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THE NUMERICAL SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS*

BY

HOWARD W. EMMONS

Harvard University

1. Introduction. Through a consideration of the fundamental aspects of the universe Newton was led to the invention of fluxions. The physical idea of "rate of change," the geometric idea of "slope of a curve" together with his newly invented mathematics proved to be very powerful in describing and predicting a wide range of phenomena of nature. Since Newton's day, an enormous number of physical phenomena have been described in terms of a few "laws of nature." Very often these laws make use of the calculus, especially when applied to a specific problem. Thus large sections of the phenomena of the physical universe are described by the solutions of differential equations for the appropriate boundary conditions.

The engineer in his attempt to make nature work his way is continually presented with problems to be solved. These problems, even when very technical in nature, usually contain an element not present in problems considered by mathematicians or physicists. A "solution" of an engineer's problem is often a numerical answer or a graph obtained in *a specified time*. A poor answer which meets the deadline date is far superior to a precise answer a week later. Thus the engineer, or any applied scientist, should, when choosing the method of attack on a problem, keep in mind the time when the answer is due.

The following methods of solution are available:

1) the answer can be guessed;
2) some experiments can be run;
3) the result can be computed from the basic laws of nature, by use of whatever mathematical methods are needed.

Obviously the first method has one certain and everlasting superiority over the second and third; it is quick. It can meet any deadline set at a future time. It, of course, has one big disadvantage. The solution is always of doubtful quantitative value even though it is of immense qualitative value. In fact, the guessing process should always be used as a guide to correct results by methods 2 or 3.

If time permits, the answer may be sought by experiment or computation, or both. At the present time computational methods, when applied to real physical systems to obtain solutions of sufficient accuracy, are often so cumbersome that the vast majority of engineering problems are solved primarily by experiment. Usually com-

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putations are confined to one dimensional approximations to the real problem, or to certain simple two or three dimensional approximations.

In the field of partial differential equations, which includes the key to a vast number of practical problems, the present day need is for methods of solution speedier and more general than the majority of analytical methods thus far discovered. Numerical methods of solution aim at securing an approximate quantitative answer to a given problem as directly as possible from the statement of the problem, so as to reduce the time required for solution. An advantage of this more direct approach is the possibility of injecting into the solution any facts known (or supposed known) by the computer. Thus, physical "facts" of any nature can be made use of, and if some of the "facts" are wrong the solution will indicate this. Obviously, this has an enormous advantage over analytical methods where generally little can be made of detailed physical observations, except as a check on the final result.

2. An elementary problem. By way of illustrating the ideas involved, let us consider the transient flow of heat in a two-dimensional homogeneous, isotropic solid for which the physical law of the conservation of energy yields the familiar differential equation

$$\frac{\partial T}{\partial t} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right), \tag{1}$$

where T denotes the temperature, x and y space coordinates, t the time, a the thermal diffusivity (a property of the material, assumed constant). The numerical methods considered here are based upon a finite difference approximation to the differential equation. The simplest of these may be obtained as follows.

By definition

$$\frac{\partial T(x, y, t)}{\partial t} = \lim_{\delta t \to 0} \frac{T(x, y, t + \delta t) - T(x, y, t)}{\delta t} \tag{2}$$

and similar expressions hold for the derivatives with respect to x and y. As an approximation, the limit operation may be omitted. If this is done the following expressions are obtained

$$\frac{\partial^2 T}{\partial x^2} = \frac{T(x - \delta x, y, t) - 2T(x, y, t) + T(x + \delta x, y, t)}{\delta x^2} = \frac{T_3 - 2T_0 + T_1}{\delta x^2} \tag{3}$$

and

$$\frac{\partial^2 T}{\partial y^2} = \frac{T(x, y - \delta y, t) - 2T(x, y, t) + T(x, y + \delta y, t)}{\delta y^2} = \frac{T_2 - 2T_0 + T_4}{\delta y^2} \tag{4}$$

where the last expressions on the right follow the notation of Fig. 1. Substitution of these approximations into Eq. (1) gives, after some rearrangement,

$$T(x, y, t + \delta t) = \frac{a\delta t}{\delta x^2}(T_3 + T_1) + \frac{a\delta t}{\delta y^2}(T_2 + T_4) + \left(1 - \frac{2a\delta t}{\delta x^2} - \frac{2a\delta t}{\delta y^2}\right)T_0, \tag{5}$$

which relates the temperature at a given point 0 at time $t+\delta t$ to the temperatures which existed in the neighborhood of 0 at time t. Since the only restriction on δt, δx, δy

is that they be small enough to render the finite difference approximation sufficiently accurate, we set

$$4a\delta t = \delta x^2 = \delta y^2 = \delta^2, \tag{6}$$

whence

$$T(x, y, t + \delta t) = \tfrac{1}{4}(T_1 + T_2 + T_3 + T_4). \tag{7}$$

This same equation can be derived directly from the physical problem by making physical assumptions only. Such an approach is very valuable, since many more or less vague physical assumptions are always made before any engineering problem can

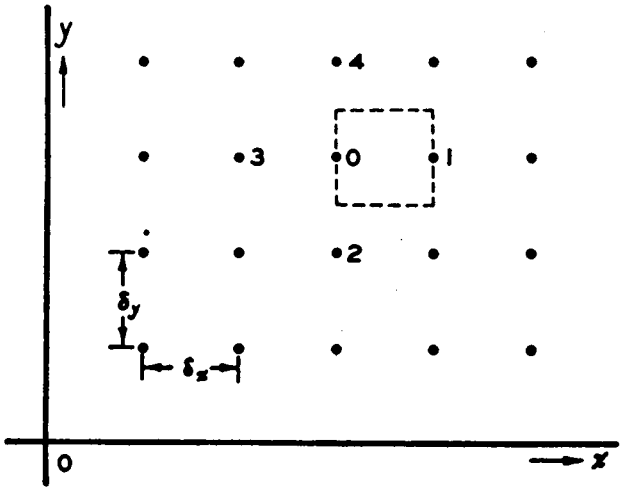

FIG. 1.

be set up mathematically. Let us assume that the material of the solid body is divided up into three sets of overlapping squares (or rectangles if desired). One set represents all heat conduction of the body in the x-direction, as indicated by the dotted square between points 0 and 1 of Fig. 1. The second set of squares represents the conduction in the y-direction and finally the third set, surrounding each point, represents all the material as far as heat capacity is concerned. The thermal energy Q_{1-0} which is conducted in unit time to point 0 along the rod $1-0$ is obtained from Fourier's heat conduction equation

$$q = -k\frac{dT}{dx} \tag{8}$$

in the form

$$Q_{1-0} = kb(T_1 - T_0), \tag{9}$$

where b is the thickness of the two dimensional body considered. The energy arriving at the point 0 from all the surrounding points is thus

$$Q_0 = kb(T_1 + T_2 + T_3 + T_4 - 4T_0). \tag{10}$$

This heat will result in an increase of temperature of the material associated with point 0, whence

$$Q_0 = c\rho b\delta^2 \frac{T(x, y, t + \delta t) - T_0}{\delta t}. \tag{11}$$

We equate (10) and (11) and rearrange terms. Thus,

$$T(x, y, t + \delta t) = \frac{k\delta t}{c\rho\delta^2}(T_1 + T_2 + T_3 + T_4) + \left(1 - \frac{4k\delta t}{c\rho\delta^2}\right)T_0, \qquad (12)$$

which reduces to (7), since the thermal diffusivity $a = k/c\rho$, and δt, δ are related by (6).

The method of use of Eq. (7) for the solution of a transient heat flow problem expressed by the differential Eq. (1) is direct. The space domain is divided into a square net of points. (The exact relationship between the required net spacing and the desired accuracy of solution has not yet been studied, to the author's knowledge.) The initial values (at $t=0$) of temperature are attached to each point and the values at successive times are computed by the averaging process. Figs. 2 illustrates such a solution.

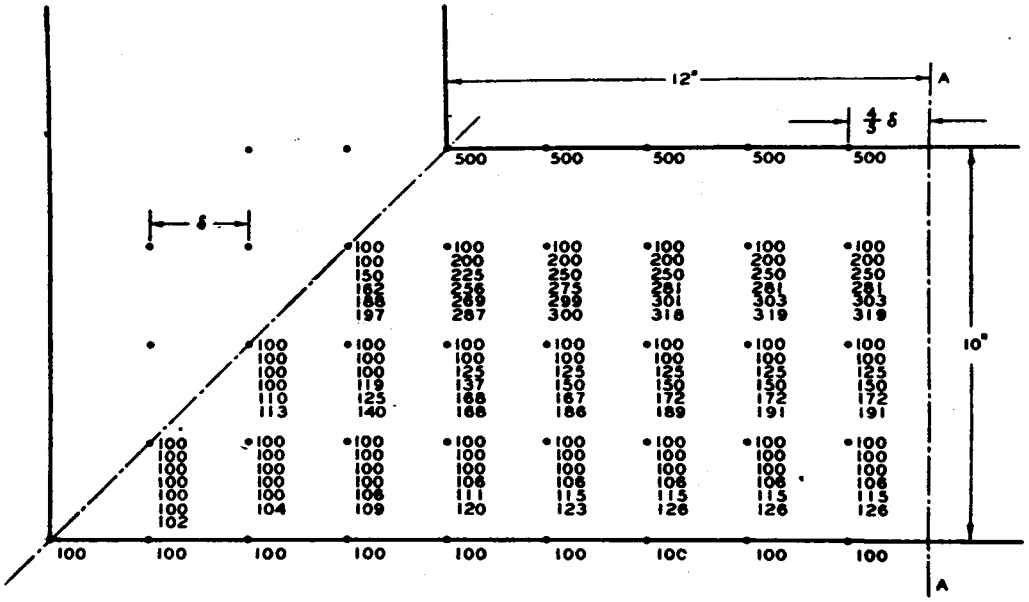

FIG. 2a. (See legend below Fig. 2c.)

By the nature of the process it is clear that the shape of the domain and the boundary conditions (generally some relation between the boundary value and the normal derivative) cause no special difficulty, such as occurs in the analytical approach, since they can be treated numerically as required for the points nearest (or on) the boundary. Boundary conditions and net spacing should be chosen of comparable accuracy as judged physically (in the absence of rigorous methods). Experience indicates that it is seldom worth the trouble to derive special boundary formulas, since generally linear extrapolation or a simple plot on graph paper will give the same accuracy much more speedily.

Let us return to Eq. (1) and the attendant physical problem. It is known that if the thermal conditions at the physical boundaries are held fixed (in time), the interior temperature will (after a sufficient period) reach a steady value (to any given

degree of approximation). In the present problem, this means that with a sufficiently large number of applications of Eq. (7) the solution of Laplace's equation will be obtained for the given domain and boundary conditions. The process proposed above, the one closely followed by nature, is numerically *very* slow unless an elaborate computing machine is used [1].*

Several schemes have been proposed for speeding up this process. By starting with Laplace's equation and making the finite difference approximations, by starting physically with Eq. (10) and $Q_0=0$, or by observing that for steady conditions $T(x, y, t+\delta t)=T_0$ in Eq. (7), one obtains

$$T_0 = \tfrac{1}{4}(T_1 + T_2 + T_3 + T_4). \tag{13}$$

When Eq. (13) is written for each of n points of a square net covering a given domain, n linear equations result. Several iteration processes have been devised for solving such a system [2, 5, 6], and the convergence of some of these has been

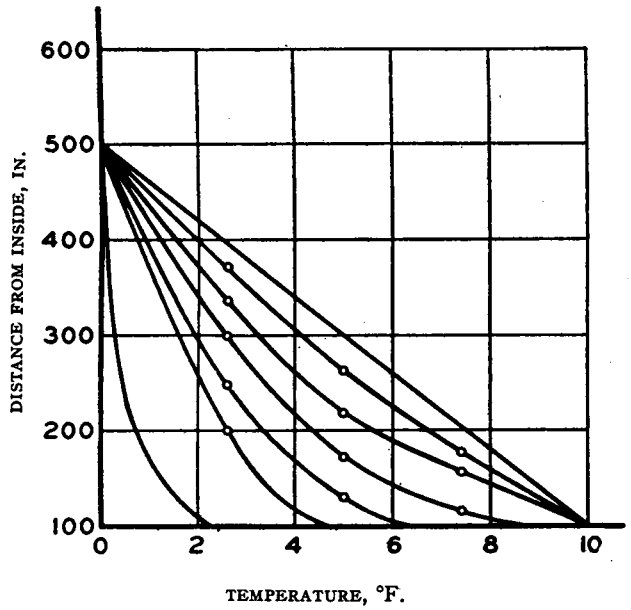

FIG. 2b.

discussed [3, 4, 7, 8]. All of these methods propose computations on the values of T by a specifically stated iteration process which can be shown rigorously to converge and can be performed by a completely automatic computing machine. In no case is it possible to add physical information after the values guessed initially are attached to each net point, without upsetting the scheme of solution.

A new scheme for solving Eq. (13) for the n points of a two-dimensional domain is given by the relaxation method [9]. This method is so superior to others in point of the time required to reach a solution of given accuracy that it will be discussed in

* The numbers in square brackets refer to the bibliography at the end of the paper.

detail. Again, the physical problem will be considered as one of heat conduction, although any other phenomena leading to Laplace's equation would serve as well. R. V. Southwell [10], who developed the relaxation method from a consideration of problems of statics, generally speaks in terms of a tension net as an approximation to a soap film or membrane.

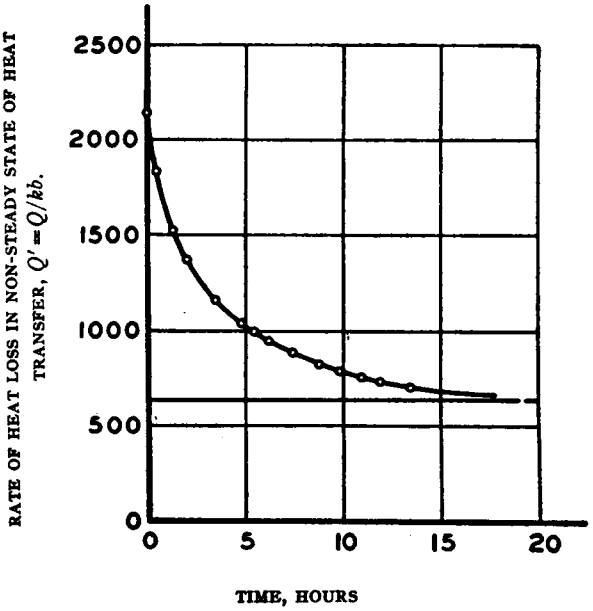

FIG. 2c.

FIGS. 2a–c. Transient heating of furnace wall (Fig. 5). At time 0 wall at uniform temperature $T = 100°$ F has inner surface temperature raised to 500° F. Wall thermal diffusivity $a = .01$ ft²/hr. $\delta_x = .208$ ft. Therefore $\delta_t = 1.083$ hrs. The 6 temperatures shown at each point are at time intervals of 1.083 hrs. Each temperature is the average of the temperatures at the 4 surrounding points at the *preceding* time. Heat loss is $\sum Q'$ along inner surface.

(From *The numerical solution of heat conduction problems.* By H. W. Emmons, Trans. A.S.M.E., 65, 607–615 (1943)).

Instead of focusing attention on the values of T and the averaging process of Eq. (13), let us return to Eq. (10). To solve a problem the domain is drawn and the net points chosen. Values of T are then attached (by guess or any information available from experiment, special solutions, prior work, field mapping, etc.) to each point. From these values the residuals

$$Q = Q_0/kb = T_1 + T_2 + T_3 + T_4 - 4T_0 \tag{14}$$

are computed and recorded. The Q, thus computed, can be thought of as interior heat sinks which must be removed. Now, instead of setting up a specific iteration process, we merely observe that if the temperature at one point (0) is altered, all others remaining fixed, the residuals will change according to the pattern in Fig. 3, the "relaxation pattern." Each change of T, at any point, effects a redistribution of the residuals,

Q, among the net points, and such changes of T are desired which will move all the sinks to the boundary.

These "operating instructions" may appear vague. Indeed they are vague. Their vagueness is the source of their great power because the computer may without any effort alter the procedure to attain more rapid approach to the final answer (of no residuals). There is only one way to appreciate fully the meaning of these remarks and that is to do a problem.

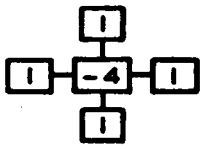

FIG. 3.

Let us consider the extremely simple problem of Fig. 4, where the solution of Laplace's equation is desired with zero boundary values. For illustrations of the various methods, only three interior points are used and the ridiculous trial solution indicated in Fig. 4 is assumed. In this simple problem all the methods are equally easy. The real time-saving advantages of the relaxation process only appear with more net points. In Table IA the transient solution [differential equation (1)] is carried out by use of the difference equation (7). In Table IB Liebman's method [2] is used, Eq. (13). In both cases, each value after the first at any point is obtained by adding four numbers and dividing by 4. This process cannot (for say 4 digit numbers) be carried out mentally. Fifteen and ten changes respectively were needed to make the error less than unity.

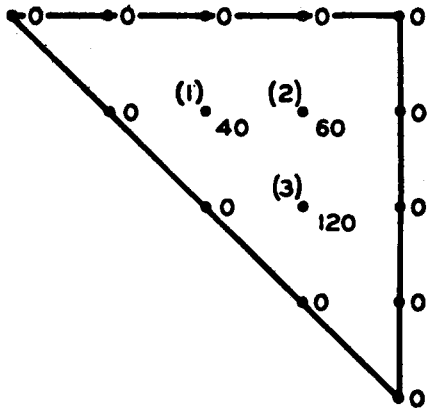

FIG. 4.

Tables IC, D, E show applications of the relaxation method and these will be followed in detail to illustrate various "tricks" which serve to speed up the elimination of residuals. In Tables IC, D, E the initial value of T and the subsequent corrections are shown at the right of each field, the values of the residuals (heat sinks) are shown to the left. The largest heat sink occurs in the vicinity of the greatest deviation of the assumed values from the correct solution, so changes are first made at this point. For purposes of illustration, each time any change is made in Tables IC, D, E a space is left at all unaffected points, so that the work can be followed. (Generally this is not done.)

Let us consider Table IC. To eliminate exactly a residual of -420 at point 3 would require a change in T_3 of -105. The making of this change is equivalent exactly to the averaging process. But why do we bother with three digit numbers? Since our residuals are very large, let us make simple large changes of about the right size. Therefore, let us change T_3 by -100. By the relaxation pattern of Fig. 3 the residuals become $Q_3 = -420+4(100) = -20$, $Q_2 = -80-100 = -180$. Now Q_2 is largest. Accordingly, we change T_2; a change of -50 was chosen and the third residuals at each point computed. We notice that the residual at the point 3, much improved at the first step, *has been spoiled again* by the following change at point 2. This always happens when a point is surrounded by other points with residuals of the same sign. We would have done better to *overshoot* zero at point 3 on the first change. On the next change, at point 1, we overshoot the zero and make the residual positive. The residual at point 2 changes from $+20$ to -20 and the previous change might well

have been larger. By not overshooting enough or by overshooting too much we do no harm except that some time is lost. A little practice improves one's guessing.

TABLE I

A) Solution by Eq. (17); 15 changes.
B) Solution by Eq. (13) (Liebman's method); 10 changes.
C) Illustrates overshooting; 5 changes. († = overshot, * = should have overshot).
D) Illustrates block relaxation; 6 changes made, first 3 at once.
E) Illustrates use of prior knowledge; 3 changes.

Point	1		2		3	
A	40		60		120	
	15		40		15	
	10		7.5		10	
	1.4		5		1.4	
	1.2		.7		1.2	
	.2		.6		.2	
B	40		60		120	
	15		33.8		8.4	
	8.4		4.2		1.0	
	1.0		.5		.1	
	.1					
C	−100	40	− 80	60	−420	120
	—		−180		− 20	−100
	−150		20	−50	− 70	
	10	−40†	− 20	*	—	
	—		− 40		10	− 20
	0		0	−10	0	
D	−100	40	− 80	60	−420	120
	50	−50	20	−50	−270	− 50
	—		− 50		10	− 70
	40		− 10	−10	0	
	0	+10	0		—	
E	−100	40	− 80	60	−420	120
	—		−200		60	−120
	−160		60	−60	0	
	0	−40	0		—	

In Table ID the block relaxation process is used, i.e., a block of points is changed simultaneously. This is desirable since all the residuals are negative. So large a change is made that their average resultant residual is about zero (or overshot if desired). A change of 50 was chosen. If one computes by the relaxation pattern [Eq. (14)] directly there is no gain by block relaxation. Instead let us consider the physical arrangement. If the temperatures at two adjacent points are both changed by the same amount there will be no change of heat flow along the connecting rod. Thus the relaxation pattern $-1, 1$ as given by Eq. (9) is used for each rod independently, and at the points of Table ID, the residuals change by $Q_1 = -100+3(50) = 50$, $Q_2 = -80+2(50) = 20$, $Q_3 = -420+3(50) = -270$. From this point the solution is continued as before.

Of course we know the solution in the present case is zero everywhere. This knowledge is used in Table IE immediately and the solution obtained in three steps. Naturally one would not have started this problem with so poor an assumed set of initial values. Also the prior knowledge is never as extensive as in the present trivial problem, but observation of trends in the solution often gives a clue to the way in which values should be changed.

It is not possible to judge the speed and ease of the relaxation method compared to the averaging methods by comparison of the various procedures illustrated in Table I. The relaxation method is far superior when a large number of points are used because of its flexibility (possibility of overshooting), its use of simple large numbers when the residuals are large and simple small numbers when they have become small, and the fact that the most difficult operation is to multiply a simple number by 4 and add the result to or subtract it from another, all of which can easily be done mentally.

Figs. 5 illustrate the solution of a somewhat more difficult problem. It should be

noted that the solution is started with a few points only. Then, for greater accuracy more points are added. These need not be added everywhere but only where greater accuracy is desired, or where some variable changes rapidly. In the illustration the points were doubled in number by adding one point at the center of each four points. The resultant net is square but diagonal to the first. Since Laplace's equation is invariant on rotation of the axes, the same average formula applies to values on the new net. Guessed values for starting the finer net are obtained by averaging the surrounding four values. If a fine net is used locally, as might have been done near the reentrant corner in the illustration, it can always be connected with the coarser net through use of the diagonal formula where the two nets meet.

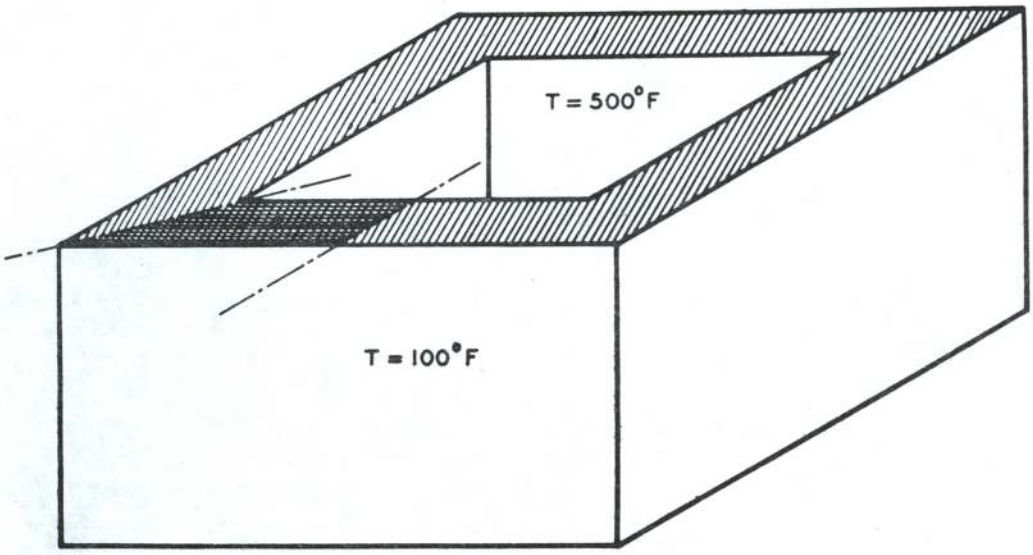

FIG. 5a. Furnace, showing section under consideration.

The results of computation of the furnace wall conduction problem are summarized in Table II. We note particularly the time required for a solution and the

TABLE II

Numerical Solution of Heat Conduction Problem of Fig. 5

Number of points used	Calculated thermal resistance	Deviation from experimental solution	Hours required for calculation
by arithmetic mean area	$\frac{.0735}{kb}$	15.3%	.05
12	$\frac{.0806}{kb}$	5%	.75
19	$\frac{.0825}{kb}$	2.8%	1.75

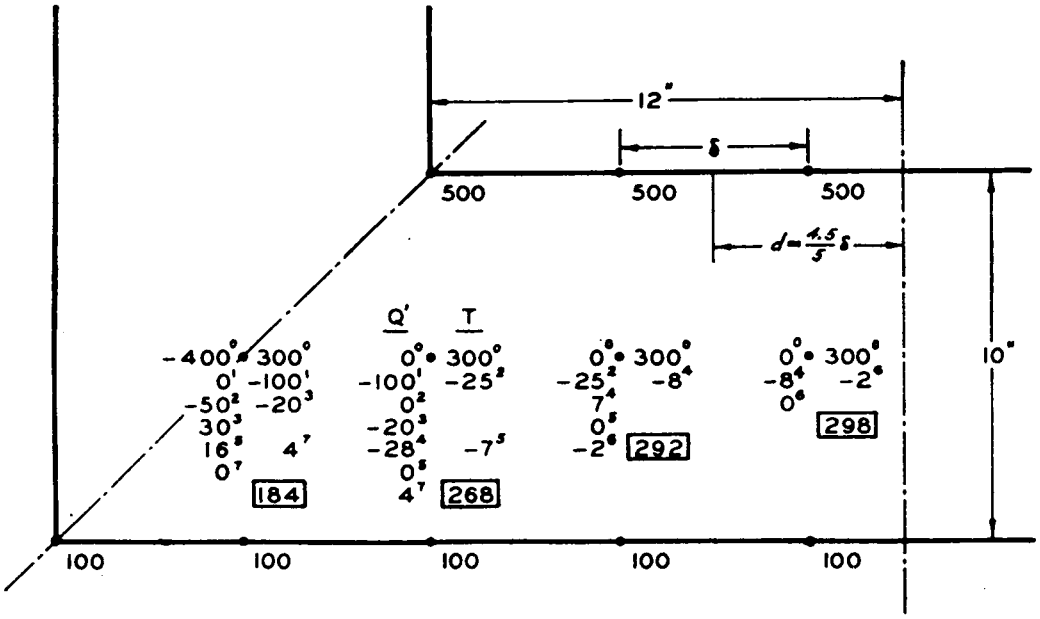

FIG. 5b. Start solution with linear temperature distribution, relax Q' to get final temperature. Heat transferred is $Q=kb[232+208+(4.5/5)202]=622\ kb$. Thermal resistance $R=\Delta T/8Q=400/(8\times622kb)=.0806/kb$. Superscripts indicate step of calculation. ^{0}=original values, $^{1,2,etc.}$=successive steps.

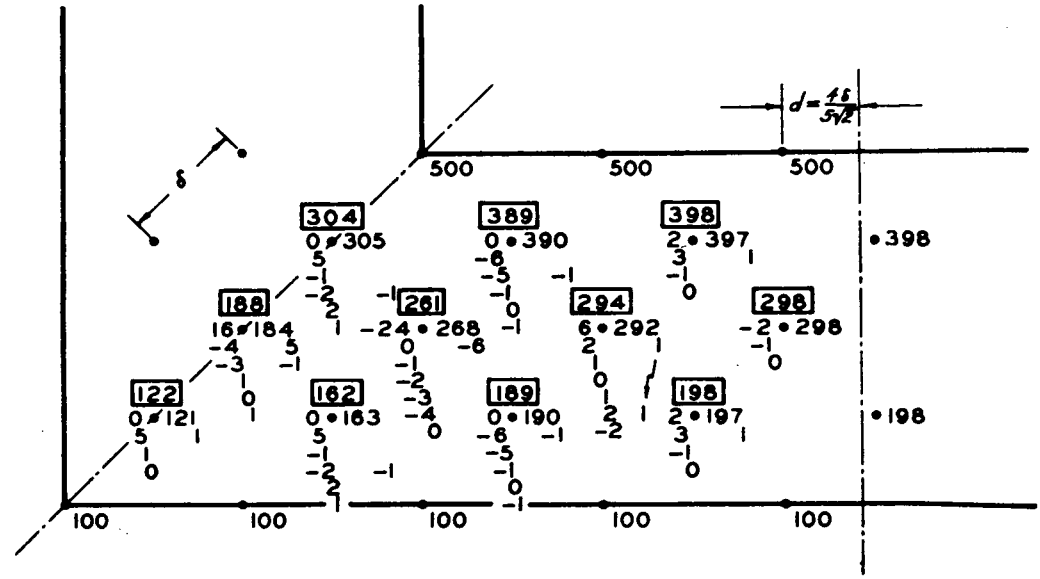

FIG. 5c. Start with solution of coarse net Fig. 3a. Relax Q' to get final temperature. Heat transferred $Q=kb[195/2+2\times111+2\times102+(4/5)102]=606\ kb$. Thermal Resistance $R=\Delta T/8Q=400/(8\times606kb)=.0825/kb$.

(From *The numerical solution of heat conduction problems*. By H. W. Emmons, Trans. A.S.M.E., **65**, 607–615 (1943)).

corresponding error. This same problem took 11 hours when solved by the averaging process.

Another problem, still more difficult, is illustrated in Fig. 6. This is the problem of the water tube of a boiler, and the boundary conditions are varied. The water inside the tube is assumed to maintain a constant wall temperature $T=0$. The lower half of the tube is embedded in insulating material, so the normal gradient of tempera-

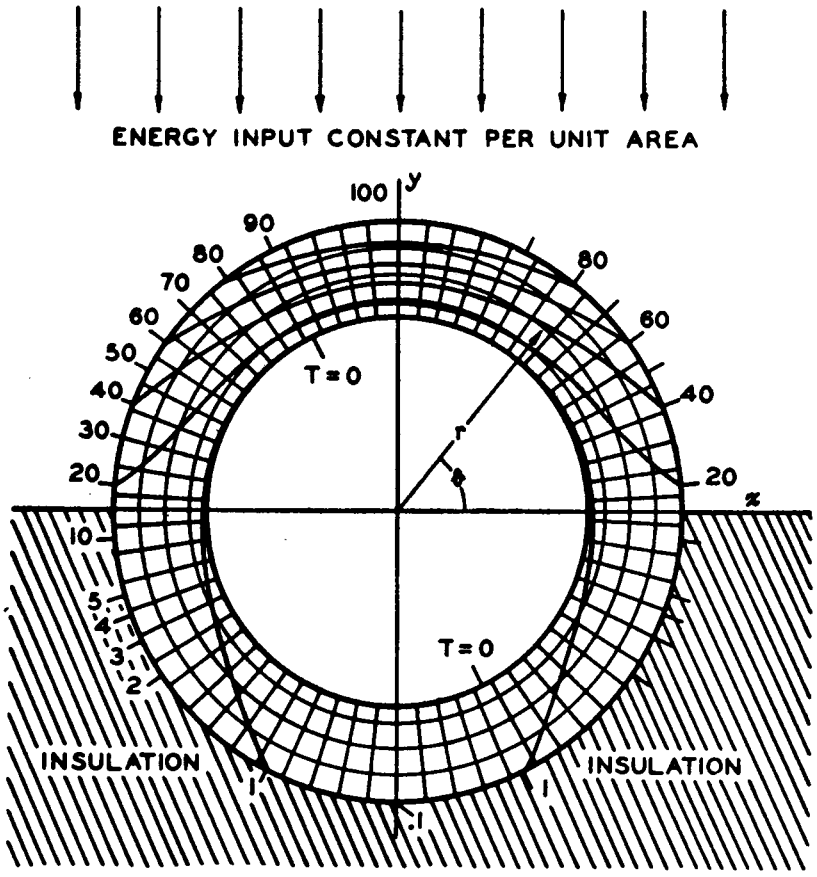

FIG. 6. Heat transfer by radiation through furnace water wall tube.

ture is there assumed to be zero. On the upper half of the tube heat is transferred by radiation in such a way that the heat input is constant per unit projected area. The normal gradient of temperature is thus proportional to sin θ. This problem was easiest to solve by a transformation from the x, y-plane to the z, θ-plane, where $\theta=\tan^{-1}y/x$, $z=\log r=\frac{1}{2}\log(x^2+y^2)$. The net points used are indicated in the x, y-plane by the intersection of the radial lines and circles. The numerical solution checked "exactly" with the analytical solution, as observed by superposing the graphed results. From an engineering point of view this solution is "exact," since it deviates much less from the analytical solution than the uncertainty of the boundary conditions.

So far we have dealt at length with the relation between the well known averaging process and the relaxation process. Indeed, it may appear that we have wasted time with trivialities of arithmetic but the author's conversations with many have indicated that it is just these minor details in point of view that make the relaxation process about five times as rapid as the iteration processes. To appreciate fully the power of the flexibility of the relaxation method, one must take pencil and paper and carry out the numerical process in all its uninteresting details. In fact, for the computer (as opposed to those who think only about the logic behind the computation methods) the relaxation method has a spirit lacking entirely from the iteration processes. The former challenges one's intellect at each step to make the best possible guess, while the latter reduces one to the status of an automatic computing machine (without the advantage of no computational errors). It should not be inferred that the relaxation process *requires* high intellectual powers. If changes are chosen in a specifiable way it reduces exactly to the iteration process. The computer can then vary from this completely specified process by whatever amount fits his own skill.

3. Other types of equations solved by the relaxation method. After the essential idea of the relaxation method is grasped, other problems may be solved by rather obvious steps. The differential equation is converted into a difference equation, some quantity (to be zero for the solution) is chosen as the residual, and the relaxation pattern is set up. Changes which make the residuals smaller are then made. We notice that no question of convergence can occur in the general relaxation process, since no specific instructions are given. If, after some steps, the residuals get worse, the intelligent computer goes back and makes changes in the opposite direction. These remarks oversimplify the problem somewhat because of two facts; first, the computer may become confused as to whether or not the residuals are really better, and secondly there is always a question of whether or not a solution with zero residuals exists (see [8]).

The following is a partial list of types of problems solved, with some brief details of their solution.

A. Poisson's equation,

$$\Delta\varphi = \varphi_{xx} + \varphi_{yy} = f(x, y). \tag{15}$$

(Subscripts denote partial differentiation.) In finite difference form, for a square net of points the approximating difference equation is (see Fig. 1 for notation)

$$\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 - 4\varphi_0 - f(x, y)\delta x^2 = Q, \tag{16}$$

where the residual Q is to be zero at each net point. Since $f(x, y)$ has a known numerical value at each net point, it enters into the value of the residuals at the start of the computation, and in no way affects the relaxation pattern used for Laplace's equation, Fig. 3. Thus the solution of Poisson's equation is precisely as easy as that of Laplace's.

B. Biharmonic equation,

$$\Delta\Delta w = \frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial^2 x \partial^2 y} + \frac{\partial^4 w}{\partial y^4} = 0. \tag{17}$$

The finite difference equation for a square net of points is most simply derived by converting the two Δ forms separately. Thus

$$\Delta w_1 + \Delta w_2 + \Delta w_3 + \Delta w_4 - 4\Delta w_0 = 0 \tag{18}$$

from which, by expansion of the remaining Δ's, one obtains the relaxation pattern shown on Fig. 7, where the solution of a problem of the deflection of a plate with clamped deflected edges is also shown. The accuracy of the present solution cannot be

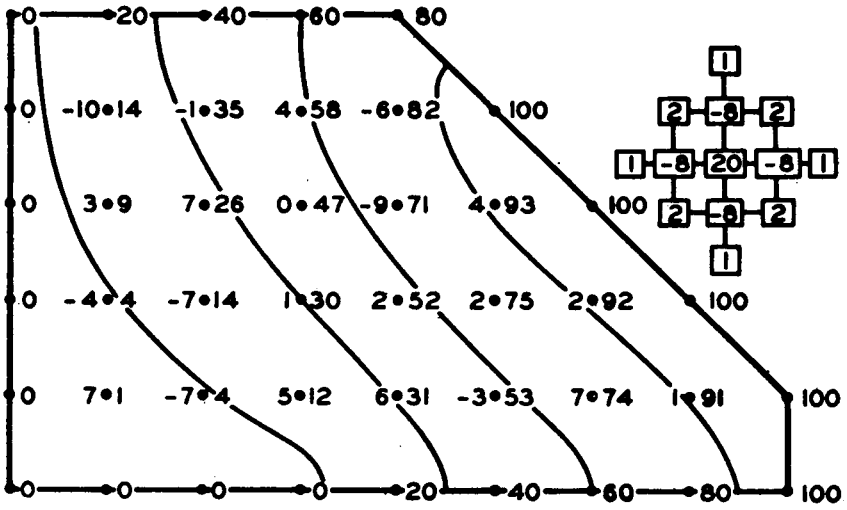

FIG. 7. Deflection of a plate with clamped deflected edges. No load. $\Delta\Delta w=0$, w given at edge, $w_n=0$ at edge. Recorded: $\Delta\Delta w \cdot w$. Solution time: 12 hours. 1st solution: 6 hours. Checking and eliminating mistakes: 6 hours.

computed, since the exact solution of the differential equation is not known for these boundary conditions. However, one's confidence in its precision is increased by observing that the finite difference solution of a square plate under uniform load is in error less than 1% in maximum deflection when only 9 interior points are used [11], rather than the present 22.

C. The equation of natural modes of a membrane,

$$\Delta w + \lambda w = 0. \tag{19}$$

In problems of the vibration of two-dimensional systems, the information sought concerns the natural frequencies and the characteristic functions. In the equation, it is required to find the permissible values of λ, which is the square of the frequency times certain physical constants. The relaxation method can be applied to this problem in several ways. For example, a value of λ could be estimated by Rayleigh's principle from an assumed deflection w,

$$\lambda = \frac{\sum (w_x^2 + w_y^2)}{\sum w^2}, \tag{20}$$

where the summation extends over all the net points. This value could then be inserted into Eq. (19) in the finite difference form,

$$w_1 + w_2 + w_3 + w_4 - (4 - \lambda\delta^2)w_0 = Q, \tag{21}$$

and the relaxation process used to reduce the Q. Periodically during the solution λ would be reevaluated and reinserted into Eq. (21). Thus new residuals would appear, which could again be reduced, the process being repeated until Eqs. (20) and (21) are satisfied with sufficient accuracy. This method of solution, as used by Southwell [12], imagines that at each net point there is a force Q applied and that these forces are to be removed by changes of the amplitudes w, and the "frequency" λ.

Another method which seems to be superior in some respects has been worked out by Dr. A. Vazsonyi, and will be published shortly.* Equation (19) is written in the form

$$\lambda = -\frac{\Delta w}{w}. \tag{22}$$

At each net point a value of the amplitude w is assumed and the corresponding values

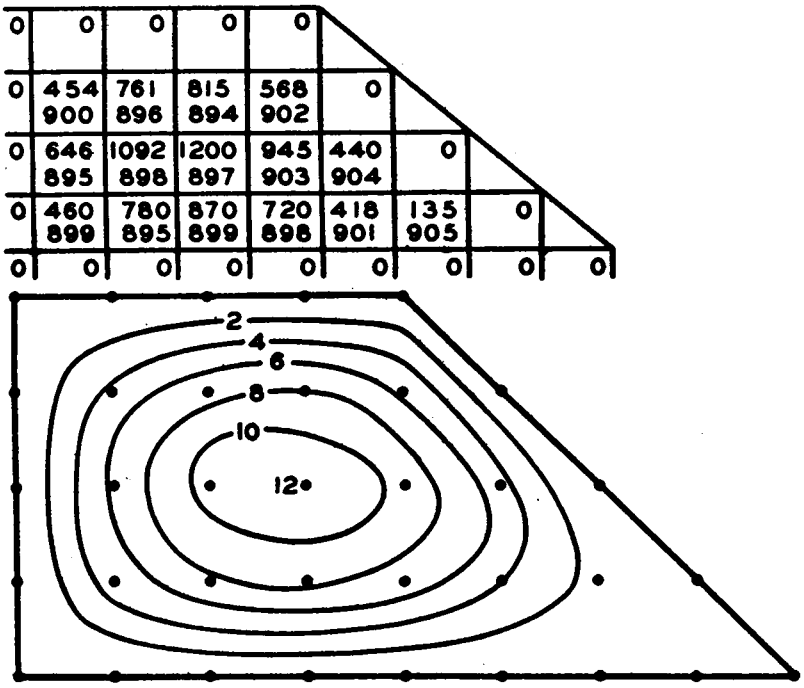

FIG. 8. Lowest frequency and natural mode of quadrangular membrane. Second approximation.

$$\Delta v + \frac{\lambda}{a} v = 0, \qquad v = 0 \text{ on edges},$$

$$\delta = 1, \qquad a = 24, \qquad \lambda_m^* = \frac{\Sigma v \Delta v}{\Sigma v^2} = .8984, \qquad \lambda = \frac{a}{\delta^2}\lambda_m^*.$$

First characteristic value: $\lambda = 21.562$.

Recorded: $\left\{\begin{array}{c} v \\ \hline -1000\dfrac{\Delta v}{v} \ (= 1000\lambda^*) \end{array}\right\}$.

* J. Appl. Physics, 15, 598–606 (1944).

of Δw and λ recorded. These assumed amplitudes constitute the solution to the problem of the vibration of a membrane of non uniform mass distribution. For a uniform mass distribution the value of λ should be the same at each point. The values of w are changed to equalize the values of λ. Whenever a w is changed the changes of Δw are immediately computed by the Laplace relaxation pattern, Fig. 3. New values of λ need not be computed *every* time. At any stage of the solution the correct λ lies between the highest and lowest value at the net points. Finally a value of λ is computed as an average of those at all the net points. The best theoretical way to compute this "average" is by the use of Eq. (20) in the form

$$\lambda = \frac{\sum w\Delta w}{\sum w^2}. \tag{23}$$

Fig. 8 shows the solution for the lowest frequency of a quadrangular membrane.

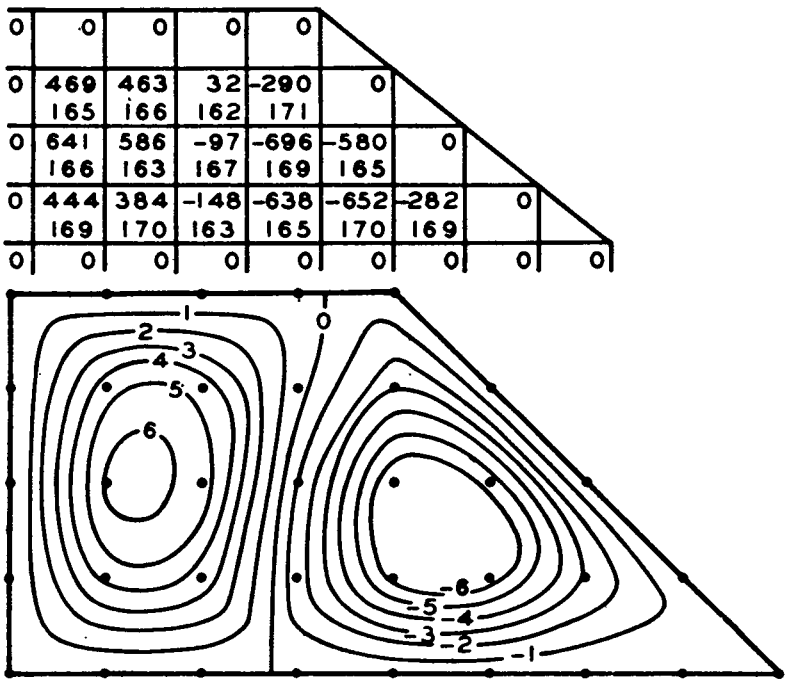

FIG. 9. Second frequency and natural mode of quadrangular membrane. Second approximation.

$$\Delta v + \frac{\lambda}{a} v = 0, \qquad v = 0 \text{ on edges}$$

$$\delta = 1, \qquad a = 24, \qquad \lambda_m^* = \frac{\Sigma v \Delta v}{\Sigma v^2} = 1.6716, \qquad \lambda = \frac{a}{\delta^2}\lambda_m^*.$$

Second characteristic value $\lambda = 40.118$.

Recorded:

$$\left\{\begin{array}{c} v \\ \hline -100\dfrac{\Delta v}{v} \; (= 100\lambda^*) \end{array}\right\}.$$

To compute higher modes, the procedure is exactly the same as outlined above for the lowest mode. The assumed amplitudes should of course include one or more nodal lines, depending upon the mode sought. All of one's information about the vibration of membranes should be used in selecting the amplitudes, and generally it is convenient to make use of the orthogonal properties of the characteristic functions. Thus, it is always possible to use any arbitrary amplitudes w and remove the portion of these arising from the first mode w_1 (obtained at the end of the first mode computation), to obtain the amplitudes w_n of the higher modes only;

$$w_n = w - aw_1, \tag{24}$$

where

$$a = \frac{\sum w_1 w}{\sum w_1^2}. \tag{25}$$

The second mode of the quadrangular membrane is shown in Fig. 9, while Table III compares the results. We note the good accuracy obtained with a few points in the square. By the use of symmetry, the 9 point approximation for the square required only 10 minutes.

TABLE III

Characteristic values of membranes

Characterization of problem	Square membrane; lowest mode			Membrane accord. to Figs. 8, 9; lowest mode		second mode	
Order of approximation	first	second	exact	first	second	first	second
Degrees of freedom of approx. system	9	49	—	7	15	7	15
Characteristic value	18.75	19.508	$2\pi^2 =$ 19.739	20.61	21.562	37.10	40.118
Number of modifications used	5	35	—	12	35	16	65
Accuracy	5%	1.2%					

It should be noted that the solution of a forced vibration problem is best carried out by the first method outlined above.

D. Equations of the type,

$$\varphi_{xx} + f(x, y, \varphi, \varphi_x, \varphi_y)\varphi_{yy} = g(x, y, \varphi, \varphi_x, \varphi_y). \tag{26}$$

Equations of this type are of frequent occurrence in engineering problems, but because of their non-linear aspects only limited assistance is offered by conventional mathematical methods. For so complicated an equation, there are many possible ways of applying the relaxation process. Only one will be mentioned here. We obtain a finite difference equation by transforming the second derivatives only;

$$\varphi_1 + \varphi_2 + f_0\varphi_3 + f_0\varphi_4 - 2(1 + f_0)\varphi_0 - g_0\delta^2 = Q, \tag{27}$$

for which the relaxation pattern is shown in Fig. 10. Thus the pattern is different

for each net point and varies during the course of solution. The luxury of an investigation of the classification of equations of the type of (26) (as to whether they are elliptic, parabolic, or hyperbolic, if such a classification is possible), of the nature of solutions, of the permissible boundary conditions, etc., is denied during war time by the urgency to get numerical results. Only casual observations have been made to date, and will not be discussed. However, it is certain that for $f>0$ [Eq. (26) of elliptic type] the process of solution described is quite easy and quick to carry out.

The domain of the problem to be solved is drawn and a solution is guessed at a net of points. From this set of values of ϕ, f_0 and g_0 and then the Q are computed at each point. By the relaxation process carried out exactly as described for Laplace's equation, except that the influence coefficients of Fig. 10 are used instead of $(-4, 1, 1, 1, 1)$, the residuals Q are reduced somewhat. Before bothering to eliminate the Q completely, we compute new values of f_0 and g_0 and hence corrected values of the residuals attached to each net point. This process of reduction and correction is continued until sufficient accuracy has been attained. Just as in previous problems, a finer net can be added for greater accuracy.

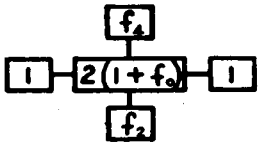

FIG. 10.

As an illustration, let us consider the distribution of electric potential in the space between two parallel planes, one of which has, standing normal to it, a right circular, cylindrical post with a hemispherical top. Fig. 11 shows the problem and the solution.

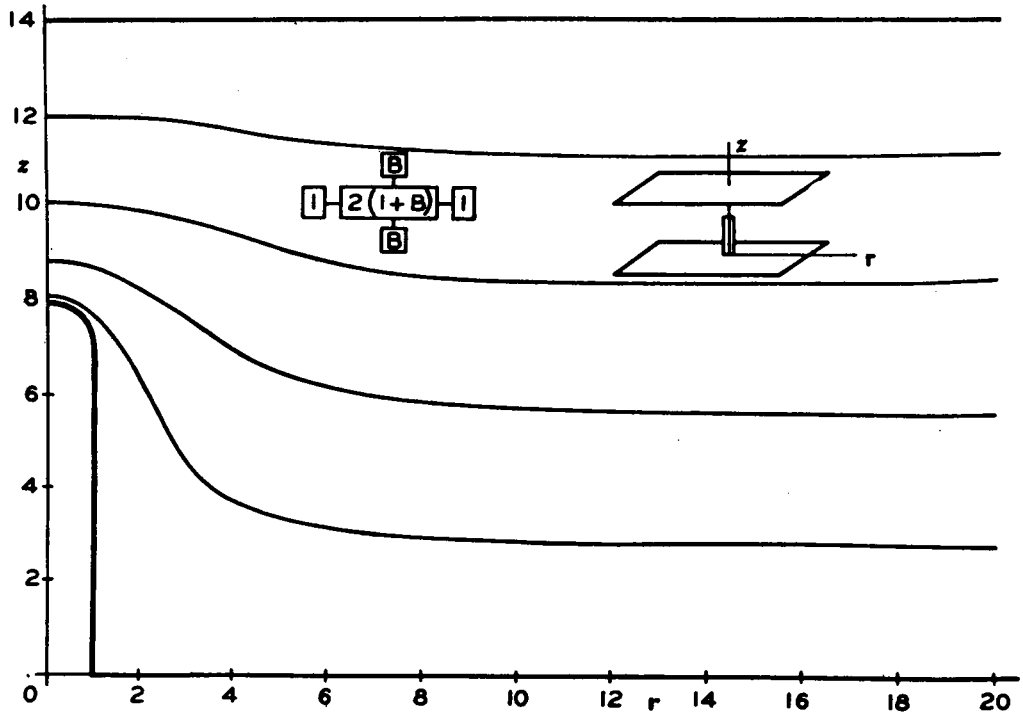

FIG. 11. Axially symmetric electric potential distribution.

$$\frac{\partial^2\varphi}{\partial z^2}+\frac{1}{r^2}\frac{\partial^2\varphi}{\partial y^2}=0, \quad \text{where } y=\log r, \varphi \text{ constant on boundaries.}$$

If cylindrical coordinates are used with the z-axis along the axis of the post and the origin at the base of the post, the potential is described by the equation

$$\frac{\partial^2\phi}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) = 0 \tag{28}$$

with the boundary conditions $\varphi = 0$ on the base plane and post, and $\varphi = 100$ on the parallel plane. Again, several alternative procedures are possible. For example, Eq. (28) can be written

$$\varphi_{zz} + \varphi_{rr} = -\frac{1}{r}\varphi_r, \tag{29}$$

and the solution can be carried out as described above for the general case.

In the case of the present problem it was decided to make the substitution

$$y = \log r, \tag{30}$$

thus converting Eq. (28) into

$$\varphi_{zz} + \frac{1}{r^2}\varphi_{yy} = 0. \tag{31}$$

Hence the finite difference equation becomes

$$\varphi_1 + \varphi_3 + B(\varphi_2 + \varphi_4) - 2(1 + B)\varphi_0 = Q, \tag{32}$$

where $B = \delta z^2/r^2\delta y^2$, δy and δz being the net spacing in the y and z directions, respectively. The net spacing was chosen so that $\delta z = 4\delta y$. In this way sufficient points appear where they are needed, i.e., near the post. In Fig. 12 the transformed plane is shown, together with the influence coefficients at the top of each column of points.

E. Equations of the type,

$$\frac{\partial}{\partial x}(\mu\varphi_x) + \frac{\partial}{\partial y}(\mu\varphi_y) = g(x, y, \varphi, \varphi_x, \varphi_y), \tag{33}$$

where $\mu = \mu(x, y, \varphi, \varphi_x, \varphi_y)$. This equation is of very general occurrence in physical problems. For example, in the case of a soap film with large deflections w under constant excess pressure p, we have [14]

$$\varphi = w, \qquad \mu = \frac{1}{\{1 + w_x^2 + w_y^2\}^{1/2}}, \qquad g = \frac{p}{2} = \text{constant}; \tag{34}$$

in the case of the plane irrotational flow of a compressible fluid with velocity potential φ, we have

$$\mu = \rho = \text{density} = \rho_0\left\{1 - \frac{\gamma - 1}{1}\frac{v^2}{a_0^2}\right\}^{\frac{1}{\gamma - 1}}, \qquad g = 0; \tag{35}$$

in the case of a steady magnetic field (φ = magnetic potential) in a non current carrying medium, we have

$$\mu = \text{magnetic permeability} = f(\varphi_x^2 + \varphi_y^2), \tag{36}$$

where the function f is given by experimental data on the material.

The flow of oil in the film lubrication of a bearing, when the viscosity is assumed independent of temperature, is described by an equation of this type [see Eq. (39)]. In general, physical problems which lead to Laplace's equation when physical parameters are constant, give equations similar to (33) when the physical properties vary. The properties may be known functions of x, y because the material is non-homogeneous, or of φ, φ_x, φ_y, etc., because of the nature of the material itself.

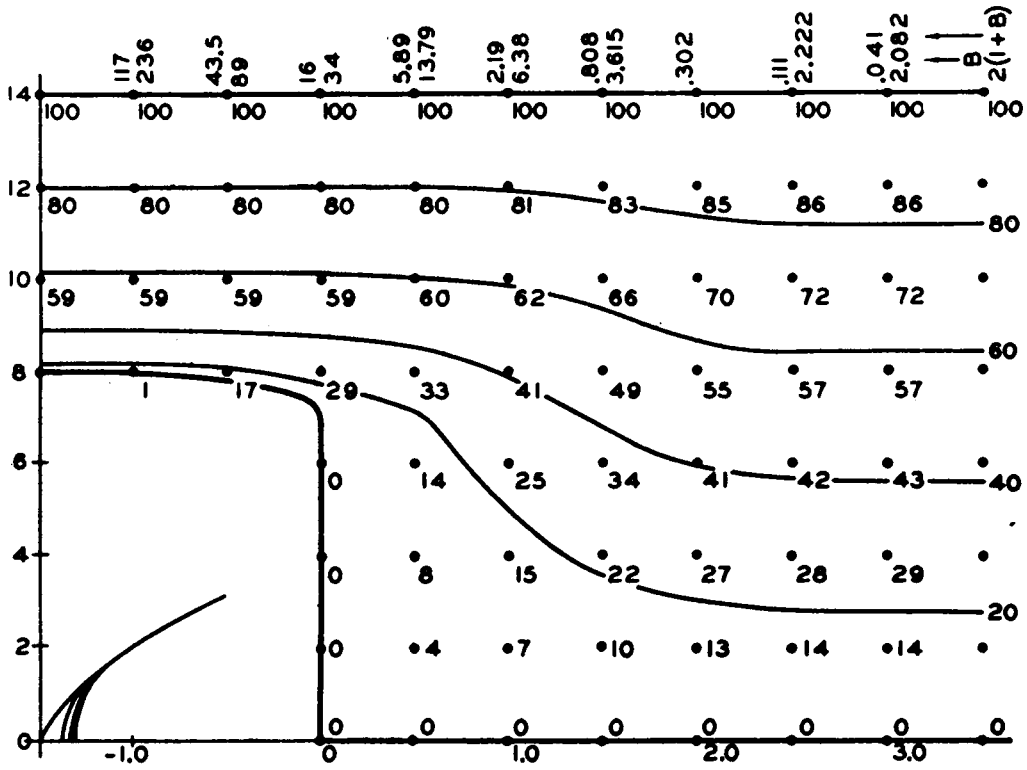

FIG. 12. Transformed y, z-plane of axially symmetric electric potential problem. Recorded: φ.

For the numerical solution of these problems it has been found convenient to use the form

$$\varphi_{xx} + \varphi_{yy} = g - (\log \mu)_x \varphi_x - (\log \mu)_y \varphi_y. \tag{37}$$

In finite differences this becomes

$$\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 - 4\varphi_0 = g_0\delta^2 + \tfrac{1}{4}\{(\varphi_1 - \varphi_3)(\log \mu_3 - \log \mu_1) + (\varphi_4 - \varphi_2)(\log \mu_2 - \log \mu_4)\} + Q. \tag{38}$$

The relaxation pattern used is that used for Laplace's equation. The Q is to be eliminated. The variable terms on the right of Eq. (38) are computed periodically as corrections. This method is well adapted to the equation as long as Eq. (33) remains of elliptic type. The "correction" terms on the right contain φ_{xx} and φ_{yy} through the relation (34). When these variations are such that Eq. (33) becomes hyperbolic, the

relaxation process becomes confusing unless a great deal of physical knowledge is available to assist the computer.

The solution of a problem of this kind is shown on Fig. 13, which shows the shape of a soap film with large deflections (Eq. 34). The maximum deflection is 100 units,

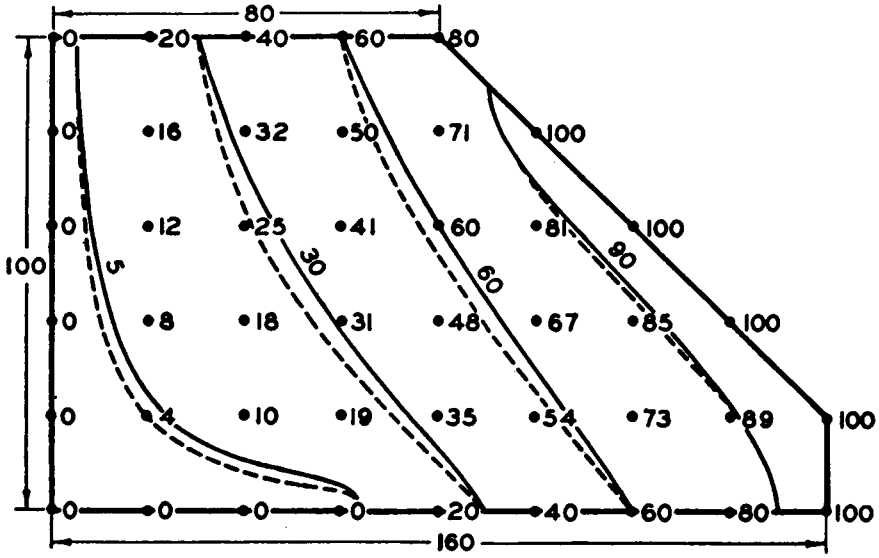

FIG. 13. Soap film with large deflections.

$$\frac{\partial}{\partial x}(\mu w_x) + \frac{\partial}{\partial y}(\mu w_y) = 0, \quad \text{where} \quad \mu = \frac{1}{(1 + w_x^2 + w_y^2)^{1/2}}.$$

Recorded: deflection w. Time to correct small deflection solution (shown by broken lines): 3 hrs.

compared with the largest dimension of 160 units for the xy projection of the boundary. We note the deflection as given by Laplace's equation and show it for comparison by dotted contours. In solving this problem, the Laplace equation solution was taken as the first approximation. Correction terms had to be computed only twice. This solution required 3 hours.

F. A more general type of equation which arises in the flow of oil in a bearing has been solved by Christopherson [13]. He solves the system of equations

$$\frac{\partial}{\partial \xi}\left[\frac{H^3}{M}\frac{\partial P}{\partial \xi}\right] + \frac{\partial}{\partial \eta}\left[\frac{H^3}{M}\frac{\partial P}{\partial \eta}\right] = \frac{\partial H}{\partial \xi}, \tag{39}$$

$$\frac{\partial T}{\partial \xi}\left[1 - \frac{H^2}{M}\frac{\partial P}{\partial \xi}\right] + \frac{\partial T}{\partial \eta}\left[1 - \frac{H^2}{M}\frac{\partial P}{\partial \eta}\right] = A\left[\frac{M}{H^2} + 3\frac{\partial P}{\partial \xi}\right], \tag{40}$$

where H is a given function of ξ, η and M is a given function of T, P; P is the pressure in the lubricant, T is the temperature of the lubricant.

G. Another very general system of non linear, integral, differential, difference equations arises in the problem of the thermal equilibrium of a nest of N coaxial, conducting cylinders of length $2b$, between which a hot gas flows. The temperature T of the cylinders varies with the number n of the cylinder and with the axial position z of the point under consideration. The temperature of the gas varies with the number n of the stream, and the position x. The equations to be solved are

$$f_1(x, n)T^4(x, n) + a\int_{-b}^{b}\{T^4(\gamma, n+1)f_2(x, n, \gamma) + T^4(\gamma, n-1)f_3(x, n, \gamma)\}d\gamma + b\{T_s(x, n) + T_s(x, n-1) - 2T(x, n)\} + c\frac{\partial^2 T(x, n)}{\partial x^2} = 0, \quad (41)$$

$$\frac{\partial T_s(x, n)}{\partial x} + \left\{T(x, n) + \frac{n+1}{n}T(x, n+1) - \frac{2n+1}{n}T_s(x, n)\right\}d = 0, \quad (42)$$

where a, b, c, d are constants

f_1, f_2, f_3 are known functions of the variables indicated. The solution of these equations, which required about 30 hours, is shown in Fig. 14. The temperature contours show only the distribution of cylinder temperatures. This analysis was actually used as the basis for redesign of the instrument involved.

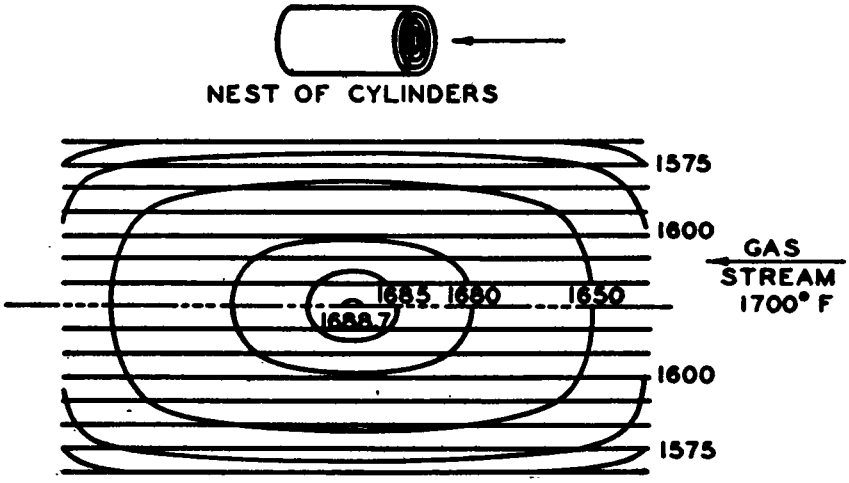

FIG. 14. Thermal equilibrium of a nest of cylinders in a hot gas stream including radiation, conduction and convection. Gas temperature: 1700° F abs. Surrounding duct temperature: 1500° F abs. Section of nest of cylinders showing cylinder temperature distribution.

4. General remarks on finite difference approximations. In this paper only the simplest approximations for rectangular nets of points have been mentioned. There is no reason why higher order approximations cannot be used, or why triangular or other point arrangements should not be considered. Indeed both of these possibilities have been used [9, 10, 13]. There is no sure way of deciding at the present time just

what net should be used with a given problem. The ultimate requirement is a sufficiently accurate answer in the shortest possible time. The author's experience to date favors the simplest possible formula, the simplest possible relaxation pattern, with accuracy obtained by extra points on a finer net where needed. Southwell and Christopherson [9, 10, 13] have occasionally found other formulas to be advantageous. Until a person is thoroughly acquainted with the details of the several processes of solution, the "bookkeeping," it is impossible to judge the relative merits in the matter of time required to reach a given accuracy.

This is particularly true in the matter of boundary conditions. Every new computer "discovers" the possibility of deriving special formulas to apply to points near (or on) the boundary of the domain, especially when the boundary itself wanders among the net points. In many cases such formulas are admirable from the point of view of accuracy for a given net size, but fail miserably when compared with linear interpolation (or extrapolation) together with a somewhat finer net (used locally). By far the best general procedure for fitting "queer" boundaries or boundary conditions is to sketch a graph for the last few points approaching the boundary and thus compute the boundary point graphically.

5. Conclusions. This paper stresses particularly the practical aspects. The finite difference approximations to partial differential equations are well known; only the detailed steps in carrying out the solutions are not yet general knowledge, and these are discussed step by step. The relaxation method, first conceived by R. V. Southwell, is the underlying process in the solution of most of the problems discussed. Once the basic idea is grasped by actually solving a problem, it is capable of enormous extension to a great many kinds of problems with only the most meagre knowledge of the current methods (if any) for solving them. From an engineering point of view this is of enormous importance, because the practical man would like to do something better than guess, and yet he cannot afford the time required to become versed in analytical procedures, procedures which too often cannot supply a numerical answer to the real physical problem with reasonable accuracy and speed.

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