THE NUMERICAL SOLUTION OF LAPLACE'S AND POISSON'S EQUATIONS*

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1. Introduction. A quite common method of solving numerically the Laplace differential equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \tag{1.1}$$

with the boundary values of V prescribed on some contour Γ bounding a region R is to approximate V by the solution u of the Laplace difference equation:

$$4u(x, y) = u(x + h, y) + u(x - h, y) + u(x, y + h) + u(x, y - h). \tag{1.2}$$

Briefly, the method of procedure, commonly called the Liebmann procedure,[1] is to cover the region R with a rectangular network of lines at distances h apart, and to assume values at the interior lattice points of this network. Using these assumed values and the known boundary values, we traverse the region R moving in some definite geometrical pattern from lattice point to lattice point, replacing the assumed values of u at each lattice point by the arithmetic average of the values of u at the four neighboring lattice points. We then repeat the traverse moving in the same pattern to obtain a second improved value of u at each lattice point; and so on until a convergent stage is reached when the values of u are no longer changed materially by continued traversing.

The purpose of this paper is to present a process which yields precisely the convergent values of u obtained by infinitely many traverses of the region. In more precise language, if u_k is the kth approximation of the value of u after k traverses, our process yields the value $u = \lim_{k\to\infty} u_k$.

2. Notation and set up of the problem. Equation (1.2) can be transformed to

$$4u(x, y) = u(x, y + 1) + u(x, y - 1) + u(x + 1, y) + u(x - 1, y) \tag{2.1}$$

by a simple transformation, and we shall concern ourselves with the solution of equation (2.1), with the values of u prescribed on the boundary lines

$$x = 0, \quad x = n, \quad y = 0, \quad \text{and} \quad y = m$$

of the rectangle R.

Unless otherwise stated the numbers m and n are fixed positive integers, and i and j will be used as variable positive integers with the range of values

$$i = 1, 2, \cdots, n - 1; \qquad j = 1, 2, \cdots, m - 1.$$

We shall denote the value of u at the point (i, j) by $u_j(i)$, and we desire to distinguish the known prescribed values of u on the boundary (which values are precisely the

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[1] For a fuller description of the process and for references to the literature on the subject, see the paper by Shortley and Weller in J. App. Phys. 9, 334–348 (1938).

same as those of V on the boundary) from the unknown values of u at the interior lattice points. Accordingly, we denote the prescribed values of u on the boundaries as follows:

$$\text{by } \bar{u}_j(0) \text{ at } (0, j); \quad \text{by } \bar{u}_j(n) \text{ at } (n, j);$$
$$\text{by } \bar{u}_0(i) \text{ at } (i, 0); \quad \text{by } \bar{u}_m(i) \text{ at } (i, m);$$

and agree that

$$u_j(0) = u_j(n) = u_0(i) = u_m(i) = 0$$

wherever these terms appear in our equations.

At each interior lattice point we can write

$$4u_j(i) = u_j(i+1) + u_j(i-1) + u_{j+1}(i) + u_{j-1}(i) + \phi_j(i), \tag{2.2}$$

where

$$\phi_j(i) = \delta_{i,1}\bar{u}_j(0) + \delta_{i,n-1}\bar{u}_j(n) + \delta_{j,1}\bar{u}_0(i) + \delta_{j,m-1}\bar{u}_m(i), \tag{2.3}$$

and δ_{ij} (or $\delta_{i,j}$) is the Kronecker delta defined by

$$\delta_{ij} = 1, \text{ if } i = j; \qquad \delta_{ij} = 0, \text{ if } i \neq j.$$

3. A special system of difference equations. We consider the solution of the system of equations

$$L_c u_j(i) \equiv c u_j(i) - u_j(i+1) - u_j(i-1) = u_{j+1}(i) + u_{j-1}(i) + \phi_j(i),$$
$$(i = 1, 2, \cdots, n-1; j = 1, 2, \cdots, m-1), \tag{3.1}$$

in which c and the $(m-1)(n-1)$ constants $\phi_j(i)$ are prescribed.[2] We assume that

$$u_j(0) = u_j(n) = u_0(i) = u_m(i) = 0,$$

and seek the values of the $(m-1)(n-1)$ unknowns $u_j(i)$.

The system (3.1) represents $(m-1)$ difference equations in the $(m-1)$ functions $u_j(x)$, $(j=1, 2, \cdots, m-1)$, whose values are desired for integral values of the argument x from 1 to $(n-1)$. It can be readily shown that system (3.1) has a unique solution, and of course this solution can be written down by Cramer's Rule, but we shall give the solution in another form.

An immediate property of the operator L_c defined by (3.1) is given by

$$L_{c+a}u(i) = (L_c + a)u(i), \tag{3.2}$$

where a is any constant. We define the inverse operator L_c^{-1} and integral powers L_c^k of the operator L_c in the usual manner.

From (3.2), we obtain

$$[L_c + a]^{-1} = L_{c+a}^{-1} \tag{3.3}$$

and

$$L_c[L_{c+a}^{-1}] = 1 - L_{c+a}^{-1}, \tag{3.4}$$

where 1 is used as the identity operator. The latter is established as follows:

$$L_c[L_{c+a}^{-1}] = [L_{c+a} - a][L_{c+a}^{-1}] = L_{c+a}L_{c+a}^{-1} - aL_{c+a}^{-1} = 1 - aL_{c+a}^{-1}.$$

[2] We assume that at least one of the $\phi_j(i)$ is different from zero; otherwise the system (3.1) has only the trivial solution $u_j(i) = 0$.

The solution of system (3.1) will be given symbolically in terms of the inverse operator L_c^{-1}, and the interpretation of the symbolic solution will depend on the constants $D_c(k)$ and $\lambda_{c,k}(i)$ defined below. To apply the solution obtained to the solution of the Laplace difference system (2.2), we have merely to observe that (2.2) is a special case of (3.1) in which $c=4$ and $\phi_j(i)$ has the value given in (2.3).

Let σ_c and ρ_c be the roots of the characteristic equation

$$c\xi - \xi^2 - 1 = 0. \tag{3.5}$$

We define $D_c(k)$ and $\lambda_{c,k}(i)$ by

$$D_c(k) = \frac{\sigma_c^k - \rho_c^k}{\sigma_c - \rho_c}, \tag{3.6}$$

$$\left.\begin{aligned} \lambda_{c,k}(i) &= \frac{D_c(k)}{D_c(n)} D_c(n - i) \quad \text{when} \quad k \leqq i, \\ \lambda_{c,k}(i) &= \frac{D_c(n - k)}{D_c(n)} D_c(i) \quad \text{when} \quad k \geqq i. \end{aligned}\right\} \tag{3.7}$$

The identities

$$D_c(n) = D_c(i)D_c(n - i + 1) - D_c(i - 1)D_c(n - i) \tag{3.8}$$

and

$$D_c(i + 1) = cD_c(i) - D_c(i - 1) \tag{3.9}$$

with $D_c(0)=0$, $D_c(1)=1$ are easily established. In terms of the operator L_c, we may write (3.9) in the form

$$L_cD_c(i) = 0. \tag{3.10}$$

Also if a is any constant, we have

$$L_cD_{c+a}(i) = [L_{c+a} - a]D_{c+a}(i) = L_{c+a}D_{c+a}(i) - aD_{c+a}(i);$$

hence

$$L_cD_{c+a}(i) = - aD_{c+a}(i). \tag{3.11}$$

We shall show that

$$L_c\lambda_{c,k}(i) = \delta_{ik}. \tag{3.12}$$

To establish (3.12) we have three cases to consider:

1) When $k \leqq i-1$, we have

$$L_c\lambda_{c,k}(i) = \frac{D_c(k)}{D_c(n)} L_cD_c(n - i) = 0, \quad \text{by} \quad (3.10).$$

2) When $k \geqq i+1$, we have

$$L_c\lambda_{c,k}(i) = \frac{D_c(n - k)}{D_c(n)} L_cD_c(i) = 0, \quad \text{by} \quad (3.10).$$

3) When $k=i$, we have

$$\begin{aligned} L_c\lambda_{c,i}(i) &= c\lambda_{c,i}(i) - \lambda_{c,i}(i+1) - \lambda_{c,i}(i-1) \\ &= [cD_c(n-i)D_c(i) - D_c(n-i-1)D_c(i) - D_c(i-1)D_c(n-i)]/D_c(n) \\ &= [D_c(i)\{cD_c(n-i) - D_c(n-i-1)\} - D_c(i-1)D_c(n-i)]/D_c(n) \\ &= [D_c(i)D_c(n-i+1) - D_c(i-1)D_c(n-i)]/D_c(n), \quad \text{by} \quad (3.9), \\ &= D_c(n)/D_c(n) = 1, \quad \text{by} \quad (3.8). \end{aligned}$$

Also useful is the relation

$$L_c\lambda_{c+a,k}(i) = \delta_{i,k} - a\lambda_{c+a,k}(i), \tag{3.13}$$

which is a consequence of (3.2) and (3.12).

4. The special cases $m=2$ and $m=3$. As an introduction to the symbolic solution, we consider first the simplest case, $m=2$, in which case there is only one equation in the system (3.1), since by hypothesis $u_0(i)=u_m(i)=0$. This equation is

$$L_cu_1(i) = \phi_1(i). \tag{4.1}$$

Its solution is given by

$$u_1(i) = \sum_{k=1}^{n-1} \lambda_{c,k}(i)\phi_1(k). \tag{4.2}$$

We also write the solution of (4.1) in the symbolic form

$$u_1(i) = L_c^{-1}[\phi_1(i)], \tag{4.3}$$

where the expression on the right side of (4.3) is to be interpreted as being equal to the right side of (4.2). Explicitly,

$$L_c^{-1}[\phi_1(i)] = \sum_{k=1}^{n-1} \lambda_{c,k}(i)\phi_1(k). \tag{4.4}$$

To make actual use of the solution given in (4.2), it is necessary to have a table of values of the constants $\lambda_{c,k}(i)$. In order not to complicate unduly the notation, the dependence of these constants on n has been omitted from our notation. A complete tabulation of these constants would require a great deal of space, since, with m fixed, they still depend on four parameters c, k, i, and n. However, for the application to the solution of (2.2), we have $c=4$, and abridged usable tables requiring only tabulations for $i=1$, 2 and varying k and n are given in Tables 1 and 2 of §9.

The entries in Table 1 give the multipliers to be applied to each of the boundary values in the calculation of $u_1(1)$. As an example let us consider a 2 by 10 rectangle. Multiply each boundary value of the first column by the multiplier which appears opposite it in the column $n=10$; add these products, and the sum is the value of $u_1(1)$. Likewise, by interchanging the arguments $(n-i)$ and i, the same multipliers can be used to calculate $u_1(9)$. Next using $u_1(1)$ and $u_1(9)$ as known boundary values and the 2 by 8 rectangle which has the points (1, 1) and (1, 9) on its ends, use the multipliers in column $n=8$ to calculate $u_1(2)$ and $u_1(8)$; and so on.

The number of points at which the values of u are to be calculated by this process can be cut in half by using Table 2. With the entries in this table, using again a 2 by

10 rectangle, the values $u_1(2)$ and $u_1(8)$ can be calculated using the multipliers in column $n=10$; then in the 2 by 6 rectangle with points (2, 1) and (8, 1) as ends, and using multipliers in column $n=6$, calculate $u_1(4)$ and $u_1(6)$. The values $u_1(1), u_1(3), \cdots, u_1(7)$ can then be obtained by the Liebmann formula, each being the average of its four neighbors. For example,

$$u_1(5) = \tfrac{1}{4}\{u_1(4) + u_1(6) + \bar{u}_0(5) + \bar{u}_2(5)\}.$$

In the case $m=3$, system (3.1) reduces to the following:

$$L_c u_1(i) = u_2(i) + \phi_1(i), \qquad L_c u_2(i) = u_1(i) + \phi_2(i), \tag{4.5}$$

with two unknown functions u_1 and u_2.

Treating the operators in (4.5) as algebraic multipliers, and solving for u_1 and u_2, we obtain symbolically

$$\left.\begin{aligned} u_1(i) &= \frac{L_c}{L_c^2 - 1}\phi_1(i) + \frac{1}{L_c^2 - 1}\phi_2(i), \\ u_2(i) &= \frac{1}{L_c^2 - 1}\phi_1(i) + \frac{L_c}{L_c^2 - 1}\phi_2(i). \end{aligned}\right\} \tag{4.6}$$

The following interpretation of the operators in the right members of (4.6) produces actual solutions. Write the operators of the right members in their partial fraction expansions, obtaining

$$\frac{L_c}{L_c^2 - 1} = \tfrac{1}{2}\left[\frac{1}{L_c - 1} + \frac{1}{L_c + 1}\right] = \tfrac{1}{2}\left[L_{c-1}^{-1} + L_{c+1}^{-1}\right],$$

$$\frac{1}{L_c^2 - 1} = \tfrac{1}{2}\left[\frac{1}{L_c - 1} - \frac{1}{L_c + 1}\right] = \tfrac{1}{2}\left[L_{c-1}^{-1} - L_{c+1}^{-1}\right].$$

The last step in each line follows from (3.3). Consequently, the explicit solution of (4.5) is given by

$$u_1(i) = \sum_{k=1}^{n-1}\left[\alpha_k(i)\phi_1(k) + \beta_k(i)\phi_2(k)\right], \qquad u_2(i) = \sum_{k=1}^{n-1}\left[\beta_k(i)\phi_1(k) + \alpha_k(i)\phi_2(k)\right]; \tag{4.7}$$

where

$$\alpha_k(i) = \tfrac{1}{2}\left[\lambda_{c-1,k}(i) + \lambda_{c+1,k}(i)\right], \qquad \beta_k(i) = \tfrac{1}{2}\left[\lambda_{c-1,k}(i) - \lambda_{c+1,k}(i)\right]. \tag{4.8}$$

To verify that (4.7) is the solution of (4.5), we use

$$L_c\alpha_k(i) = \delta_{ik} + \beta_k(i), \qquad L_c\beta_k(i) = \alpha_k(i),$$

which follow from (4.8) and (3.13). Consequently,

$$\begin{aligned} L_c u_1(i) &= \sum_{k=1}^{n-1}\left\{\left[L_c\alpha_k(i)\right]\phi_1(k) + \left[L_c\beta_k(i)\right]\phi_2(k)\right\} \\ &= \sum_{k=1}^{n-1}\delta_{ik}\phi_1(k) + \sum_{k=1}^{n-1}\left[\beta_k(i)\phi_1(k) + \alpha_k(i)\phi_2(k)\right] \\ &= \phi_1(i) + u_2(i). \end{aligned}$$

Tables 3 and 4 of §9 are useful in calculating values of $u_1(1)$ and $u_1(2)$ for the case $m=3$, again using $c=4$ which is appropriate for the Laplace equation.

These tables are used in a similar manner as Tables 1 and 2. However, for each value of n, there are two numbers side by side, and also two boundary values in the first column. The interpretation is to multiply the left one of the two boundary values of each line by the left one of the two multipliers of the same line in the appropriate n column.

For a given n, calculate $u_1(1)$ using multipliers of Table 3; then interchanging the subscripts 1 and 2 and the subscripts 0 and 3, calculate $u_2(2)$ using multipliers of Table 4. The value $u_2(1)$ is then obtained by the Liebmann formula

$$u_2(1) = \tfrac{1}{4}[\bar{u}_2(0) + \bar{u}_3(1) + u_1(1) + u_2(2)].$$

As in the case $m=2$, we can also calculate from the end $x=n$ of the rectangle.

5. The general case. We now consider the general system (3.1) for any value of m, and show that its solution can be written symbolically in terms of the polynomial operators defined by

$$P_0 = 0, \qquad P_1 = 1, \qquad P_k = L_cP_{k-1} - P_{k-2} \quad \text{for} \quad k > 1. \tag{5.1}$$

The operator P_k is a polynomial in L_c of the $(k-1)$st degree, which is precisely the same function of L_c as $D_c(k)$ is of c. From this observation, the following analogue of (3.8) can be seen to be valid,

$$P_n = P_iP_{n-i+1} - P_{i-1}P_{n-i}. \tag{5.2}$$

By solving the system (3.1) treating L_c as an algebraic multiplier, we obtain symbolically

$$u_j(i) = \sum_{k=1}^{j} \frac{P_{m-j}P_k}{P_m}\,\phi_k(i) + \sum_{k=j+1}^{m-1} \frac{P_jP_{m-k}}{P_m}\,\phi_k(i), \qquad (j = 1, 2, \cdots, m-1), \tag{5.3}$$

in which the operators of the right member are to be interpreted similarly to those of (4.6); that is they are to be expanded into partial fractions. Since each of the operator coefficients in the right member of (5.3) is a proper fraction, their expansions will have the form

$$\frac{P_jP_k}{P_m} = \sum_{l=1}^{m-1} \frac{b_l(j,\,k)}{L_c - a_l} = \sum_{l=1}^{m-1} b_l(j,\,k)L_{c-a_l}^{-1}, \tag{5.4}$$

where $a_1, a_2, \cdots, a_{m-1}$ are the roots of $P_m(\xi)=0$ and the numbers $b_l(j, k)$ are uniquely determined. The actual solution of (3.1) is thus given in terms of the operators $L_{c-a_l}^{-1}$ which have the meaning given in (4.4).

That the foregoing actually yields the solution of (3.1) can be established by operating on each side of (5.3) with the operator L_c and using the relations (5.1) and (5.2).

6. Application to solutions of Laplace's equation. We have already observed that (2.2) is a special case of the system (3.1) in which $c=4$ and $\phi_j(i)$ has the value given in (2.3). To apply the preceding results, it becomes necessary to calculate the coefficients of $\phi_k(i)$ in the solution of (3.1) for various values of m. The polynomial operators P_m must be factored, and the operators appearing in (5.3) must be written in the more useful form (5.4).

We have already considered in detail the cases $m=2$, $m=3$ in §4. We now particularize the general solution of §5 to the case $m=4$.

From (5.3) we have symbolically:

$$\left.\begin{aligned} u_1 &= \frac{P_1P_3}{P_4}\phi_1 + \frac{P_1P_2}{P_4}\phi_2 + \frac{P_1P_1}{P_4}\phi_3, \\ u_2 &= \frac{P_1P_2}{P_4}\phi_1 + \frac{P_2P_2}{P_4}\phi_2 + \frac{P_1P_2}{P_4}\phi_3, \\ u_3 &= \frac{P_1P_1}{P_4}\phi_1 + \frac{P_1P_2}{P_4}\phi_2 + \frac{P_1P_3}{P_4}\phi_3, \end{aligned}\right\} \tag{6.1}$$

where

$$\left.\begin{aligned} \frac{P_1P_1}{P_4} &= \frac{1}{L_c^3 - 2L_c} = \frac{1}{4}\left[\frac{1}{L_c-\sqrt{2}} + \frac{1}{L_c+\sqrt{2}} - \frac{2}{L_c}\right] = \frac{1}{4}\left[L_{c-\sqrt{2}}^{-1} + L_{c+\sqrt{2}}^{-1} - 2L_c^{-1}\right], \\ \frac{P_1P_2}{P_4} &= \frac{L_c}{L_c^3 - 2L_c} = \frac{1}{2\sqrt{2}}\left[\frac{1}{L_c-\sqrt{2}} - \frac{1}{L_c+\sqrt{2}}\right] = \frac{1}{2\sqrt{2}}\left[L_{c-\sqrt{2}}^{-1} - L_{c+\sqrt{2}}^{-1}\right], \\ \frac{P_1P_3}{P_4} &= \frac{L_c^2-1}{L_c^3 - 2L_c} = \frac{1}{4}\left[\frac{1}{L_c-\sqrt{2}} + \frac{1}{L_c+\sqrt{2}} + \frac{2}{L_c}\right] = \frac{1}{4}\left[L_{c-\sqrt{2}}^{-1} + L_{c+\sqrt{2}}^{-1} + 2L_c^{-1}\right], \\ \frac{P_2P_2}{P_4} &= \frac{L_c^2}{L_c^3 - 2L_c} = \frac{1}{2}\left[\frac{1}{L_c-\sqrt{2}} + \frac{1}{L_c+\sqrt{2}}\right] = \frac{1}{2}\left[L_{c-\sqrt{2}}^{-1} + L_{c+\sqrt{2}}^{-1}\right]. \end{aligned}\right\} \tag{6.2}$$

The explicit form of the solution is given by

$$\left.\begin{aligned} u_1(i) &= \sum_{k=1}^{n-1}\left[Q_k(i)\phi_1(k) + S_k(i)\phi_2(k) + R_k(i)\phi_3(k)\right], \\ u_2(i) &= \sum_{k=1}^{n-1}\left[S_k(i)\left\{\phi_1(k) + \phi_3(k)\right\} + T_k(i)\phi_2(k)\right], \\ u_3(i) &= \sum_{k=1}^{n-1}\left[R_k(i)\phi_1(k) + S_k(i)\phi_2(k) + Q_k(i)\phi_3(k)\right], \end{aligned}\right\} \tag{6.3}$$

where

$$\left.\begin{aligned} Q_k(i) &= \tfrac{1}{4}\left[\lambda_{c-\sqrt{2},k}(i) + \lambda_{c+\sqrt{2},k}(i) + 2\lambda_{c,k}(i)\right], \\ R_k(i) &= \tfrac{1}{4}\left[\lambda_{c-\sqrt{2},k}(i) + \lambda_{c+\sqrt{2},k}(i) - 2\lambda_{c,k}(i)\right], \\ S_k(i) &= \frac{1}{2\sqrt{2}}\left[\lambda_{c-\sqrt{2},k}(i) - \lambda_{c+\sqrt{2},k}(i)\right], \\ T_k(i) &= \tfrac{1}{2}\left[\lambda_{c-\sqrt{2},k}(i) + \lambda_{c+\sqrt{2},k}(i)\right], \end{aligned}\right\} \tag{6.4}$$

and

$$\left.\begin{aligned} \phi_1(k) &= \delta_{k,1}\bar{u}_1(0) + \delta_{k,n-1}\bar{u}_1(n) + \bar{u}_0(k), \\ \phi_2(k) &= \delta_{k,1}\bar{u}_2(0) + \delta_{k,n-1}\bar{u}_2(n), \\ \phi_3(k) &= \delta_{k,1}\bar{u}_3(0) + \delta_{k,n-1}\bar{u}_3(n) + \bar{u}_4(k). \end{aligned}\right\} \tag{6.5}$$

Multipliers for calculating $u_2(1)$, $u_2(2)$, and $u_1(2)$ are given in Tables 5, 6, and 7. For a 4 by n rectangle, calculate $u_2(1)$ using Table 5, $u_1(2)$ using Table 7, $u_3(2)$ using

Table 7 with subscripts 0 and 4 and subscripts 1 and 3 interchanged. Then calculate $u_1(1)$, $u_3(1)$ by the Liebmann Formula. Then using known values $u_1(1)$, $u_2(1)$, $u_3(1)$ and treating them as known boundary values for the rectangle one unit shorter, calculate $u_2(3)$ using Table 6; $u_2(2)$ can then be obtained by use of the Liebmann Formula. Then calculate $u_1(4)$, $u_3(4)$ using Table 7; and so on. Of course, as in the case $m=2$ and $m=3$, we can also calculate from the end $x=n$ of the rectangle. By this process the value at only every other point on each line is calculated by the use of the tabular values.

For values of m larger than 4, it is only convenient to use composite values of m. It can be easily shown that when m is composite having q for one factor then P_m contains P_q as a factor. However, for larger values of m, the necessary tables occupy more space and require a tremendous amount of time in their preparation. Theoretically the complete solution for any value of m is given by (5.3), but practically it is sufficient to use tables with $m=4$. A given rectangle can be covered by a lattice 4 units wide and the values of the function u at the interior lattice points calculated by the methods indicated. When the values of u at these lattice points are obtained, each rectangle can then be subdivided again by a finer network and the first calculated values are then used as approximate values for the finer network, and can in turn be improved either by traversing or by the use of the tables given here.

7. The Poisson Equation. The preceding methods and results can be extended with slight modification to apply to the numerical solution of the more general Poisson equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = F(x, y), \tag{7.1}$$

in which $F(x, y)$ is defined in the interior of the region R. The approximating difference equation in this case is

$$4u(x, y) = u(x + h, y) + u(x - h, y) + u(x, y + h) + u(x, y - h) - h^2F(x, y). \tag{7.2}$$

Employing our former notation, we can write at each interior lattice point

$$\begin{aligned} 4u_j(i) = {} & u_j(i + 1) + u_j(i - 1) + u_{j+1}(i) + u_{j-1}(i) - h^2F_j(i) + \delta_{i,1}\bar{u}_j(0) + \delta_{i,n-1}\bar{u}_j(n) \\ & + \delta_{j,1}\bar{u}_0(i) + \delta_{j,m-1}\bar{u}_m(i), \quad (i = 1, 2, \cdots, n - 1; j = 1, 2, \cdots, m - 1), \end{aligned} \tag{7.3}$$

and again consider

$$u_j(0) = u_j(n) = u_0(i) = u_m(i) = 0.$$

The system (7.3) is again a special case of system (3.1) in which the known function $\phi_j(i)$ is now given by

$$\phi_j(i) = \delta_{i,1}\bar{u}_j(0) + \delta_{i,n-1}\bar{u}_j(n) + \delta_{j,1}\bar{u}_0(i) + \delta_{j,m-1}\bar{u}_m(i) - h^2F_j(i). \tag{7.4}$$

Consequently, the general solution given in §5 applies at once.

In the case $m=2$, for example, we have

$$u_1(i) = \sum_{k=1}^{n-1} \lambda_{c,k}(i)\left[\delta_{k,1}\bar{u}_1(0) + \delta_{k,n-1}\bar{u}_1(n) + \bar{u}_0(k) + \bar{u}_2(k) - h^2F_1(k)\right]. \tag{7.5}$$

To apply Tables 1 and 2 to obtain the values of $u_1(1)$ and $u_1(2)$, first calculate $h^2F_1(k)$, $k=1, 2, \cdots, n-1$, at each interior lattice point. Then apply to these values the same multipliers as are applied to $\bar{u}_0(k)$ and $\bar{u}_2(k)$. If Table 2 is used to calculate $u_1(2)$, the value $u_1(1)$ can be obtained from the associated Liebmann equation

$$u_1(1) = \tfrac{1}{4}[u_1(2) + \bar{u}_1(0) + \bar{u}_0(1) + \bar{u}_2(1) - h^2F_1(1)]. \tag{7.6}$$

For the case $m=3$, multiply $h^2F_1(k)$ and $h^2F_2(k)$, respectively, by the same multipliers as are used for $\bar{u}_0(k)$ and $\bar{u}_3(k)$.

For the case $m=4$, additional tables are required. The explicit solution in this case is formally the same as that given in (6.3), but $\phi_1(k)$, $\phi_2(k)$, and $\phi_3(k)$ have the following values:

$$\left.\begin{aligned}\phi_1(k) &= \delta_{k,1}\bar{u}_1(0) + \delta_{k,n-1}\bar{u}_1(n) + \bar{u}_0(k) - h^2F_1(k),\\ \phi_2(k) &= \delta_{k,1}\bar{u}_2(0) + \delta_{k,n-1}\bar{u}_2(n) - h^2F_2(k),\\ \phi_3(k) &= \delta_{k,1}\bar{u}_3(0) + \delta_{k,n-1}\bar{u}_3(n) + \bar{u}_4(k) - h^2F_3(k).\end{aligned}\right\} \tag{7.7}$$

The multipliers $Q_k(2)$ and $R_k(2)$ appear in Table 7 and are respectively those multipliers applied to $\bar{u}_0(k)$ and $\bar{u}_4(k)$ in the calculation of $u_1(2)$. The multipliers $S_k(1)$ appear in Table 5 and are those multipliers applied to both $\bar{u}_0(k)$ and $\bar{u}_4(k)$ in the calculation of $u_2(1)$. The multipliers $S_k(2)$ are those multipliers in Table 6 which are applied to both $\bar{u}_0(k)$ and $\bar{u}_4(k)$ in the calculation of $u_2(2)$.

The multipliers $T_k(1)$ and $T_k(2)$ which must be applied to $h^2F_2(k)$ in the calculation of $u_2(1)$ and $u_2(2)$ appear in Tables 8 and 9 respectively.

8. Irregular cases and non-rectangular boundaries. The preceding solutions both in the case of the Laplace equation and the Poisson equation apply only to rectangular boundaries whose dimensions are integral multiples of the lattice unit h. To apply Tables 5, 6, and 7 in the solution of the Laplace equation for a rectangular boundary, divide the smaller dimension of the rectangle by 4 to obtain the lattice unit h. If the longer dimension of the rectangle is an integral multiple of h, the process here outlined for the solution applies directly. We call this the regular case. If the longer dimension is not an integral multiple of h, we call this the irregular case, and a modification of the process here outlined is required. We need an analogue of the Liebmann Formula to express the value of a harmonic function approximately in terms of the values of the function at four non-equidistant neighbors.

Let $H(x, y)$ be an arbitrary harmonic function whose value H_0 at (x_0, y_0) is to be expressed approximately in terms of H_1, H_2, H_3, H_4 which are the values of $H(x, y)$ at the points (x_0+r_1h, y_0), (x_0-r_2h, y_0), (x_0, y_0+r_3h), (x_0, y_0-r_4h) where r_1, r_2, r_3, r_4, and h are positive. When $r_1=r_2=r_3=r_4=1$, and $H(x, y)$ is approximated by its T. S. (Taylor Series) expansion about (x_0, y_0) up to terms including those of the third degree in h, H_0 is found to satisfy the Liebmann equation

$$H_0 = \tfrac{1}{4}(H_1 + H_2 + H_3 + H_4). \tag{8.1}$$

When r_1, r_2, r_3, and r_4 are not equal, and $H(x, y)$ is approximated by its T.S. up to terms including those of the second degree in h, we find[3]

$$H_0 = a_1H_1 + a_2H_2 + a_3H_3 + a_4H_4 \tag{8.2}$$

where

[3] This relation is also given by Shortley and Weller, ibid.; but their method of derivation is slightly different from ours.

$$
\begin{aligned}
a_1 &= r_2r_3r_4/(r_1 + r_2)(r_1r_2 + r_3r_4), & a_2 &= r_1r_3r_4/(r_1 + r_2)(r_1r_2 + r_3r_4), \\
a_3 &= r_1r_2r_4/(r_3 + r_4)(r_1r_2 + r_3r_4), & a_4 &= r_1r_2r_3/(r_3 + r_4)(r_1r_2 + r_3r_4).
\end{aligned} \tag{8.3}
$$

If $H(x, y)$ is approximated by its T.S. up to terms of the first degree in h, we find

$$H_0 = \sum_{k=1}^{4} b_kH_k \quad \text{with} \quad b_k = r_k^{-1}/(r_1^{-1} + r_2^{-1} + r_3^{-1} + r_4^{-1}). \tag{8.4}$$

Either (8.2) or (8.4) expresses H_0 as a weighted average of its four neighbors, and although (8.4) is easier to use, presumably (8.2) gives a better approximation to the value of H_0. However in the application to the irregular case of the rectangle, we require only the simpler forms to which (8.3) and (8.4) reduce when three of the r_k ($k=1, 2, 3, 4$) are equal to unity.

To determine the values of a harmonic function at the interior lattice points of a rectangle whose dimensions are $4h$ by $(n+r)h$ where n is an integer and $0<r<1$, let x, y, z, etc. be the values of the harmonic function at the points indicated in Fig. 1. By means of Tables 5, 6, 7 we can express u, v, and w as linear functions of x, y, z

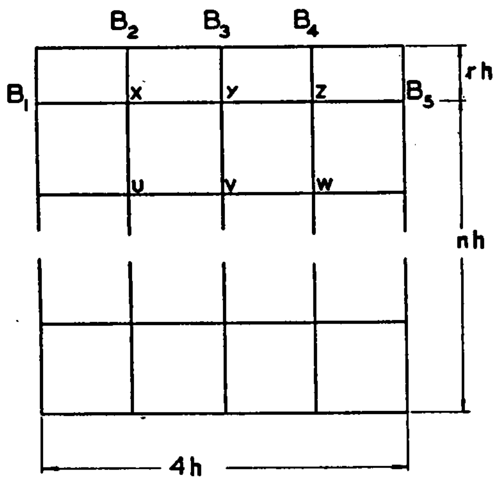

FIG. 1.

and the boundary values. Then by means of (8.2) or (8.4) or any other approximation method determine x, y, and z in terms of u, v, w, and the boundary values B_k ($k=1, 2, 3, 4, 5$) at the indicated points of the figure. This process leads to three linear algebraic equations for the determination of x, y, and z. When these values are determined, the values of the harmonic function at the other interior lattice points can be obtained as the problem is now reduced to the regular case.

A similar method can of course be used for the Poisson equation. In this case the analogue of (8.2) is

$$H_0 = a_1H_1 + a_2H_2 + a_3H_3 + a_4H_4 - a_0F_0 \tag{8.5}$$

where

$$a_0 = \tfrac{1}{2}h^2\left(\frac{r_1r_2r_3r_4}{r_1r_2 + r_3r_4}\right), \tag{8.6}$$

F_0 denotes $F(x_0, y_0)$, and a_1, a_2, a_3, a_4 are given by (8.3).

The foregoing method applies equally well if the top and bottom boundaries of the figure are not straight. We postpone for a later paper the procedure which can be applied for non-rectangular boundaries in general. In this subsequent paper we shall also show the application of our methods to an extension of the Liebmann process in which the values at certain of the interior lattice points are calculated from the arithmetic average of the values at their four normal neighbors while the values at the other lattice points are calculated from the values at their four diagonal neighbors.

9. Tables. The entries in the following tables were rounded off to four decimal places from calculations carried out to a higher number of decimal places. In the tables, the decimal points are not printed but are to be understood to be present just before the first digit. In the compilation of these tables, the values of $D_c(k)$, defined in (3.6), were required for the values $c=4, 3, 5, 4-\sqrt{2}$, and $4+\sqrt{2}$. These were calculated from the recurrence relation (3.9), and are integers only when c is an integer.

The values $\lambda_{4,k}(1)$ and $\lambda_{4,k}(2)$, defined in (3.7) are the entries in Tables 1 and 2, respectively. The entries in Tables 3 and 4 are the values $\alpha_k(i)$ and $\beta_k(i)$ for $i=1$ and 2, respectively; these were calculated from $\lambda_{3,k}(i)$ and $\lambda_{5,k}(i)$ for $i=1$ and 2 using the relations (4.8). The entries of Tables 5 to 9 were calculated from the relations (6.4).

As one convenient check on the accuracy of Tables 1 to 7, the sum of the multipliers used on all of the boundary values must be equal to unity; this check was applied. The author would be happy to know that no errors were made in the many calculations required in the preparation of these tables.

TABLE 1.—To calculate $u_1(1)$; $m=2$

Boundary values	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n\geqq 9$
$\bar{u}_1(0)$	2667	2679	2679	2679	2679	2679	2679
$\bar{u}_0(1)+\bar{u}_2(1)$	2667	2679	2679	2679	2679	2679	2679
$\bar{u}_0(2)+\bar{u}_2(2)$	0667	0714	0718	0718	0718	0718	0718
$\bar{u}_0(3)+\bar{u}_2(3)$		0179	0191	0192	0192	0192	0192
$\bar{u}_0(4)+\bar{u}_2(4)$			0048	0051	0052	0052	0052
$\bar{u}_0(5)+\bar{u}_2(5)$				0013	0014	0014	0014
$\bar{u}_0(6)+\bar{u}_2(6)$					0003	0004	0004
$\bar{u}_0(7)+\bar{u}_2(7)$						0001	0001
$\bar{u}_0(8)+\bar{u}_2(8)$							0000
$\bar{u}_1(n)$	0667	0179	0048	0013	0003	0001	0000

TABLE 2.—To calculate $u_1(2)$; $m=2$

Boundary values	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n\geqq 10$
$\bar{u}_1(0)$	0714	0718	0718	0718	0718	0718	0718
$\bar{u}_0(1)+\bar{u}_2(1)$	0714	0718	0718	0718	0718	0718	0718
$\bar{u}_0(2)+\bar{u}_2(2)$	2857	2871	2872	2872	2872	2872	2872
$\bar{u}_0(3)+\bar{u}_2(3)$	0714	0766	0769	0770	0770	0770	0770
$\bar{u}_0(4)+\bar{u}_2(4)$		0191	0205	0206	0206	0206	0206
$\bar{u}_0(5)+\bar{u}_2(5)$			0051	0055	0055	0055	0055
$\bar{u}_0(6)+\bar{u}_2(6)$				0014	0015	0015	0015
$\bar{u}_0(7)+\bar{u}_2(7)$					0004	0004	0004
$\bar{u}_0(8)+\bar{u}_2(8)$						0001	0001
$\bar{u}_0(9)+\bar{u}_2(9)$							0000
$\bar{u}_1(n)$	0714	0191	0051	0014	0004	0001	0000

TABLE 3.—To calculated $u_1(1)$; $m=3$

Boundary values		$n=2$		$n=3$		$n=4$		$n=5$		$n=6$	
$\bar{u}_1(0)$	$\bar{u}_2(0)$	2667	0667	2917	0833	2948	0861	2953	0866	2953	0866
$\bar{u}_0(1)$	$\bar{u}_3(1)$	2667	0667	2917	0833	2948	0861	2953	0866	2953	0866
$\bar{u}_0(2)$	$\bar{u}_3(2)$			0833	0417	0932	0497	0945	0509	0947	0511
$\bar{u}_0(3)$	$\bar{u}_3(3)$					0282	0195	0318	0227	0323	0232
$\bar{u}_0(4)$	$\bar{u}_3(4)$							0100	0082	0114	0095
$\bar{u}_0(5)$	$\bar{u}_3(5)$									0037	0033
$\bar{u}_1(n)$	$\bar{u}_2(n)$	2667	0667	0833	0417	0282	0195	0100	0082	0037	0033

TABLE 3.—(continued)

Boundary values		$n=7$		$n=8$		$n=9$		$n=10$		$n\geqq 11$	
$\bar{u}_1(0)$	$\bar{u}_2(0)$	2953	0866	2953	0866	2953	0866	2953	0866	2953	0866
$\bar{u}_0(1)$	$\bar{u}_3(1)$	2953	0866	2953	0866	2953	0866	2953	0866	2953	0866
$\bar{u}_0(2)$	$\bar{u}_3(2)$	0947	0512	0947	0512	0947	0512	0947	0512	0947	0512
$\bar{u}_0(3)$	$\bar{u}_3(3)$	0324	0233	0324	0233	0324	0233	0324	0233	0324	0233
$\bar{u}_0(4)$	$\bar{u}_3(4)$	0116	0097	0116	0097	0116	0097	0116	0097	0116	0097
$\bar{u}_0(5)$	$\bar{u}_3(5)$	0042	0038	0043	0039	0043	0039	0043	0039	0043	0039
$\bar{u}_0(6)$	$\bar{u}_3(6)$	0014	0013	0016	0015	0016	0015	0016	0015	0016	0015
$\bar{u}_0(7)$	$\bar{u}_3(7)$			0005	0005	0006	0006	0006	0006	0006	0006
$\bar{u}_0(8)$	$\bar{u}_3(8)$					0002	0002	0002	0002	0002	0002
$\bar{u}_0(9)$	$\bar{u}_3(9)$							0001	0001	0001	0001
$\bar{u}_0(10)$	$\bar{u}_3(10)$									0000	0000
$\bar{u}_1(n)$	$\bar{u}_2(n)$	0014	0013	0005	0005	0002	0002	0001	0001	0000	0000

TABLE 4.—To calculate $u_1(2)$; $m=3$

Boundary values		$n=3$		$n=4$		$n=5$		$n=6$		$n=7$	
$\bar{u}_1(0)$	$\bar{u}_2(0)$	0833	0417	0932	0497	0945	0509	0947	0511	0947	0512
$\bar{u}_0(1)$	$\bar{u}_3(1)$	0833	0417	0932	0497	0945	0509	0947	0511	0947	0512
$\bar{u}_0(2)$	$\bar{u}_3(2)$	2916	0834	3230	1056	3271	1093	3277	1098	3277	1099
$\bar{u}_0(3)$	$\bar{u}_3(3)$			0932	0497	1045	0591	1061	0606	1063	0608
$\bar{u}_0(4)$	$\bar{u}_3(4)$					0318	0227	0360	0265	0366	0271
$\bar{u}_0(5)$	$\bar{u}_3(5)$							0114	0095	0129	0109
$\bar{u}_0(6)$	$\bar{u}_3(6)$									0042	0038
$\bar{u}_1(n)$	$\bar{u}_2(n)$	2916	0834	0932	0497	0318	0227	0114	0095	0042	0038

TABLE 4.—(continued)

Boundary values		$n=8$		$n=9$		$n=10$		$n=11$		$n\geqq 12$	
$\bar{u}_1(0)$	$\bar{u}_2(0)$	0947	0512	0947	0512	0947	0512	0947	0512	0947	0512
$\bar{u}_0(1)$	$\bar{u}_3(1)$	0947	0512	0947	0512	0947	0512	0947	0512	0947	0512
$\bar{u}_0(2)$	$\bar{u}_3(2)$	3277	1099	3277	1099	3277	1099	3277	1099	3277	1099
$\bar{u}_0(3)$	$\bar{u}_3(3)$	1063	0609	1063	0609	1063	0609	1063	0609	1063	0609
$\bar{u}_0(4)$	$\bar{u}_3(4)$	0367	0272	0367	0272	0367	0272	0367	0272	0367	0272
$\bar{u}_0(5)$	$\bar{u}_3(5)$	0131	0112	0132	0112	0132	0112	0132	0112	0132	0112
$\bar{u}_0(6)$	$\bar{u}_3(6)$	0048	0044	0049	0044	0049	0045	0049	0045	0049	0045
$\bar{u}_0(7)$	$\bar{u}_3(7)$	0016	0015	0018	0017	0018	0017	0018	0017	0018	0017
$\bar{u}_0(8)$	$\bar{u}_3(8)$			0006	0006	0007	0007	0007	0007	0007	0007
$\bar{u}_0(9)$	$\bar{u}_3(9)$					0002	0002	0003	0003	0003	0003
$\bar{u}_0(10)$	$\bar{u}_3(10)$							0001	0001	0001	0001
$\bar{u}_0(11)$	$\bar{u}_3(11)$									0000	0000
$\bar{u}_1(n)$	$\bar{u}_2(n)$	0016	0015	0006	0006	0002	0002	0001	0001	0000	0000

TABLE 5.—To calculate $u_2(1)$; $m=4$

Boundary values	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$	$n=11$	$n=12$	$n\geqq 13$
$\bar{u}_2(0)$	2857	3230	3304	3320	3323	3324	3324	3324	3324	3324	3324	3324
$\bar{u}_1(0)+\bar{u}_3(0)$	0714	0932	0982	0993	0996	0997	0997	0997	0997	0997	0997	0997
$\bar{u}_0(1)+\bar{u}_4(1)$	0714	0932	0982	0993	0996	0997	0997	0997	0997	0997	0997	0997
$\bar{u}_0(2)+\bar{u}_4(2)$		0497	0625	0654	0661	0662	0663	0663	0663	0663	0663	0663
$\bar{u}_0(3)+\bar{u}_4(3)$			0268	0332	0346	0349	0350	0350	0350	0350	0350	0350
$\bar{u}_0(4)+\bar{u}_4(4)$				0133	0164	0171	0172	0173	0173	0173	0173	0173
$\bar{u}_0(5)+\bar{u}_4(5)$					0064	0079	0082	0083	0083	0083	0083	0083
$\bar{u}_0(6)+\bar{u}_4(6)$						0031	0038	0039	0040	0040	0040	0040
$\bar{u}_0(7)+\bar{u}_4(7)$							0015	0018	0019	0019	0019	0019
$\bar{u}_0(8)+\bar{u}_4(8)$								0007	0008	0009	0009	0009
$\bar{u}_0(9)+\bar{u}_4(9)$									0003	0004	0004	0004
$\bar{u}_0(10)+\bar{u}_4(10)$										0002	0002	0002
$\bar{u}_0(11)+\bar{u}_4(11)$											0001	0001
$\bar{u}_0(12)+\bar{u}_4(12)$												0000
$\bar{u}_1(n)+\bar{u}_3(n)$	0714	0497	0268	0133	0064	0031	0015	0007	0003	0002	0001	0000
$\bar{u}_2(n)$	2857	1056	0446	0226	0093	0044	0021	0010	0005	0002	0001	0000

TABLE 6.—To calculate $u_2(2)$; $m=4$

Boundary values	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$	$n=11$	$n=12$	$n=13$	$n=14$	$n\geqq 15$
$\bar{u}_2(0)$	1056	1250	1292	1301	1303	1304	1304	1304	1304	1304	1304	1304	1304
$\bar{u}_1(0)+\bar{u}_3(0)$	0497	0625	0654	0661	0662	0663	0663	0663	0663	0663	0663	0663	0663
$\bar{u}_0(1)+\bar{u}_4(1)$	0497	0625	0654	0661	0662	0663	0663	0663	0663	0663	0663	0663	0663
$\bar{u}_0(2)+\bar{u}_4(2)$	0932	1250	1325	1342	1346	1347	1347	1347	1347	1347	1347	1347	1347
$\bar{u}_0(3)+\bar{u}_4(3)$		0625	0788	0825	0833	0835	0835	0835	0835	0835	0835	0835	0835
$\bar{u}_0(4)+\bar{u}_4(4)$			0332	0410	0428	0432	0433	0433	0433	0433	0433	0433	0433
$\bar{u}_0(5)+\bar{u}_4(5)$				0164	0202	0210	0212	0212	0212	0212	0212	0212	0212
$\bar{u}_0(6)+\bar{u}_4(6)$					0079	0097	0101	0102	0102	0102	0102	0102	0102
$\bar{u}_0(7)+\bar{u}_4(7)$						0038	0046	0048	0048	0049	0049	0049	0049
$\bar{u}_0(8)+\bar{u}_4(8)$							0018	0022	0023	0023	0023	0023	0023
$\bar{u}_0(9)+\bar{u}_4(9)$								0008	0010	0011	0011	0011	0011
$\bar{u}_0(10)+\bar{u}_4(10)$									0004	0005	0005	0005	0005
$\bar{u}_0(11)+\bar{u}_4(11)$										0002	0002	0002	0002
$\bar{u}_0(12)+\bar{u}_4(12)$											0001	0001	0001
$\bar{u}_0(13)+\bar{u}_4(13)$												0000	0001
$\bar{u}_0(14)+\bar{u}_4(14)$													0000
$\bar{u}_1(n)+\bar{u}_3(n)$	0932	0625	0332	0164	0079	0038	0018	0008	0004	0002	0001	0000	0000
$\bar{u}_2(n)$	3230	1250	0539	0245	0114	0054	0025	0012	0006	0003	0001	0001	0000

TABLE 7.—To calculate $u_1(2)$; $m=4$

Boundary values		$n=3$		$n=4$		$n=5$		$n=6$		$n=7$		$n=8$	
$\bar{u}_2(0)$		0497		0625		0654		0661		0662		0663	
$\bar{u}_1(0)$	$\bar{u}_3(0)$	0861	0195	0982	0268	1005	0287	1010	0292	1011	0293	1011	0293
$\bar{u}_0(1)$	$\bar{u}_4(1)$	0861	0195	0982	0268	1005	0287	1010	0292	1011	0293	1011	0293
$\bar{u}_0(2)$	$\bar{u}_4(2)$	2948	0282	3304	0446	3365	0494	3377	0506	3380	0508	3381	0509
$\bar{u}_0(3)$	$\bar{u}_4(3)$			0982	0268	1129	0364	1158	0389	1164	0394	1165	0396
$\bar{u}_0(4)$	$\bar{u}_4(4)$					0365	0174	0429	0224	0442	0236	0445	0239
$\bar{u}_0(5)$	$\bar{u}_4(5)$							0148	0097	0177	0122	0183	0128
$\bar{u}_0(6)$	$\bar{u}_4(6)$									0064	0050	0077	0062
$\bar{u}_0(7)$	$\bar{u}_4(7)$											0029	0025
$\bar{u}_1(n)$	$\bar{u}_3(n)$	2948	0282	0982	0268	0365	0174	0148	0097	0064	0050	0029	0025
$\bar{u}_2(n)$		0932		0625		0332		0164		0079		0038	

TABLE 7.—(continued)

Boundary values		$n=9$		$n=10$		$n=11$		$n=12$		$n=13$		$n\geqq 14$	
$\bar{u}_2(0)$		0663		0663		0663		0663		0663		0663	
$\bar{u}_1(0)$	$\bar{u}_3(0)$	1011	0293	1011	0293	1011	0293	1011	0293	1011	0293	1011	0293
$\bar{u}_0(1)$	$\bar{u}_4(1)$	1011	0293	1011	0293	1011	0293	1011	0293	1011	0293	1011	0293
$\bar{u}_0(2)$	$\bar{u}_4(2)$	3381	0509	3381	0509	3381	0509	3381	0509	3381	0509	3381	0509
$\bar{u}_0(3)$	$\bar{u}_4(3)$	1166	0396	1166	0396	1166	0396	1166	0396	1166	0396	1166	0396
$\bar{u}_0(4)$	$\bar{u}_4(4)$	0446	0240	0446	0240	0446	0240	0446	0240	0446	0240	0446	0240
$\bar{u}_0(5)$	$\bar{u}_4(5)$	0184	0129	0185	0129	0185	0130	0185	0130	0185	0130	0185	0130
$\bar{u}_0(6)$	$\bar{u}_4(6)$	0080	0065	0081	0066	0081	0066	0081	0066	0081	0066	0081	0066
$\bar{u}_0(7)$	$\bar{u}_4(7)$	0035	0031	0036	0032	0036	0032	0037	0033	0037	0033	0037	0033
$\bar{u}_0(8)$	$\bar{u}_4(8)$	0013	0012	0016	0015	0017	0016	0017	0016	0017	0016	0017	0016
$\bar{u}_0(9)$	$\bar{u}_4(9)$			0006	0006	0007	0007	0008	0008	0008	0008	0008	0008
$\bar{u}_0(10)$	$\bar{u}_4(10)$					0003	0003	0003	0003	0004	0004	0004	0004
$\bar{u}_0(11)$	$\bar{u}_4(11)$							0001	0001	0002	0002	0002	0002
$\bar{u}_0(12)$	$\bar{u}_4(12)$									0001	0001	0001	0001
$\bar{u}_0(13)$	$\bar{u}_4(13)$											0000	0000
$\bar{u}_1(n)$	$\bar{u}_3(n)$	0013	0012	0006	0006	0003	0003	0001	0001	0001	0001	0000	0000
$\bar{u}_2(n)$		0018		0008		0004		0002		0001		0000	

TABLE 8.—Values of $T_k(1)$

k	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$	$n=11$	$n=12$	$n=13$	$n\geqq 14$
1	2857	3230	3304	3320	3323	3324	3324	3324	3324	3324	3324	3324	3324
2		1056	1250	1292	1301	1303	1304	1304	1304	1304	1304	1304	1304
3			0446	0539	0560	0564	0565	0565	0566	0566	0566	0566	0566
4				0201	0245	0255	0257	0258	0258	0258	0258	0258	0258
5					0093	0114	0119	0120	0120	0120	0120	0120	0120
6						0044	0054	0056	0056	0056	0057	0057	0057
7							0021	0025	0026	0027	0027	0027	0027
8								0010	0012	0012	0013	0013	0013
9									0005	0006	0006	0006	0006
10										0002	0003	0003	0003
11											0001	0001	0001
12												0000	0001
13													0000

TABLE 9.—Values of $T_k(2)$

k	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$	$n=11$	$n=12$	$n=13$	$n=14$	$n\geqq 15$
1	1056	1250	1292	1301	1303	1304	1304	1304	1304	1304	1304	1304	1304
2	3230	3750	3859	3883	3888	3890	3890	3890	3890	3890	3890	3890	3890
3		1250	1493	1546	1558	1561	1561	1562	1562	1562	1562	1562	1562
4			0539	0653	0678	0684	0685	0686	0686	0686	0686	0686	0686
5				0245	0299	0311	0314	0314	0314	0314	0314	0314	0314
6					0114	0140	0145	0146	0147	0147	0147	0147	0147
7						0054	0066	0068	0069	0069	0069	0069	0069
8							0025	0031	0032	0033	0033	0033	0033
9								0012	0015	0015	0015	0015	0015
10									0006	0007	0007	0007	0007
11										0003	0003	0003	0003
12											0001	0002	0002
13												0001	0001
14													0000