Reprint from *Not for sale*

APPLIED MECHANICS

PROCEEDINGS OF THIRTEENTH INTERNATIONAL CONGRESS OF THEORETICAL AND APPLIED MECHANICS · MOSCOW, AUGUST 1972

Edited by E. BECKER, Darmstadt, F. R. Germany and G. K. MIKHAILOV, Moscow, USSR

(Springer-Verlag, Berlin · Heidelberg · New York 1973)
(Printed in Germany)

The numerical analysis of structures

B. Fraeijs de Veubeke

Liège, Belgium

The numerical analysis of structures

B. Fraeijs de Veubeke

Liège, Belgium

Introduction

Elasticity theory, even in its simplest guise, linear with respect to geometrical deformations and with respect to material properties, has few so-called "closed form" solutions. The need for solving numerically elasticity problems and the advent of fast and powerful computing machines seem even to have partly robbed such types of solutions of their interest. It is now often easier and faster to produce numerical results of a given problem by using modern discretization methods than to have to compute the special functions involved in a closed form presentation. In the end the value of such presentations lies in the ease with which they can be manipulated to exhibit the influence of the design parameters, whereas a similar computer study of the influence of design is still a long and costly process. Structural optimization, which is the goal of such studies, receives nowadays a good deal of attention; its costeffective implementation will largely depend on the speed and accuracy with which the more classical problems can be solved, namely the determination, for a given design, of its statical and dynamical response to external loads or displacements of supports.

Matrix structural analysis

Not so long ago the concept of a "structure" was essentially that of an assemblage of simple types of elements, mostly bars and beams capable of transmitting axial loads, transverse shear loads, bending and possibly torsion moments. For such elements the classical theory of elasticity provided accurate foundations for good engineering approximations, allowing their behavior as "transmitters" to be described by a finite number of displacement and conjugate force coordinates. Thus a structure could be naturally conceived as a system with a finite, although possibly large, number of degrees of freedom, and design in force transmission problems, like bridge engineering, consisted mainly in adopting a more or less complicated topology of interconnexions between the constitutive elements.

Matrix structural analysis is essentially concerned with the mathematical formulation of the interconnexions and the resultant properties of the structure in terms of the kinematics of straining and external and internal equilibrium. Its extensive development was stimulated by its natural compatibility with electronic computer software. Its achievements remain of utmost importance in the more general context of the finite element method.

Indeed it may be suspected that the finite element idea emerged from a desire to adapt the available arsenal of matrix methods to structures of less particular nature. Many structures are determined primarily by constraints due to the functions they perform; the shape of aircraft wings is largely dominated by aerodynamic considerations: torsional stiffness requirements are generally more efficiently met by using stressed-skin construction; containers and roofs lead naturally to shell-type construction. Thus, to save the concept of a structure as a finite-dimensional system made of interconnected elements, it becomes necessary to conceive the

element as a discretized model of a piece of elastic continuum and to generalize the concept of connexion to extensive, and no more punctual, interfaces.

There are of course other advantages to this scheme that explain its growing success. Modern aerospace structures have very complex geometries; this does not inhibit the finite element method, whose pieces can be taylored to fit all the geometrical accidents. Furthermore the simpler the individual pieces, the easier a discrete representation of their elastic behaviour in terms of shaping functions. Local changes in the nature of the material or modifications in its behaviour are also easier to take into account by the subdivision process. Finally, in the presence of unsatisfactory results, it is often possible to localize the defects in idealization or discretization and to correct them while saving a large part of the previous calculations.

Force and displacement method

One of the distinctions that matrix structural analysis contributed to underline is that in the choice of either displacements or redundant forces as basic unknowns to solve for. Denoting by

q the column vector of generalized displacements
p the column vector of conjugate external forces,
e the column vector of generalized strains,
s a column vector of conjugate generalized stresses,

we may write the basic equations of matrix structural analysis as:

$$\text{kinematics} \qquad S^T q = e, \tag{1}$$

$$\text{statics} \qquad Ss = p, \tag{2}$$

$$\text{stress-strain} \qquad s = Je, \quad e = Fs. \tag{3}$$

Equation (1) is a discretized form of the linearized strain measures of a continuum

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} D_1 & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & D_3 \\ 0 & D_3 & D_2 \\ D_3 & 0 & D_1 \\ D_2 & D_1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad \text{or} \quad \varepsilon = Du \tag{4}$$

presented here in cartesian coordinates (x_1, x_2, x_3) and introducing the matrix differential operator D whose elements are partial differential operators $D_i = \partial/\partial x_i$.

Equation (2) is a discretized form of the equilibrium equations of a continuum: volume equilibrium

$$D^T \sigma + \phi = 0 \tag{5}$$

with σ the stress vector conjugate to ε and ϕ the vector of body loads, and surface equilibrium

$$N^T \sigma = t \tag{6}$$

with t the vector of surface tractions. N is derived from the operator D by substituting the direction cosines n_i of the outward normal in place of D_i. Equation (3) is a discretized form of the stress-strain relations

$$\sigma = H\varepsilon, \tag{7}$$

where H is a Hookean matrix of elasticity moduli. The duality between kinematics and statics in a continuum is prefigurated by the appearance of the same differential operator D in (4) and (5), and this is confirmed by the equality between internal and external virtual work:

$$\int_R \sigma^T \varepsilon \, dV = \int_R \sigma^T Du \, dV = \int_S (N^T \sigma)^T u \, dS - \int_R (D^T \sigma)^T u \, dV = \int_S t^T u \, dS + \int_R \phi^T u \, dV \tag{8}$$

by, in succession, using (4), integrating by parts and then using (5) and (6). If the region R over which the integration is performed is subdivided into finite elements, the result implies that there is no contribution to virtual work from the interfaces. This will be the case if we adhere

to the exact transition conditions between face + and face − of an interface:

$$u^+ = u^-, \tag{9}$$

$$t^+ + t^- = 0 \tag{10}$$

expressing respectively continuity of displacements and reciprocity of surface tractions. This explains why the discretized forms (1) and (2) show the same duality by the presence of a single matrix operator S^T, the *structural kinematical matrix*. Passing from internal to external virtual work in discretized form

$$e^T s = (S^T q)^T s = q^T S s = q^T p \tag{11}$$

we can see that their equality implies the truth of (2) if (1) is known and conversely.

The study of kinematical and statical properties of a structure is fundamentally related to the problems of inverting Eqs. (1) and (2). The solutions to the homogeneous problems introduce important concepts

$$S^T q = 0 \rightarrow q = Rr \qquad r \text{ arbitrary}, \tag{12}$$

$$Ss = 0 \rightarrow s = Xx \qquad x \text{ arbitrary}. \tag{13}$$

In (12) R is a base matrix of kinematical modes of the structure (all modes involving no strain such as rigid body modes and possible mechanisms). In (13) X is a base matrix of self-stressing modes of the structure (all modes involving no external forces). The columns of R and X are maximal sets of linearly independent solutions. Between the number of elements of the different vectors

$$\nu = n(q), \quad \sigma = n(s), \quad \varrho = n(r) \text{ and } \xi = n(x)$$

the following relation holds

$$\nu - \varrho = \sigma - \xi. \tag{14}$$

From (12) and (13) follow, by definition

$$S^T R = 0, \tag{15}$$

$$SX = 0. \tag{16}$$

Since problem (1) for q is singular, there are existence conditions for a solution

$$X^T S^T q = 0 \rightarrow X^T e = 0. \tag{17}$$

They are, in discretized form the compatibility equations for strains. Similarly the existence conditions for a solution of problem (2) are

$$R^T S s = 0 \rightarrow R^T p = 0. \tag{18}$$

They are the global equilibrium conditions for the applied external forces. Assuming the existence conditions to be satisfied, the problems have the following general solutions

$$q = S^{T*} e + Rr, \tag{19}$$

$$S = S^* p + Xx, \tag{20}$$

where the first terms are particular solutions. Computing virtual works

$$s^T e = p^T S^{*T} e + x^T X^T e = p^T S^{*T} e$$

by virtue of (17), and

$$p^T q = p^T S^{T*} e + p^T R r = p^T S^{T*} e$$

by virtue of (18).

As strains are compatible and external forces in equilibrium, both expressions must be equal. Consequently if a pseudo-inverse matrix S^* has been found for problem (2), its transpose S^{*T} can serve as pseudo-inverse S^{T*} for the solution of problem (1) and conversely. The automatic computation of the matrices involved in the solution (19) and (20) will be discussed in the next section. *The displacement method* consists in presenting the elastic problem in the form

$$p = Ss = SJe = (SJS^T)\, q = Kq \tag{21}$$

resulting from a successive application of (1), (3) and (2).

$$K = SJS^T$$

is the symmetrical *structural stiffness matrix*. Its *core* J is normally positive definite so that the solutions of the homogeneous problem $Kq = 0$ are the same as those of (12), and the existence conditions are the same as (18): global equilibrium of the applied loads. In most applications $R = 0$, that is the structure is at least isostatically restrained, in which case K is non singular and invertible. In practice the solution is carried out on computer by any form of Gaussian elimination procedure or by conjugate gradient methods. For very large problems recourse has to be made to substructure techniques. If the structure has kinematical freedoms, the pseudo-inversion of the now singular problem (21), leading to a solution of type

$$q = K^* p + Rr \tag{22}$$

akin to (19), can be carried out by the same techniques as will be discussed in the next section. Once q is determined, the strains and stresses follow from (1) and (3).

The force method is most easily presented as an application of the principle of minimum complementary energy. The most general state of stress of the structure being given by (20), the stress energy is

$$\frac{1}{2} s^T F s = \frac{1}{2} p^T S^{*T} F S^* p + x^T X^T F S^* p + \frac{1}{2} x^T X^T F X x$$

the first part of which may be ignored as independent of the unknown redundancy coordinates x.

Minimization of the energy with respect to x gives

$$X^T F X x = -X^T F S^* p \tag{23}$$

and this can be inverted because $X^T F X$ is positive definite (F is positive definite and X has linearly independent columns). Backsubstitution of the solution into (20) solves the problem of stress determination. If displacements are needed, they can be found through (3) and (19).

The combined method consists in minimizing the total complementary energy

$$\frac{1}{2} s^T F s - p^T q$$

thus assuming the displacements to be imposed and incorporating the equilibrium constraints (2) by means of a Lagrangian vector multiplier λ:

$$\frac{1}{2} s^T F s + \lambda^T (p - Ss) - p^T q = \min!$$

Minimization with respect to p identifies $\lambda - q$. Minimization with respect to s yields

$$Fs = S^T \lambda = S^T q$$

a relation which is directly obvious from (1) and (3). This and the equilibrium Eq. (2) are then solved together

$$\begin{bmatrix} F & S^T \\ S & 0 \end{bmatrix} \begin{bmatrix} s \\ -q \end{bmatrix} = \begin{bmatrix} 0 \\ p \end{bmatrix}. \tag{24}$$

The partioned matrix remains symmetrical, and is positive semi-definite.

The Gauss-Jordan algorithm

For structures made of assembled beams the determination of redundancies was traditionally done by *releases*, that is by cutting members or at least some of their load transmission capabilities, until the internal forces in the released structure could be determined by statics. This procedure which largely relies for its success on physical intuition becomes too slow, tedious and prone to error for the large arrays of more complicated finite elements. The automatic determination of all the self-stressing states by the computer itself was a considerable step towards salvation of the force method, it is due independently to *Denke* [12] and *Robinson* [18]. As will be shown the method used is still richer in the answers it provides, being capable of determining simultaneously the kinematical degrees of freedom that can play an important role in dynamic analysis, and the pseudo-inverse matrices of singular problems. The Jordan

version of the Gauss elimination algorithm can be presented as follows when applied to the transpose of the structural kinematical matrix: a $\nu \times \nu$ non singular premultiplier P is determined step by step by selection of pivots to transform S into a matrix

$$PS = \begin{bmatrix} I_{\sigma-\xi} & Q \\ 0 & 0 \end{bmatrix} \tag{25}$$

having ϱ last rows of zeros (whereby ϱ is actually determined by the algorithm and consequently $\xi = \sigma - \nu + \varrho$) and where $I_{\sigma-\xi}$ is the identity matrix in $\sigma - \xi = \nu - \varrho$ dimensions, Q being a $(\sigma - \xi) \times \xi$ matrix. P itself is structured as

$$P = \begin{bmatrix} N & 0 \\ T & L \end{bmatrix}, \tag{26}$$

where N is a non singular $(\nu - \varrho) \times (\nu - \varrho)$ matrix, L is non singular $\varrho \times \varrho$ and lower triangular, T is $\varrho \times (\nu - \varrho)$. Problem (2) becomes equivalent to

$$s_{(1)} + Q s_{(2)} = N p_{(1)}, \tag{27}$$

$$0 = T p_{(1)} + L p_{(2)}, \tag{28}$$

where s and p have been split to suit the partitioning of P and PS. The homogeneous problem (13) has clearly, from (27), the solution $s_{(2)} = x$ arbitrary $s_{(1)} = -Qx$ and we can take

$$X = \begin{bmatrix} -Q \\ I_\xi \end{bmatrix}. \tag{29}$$

Equation (28) is also clearly the existence conditions (18) and we can take

$$R^T = (T\ L). \tag{30}$$

Finally the complete Eq. (27) compared to (20) yields

$$S^* = \begin{bmatrix} N & 0 \\ 0 & 0 \end{bmatrix} \tag{31}$$

and it is easily verified that by using in (19), (30) and

$$S^{T*} = S^{*T} = \begin{bmatrix} N^T & 0 \\ 0 & 0 \end{bmatrix} \tag{32}$$

Eq. (1) is correctly satisfied.

The Gauss-Jordan algorithm gives at least one computationally automatic answer to the inversion problems of (1) and (2) and consequently to the use of a Force program. It does not however provide the answer to the search of *compact* self-stressing states [6]. Compact self-stressing states are those which concern a minimum number of elements within the structure, their use minimizes coupling between self-stressing states and provides optimum conditioning of Eq. (23). Again, if the structure is dismantled and then reassembled by progressive addition of elements, the compact self-stressings can be discovered, the problem is to replace tedious manual processes of search by some improved version of the Gauss-Jordan algorithms. Research has proceeded in the direction of weighting of the matrix elements and pivot choices [25], proper sequencing of the elements is certainly one of the important factors.

The option between displacements methods and force methods depends at first glance on the relative sizes of the systems of Eqs. (21) or (23) and this, in turn, depends primarily on the topology of interelement connexions. There are other important factors to consider. The displacement method can bypass the construction of the structural kinematical matrix by setting up directly the structural stiffness matrix and this may be viewed as economical. On the other hand the structural kinematical matrix is independent of the elasticity characteristics of the elements, hence of those dimensions, thicknesses ... areas, that do not affect the geometry of interconnexions. A single Gauss-Jordan analysis remains valid in optimization studies with respect to such design parameters. This may give the force method a regain of favour.

Displacement and equilibrium models

Except for the early introduction of the concept of shear-panel to deal in a simple manner with stressed-skin construction, a scientific elaboration of finite element models did not start before 1960, about 10 years after matrix methods were already well established.

The variational principles of elasticity theory played an important role in the mathematical theory of finite element models.

Discretization of the displacement field by a Rayleigh-Ritz process was of course directly applicable to each element, the new problem was to establish the correct transitional conditions at interfaces. The simplest concept in theory, not always so easy to realize in practice, is to adopt the continuity condition (9) suggested by the continuum approach. This was certainly intuitive in the first real two-dimensional finite element of the literature [10].

The property of continuity of displacements at interfaces is called *conformity*. Accordingly, *conforming displacement models* have their displacement patterns defined by local (nodal) displacements on their boundaries in such a way that identification of nodal displacements on an interface produces complete interface continuity.

The continuous, piecewise differentiable approximation to the displacement field, provided by conforming displacement models, has the interesting property of underestimating the strain energy of a structure submitted to prescribed loads under zero displacements of supports. It overestimates it when the loads are removed but the supports are given prescribed displacements.

Another, rather intuitive, property that is usually incorporated in displacement models is the existence of rigid body modes amongst the displacement degrees of freedom. Although experience has shown that this requirement is not necessary for convergence of the displacement field with reduction in element size, its satisfaction has invariably produced better results.

The idea that another mathematical model of finite element could produce reverse bounds to the strain energy was investigated later [14, 17] on the basis of the variational principle of stresses, or minimum of the complementary energy. Indeed the work of *Friedrichs* [2], showing how the minimum problem of one functional could be transformed into the maximum of another, then that of *Prager* and *Synge* on approximations in function space [5, 11] generalizing early considerations by *Trefftz* [1], led immediately to this possibility. Its application to finite elements required again the correct type of transitional conditions at interfaces. The one that was directly suggested by continuum theory was the reciprocity condition (10). When it is enforced, finite elements are called *diffusive*.

Accordingly *diffusive equilibrium models* have their discretized stress field, in equilibrium with body loads, defined entirely by stress resultants (the generalized forces) of boundary surface tractions.

Reciprocity of the stress resultants entails complete reciprocity of the boundary surface tractions.

Building up a library of elements of both types, which conform to displacement and diffusive equilibrium, enables the same structure to be analyzed twice with complementary effects; it is the principle of dual analysis [27].

The exact and unknown value of the strain energy can be bracketed between the numerical estimates provided by each finite element analysis, thus one has available a quantitative estimate of *energy convergence*. Numerical brackets for local displacements and even for local stress values are obtainable in principle by introducing corresponding singularities in the problem [9]. This procedure has however up to now shown itself to be too expensive for current use.

Because the displacement method and conforming displacement models are both intimately related to the variational principle of displacements, or minimum total potential, and similarly with respect to the variational principle of stresses for the force method and diffusive equilibrium models, there has been an unfortunate tendency to obliterate the distinction between models and methods. Let it be stressed that diffusive equilibrium models are easily provided with a stiffness matrix so that their assemblages can be solved, and have been solved, using a

displacement method. Conversely, self-stressing states of arrays of conforming displacement models can be found by Gauss-Jordan techniques and a force program used for their numerical evaluation. This allows dual analysis to be performed with only one type of computer software.

There is however the interesting observation that the topology of element connexions has a tendency to favour the force method for equilibrium models and the displacement method for displacement models, although the differences in problem sizes diminish with increasing sophistication of the models.

Strong and weak nodal connexions

Another important distinction brought forward by the nature of the models is the so-called strong or weak character of the generalized variables. In displacement models the nodal displacements are called strong because, once their numerical values are known, any local value of the displacement field is determined. On the contrary the equivalent nodal forces are weak because they are determined from virtual work considerations. If q denotes the boundary nodal displacements of an isolated element, the cartesian components of the displacement field of that element is representable in the form

$$\begin{bmatrix} u_1(x) \\ u_2(x) \\ u_3(x) \end{bmatrix} = u(x) = Q(x)\, q + B(x)\, b, \tag{33}$$

where $Q(x)$ is a matrix of assumed functions (generally low order polynomials) called *shaping functions* or weighting functions and $B(x)$ a matrix of *bubble functions*, so-called because they vanish on the boundary and do not consequently influence the motion of the element boundaries. The bubble coordinates b can obviously be determined by energy minimization at the element level, since they are unconnected to the neighboring elements. The generalized boundary forces g, conjugate to q, and the generalized body loads f, conjugate to the bubble coordinates, are defined by virtual work computation over the domain E of the elements and its boundary ∂E:

$$\int_E u^T \phi \, dV + \int_{\partial E} u^T t \, dS = q^T g + b^T f \tag{34}$$

with, as a consequence of (33),

$$g = \int_E Q^T \phi \, dV + \int_{\partial E} Q^T t \, dS, \tag{35}$$

$$f = \int_E B^T \phi \, dV. \tag{36}$$

Thus the generalized loads are weighted averages of the real body load and surface traction distributions. The knowledge of their numerical values determines the distributions only within equivalence classes.

Recognition of this weak character of generalized loads suggests at once that the stress information might be of the same nature. As a matter of fact the strong stress information deducible by differentiation of the displacement field through (4) and (7) is sometimes difficult to interpretate.

In correspondance with (35) and (36) it is easily demonstrated that the effect of discretizing displacements is to replace the local equilibrium Eqs. (5) and (6) by their weighted averages using the same weighting functions. Thus, as is well known, the Rayleigh-Ritz process becomes identical with a Galerkin process of weighted residuals of the equilibrium equations. This has led to the obvious extension of using the more general Galerkin approach with finite elements in field problems where no variational principle is known to exist.

In equilibrium models the situation is reversed. The local stresses are completely fixed by the generalized forces and consequently the body loads and surface tractions with which they are in equilibrium. The virtual work process now yields weighted averages for the conjugate

displacements which become the weak variables. Perhaps the idea that from the kinematical viewpoint the elements are only weakly interconnected does not appeal to engineers, which might explain the relative dearth of equilibrium elements in present day literature. However it would seem that in the majority of applications the engineer is more concerned about stresses and that a certain fuzziness in the displacement information is no repealing disadvantage. While equilibrium equations are here exactly satisfied, the Galerkin type of weighted residual approach applies to the compatibility equations for strains.

Hybrid models

The Friedrichs transformations, when applied to elasticity theory furnish a host of more general types of variational principles. From the finite element viewpoint the liberation from transitional constraints like (8) and (9) by Lagrangian multipliers is of equal importance as the liberation from the strain displacement constraints (4) inside of the element. The different possible combinations lead to a bewildering variety of hybrid elements, only few of which have been experimented. In this connexion one should mention the contributions of *Pian* [16], *Jones* [15] and *Prager* [19].

Elements of hybrid character may possess interesting convergence properties, unfortunately they do no more provide bound estimates for the strain energy.

Plate bending models. Static-geometric analogies

The acceptance of the Kirchhoff-Love assumption in the bending theory of thin plates or shells has posed real difficulties in attempts to enforce conformity. Not only the transverse deflection of the plate but also its normal slope must be continuous at an interface. To obtain this it is necessary to construct a superelement, that is a small aggregate of elements whose outer boundary motion can be determined in terms of local deflections and slopes ensuring conformity [22].

It is characteristic of such elements that, at their corners, the curvature tensor is not single-valued. Several authors have produced conforming plate or shell bending elements of apparently simpler type. However they use higher order connexions, adding partly or in toto local values of the curvature tensor. Not only does this preclude loading modes associated with curvature discontinuities but it makes it impossible to connect rationally elements which are not coplanar. This limits severely their use in complex geometrical situations. By contrast, the plate bending diffusive type of equilibrium element is quite simple [20, 23].

Kirchhoff plate bending and plate stretching are known to be related by static geometric analogies. Exchanging the roles of stresses and strains, the equilibrium equations of bending go over into the compatibility equations of stretching and conversely. As a result the transverse deflection distribution of bending can serve as Airy function to give an equilibrium state in stretching. The cartesian component of displacement in stretching can serve as stress-functions to satisfy the bending equilibrium equations. The recognition that conformity and diffusitivity for finite elements also exchange their roles in the analogies [21] led directly to the concept of setting up diffusive equilibrium models from their analogues, that is discretizing directly the stress-functions. As a matter of fact the discretization of stress functions brings us back to the "Gegenstück-Verfahren" of *Trefftz*, where it was applied to the Prandtl torsion function. Because the volume equilibrium equations were automatically satisfied by the very existence and differentiability of the torsion function, *Trefftz* was able to show that an underestimate of the torsional rigidity occured when the torsion function satisfied simply the boundary conditions, that in fact satisfies surface equilibrium. His method for obtaining an overestimate is however too complicated. Instead of requiring satisfaction of the differential equation governing the torsion function, which is tantamount to requiring compatibility of strains, discretization of the conjugate warping function in the conjugate variational principle of displacements yields the same overestimates with much less effort.

Conclusions

In this brief survey, there are many topics that could not be touched upon by lack of time and space. The isoparametric elements developped mainly by the finite element group at Swansea [30]; they furnish conforming displacement models with curved boundaries. Elastodynamics, where the extension to a continuum of a variational principle of *Toupin* [8], the complementary energy analogue to Hamilton's principle, allows diffusive equilibrium elements to prove their efficiency as eigenvalue economizers [26]. The recent penetration of finite element procedures into geometrically non-linear elasticity problems and in the much wider field of material non-linearities. Finally the controversial subjects of convergence and finite differences versus finite element methods. Those are important aspects, where conjugate efforts by mathematicians and engineers are needed to bring in a far more mature state of the art some fundamental guidelines, that satisfy both mathematical rigour and physical understanding.

Lack of space is also responsible for the limitations in the list of references. A much wider cross-section of the enormous body of papers available can be found by consulting the references [18, 24, 28 and 29].

References

1. Trefftz, E.: Proc. 2nd Int. Cong. Appl. Mech. Zürich, 1927, 131.
2. Friedrichs, K.: Nachr. der Ges. d. Wiss. zu Göttingen, 1929, 13—20.
3. Courant, R., Hilbert, D.: Methoden der Mathematischen Physik IV. Die Grundtatsachen der Variationsrechnung, 139—233, Berlin: Springer 1931.
4. Ebner, H., Koeller, H.: Luftfahrtforschung 14 (1937) 12.
5. Prager, W., Synge, J. L.: Quart. Appl. Math. 5 (1947) 241—269.
6. de Veubeke, B. F.: Bull. 24 du Service Technique de l'Aéronautique, Bruxelles 1951.
7. Langefors, B.: J. Aer. Sci. 19 (1952) 451—458.
8. Toupin, R. A.: J. Appl. Mech., June (1952) 151—152.
9. Washizu, K.: J. Math. Phys. 32 (1953) 117—128.
10. Turner, M. J., Clough, R. W., Martin, M. C., Topp, L. J.: J. Aero. Sci. 23 (1956) 805—823.
11. Synge, J. L.: The hypercircle in mathematical physics, Cambridge: Cambridge University Press 1957.
12. Denke, P. H.: NASA T.N. D-1666, 1959.
13. Argyris, J. H., Kelsey, S.: Energy theorems and structural analysis, London: Butterworth 1960.
14. de Veubeke, B. F.: AGARDograph 72, Oxford/London: Pergamon Press, 1963, 165—201.
15. Jones, R. E.: AIAA J. 2 (1964) 821—826.
16. Pian, T. H. H.: AIAA J. 2 (1964) 1232—1336.
17. de Veubeke, B. F.: Displacement and equilibrium models in the finite element method, Ch. 9, Stress Analysis, ed. O. C. Zienkiewicz and G. Holister, New York: Wiley 1965.
18. Robinson, J.: Structural matrix analysis for the engineer, New York: Wiley 1966.
19. Prager, W.: Recent Progr. in Appl. Mech. The Folke Odqvist Volume, 1967, 463—474.
20. Morley, L. S. D.: J. Roy. Aero. Soc. 71 (1967) 715—719.
21. de Veubeke, B. F., Zienkiewicz, O. C.: J. Strain Analysis 2 (1967) 265—271.
22. de Veubeke, B. F.: Int. J. Solids and Str. 4 (1968) 95—108.
23. de Veubeke, B. F., Sander, G.: Int. J. Solids and Str. 4 (1968) 447—468.
24. Przemieniecki, J. S.: Theory of matrix structural analysis. New York: McGraw-Hill 1968.
25. Robinson, J., Haggenmacher, G. W.: Some new developments in matrix force analysis. Japan-US seminar on matrix methods of structural analysis and design, Tokyo 1969.
26. Geradin, M.: Proc. IUTAM Coll. on High Speed Computing of Elastic Structure Liège. Congrès et Colloques de l'Université de Liège, Place du XX Août 16 (1970) 589—623.
27. Sander, G.: Proc. IUTAM Coll. on High Speed Computing of Elastic Structures Liège, Congrès et Colloques de l'Université de Liège, Place du XX Août 16 (1970) 167— 207.
28. Zienkiewicz, O. C.: Appl. Mech. Rev. (1970) 249—256.
29. Zienkiewicz, O. C.: The finite element method in Engineering Science, New York: McGraw-Hill 1971.
30. Irons, B. M., Zienkiewicz, O. C.: Proc. Conf. Recent Advances in Stress Analysis, Roy. Aero. Soc. (1968).