine $A'B$. For each $a \in A'$, $F(a) \geqq t - \delta$; so there exists $x \in [n]$, $|G_{ax}| \geqq t - \delta$. Now aB may be determined by selecting x (in at most n ways), G_{ax} (in $\sum_{i=t-\delta}^{t} \binom{t}{i} \leqq 1 + t^\delta \leqq n^\delta$ ways), and determining ab for $b \in B - G_{ax}$ (in $n^{|B|-|G_{ax}|} \leqq n^\delta$ ways.) Thus each row has at most $n^{2\delta+1}$ possibilities, and therefore $A'B$ has at most $n^{(2\delta+1)|A'|} \leqq n^{(2\delta+1)t}$ possibilities.

(iv) Determine gA. For each $a \in A$ we have $ga \in B$ (by property E). There are at most t^{n-t} possible gA.

(v) Define an equivalence class on A by $a \equiv b$ if $ga = gb$. Since $gA \subseteq B$ there are at most t equivalence classes and we select representatives $g = a_1, \ldots, a_s$, $s \leqq t$, of the classes in an arbitrary fashion. For $2 \leqq i \leqq s$ we determine a_iA, each in at most t^{n-t} ways, for a total of $t^{(n-t)(s-1)}$ possibilities.

(vi) Determine the remainder of AA. Let $a, c \in A$, $a \neq a_1, \ldots, a_s$. Then $a \equiv a_i$ for some i, $1 \leqq i \leqq s$. Then

$$g(ac) = (ga)c = (ga_i)c = g(a_ic) \quad \text{and} \quad ac \in G_{g,g(a_ic)}.$$

Now a_ic has already been determined, so $g(a_ic)$ has been determined, as has $G_{g,g(a_ic)}$. From the definition of δ,

$$|G_{g,g(a_ic)}| \leqq t - \delta.$$

The product ac can be determined in at most $t - \delta$ ways. This holds for $(n - t)(n - t - s)$ pairs (a, c), giving $(t - \delta)^{(n-t)(n-t-s)}$ possibilities.

By Claim 1 the semigroup is now determined.

$$N_\delta^*(A) \leqq nn^t n^{(2\delta+1)t} t^{(n-t)s} (t - \delta)^{(n-t)(n-t-s)} \tag{4.4}$$

$$\leqq n^{1+(2\delta+2)t} (1 - \delta/t)^{n^2(1+o(1))} t^{(n-t)^2} \tag{4.5}$$

and

$$\sum_{\delta=1}^{t-1} N_\delta^*(A) = o\left[t^{1+(n-t)^2}\right] \tag{4.6}$$

by an elementary calculation (using $t \sim n/(2 \ln n)$). That is, *almost all semigroups in* $\mathfrak{T}^*(A)$ *have* $\delta = 0$. Intuitively, for $\delta \geqq 1$, $N_\delta^*(A) = o(t^{(n-t)^2})$ because for most $a, c \in A$ the product ac can take at most $(t - \delta)$ values versus t in the $\delta = 0$ case.

Call an $SG \in \mathfrak{T}^*(A)$ *absurd* if $\delta = 0$ but is not trivial. Since $\delta = 0$ each aB, $a \in A$, is constant.

Claim 2. If $SG \in \mathfrak{T}^*(A)$ and $AB =$ constant, then SG is trivial.

PROOF. Say $AB = e$. For any $a \in A$, $aa \in B$ so $aaa = e$. Now $e \notin A$, since, if it were, $A - \{e\}$ would generate e and, therefore, $[n]$, contradicting the minimality of $|A|$. So $e \in B$. Let $b \in B$, $x \in [n]$. Since A generates $[n]$, b

$= a_1 \cdots a_s$ $(s \geqq 2)$, $x = a_{s+1} \cdots a_{s+t}$ $(t \geqq 1)$ so $bx = a_1 \cdots a_{s+t} = e$ as $s + t \geqq 3$. That is, $B[n] = e$ so SG is trivial.

Claim 3. Let $SG \in \mathfrak{T}^*(A)$, $\delta = 0$, $a_1, a_2 \in A$, $a_1 B = e_1 \neq e_2 = a_2 B$. Then $a_1 A \cap a_2 A = \varnothing$.

PROOF. Suppose $x, y \in A$, $a_1 x = a_2 y$. For any $z \in A$, $xz, yz \in B$ so $e_1 = a_1(xz) = (a_1 x)z = (a_2 y)z = a_2(yz) = e_2$, a contradiction.

The conditions on absurd $SG \in \mathfrak{T}^*(A)$ imposed by Claim 3 are sufficiently stringent that we may easily show (details omitted) that they are "small" in number (i.e. $o(t^{1+(n-t)^2})$). Hence, almost all $SG \in \mathfrak{T}^*(A)$ are trivial, yielding (4.2).

5. **The general case–outline.** Theorem 1 is implied by (4.1). We give a brief outline of the proof of (4.1).

We let $\mathfrak{T}(A, \delta, \ldots)$ and $N(A, \delta, \ldots)$ denote the set, and number, of semigroups with parameters A, δ, ... By $N(A, \delta, \ldots)$ "small" we always mean in comparison to $t^{(n-t)^2}$.

For a particular counting scheme let $\nu(a)$ denote the number of possible rows aA and $\mu(a) = \nu(a)/t^{n-t}$. By (3.2) all $\nu(a) \leqq (t+2)^{n-t}$ so $\mu(a) \leqq n^4$.

For $a \in A$ set

$$S_a = \{x \in A: ax = x\} \quad \text{and} \quad L = \{a \in A: |S_a| \geqq .01n\}, \qquad l = |L|.$$

If $a \in L$ there are less than 2^n choices for S_a and $n^{.99n}$ choices for $a([n] - S_a)$. There are less than $2^n n^{.99n} = t^{(.99+o(1))n}$ choices for $a[n]$. We may determine $SG \in \mathfrak{T}(A, L)$ by determining LA, then $(A - L)\,A$, then defining A' and determining $A'B$. If l is "large", $N(A, L)$ is "small". Most $SG \in \mathfrak{T}(A)$ have $l = o(n)$ –which we assume for the duration.

We modify the bounding of $N_\delta^*(A)$ in §4 by (4.4) to $N(A, L, \delta)$. We need a slightly different definition for δ:

$$\delta = t - \min_{a \in A - L} F(a). \tag{5.1}$$

We fix $g \in A - L$ so that $F(g) = t - \delta$. We define A' as the minimal set so that $(A' \cup L)A \supseteq AA \cap B$. We bound $N(A, L, \delta)$ by first determining $L[n]$ and then following steps (i), ..., (vi) of §4. The equivalence classes, in step (v), are defined on $A - L$ and there will be $s \leqq t + 1 + .01n$ such classes (when $ga = a$, $\{a\}$ is an equivalence class). We find

$$\begin{aligned} N(A, L, \delta) \leqq\ & t^{(.99+o(1))nl} n n^t n^{(2\delta+1)t} \\ & \cdot (t+2)^{s(n-t)} (t + 2 - \delta)^{(n-t-l-s)(n-t)}. \end{aligned} \tag{5.2}$$

For $\delta \geqq 3$, $N(A, L, \delta)$ is "small". We assume $\delta \leqq 2$ for the duration. Note this implies $|aB| \leqq 3$ for all $a \in A - L$.

Set

$$I = \{a \in A - L: aa = a\},$$

$$V = \{a \in A - L - I: \exists x \in A - L, a \neq x, ax = a \text{ or } xa = a\},$$

$$|I| = i \, |V| = v.$$

For $a \in I$, $x \in A - \{a\}$,

$$ax = (aa)x = a(ax) \in \{a, x\} \cup aB,$$

at most 5 possibilities, so $\nu(a) \leqq 5^{n-t}$, $\mu(a) = t^{n(-1+o(1))}$.

Suppose $a \in V$, $x \in A - L$, $ax = a$ or $xa = a$ and xA has already been determined.

Case 1. $xa = a$. For all $c \in A$,

$$ac = (xa)c = x(ac) \in \{xa, xc\} \cup xB$$

at most 5 possibilities. Then $\nu(a) \leqq 5^{n-t}$, $\mu(a) \leqq t^{n(-1+o(1))}$.

Case 2. $ax = a$. For $c \in A$, $xc \neq c$ we have

$$ac = (ax)c = a(xc) \in \{ax\} \cup aB,$$

at most 4 possibilities. We determine aB (at most n^t ways), then aA. Since $x \notin L$, at most $.01n$ c's have $xc = c$. For these there are n possible ac. Then $\nu(a) \leqq n^t 5^n n^{.01n} = n^{n(.01+o(1))}$, so $\mu(a) \leqq t^{n(-.99+o(1))}$.

If i is large we may bound $N(A, I)$ by determining IA, then $(A - I)A$, then $A'A$. We assume $i = o(n)$ for the duration.

If v is large we may bound $N(A, V)$ by determining $(A - V)A$, then VA, then $A'A$. (A technical problem arises. For $a \in V$, $ax = a$ or $xa = a$, we want to determine aA after xA. But perhaps $x \in V$. One may order V (trying any ordering and its reverse) so at least half the $a \in V$ come after their respective x.) We may assume $v = o(n)$ for the duration.

We bound $N(A, L, I, V, \delta)$, where $l, i, v = o(n)$. We determine xA for $x \in L \cup I \cup V$–and when xA is needed before aA, $a \in V$. We define equivalence classes on the remaining $x \in A$, determining xA first for x a representative, then for the remainder of A. We achieve an expression analogous to (5.2) with a factor of $(t - \delta)^{m(n-t)}$ where m, the number of "remaining" A, is at least $(.99 + o(1))n$. The expression is "small" for $\delta > 0$. We assume $\delta = 0$.

If $L \cup I \cup V \neq \varnothing$ we have factors $\mu(a) \leqq t^{(-.99+o(1))n}$ that are not adequately counterbalanced so that $N(A, \delta, L, I, V)$ is "small". Most semigroups have $\delta = 0$, $L = I = V = \varnothing$ and are trivial.

(A final note on "filling in details". One shows $N(A, \delta, L, \ldots) \leqq t^{-.99n} t^{(n-t)^2}$ and there are less than, say, 5^n possible $\delta, L, \ldots$ so that $\sum N(A, \delta, L, \ldots)$ taken over *all* δ, L, $\ldots$, except $\delta = 0$, $L = \cdots = \varnothing$, is small.)

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