Research Article

# The Numerical Class of a Surface on a Toric Manifold 

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In this paper, we give a method to describe the numerical class of a torus invariant surface on a projective toric manifold. As applications, we can classify toric 2-Fano manifolds of the Picard number 2 or of dimension at most 4.

## 1. Introduction

The classification of smooth toric Fano $d$-folds is an important and interesting problem. They are classified for $d=3$ by [1,2], for $d=4$ by [3,4], and for $d=5$ by [5]. In Øbro's recent excellent paper [6], an algorithm which classifies all the smooth toric Fano $d$-folds for any given natural number $d$ was constructed. So, we can say that the classification of smooth toric Fano varieties is completed.

On the other hand, de Jong and Starr defined a special class of Fano manifolds called 2-Fano manifolds in [7] (see Definition 4.2). So, we consider the problem of the classification of toric 2-Fano manifolds as a next step. For this classification, we give a method to describe the numerical class of a 2-cycle on projective toric manifolds (see Section 3). This method makes calculations of intersection numbers much easier. As results, we obtain the classification of toric 2-Fano manifolds for the case of the Picard number $\rho(X)=2$ and for the case of $\operatorname{dim}(X) \leq 4$. We remark that Nobili classified smooth toric 2-Fano 4-folds in [8] by using a Maple program.

The contents of this paper are as follows. In Section 2, we define the basic notation such as nef 2-cocycle and 2-Mori cone for our theory. In Section 3, we define a polynomial $I_{Y / X}$ for a torus invariant subvariety $Y \subset X$. This polynomial has all the information of intersection numbers of $Y$ on $X$. So, we can consider this polynomial as the numerical class
of $Y$. For a some special surface $S, I_{S / X}$ has a good property to calculate intersection numbers (see Theorems 3.4 and 3.5). As applications, we classify toric 2-Fano manifolds under some assumptions in Section 4.

Notation. We will work over an algebraically closed field $k$ throughout this paper. We denote a projective toric $d$-fold by $X=X_{\Sigma}$, where $\Sigma$ is the associated fan in $N:=\mathbb{Z}^{d} . G(\Sigma) \subset N$ is the set of the primitive generators for the 1-dimensional cones in $\Sigma$.

## 2. Preliminaries

In this section, we explain the notation and some basic facts of the toric geometry and the birational geometry used in this paper. See [9-11] for the details.

Let $X$ be a smooth projective toric $d$-fold. Put $Z^{2}(X)$ to be the free $\mathbb{Z}$-module of 2 cocycles on $X$ and $Z_{2}(X)$ the free $\mathbb{Z}$-module of 2 -cycles on $X$. We define the numerical equivalence " $\equiv$ " on $Z^{2}(X)$ and $Z_{2}(X)$. A 2-cocycle $E \in Z^{2}(X)$ is numerically equivalent to 0 ; that is, $E \equiv 0$ if the intersection number $(E \cdot S)=0$ for any 2-cycle $S \in Z_{2}(X)$, while a 2-cycle $S \in \mathrm{Z}_{2}(X)$ is numerically equivalent to 0 ; that is, $S \equiv 0$ if the intersection number $(E \cdot S)=0$ for any 2-cocycle $E \in Z^{2}(X)$. We define $N^{2}(X):=\left(Z^{2}(X) / \equiv\right) \otimes \mathbb{R}$ and $N_{2}(X):=\left(Z_{2}(X) / \equiv\right) \otimes \mathbb{R}$.

The following definitions are similar to the case of divisors and curves.
Definition 2.1. A 2-cocycle $E \in Z^{2}(X)$ is a nef 2-cocycle if $(E \cdot S) \geq 0$ for any effective 2-cycle $S \in Z_{2}(X)$.

Definition 2.2. For a projective toric manifold $X$, let $\mathrm{NE}_{2}(X) \subset N_{2}(X)$ be the cone of effective 2-cycles; namely,

$$
\begin{equation*}
\mathrm{NE}_{2}(X):=\left\{\left[\sum_{i} a_{i} S_{i}\right] \in \mathrm{N}_{2}(X) \mid a_{i} \geq 0\right\} . \tag{2.1}
\end{equation*}
$$

One calls $\mathrm{NE}_{2}(X) \subset \mathrm{N}_{2}(X)$ the 2-Mori cone of $X$.
We should remark that $N^{l}(X), N_{l}(X)$, and $\mathrm{NE}_{l}(X)$ can be defined for any $1 \leq l \leq d$ similarly.

The following is an immediate consequence of the projectivity of $X$.
Proposition 2.3. $N E_{2}(X)$ is a strongly convex cone.
Proof. Let $D$ be an ample divisor on $X$. Then, for any $S \in \mathrm{NE}_{2}(X) \backslash\{0\}$, we have $\left(D^{2} \cdot S\right)>0$; namely, $\mathrm{NE}_{2}(\mathrm{X})$ is strongly convex.

On the other hand, for the toric case, the following is obvious.
Proposition 2.4. Let $X$ be a smooth projective toric $d$-fold. Then, $N E_{2}(X)$ is a polyhedral cone.
Thus, $\mathrm{NE}_{2}(X)$ is a strongly convex polyhedral rational cone similarly as $\mathrm{NE}(X)$.

We end this section by giving the following simple examples.
Example 2.5. (1) If $X=\mathbb{P}^{d}$, then

$$
\begin{equation*}
\mathrm{NE}_{2}(X)=\mathbb{R}_{\geq 0}[S] \tag{2.2}
\end{equation*}
$$

where $S$ is a plane in $X$.
(2) If $X=\mathbb{P}^{1} \times \mathbb{P}^{3}$, then

$$
\begin{equation*}
\mathrm{NE}_{2}(X)=\mathbb{R}_{\geq 0}\left[(\text { a point }) \times \mathbb{P}^{2}\right]+\mathbb{R}_{\geq 0}\left[\mathbb{P}^{1} \times \mathbb{P}^{1}\right] \tag{2.3}
\end{equation*}
$$

(3) If $X=\mathbb{P}^{2} \times \mathbb{P}^{2}$, then

$$
\begin{equation*}
\mathrm{NE}_{2}(X)=\mathbb{R}_{\geq 0}\left[(\text { a point }) \times \mathbb{P}^{2}\right]+\mathbb{R}_{\geq 0}\left[\mathbb{P}^{1} \times \mathbb{P}^{1}\right]+\mathbb{R}_{\geq 0}\left[\mathbb{P}^{2} \times(\text { a point })\right] \tag{2.4}
\end{equation*}
$$

## 3. Combinatorial Descriptions

In this section, we establish a method to describe the numerical class of a torus invariant subvariety. We assume that $X=X_{\Sigma}$ is a smooth projective toric variety.

Let $Y=Y_{\sigma} \subset X$ be a torus invariant subvariety of $\operatorname{dim} Y=l$ associated to a cone $\sigma \in \Sigma$ and $G(\Sigma)=\left\{x_{1}, \ldots, x_{m}\right\}$. Put

$$
\begin{align*}
I_{Y / X}=I_{Y / X}\left(X_{1}, \ldots, X_{m}\right) & :=\sum_{1 \leq i_{1}, \ldots, i_{l} \leq m}\left(D_{x_{i_{1}}} \cdots D_{x_{i_{l}}} \cdot Y\right) X_{i_{1}} \cdots X_{i_{l}}  \tag{3.1}\\
& \in \mathbb{Z}\left[X_{1}, \ldots, X_{m}\right]
\end{align*}
$$

where $D_{x_{i}}$ is the torus invariant prime divisor corresponding to $x_{i}$, while $X_{i}$ is defined to be the independent variable corresponding to $x_{i}$. We will use this notation throughout this paper.

Remark 3.1. $I_{Y / X}$ has all the informations of intersection numbers of $Y$ on $X$. So, we can consider $I_{Y / X}$ as the numerical class of $Y \in N_{l}(X)$.

Example 3.2. Let $C=C_{\tau} \subset X$ be a torus invariant curve, where $\tau$ is a ( $d-1$ )-dimensional cone, that is, a wall in $\Sigma$. In this case,

$$
\begin{equation*}
I_{C / X}=\sum_{i}\left(D_{i} \cdot C\right) X_{i} \tag{3.2}
\end{equation*}
$$

is a polynomial of degree 1 . On the other hand,

$$
\begin{equation*}
\sum_{i}\left(D_{i} \cdot C\right) x_{i}=0 \tag{3.3}
\end{equation*}
$$

is the so-called Reid's wall relation associated to the wall $\tau$ (see [12]); namely, $I_{C / X}$ is calculated from the wall relation immediately.

Example 3.3. When $Y=X, I_{X / X}$ sometimes becomes a simple shape as follows.
(1) Projective spaces. Let $X$ be the $d$-dimensional projective space $\mathbb{P}^{d}$ and $G(\Sigma)=\left\{x_{1}:=\right.$ $\left.e_{1}, \ldots, x_{d}:=e_{d}, x_{d+1}:=-\left(e_{1}+\cdots+e_{d}\right)\right\}$. Then,

$$
\begin{equation*}
I_{X / X}=\left(X_{1}+\cdots+X_{d+1}\right)^{d} \tag{3.4}
\end{equation*}
$$

(2) Hirzebruch surfaces. Let $X$ be the Hirzebruch surface $F_{\alpha}$ of degree $\alpha$ and $G(\Sigma)=$ $\left\{x_{1}:=e_{1}, x_{2}:=e_{2}, x_{3}:=-e_{1}+\alpha e_{2}, x_{4}=-e_{2}\right\}$. Then,

$$
\begin{equation*}
I_{X / X}=\alpha\left(X_{2}+X_{4}\right)^{2}+2\left(X_{2}+X_{4}\right)\left(X_{1}+X_{3}-\alpha X_{2}\right) \tag{3.5}
\end{equation*}
$$

Let $X$ be a smooth projective toric variety and $S \subset X$ a torus invariant surface. For some special cases, $I_{S / X}$ is simply calculated as follows. These are the main theorems of this paper.

Theorem 3.4. Suppose $S \cong \mathbb{P}^{2}$. Let $C \subset S$ be a torus invariant curve. Then, $I_{S / X}=\left(I_{C / X}\right)^{2}$.
Proof. Let $\tau=\mathbb{R}_{\geq 0} x_{1}+\cdots+\mathbb{R}_{\geq 0} x_{d-2} \in \Sigma$ be the $(d-2)$-dimensional cone associated to $S=S_{\tau}$, where $\tau \cap G(\Sigma)=\left\{x_{1}, \ldots, x_{d-2}\right\}$. Then, there exist exactly three maximal cones $\tau+\mathbb{R}_{\geq 0} y_{1}, \tau+$ $\mathbb{R}_{\geq 0} y_{2}$, and $\tau+\mathbb{R}_{\geq 0} y_{3} \in \Sigma$ which contain $\tau$. Put

$$
\begin{equation*}
y_{1}+y_{2}+y_{3}+a_{1} x_{1}+\cdots+a_{d-2} x_{d-2}=0 \tag{3.6}
\end{equation*}
$$

to be the wall relation corresponding to $C$. For the proof, it is sufficient to show that

$$
\begin{equation*}
D_{z} D_{w} S=a_{z} a_{w} \tag{3.7}
\end{equation*}
$$

for any $z, w \in G(\Sigma)$, where $D_{z}$ is the prime torus invariant divisor corresponding to $z$, while $a_{z}$ is the coefficient of $z$ in the above wall relation.

Suppose that $z$ or $w \notin\left\{x_{1}, \ldots, x_{d-2}, y_{1}, y_{2}, y_{3}\right\} ;$ namely, $a_{z}=0$ or $a_{w}=0$. In this case, trivially, $D_{z} S=0$ or $D_{w} S=0$. So, $D_{z} D_{w} S=a_{z} a_{w}=0$.

For any $1 \leq i, j \leq 3$,

$$
\begin{equation*}
D_{y_{i}} D_{y_{j}} S=\left(\left.D_{y_{i}}\right|_{S}\right)\left(\left.D_{y_{j}}\right|_{S}\right)=C^{2}=1 \tag{3.8}
\end{equation*}
$$

So, the remaining case is $z$ or $w \in\left\{x_{1}, \ldots, x_{d-2}\right\}$. By calculating the rational functions associated to a $\mathbb{Z}$-basis $\left\{x_{1}, \ldots, x_{d-2}, y_{1}, y_{2}\right\}$ for $N$, we have the relations

$$
\begin{gather*}
D_{x_{1}}-a_{1} D_{y_{3}}+E_{1}=0, \ldots, D_{x_{d-2}}-a_{d-2} D_{y_{3}}+E_{d-2}=0  \tag{3.9}\\
D_{y_{1}}-D_{y_{3}}+E_{d-1}=0, \quad D_{y_{1}}-D_{y_{3}}+E_{d}=0
\end{gather*}
$$

in $\operatorname{Pic} X$, where $E_{1}, \ldots, E_{d}$ are torus invariant divisors such that Supp $E_{i} \cap S=\emptyset$ for any $1 \leq i \leq$ d. Therefore, we have

$$
\begin{equation*}
D_{x_{1}} S=a_{1} D_{y_{3}} S, \ldots, D_{x_{d-2}} S=a_{d-2} D_{y_{3}} S . \tag{3.10}
\end{equation*}
$$

By these relations, the equality $D_{z} D_{w} S=a_{z} a_{w}$ is obvious.
Theorem 3.5. Suppose $S \cong F_{\alpha}$, that is, a Hirzebruch surface of degree $\alpha$. Let $C_{\text {fib }} \subset S$ be a fiber of the projection $S=F_{\alpha} \rightarrow \mathbb{P}^{1}$, while let $C_{\text {neg }}$ be the negative section of $S$. Then, $I_{S / X}=\alpha\left(I_{C_{\text {fib }} / X}\right)^{2}+$ $2 I_{C_{\text {fib }} / X} I_{C_{\text {neg }} / X}$.

Proof. Let $\tau=\mathbb{R}_{\geq 0} x_{1}+\cdots+\mathbb{R}_{\geq 0} x_{d-2} \in \Sigma$ be the $(d-2)$-dimensional cone associated to $S=S_{\tau}$, where $\tau \cap G(\Sigma)=\left\{x_{1}, \ldots, x_{d-2}\right\}$. Then, there exist exactly four maximal cones $\tau+\mathbb{R}_{\geq 0} y_{1}, \tau+$ $\mathbb{R}_{\geq 0} y_{2}, \tau+\mathbb{R}_{\geq 0} y_{3}$, and $\tau+\mathbb{R}_{\geq 0} y_{4} \in \Sigma$ which contain $\tau$. Put

$$
\begin{equation*}
y_{1}+y_{3}-\alpha y_{2}+a_{1} x_{1}+\cdots+a_{d-2} x_{d-2}=0 \tag{3.11}
\end{equation*}
$$

to be the wall relation corresponding to $C_{\text {neg, }}$, while

$$
\begin{equation*}
y_{2}+y_{4}+b_{1} x_{1}+\cdots+b_{d-2} x_{d-2}=0 \tag{3.12}
\end{equation*}
$$

to be the wall relation corresponding to $C_{\text {fib }}$. As in the proof of Theorem 3.4, by calculating the rational functions associated to a $\mathbb{Z}$-basis $\left\{x_{1}, \ldots, x_{d-2}, y_{1}, y_{2}\right\}$ for $N$, we have the relations

$$
\begin{gather*}
D_{x_{1}} S=a_{1} D_{y_{3}} S+b_{1} D_{y_{4}} S, \ldots, D_{x_{d-2}} S=a_{d-2} D_{y_{3}} S+b_{d-2} D_{y_{4}} S, \\
D_{y_{1}} S=D_{y_{3}} S, \quad D_{y_{2}}=-\alpha D_{y_{3}} S+D_{y_{4}} S . \tag{3.13}
\end{gather*}
$$

First, we remark that, for any $1 \leq i, j \leq 4$,

$$
\begin{equation*}
D_{y_{i}} D_{y_{j}} S=\left(\left.D_{y_{i}}\right|_{S}\right)\left(\left.D_{y_{j}}\right|_{S}\right) \tag{3.14}
\end{equation*}
$$

on $S$. So, these intersection numbers can be recovered from $I_{S / S}$ (see Example 3.3).
The above relations say that, for any $1 \leq i, j \leq d-2$,

$$
\begin{equation*}
D_{x_{i}} D_{x_{j}} S=\alpha b_{i} b_{j}+a_{i} b_{j}+a_{j} b_{i} \tag{3.15}
\end{equation*}
$$

while for any $1 \leq i \leq d-2$,

$$
\begin{equation*}
D_{y_{1}} D_{x_{i}}=b_{i}, \quad D_{y_{2}} D_{x_{i}}=a_{i}, \quad D_{y_{3}} D_{x_{i}}=b_{i}, \quad D_{y_{4}} D_{x_{i}}=a_{i}+\alpha b_{i} \tag{3.16}
\end{equation*}
$$

On the other hand, put $f_{1}=f_{1}\left(X_{1}, \ldots, X_{d-2}\right):=a_{1} X_{1}+\cdots+a_{d-2} X_{d-2}$ and $f_{2}=$ $f_{2}\left(X_{1}, \ldots, X_{d-2}\right):=b_{1} X_{1}+\cdots+b_{d-2} X_{d-2}$. Then,

$$
\begin{align*}
\alpha\left(I_{C_{\text {fib }} / X}\right)^{2}+2 I_{C_{\text {fib }} / X} I_{C_{\text {neg }} / X}= & \alpha\left(Y_{2}+Y_{4}+f_{1}\right)^{2}+2\left(Y_{2}+Y_{4}+f_{1}\right)\left(Y_{1}+Y_{3}-\alpha Y_{2}+f_{2}\right) \\
= & I_{S / S}\left(Y_{1}, Y_{2}, Y_{3}, Y_{4}\right)+\alpha f_{2}^{2}+2 f_{1} f_{2}  \tag{3.17}\\
& +2 Y_{1} f_{2}+2 Y_{2} f_{1}+2 Y_{3} f_{2}+Y_{4}\left(2 f_{1}+2 \alpha f_{2}\right) .
\end{align*}
$$

This coincides with $I_{S / X}$ by the above calculations.

## 4. 2-Fano Manifolds

As an application of Section 3, we study on toric 2-Fano manifolds in this section. The notion of 2-Fano manifolds was introduced in [7].

Definition 4.1. A smooth projecive algebraic variety $X$ is a Fano manifold if its first Chern $\operatorname{class}_{1}(X)=-K_{X}$ is an ample divisor.

Definition 4.2 (see [7]). A Fano manifold $X$ is a 2 -Fano manifold if its second Chern character $\operatorname{ch}_{2}(X)=(1 / 2)\left(c_{1}(X)^{2}-2 c_{2}(X)\right)$ is a nef 2-cocycle.

Remark 4.3. Since a 2-Fano manifold is a Fano manifold by the definition, for the classification of toric 2-Fano manifolds, all we have to do is to check the list of toric Fano manifolds. The classification of toric Fano manifolds can be done by the algorithm of Øbro [6] for any dimension.

For a projective toric manifold $X$, one can easily see that $c h_{2}(X)=(1 / 2) \sum_{i=1}^{m} D_{i}^{2}$, where $D_{1}, \ldots, D_{m}$ are the torus invariant prime divisors. So, the following is immediate.

Proposition 4.4. For a torus invariant surface $S \subset X$, put $I_{S / X}:=\sum_{i, j} a_{i j} X_{i} X_{j}$. Then, $\left(\operatorname{ch}_{2}(X) \cdot S\right)=$ $(1 / 2) \sum_{i=1}^{m} a_{i i}$.

First of all, we classify toric 2-Fano manifolds of Picard number 2. So, let $X$ be a complete toric manifold of $\rho(X)=2$. In this case, the structure of $X$ is very simple as follows.

Theorem 4.5 (see [13]). Every complete toric manifold of the Picard number 2 is a projective space bundle over a projective space.

By Theorem 4.5, we can put

$$
\begin{equation*}
X=X_{\Sigma}=\mathbb{P}_{\mathbb{P}^{n-1}}\left(\mathcal{O} \oplus \mathcal{O}\left(a_{1}\right) \oplus \cdots \oplus \mathcal{O}\left(a_{m-1}\right)\right) \tag{4.1}
\end{equation*}
$$

where $a_{1} \geq \cdots \geq a_{m-1} \geq 0, m+n-2=d:=\operatorname{dim} X$. Let

$$
\begin{gather*}
x_{1}+\cdots+x_{m}=0  \tag{4.2}\\
y_{1}+\cdots+y_{n}=a_{1} x_{1}+\cdots+a_{m-1} x_{m-1} \tag{4.3}
\end{gather*}
$$

be the wall relations of $\Sigma$ which correspond to the extremal rays of $\mathrm{NE}(X)$, where

$$
\begin{equation*}
G(\Sigma)=\left\{x_{1}, \ldots, x_{m}, y_{1}, \ldots, y_{n}\right\} \tag{4.4}
\end{equation*}
$$

Let $C_{1}$ and $C_{2}$ be the extremal torus invariant curves corresponding to the wall relations (4.2) and (4.3), respectively.

First, we determine the extremal rays of $\mathrm{NE}_{2}(X)$. By calculating the rational functions for a $\mathbb{Z}$-basis $\left\{x_{1}, \ldots, x_{m-1}, y_{1}, \ldots, y_{n-1}\right\}$, we have the relations

$$
\begin{gather*}
D_{1}-D_{m}+a_{1} E_{n}=0, \ldots, D_{m-1}-D_{m}+a_{m-1} E_{n}=0 \\
E_{1}-E_{n}=0, \ldots, E_{n-1}-E_{n}=0 \tag{4.5}
\end{gather*}
$$

in $\mathrm{N}^{1}(X)$, where $D_{1}, \ldots, D_{m}, E_{1}, \ldots, E_{n}$ are torus invariant prime divisors corresponding to $x_{1}, \ldots, x_{m}, y_{1}, \ldots, y_{n}$. Therefore, for $1 \leq i, j \leq m-1$,

$$
\begin{gather*}
D_{j}=D_{i}+\left(a_{i}-a_{j}\right) E_{n} \\
E_{1}=E_{2}=\cdots=E_{n} \tag{4.6}
\end{gather*}
$$

On the other hand, every $(d-2)$-dimensional cone $\tau \in \Sigma$ is expressed as

$$
\begin{equation*}
\tau=\mathbb{R}_{\geq 0} x_{i_{1}}+\cdots+\mathbb{R}_{\geq 0} x_{i_{k}}+\mathbb{R}_{\geq 0} y_{j_{1}}+\cdots+\mathbb{R}_{\geq 0} y_{j_{l}} \tag{4.7}
\end{equation*}
$$

for some $1 \leq i_{1}<\cdots<i_{k} \leq m, 1 \leq j_{1}<\cdots<j_{l} \leq n$ such that $k<m, l<n$, and $k+l=d-2$. So, the corresponding torus invariant surface $S_{\tau}$ is expressed as

$$
\begin{equation*}
S_{\tau}=D_{i_{1}} \cdots D_{i_{k}} E_{j_{1}} \cdots E_{j_{l}} \in \mathrm{~N}_{2}(X) . \tag{4.8}
\end{equation*}
$$

By using (4.6), any $S_{\tau}$ is expressed as a linear combination of 2-cycles:

$$
\begin{equation*}
D_{1} \cdots D_{p} E^{q} \quad(p \leq m-1, q \leq n-1, p+q=d-2) \tag{4.9}
\end{equation*}
$$

whose coefficients are nonnegative, because $i<j$ implies $a_{i}-a_{j} \geq 0$. Moreover, since $D_{1} \cdots D_{m}=E_{1} \cdots E_{n}=0$ by wall relations (4.2) and (4.3), the possibilities for the generators of $\mathrm{NE}_{2}(X)$ are

$$
\begin{gather*}
S_{1}:=D_{1} \cdots D_{m-3} E^{n-1}, \quad S_{2}:=D_{1} \cdots D_{m-2} E^{n-2}, \quad \text { or } \\
S_{3}:=D_{1} \cdots D_{m-1} E^{n-3} . \tag{4.10}
\end{gather*}
$$

In fact, the following hold:

$$
\begin{gather*}
\mathrm{NE}_{2}(X)=\mathbb{R}_{\geq 0} S_{1}+\mathbb{R}_{\geq 0} S_{2}+\mathbb{R}_{\geq 0} S_{3} \quad \text { if } m \geq 3, n \geq 3 . \\
\mathrm{NE}_{2}(X)=\mathbb{R}_{\geq 0} S_{2}+\mathbb{R}_{\geq 0} S_{3} \quad \text { if } m=2, n \geq 3 .  \tag{4.11}\\
\mathrm{NE}_{2}(X)=\mathbb{R}_{\geq 0} S_{1}+\mathbb{R}_{\geq 0} S_{2} \quad \text { if } m \geq 3, n=2 .
\end{gather*}
$$

For each case, $\operatorname{dim} \mathrm{N}_{2}(X)=3, \operatorname{dim} \mathrm{~N}_{2}(X)=2$, and $\operatorname{dim} \mathrm{N}_{2}(X)=2$, respectively. So, $\mathrm{NE}_{2}(X)$ is a simplicial cone for each case, and $S_{1}, S_{2}$, and $S_{3}$ are extremal surfaces.

Next, we will check when $X$ becomes a 2-Fano manifold.
So, let $C_{2}$ be the torus invariant curve which generates the extremal ray corresponding to the wall relation (4.3). Then,

$$
\begin{equation*}
\left(-K_{X} \cdot C_{2}\right)=n-\left(a_{1}+\cdots+a_{m-1}\right) \tag{4.12}
\end{equation*}
$$

Therefore, $X$ is a Fano manifold if and only if

$$
\begin{equation*}
n-\left(a_{1}+\cdots+a_{m-1}\right)>0 \tag{4.13}
\end{equation*}
$$

Since $S_{1} \cong S_{3} \cong \mathbb{P}^{2},\left(\operatorname{ch}_{2}(X) \cdot S_{1}\right) \geq 0$ and $\left(\operatorname{ch}_{2}(X) \cdot S_{3}\right) \geq 0$ are trivial by Theorem 3.4. On the other hand, we can easily check that $S_{2} \cong F_{a_{m-1}}$. By Theorem 3.5, we have

$$
\begin{align*}
I_{S_{2}}= & a_{m-1}\left(I_{C_{1}}\right)^{2}+2 I_{C_{1}} I_{C_{2}}=a_{m-1}\left(X_{1}+\cdots+X_{m}\right)^{2}  \tag{4.14}\\
& +2\left(X_{1}+\cdots+X_{m}\right)\left(Y_{1}+\cdots+Y_{n}-\left(a_{1} X_{1}+\cdots+a_{m-1} X_{m-1}\right)\right)
\end{align*}
$$

So, we obtain

$$
\begin{equation*}
\left(\operatorname{ch}_{2}(X) \cdot S_{2}\right)=m a_{m-1}-2\left(a_{1}+\cdots+a_{m-1}\right) \tag{4.15}
\end{equation*}
$$

In (4.15), suppose that $m \geq 3$ and $\left(\operatorname{ch}_{2}(X) \cdot S_{2}\right) \geq 0$. Then,

$$
\begin{equation*}
\left(\operatorname{ch}_{2}(X) \cdot S_{2}\right)=(m-2) a_{m-1}-2\left(a_{1}+\cdots+a_{m-2}\right) . \tag{4.16}
\end{equation*}
$$

The assumption $a_{1} \geq \cdots \geq a_{m-1} \geq 0$ says that $a_{1}=\cdots=a_{m-1}=0$; that is, $X \cong \mathbb{P}^{m-1} \times \mathbb{P}^{n-1}$. On the other hand, suppose that $m=2$ in (4.15). Then, $\left(\operatorname{ch}_{2}(X) \cdot S_{2}\right)=0$; that is, $\operatorname{ch}_{2}(X)$ is nef.

By (4.13), we can summarize as follows.
Theorem 4.6. If $X$ is a toric 2-Fano manifold of the Picard number 2, then $X$ is one of the following:
(1) a direct product of projective spaces,
(2) $\mathbb{P}_{\mathbb{P}^{d-1}}(\mathcal{O} \oplus \mathcal{O}(a))(1 \leq a \leq d-1)$.

Remark 4.7. This calculation shows that there exist infinitely many projective toric manifolds of fixed dimension d whose second Chern character is nef.

Next, we consider the classification of toric 2-Fano manifolds of a fixed dimension $d$. For $d \leq 4$, fortunately, these classifications can be done by only Theorems 3.4 and 3.5. Table 1 is the classification list (see [8] for the detail).

Since there exist 124 smooth toric Fano 4 -folds, it is hard to check all the smooth toric Fano 4 -folds. However, by using the following trivial Lemma 4.8, we can omit a large part of the calculations.

Table 1

| $d$ | 1 | 2 | 3 | 4 |
| :--- | :--- | :---: | :---: | :---: |
| \# of toric Fano | 1 | 5 | 18 | 124 |
| $\#$ of toric 2-Fano | 1 | 3 | 8 | 25 |

Lemma 4.8. Let $X$ be a 4-dimensional toric 2-Fano manifold. Then,

$$
\begin{equation*}
c_{1}^{4}(X)-2 c_{1}^{2}(X) c_{2}(X) \geq 0 \tag{4.17}
\end{equation*}
$$

For any smooth toric Fano 4-fold $X, c_{1}^{4}(X)$ and $c_{1}^{2}(X) c_{2}(X)$ are calculated in [3]. One can see that for 52 smooth toric Fano 4 -folds, they are not 2-Fano manifolds by Lemma 4.8.

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## References

[1] V. Batyrev, "Toroidal Fano 3-folds," Math. USSR-Izv, vol. 19, pp. 13-25, 1982.
[2] K. Watanabe and M. Watanabe, "The classification of Fano 3-folds with torus embeddings," Tokyo Journal of Mathematics, vol. 5, no. 1, pp. 37-48, 1982.
[3] V. V. Batyrev, "On the classification of toric Fano 4-folds," Journal of Mathematical Sciences, vol. 94, no. 1, pp. 1021-1050, 1999.
[4] H. Sato, "Toward the classification of higher-dimensional toric Fano varieties," The Tohoku Mathematical Journal, vol. 52, no. 3, pp. 383-413, 2000.
[5] M. Kreuzer and B. Nill, "Classification of toric Fano 5-folds," Advances in Geometry, vol. 9, no. 1, pp. 85-97, 2009.
[6] M. Øbro, "An algorithm for the classification of smooth Fano polytopes," arXiv, Article ID 0704.0049, 17 pages, 2007.
[7] A. J. de Jong and J. Starr, "Higher Fano manifolds and rational surfaces," Duke Mathematical Journal, vol. 139, no. 1, pp. 173-183, 2007.
[8] E. Ervilha Nobili, "Classification of toric 2-Fano 4-folds," Bulletin of the Brazilian Mathematical Society, vol. 42, no. 3, pp. 399-414, 2011.
[9] O. Fujino and H. Sato, "Introduction to the toric Mori theory," The Michigan Mathematical Journal, vol. 52, no. 3, pp. 649-665, 2004.
[10] W. Fulton, Introduction to Toric Varieties, vol. 131 of Annals of Mathematics Studies, The William H. Roever Lectures in Geometry, Princeton University Press, Princeton, NJ, USA, 1993.
[11] T. Oda, Convex Bodies and Algebraic Geometry. An Introduction to the Theory of Toric Varieties, vol. 15 of Results in Mathematics and Related Areas (3), Springer, Berlin, Germany, 1988.
[12] M. Reid, "Decomposition of Toric Morphisms," in Arithmetic and Geometry, Vol. II, vol. 36 of Progress in Mathematics, pp. 395-418, Birkhäauser, Boston, Mass, USA, 1983.
[13] P. Kleinschmidt, "A classification of toric varieties with few generators," Aequationes Mathematicae, vol. 35, no. 2-3, pp. 254-266, 1988.


