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THE NUMERICAL RANGE OF LINEAR OPERATORS

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Abstract

We offer a simple proof that convexoid operators on Hilbert space are normaloid, and given an example to show that the converse fails.

1 Introduction

Let H be a Hilbert space equipped with the inner product (x, y), and let $B(H)$ be the algebra of bounded linear operators acting on H. The numerical range (also known as the field of values) $W(A)$ of $A \in B(H)$ is the collection of all complex numbers of the form (Ax, x), where x is a unit vector in H, and the numerical radius $r(A)$ of A is the radius of the smallest circle centered at the origin containing $W(A)$. The study of the numerical range and numerical radius has an extensive history, and there is a great deal of current research on these concepts and their generalizations. In particular, the subject has connections and applications to various areas including C^*- algebras, iterations methods, several operator theory, dilation theory, Krein space operators, factorizations of matrix polynomials, unitary similarity,etc.(e.g., see[1, 4, 5, 6, 7, 8, 9, 10, 13, 14, 15, 16, 18, 19, 20, 21, 22], and their reference). All this constitutes a very active field of research in operator theory. The numerical range of an operator, like the spectrum, is a subset of the complex plane whose geometrical properties should says something about that operator. The goal of this paper is to give some idea of what this "something" might be. A major theme will be to compare the properties and utility of the numerical range and the spectrum. In [27] J.P.Williams showed that an operator $A \in B(H)$ is normaloid if and only if it is convexoid. It is known that the part if in J.P.Williams' result is not true as it mentioned in the review MR0264445. In this paper we will present an example which contractics the part " if " in J.P. Williams' result. We also give a simple proof of the part "only if" of this result. A necessary and sufficient condition for an operator $A \in B(H)$ to be convexoid is also given.

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2 Numerical range in a Banach algebra

We begin by the following definitions and well-known propositions.

Definition 2.1. *Let $\mathcal{A}$ be a C^*-algebra with identity 1, a linear functional φ on $\mathcal{A}$ is positive if, $\varphi(a^*a) \geq 0$, for all $a \in \mathcal{A}$ (denoted $\varphi \geq 0$), a state if $\varphi \geq 0$ and $||\varphi|| = 1$. The set of all states of $\mathcal{A}$ is denoted by $\mathcal{P}(\mathcal{A}) = \mathcal{P}$.*

Recall that
(i) $\varphi \geq 0 \Rightarrow \varphi = \varphi*$, $||\varphi|| = 1$,
(ii) $\varphi \in \mathcal{A}^{'}$, $\exists a \in \mathcal{A}$, $a \leq 0$ such that $\varphi(a) = ||\varphi||||a|| \Rightarrow \varphi \leq 0$.
(iii) for all $a \in \mathcal{A}$, $||a|| = \sup\{|\varphi(a)|, \varphi \in \mathcal{P}\}$,
(iv) $\mathcal{P}$ is nonvoid, since by the Hahn-Banach theorem, there exists $f \in \mathcal{A}^{'}$ such that $f(e) = 1$ and $||f|| = 1$.
(v) $\mathcal{P}$ is a convex and compact set for the w^*-topology

Definition 2.2. *The numerical range $V(a)$ of an element $a \in \mathcal{A}$ is defined by*

$$V(a) = \{f(a),\, f \in \mathcal{P}\}.$$

Proposition 2.1. *$V(a)$ is a non empty convex and compact set.*

Definition 2.3. *Let a be an element of the Banach algebra $\mathcal{A}$. The spectrum of a, $\sigma(a)$ is defined by*

$$\{\lambda \in \mathbb{C} : a - \lambda \text{ is not invertible}\}.$$

Proposition 2.2. $\sigma(a) \subset V(a)$.

Remark 2.1. *As every convex compact subset of $\mathbb{C}$, $V(a)$ is determined by its support lines, $V(\alpha a + \beta) = \alpha V(a) + \beta$ by the definition of V. Hence it suffices to identify one support vertical line to know how to identify them all. This justifies the following proposition which is Theorem 2.5 in [8] and holds for general Banach algebras.*

Proposition 2.3.

$$\sup\{Re\mu,\, \mu \in V(a)\} = \inf_{t>0}(\frac{1}{t}(||1 + ta|| - 1)) = \lim_{t\to 0^+}(\frac{1}{t}(||1 + ta|| - 1)).$$

Proof. Let

$$h = \sup\{Re\mu,\, \mu \in V(a)\} = \sup\{Ref(a),\ f \in \mathcal{P}\}.$$

We have

$$\forall y > 0,\, 1 + tRef(a) \leq |1 + tf(a)| = |f(1 + ta)| \leq ||1 + ta||.$$

Hence

$$h \leq \inf_{t>0}\{(\frac{1}{t}(||1 + ta|| - 1))\}.$$

If we put $m = \inf_{t>0}\{(\frac{1}{t}(||1+ta||-1))\}$, then $h \le m$.

For the inverse inequality, let $x \in \mathcal{A}$ be such that $||x|| = 1$, $f \in \mathcal{A}^{'}$ and $f(x) = ||f|| = 1$. Let's consider the map $g : \mathcal{A} \mapsto \mathbb{C}$ defined by $g(y) = f(yx)$. We have $g \in \mathcal{P}$. Indeed, it is clear that g is linear and $|g(y)| \le ||y||$, $g(1) = 1$. Hence $g \in \mathcal{P}$ and we have

$$||(1-ta)x|| \ge f((1-ta)x) \ge 1 - tRef(ax) = 1 - tReg(a) \ge 1 - th.$$

Hence,

$$||(1-ta)x|| \ge (1-th)||x||, \forall\, x \in H.$$

Choose $x = 1 + ta$, we get

$$(1-th)||1+ta|| \le ||(1-ta)(1+ta)|| \le (1+t^2)||a||^2.$$

By choosing $t \le h^{-1}$, we obtain

$$\begin{aligned} m &\le t^{-1}(||1+ta||-1) \le \frac{h+t||a||^2}{1-th} \,\text{and}\, m \le \inf_{t>0}\frac{h+t||a||^2}{1-th} = h \\ &= \lim_{t\to 0^+}\frac{h+t||a||^2}{1-th}. \end{aligned}$$

Which establishes also the existence of $\lim_{t\to 0^+} t^{-1}(||1+ta||-1)$. □

Proposition 2.4. *[27] Let $a \in \mathcal{A}$. Then the following properties hold*
(i) $0 \in V(a) \Leftrightarrow \forall\lambda \in \mathbb{C},\ |\lambda| \le ||a-\lambda||$.
(ii) $V(a) = \cap\{\{\mu \in \mathbb{C} :\ |\lambda^{'} - \mu| \le ||a-\lambda^{'}||,\ \lambda^{'} \in \mathbb{C}\}\}$.

Remark 2.2. *1) The previous proposition (i) implies that $\{a \in \mathcal{A} :\ 0 \in V(a)\}$ is closed in $\mathcal{A}$. Indeed, if $a = \lim_n a_n$, $|\lambda| \le ||a_n - \lambda|| \Rightarrow |\lambda| \le ||a-\lambda||$.*

2) We use in a Banach algebra $\mathcal{A}$ the notion ($x\,\text{orthogonal to}\,y$), $x \perp y$, if

$$\forall\lambda \in \mathbb{C},\ ||\lambda y|| \le ||x-\lambda y||.$$

Thus the previous proposition (i) implies that $0 \in V(a) \Leftrightarrow a\perp 1$. According to J.P.Williams' definition the set of elements a of a Banach algebra such that $a\perp 1$ are called finite. Thus one of the important application of the numerical range is the characterization of those finite elements.

Proposition 2.5. *Let $a \in \mathcal{A}$ be such that: for all $\lambda^{'} \in \mathbb{C}$, $||a-\lambda^{'}|| = r(a-\lambda^{'})$. Then $V(a)$=conv$\sigma(a)$.*

Proof. It is a simple consequence of the previous proposition (ii):

$$\text{conv}\,\sigma(A) = \cap\{\text{disk centred at}\,\lambda' \,\text{ and of radius}\,||a-\lambda^{'}||,\ \lambda^{'} \in \mathbb{C}\}.$$

□

Definition 2.4. *The numerical range of the operator $A \in B(H)$ is defined by*

$$W(A) = \{(Ax, x) :\ ||x|| = 1,\ x \in H\}.$$

Proposition 2.6. *Let $\mathcal{A} = B(H)$, where H is a complex Hilbert space and $a = A \in B(H)$. Then $V(A) = \overline{W(A)}$.*

Proof. It suffices to apply Proposition 3.2 and the previous remark for $W(A)$ to show that

$$\sup\{Re\mu,\ \mu \in W(A)\} = \lim_{t\to 0^+}(\frac{1}{t}(||1+tA||-1)).$$

Since the map

$$x^\wedge : B \mapsto (Bx, x),\ B(H) \mapsto \mathbb{C}$$

is an element of $\mathcal{P}$, by the same way as in the first part of the proof of Proposition 3.2. If

$$h_1 = \sup\{Re\mu,\ \mu \in W(A)\},\ m = \lim_{t\to 0^+}(\frac{1}{t}(||1+tA||-1)),$$

then $h_1 \leq m$. For the inverse inequality we have

$$||(I-tA)x||||x|| \geq Re((I-tA)x, x) \geq (I-th_1)||x||^2.$$

Choose t small enough to get $I - th_1 > 0$. Let $x = (I+tA)y$ with $||y|| = 1$. Then

$$||(I-t^2A^2)y|| \geq (1-th_1)||(I+tA)y||,$$

hence

$$(1-th_1)||I+tA|| \leq 1 + t^2||A||^2.$$

By this last inequality as in the proof of Proposition 3.2, we get $m \leq h_1$. □

Remark 2.3. *In [27, Corollary 2], Proposition 2.5 appeared by the equivalence,i.e.*

$$W(a) = co\sigma(a) \Leftrightarrow ||a - \lambda'|| = r(a - \lambda'),\ \lambda' \in \mathbb{C}.$$

The following example contradicts the part ($\Rightarrow$).

Example 2.1. *Let $\{e_1, e_2, ..., e_6\}$ be an orthonormal basis in H and let A be such that*

$$Ae_1 = \sqrt{2}e_1, Ae_2 = i\sqrt{2}e_2, Ae_3 = \sqrt{2}e_2, Ae_4 = -i\sqrt{2}e_4, Ae_5 = 0, Ae_6 = 2e_5.$$

It is easily seen that

$$\sigma(A) = \{\sqrt{2}, i\sqrt{2}, -\sqrt{2}, -i\sqrt{2}\}.$$

$$W(A) = co\{\sqrt{2}, i\sqrt{2}, -\sqrt{2}, -i\sqrt{2}, \textit{disk (center 0, radius 1)}\}.$$

Hence $W(A) = co\sigma(A)$, $r(A) = \sqrt{2}$ and $||A|| = \max\{\sqrt{2}, A|Vect\{e_5, e_6\}\} = 2$. Remark that $||A|| > r(A)$, then we have $W(A) = co\sigma(A)$. But for $\lambda' = 0$, $r(A - \lambda') < ||A - \lambda'||$.

Definition 2.5. *An operator $A \in B(H)$ is said to be convexoid if $\overline{W(A)} = co\sigma(A)$. This class of operators is denoted by $\mathcal{C}$.*

Definition 2.6. *An operator $A \in B(H)$ is said to be normaloid if $r(A) = ||A||$, and A is transaloid if*

$$\forall \lambda \in \rho(A), r(R_\lambda(A)) = ||R_\lambda(A)||,\ (R_\lambda(A) = (A - \lambda I)^{-1}),$$

this class is denoted by $\mathcal{T}$

An operator $A \in B(H)$ is said to be paranormal if

$$||Ax||^2 \leq ||A^2x||||x||$$

for all $x \in H$. We say that A is algebraically paranormal if there exists a nonconstant complex polynomial p such that $p(A)$ is paranormol. In general

$$\text{hyponormal} \subset \text{p} - \text{hyponormal} \subset \text{paranormal} \subset \text{normaloid}.$$

A is said to be log-hyponormal if A is invertible and satisfies the following equality

$$log(A^*A) \geq log(AA^*).$$

It is known that invertible p-hyponormal operators are log-hyponormal operators but the converse is not true [23]. However it is very interesting that we may regards log-hyponormal operators as 0-hyponormal operators [23, 24]. The idea of log-hyponormal operator is due to Ando [2] and the first paper in which log-hyponormality appeared is [11]. See [3, 23, 24, 26] for properties of log-hyponormal operators.

We say that an operator $A \in B(H)$ belongs to the class A if $|A^2| \geq |A|^2$. Class A was first introduced by Furuta-Ito-Yamazaki [12] as a subclass of paranormal operators which includes the classes of p-hyponormal and log-hyponormal operators. The following theorem is one of the results associated with a class A operator.

Theorem 2.1. *[12]*

(1) Every log-hyponormal operator is a class A operator.

(2) Every class A operator is a paranormal operator.

Proposition 2.7. *Let $A \in B(H)$. Then the following properties hold*

(i) $\overline{W(A)}$ =conv $\sigma(A) \Leftrightarrow \forall \lambda \notin$ conv $\sigma(A)$, $||R_\lambda(A)|| \leq (d(\lambda,$ conv $\sigma(A))$, where d is the usual distance on $\mathbb{C}$.

(ii) Every transaloid operator is convexoid,i.e., $\mathcal{T} \subset \mathcal{C}$.

(iii) Every paranormal operator is convexoid.

Proof. By applying the transformation

$$A \mapsto \alpha A + \beta$$

we can suppose that

$$[\lambda < 0, 0 \in \text{conv}\sigma(A) \subset \{z \in \mathbb{C} : \Re z \geq 0\}].$$

Let $\overline{W(A)}$=conv$\sigma(A)$. We have for all $x \in H$,

$$||(A-\lambda)x||^2 = ||Ax||^2 - \lambda[(Ax,x)+(x,Ax)] + \lambda^2||x||^2 \geq \lambda^2||x||^2.$$

Since $(A-\lambda)$ is invertible, we have for all $x \in H$,

$$||x||^2 \geq \lambda^2||(A-\lambda)^{-1}x||^2.$$

Hence $|\lambda|^{-1} \geq ||(A-\lambda)^{-1}x||$, or $|\lambda| = d(\lambda, \text{conv } \sigma(A))$.

Conversely, assume that $\forall\lambda \notin co\sigma(A), ||R_\lambda(A)|| \leq (d(\lambda, co\sigma(A))$. We have to prove $\overline{W(A)} = co\sigma(A)$. In other words, we prove that if

$$\lambda \notin co\sigma(A),$$

then

$$\lambda \notin \overline{W}(A).$$

By applying the transformation

$$A \mapsto \alpha A + \beta$$

we can suppose that

$$[\lambda < 0, 0 \in co\sigma(A) \subset \{z \in C : \Re z \geq 0\}], \forall c < 0.$$

The estimate

$$dist(c, co\sigma(A)) \geq |c|$$

implies

$$\left\|(A-c)^{-1}\right\| \leq |c|^{-1},$$

so

$$c^2 \|x\|^2 \leq ((A-c)\,x \mid (A-c)\,x).$$

This implies (after letting c tend to minus infinity) that

$$(Ax \mid x) + (x \mid Ax) \geq 0.$$

Hence,

$$\overline{W(A)} \subset \{z \in \mathbb{C} : \Re z \geq 0\},$$

that is,

$$\lambda \notin \overline{W(A)}.$$

(ii) Let A be such that for all $\lambda \in \rho$, $||R_\lambda(A)|| = r(R_\lambda(A))$. Since $r(R_\lambda(A)) = \{d(\lambda, \sigma(A)\}^{-1}$ and $d(\lambda, co\sigma(A)) \leq d(\lambda, \sigma(A))$. By applying (i) we get $A \in \mathcal{C}$.

(iii) It is easily seen that the class of paranormal operators is preserved under the transformation $A \mapsto A - \lambda$, $\lambda \in \mathbb{C}$. Also if A paranormal and invertible, then A^{-1} is paranormal. Endeed, if we let $y = A^{-1}x$ and $z = A^{-1}y$, then $Az = y$ and $A^2 z = x$. Therefore

$$||A^{-1}x||^2 = ||y||^2 = ||Az||^2 \leq ||A^2z||||z|| = ||x||||A^{-2}x|| = ||(A^{-1})^2x||||x||.$$

Hence to prove (iii) it suffices to show that every paranormal operator is normaloid. But it is known that a paranormal operator is normaloid. This completes the proof. □

References

[1] T. Ando and C-K. Li (Special editors), The Numerical Range and Numerical Radius, Linear and Multilinear Algebra Vol. 37, nos 1–3, 1994.

[2] T. Ando, Operators with a norm condition, Acta. Sci. Math (Szeged) 33 (1972), 169-178.

[3] Aluthge, On p-hyponormal operators for $0 < p < 1$, Integr. Equat. Oper. Th., 13(1990), 307-315.

[4] W. Arveson, Subalgebras of C*-algebras II, Acta Math. 128 (1972), 271-308.

[5] O. Axelsson et. al., On the numerical radius of matrices and its application for iterative solution methods, Linear and Multilinear Algebra 37 (1994), 225-238.

[6] T. Bayasgalan, The numerical range of linear operators in spaces with an indefinite metric (Russian), Acta Math. Hungar. 57 (1991), 157-168.

[7] P. Binding, D.R. Farenick and C.K. Li, A dilation and norm in several variable operator theory, Canad. J. Math. 47 (1995), 449-461.

[8] F. F. Bonsall and Duncan, Numerical ranges, Vol. I and II, Cambridge University Press, 1971 and 1973.

[9] F. F. Bonsall and Duncan, Studies in Functional Analysis – Numerical ranges, Studies in Mathematics Vol. 21, The Mathematical Association of America, 1980.

[10] D. Farenick, Matricial Extensions of the numerical range: A brief survey, Linear and Multilinear Algebra 34 (1993), 197-211.

[11] M.Fujii, C.Himeji and A.Matsumoto, Theorems of Ando and Saito for p-hyponormal operators, Math. Japonica 39(1994), 595-598.

[12] T.Furuta, M.Ito and T. Yamazaki, A subclass of paranormal operators including class of log-hyponormal and severel related classes., Scientiae Mathematicae 1(1998), 389-403.

[13] P. R. Halmos, Normal dilations and extensions of operators, Summa Brasil. Math. 2 (1950), 125-134.

[14] P. R. Halmos, A Hilbert Space Problem Book, Second Ed., Springer-Verlag, New York, 1982.

[15] R. A. Horn and C.R. Johnson, Topics in Matrix Analysis, Cambridge Univ. Press, Cambridge, 1991.

[16] B. Istratescu, Introduction to Linear Operator Theory, Marcel Dekker, New York, 1981.

[17] C. K. Li , and H. Woerdeman, A problem on the $(1, k)$ numerical radius, Linear and Multilinear Algebra, to appear.

[18] C.K. Li and L. Rodman, Numerical range of matrix polynomials, SIAM J. of Matrix Analysis Appl. 15 (1994), 1256-1265.

[19] C.K. Li , N.K. Tsing and F. Uhlig, Numerical range of an operator on an indefinite inner product space, submitted.

[20] P.B. Morenz, The Structure of C^*-convex Sets, Ph.D. Thesis, University of Toronto, 1992.

[21] S.K. Parrott, Unitary dilations for commuting contractions, Pac. J. Math. 34 (1970), 481-490.

[22] H. Shapiro, A survey of canonical forms and invariants for unitary similarity, Linear Algebra Appl. 147 (1991), 101-167.

[23] K.Tanahashi, On log-hyponormal operators, Integral equations Operator Theory., 34(1999), 364-372.

[24] K.Tanahashi, Putnam's inequality for log-hyponormal operators, Integral equations Operator Theory, to appear.

[25] K. Tanahashi, A.Uchiyama and Muneo Cho, Isolated points of spectrum of (p, k)-quasihyponormal operators, Linear Algebra Appl., 382(2004), 221-229.

[26] A.Uchiyama, Inequalities of Purnam and BergerShaw for p-quasihyponormal operators, Integr.Equat. Oper. Th., 34(1999), 179-180.

[27] J.P.Williams, Finite operators, Proc.amer.Math.Soc., 26(1970) 129-136.

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